UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA
(DOCTORAL THESIS)

O EKSTREMNIH GRAFIH Z DANO STOPNJO IN PREMEROM/OŽINO

(ON EXTREMAL GRAPHS OF GIVEN DEGREE AND DIAMETER/GIRTH)

SLOBODAN FILIPOVSKI

KOPER, 2018

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Abstract

This PhD thesis consists of three interrelated parts: Cage problem, Degree/Diameter problem for undirected and Degree/Diameter problem for directed graphs. The central common theme among the three parts is the study of cages. We address the problem of cages improving the lower bounds on the order of the cages of even girth, proving the non-existence of almost all potential antipodal cages of even girth and small excess and proving that the excess of almost all vertex-transitive cages is arbitrary large.

In the thesis we present a connection of a question of Bermond and Bollobás concerning the degree/diameter problem for undirected graphs and the class of Ramanujan graphs proving that the negative answer to the question of Bermond and Bollobás would imply a positive answer to the open question whether infinitely many k-regular non-bipartite Ramanujan graphs exist for any degree k.

Considering the degree/diameter problem for directed graphs we prove that the largest possible order of the (d, k)-digraphs of given maximum out-degree d and diameter k $n_{d,k}$ is a monotonically increasing function in d and in k. Finally, we prove the non-existence of infinitely many families of (d, k, δ)-digraphs containing only selfrepeat vertices.

Math. Subj. Class (2010): 05C12, 05C20, 05C35, 05C38, 05C50

Keywords: adjacency matrix, antipodal graphs, cages, excess, defect, Ramanujan graphs, selfrepeats, degree/diameter problem, spectrum, Moore graphs, asymptotic density, distance matrices, Bermond and Bollobás problem

Povzetek

Doktorska disertacija je sestavljena iz treh povezanih delov. V prvem delu obravnavamo preblem kletk, v drugem problem stopnje in premera za neusmerjene grafe, ter v tretjem problem stopnje in premera za usmerjene grafe. Glavna skupna tema teh treh problemov je proučevanje kletk. V disertacij izboljšamo spodnjo mejo za red kletk sode ožine in dokažemo neobstoj skoraj vseh potencialnih antipodnih kletk sode ožine. Dokažemo, da je presežek skoraj vseh vozliščno-tranzitivnih kletk poljubno velik.

V disertacij predstavimo povezavo med vprašanjem Bermonda in Bollobása, glede problema stopnje in premera za neusmerjene grafe, in razredom Ramanujan-ovih grafov. S tem dokažemo, da bi negativen odgovor na vprašanje Bermonda in Bollobása impliciral pozitiven odgovor na odprto vprašanje, ali obstaja neskončno mnogo k-regularnih ne-dvodelnih Ramanujan-ovih grafov za poljubno stopnjo k.

Glede problema stopnje in premera za usmerjene grafe pa dokažemo, da je največji možen red (d, k)-digrafa dane maksimalne izhodne stopnje d in premera k (ki ga označimo z $n_{d,k}$) monotono naraščajoča funkcija spremenljivk d in k. Za konec pa dokažemo še neobstoj neskončno mnogo družin (d, k, δ)-digrafov, ki bi vsebovali samoponavljajoča vozlišča.

Math. Subj. Class (2010): 05C12, 05C20, 05C35, 05C38, 05C50

Ključne besede: matrika sosednosti, antipodni grafi, kletke, presežek, defekt, Ramanujan-ovih grafi, samoponavljanje, problem stopnje in premera, spekter, Moorovi grafi, asimptotična gostota, razdaljne matrike, problem Bermonda in Bollobása

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Chapter 1

Introduction

The topology of a network (such as a telecommunication, multiprocessor, or local area network, to name just a few) is usually modelled by a graph in which vertices represent 'the nodes' (stations or processors) while undirected or directed edges stand for 'links' or other types of connections, there are a number of features that must be taken into account. The most common ones, seem to be limitations on the vertex degrees and on the diameter. The network interpretation of these two parameters is obvious: The degree of a vertex is the number of the connections attached to a node, while the diameter indicates the largest number of links that must be traversed in order to transmit a message between any two nodes. What is then the largest number of nodes in a network with a limited degree and diameter? If links are modelled by undirected edges, this leads to the

- Degree/Diameter Problem: Given natural numbers k and d, find the largest possible number of vertices $n(k, d)$ in a graph of maximum degree k and diameter d.

The statement of the directed version of the problem differs only in the degree is replaced by out-degree. We recall that the out-degree of a vertex in a digraph is the number of directed edges leaving the vertex. We thus arrive at the

- (Directed) Degree/Diameter Problem: Given natural numbers d and k, find the largest possible number of vertices $n_{d,k}$ in a digraph of maximum out-degree d and diameter k.

The problem known as the Cage problem or the Degree/Girth problem, is closely related to the Degree/Diameter problem.

- Cage Problem (Degree/Girth Problem): Given natural numbers k and g, find the smallest possible number of vertices $n(k, g)$ in a graph of degree k and girth g.

In the PhD thesis we focus on open questions and problems which concern the Cage problem and the Degree/Diameter problem for undirected and directed graphs.

The well-known Moore bound $M(k, g)$ serves as a universal lower bound for the order of k-regular graphs of girth g. The excess e of a k-regular graph G of girth g and order n is the difference between its order n and the corresponding Moore bound, $e = n - M(k, g)$.

In Chapter 3 we present a number of formulas for counting cycles of lengths close to the girth in k-regular graphs of girth g and excess not exceeding 3. Based on these formulas, we attempt to exclude the existence of graphs with small excess for infinite families of degree-girth pairs. Perhaps surprisingly, we observe that counting cycles does not exclude too many families, an observation made previously in the setting of strongly regular graphs by Vašek Chvátal, [22]. Based on the same methodology in Chapter 4 we find infinite families of parameters (k, g), $g > 6$ and even, for which we show that the excess of any k-regular graph of girth g is larger than 4, see Theorem 4.7. This yields new improved lower bounds on the order of k-regular graphs of girth g of smallest possible order; the so-called (k, g)-cages. We also show that the excess of k-regular graphs of girth g can be arbitrarily large for a restricted family of (k, g)-graphs satisfying an additional structural property and large enough k and g, see Theorem 4.10.

In Chapter 5 we consider the existence of (k, g)-bipartite graphs of excess 4 by studying spectral properties of their adjacency matrices. We observe that the eigenvalues other than $\pm k$ of these graphs are roots of the polynomials $H_{d-1}(x) + \lambda$, where λ is an eigenvalue of E, $E = A_{d+1}$ is the adjacency matrix of a union of vertex-disjoint cycles, and $H_{d-1}(x)$ is the Dickson polynomial of the second kind with parameter $k-1$ and degree $d-1$, see Theorem 5.3. Based on the irreducibility of $H_{d-1}(x) \pm 2$, we give necessary conditions for the existence of these graphs, see Theorem 5.6. If E is the adjacency matrix of a cycle of order n, we call the corresponding graphs *graphs with cyclic excess*; if E is the adjacency matrix of a disjoint union of two cycles, we call the corresponding graphs *graphs with bicyclic excess*. Related to this, we prove the non-existence of (k, g)-graphs with cyclic excess 4 if $k \geq 6$ and $k \equiv 1 \pmod 3$, $g = 8, 12, 16$ or $k \equiv 2 \pmod 3$, $g = 8$; and the non-existence of (k, g)-graphs with bicyclic excess 4 if $k \geq 7$ is an odd number and $g = 2d$ such that $d \geq 4$ is even, see Theorems 5.9 and 5.11.

Biggs and Ito in [15] proved that any (k, g)-cage of even girth $g = 2d \geq 6$ and excess $e \leq k - 2$ is a bipartite graph of diameter $d + 1$. It is known that some of these cages are potential antipodal graphs. Based on specral analysis, in Chapter 6 we prove the non-existence of antipodal (k, g)-cages of excess e, for $k \geq e + 2 \geq 6$ and $g = 2d \geq 8$, see Theorem 6.12.

Vertex-transitive graphs constitute a significant part of the known cages and the smallest known (k, g)-graphs. The role of vertex-transitivity in the Cage Problem is still poorly understood, and in some cases the order of the smallest vertex-transitive (k, g)-graph exceeds the order of the smallest (k, g)-graph by a significant amount (for example, while the order of the smallest $(3, 11)$-graph is 112, the order of the smallest vertex-transitive $(3, 11)$-graph is 192 [69]). In Chapter 7 we consider a restriction of the Cage Problem to the class of vertex-transitive graphs. Counting cycles to obtain necessary arithmetic conditions on the parameters (k, g), we extend previous results of Biggs, Theorem 7.1, and prove that, for any given excess e and any given degree $k \geq 3$, the asymptotic density of the set of girths g for which there exists a vertex-transitive (k, g)-cage with excess not exceeding e is 0, see Theorems 7.6 and 7.12.

In the second part of the PhD thesis we investigate in the degree/diameter problem for undirected graphs. If we let $n(k, d)$ denote the order of the largest undirected graphs of maximum degree k and diameter d, and let $M(k, d)$ denote the corresponding Moore bound, then $n(k,d) \leq M(k,d)$, for all $k \geq 3, d \geq 2$. While the inequality has been proved strict for all but very few pairs k and d, the exact relation between the values $n(k,d)$ and $M(k,d)$ is unknown, and the uncertainty of the situation is captured by an open question of Bermond and Bollobás given in [11] who asked whether it is true that for any positive integer $c > 0$ there exist a pair k and d, such that $n(k,d) \leq M(k,d) - c$. In Chapter 8 we present a connection of this question to the value $2\sqrt{k-1}$, which is also essential in the definition of the Ramanujan graphs defined as k-regular graphs whose second largest eigenvalue (in modulus) does not exceed $2\sqrt{k-1}$. We further reinforce this surprising connection by showing that if the answer to the question of Bermond and Bollobás were negative and there existed a $c > 0$ such that $n(k,d) \geq M(k,d) - c$, for all $k \geq 3, d \geq 2$, then, for any fixed k and all sufficiently large even d's, the largest undirected graphs of degree k and diameter d would have to be Ramanujan graphs, see Theorem 8.3. This would imply a positive answer to the open question whether infinitely many k-regular non-bipartite Ramanujan graphs exist for any degree k.

In the last part of the thesis we consider digraphs. Let $n_{d,k}$ be the largest order of a directed graph (digraph) with given maximum out-degree d and diameter k. In Chapter 9 we give a positive answer to the open question concerning the degree/diameter problem for digraphs asked in [60]: is $n_{d,k}$ monotonic in d and k?

In Chapter 10 we generalize the concept of a selfrepeat vertex in order to be used in the study of the existence of (d,k,δ)-digraphs, with $\delta \geq 2$. We derive a formula for calculating multiplicities of the eigenvalues of (d,k,δ)-digraphs containing only selfrepeat vertices and we prove the non-existence of such digraphs for $d \geq \delta \geq k+1 \geq 4$, showing that some of their multiplicities are not integer numbers. Also, following the same methodology that we use for studying digraphs containing only selfrepeats, we give another proof of the well-known result concerning the non-existence of Moore digraphs; for $k > 1$ and $d > 1$, there exist no (d,k)-directed Moore graph; this result was obtained by Plesník and Znám in [68] and later by Bridges and Toueg in [16].

The results of this PhD thesis are published in the following articles:

- T. B. Jajcayova, S. Filipovski and R. Jajcay. Counting cycles in graphs with small excess. *Lecture Notes of Seminario Interdisciplinare di Matematica* Vol. 14 (2016) 17-36.

- T. B. Jajcayova, S. Filipovski and R. Jajcay. Improved lower bounds for the orders of even girth cages. *The Electronic Journal of Combinatorics* 23(3) (2016) #P3.55.

- S. Filipovski. On bipartite cages of excess 4. *The Electronic Journal of Combinatorics* 24(1) (2017) #P1.40.

- S. Filipovski. On the non-existence of antipodal cages of even girth. *Linear Algebra and its Applications* 546 (2018) 261-273.

- S. Filipovski and R. Jajcay. On the excess of vertex-transitive graphs of given degree and girth. *Discrete Mathematics* 341 (2018) 772-780.

- S. Filipovski and R. Jajcay. A connection between a question of Bermond and Bollobás and Ramanujan graphs. *Combinatorica*, submitted.

- S. Filipovski. A note on degree/diameter monotonicity of digraphs. *Australasian Journal of Combinatorics* Volume 70(1) (2018).
- S. Filipovski and R. Jajcay. On the non-existence of families of (d, k, δ)-digraphs containing only selfrepeat vertices. *Linear Algebra and its Applications* 563 (2019) 302-312.

Chapter 2

Background

2.1 Graphs

A *graph* is an ordered pair $G = (V, E)$, where V is a nonempty finite set, called the set of *vertices* of G, and E is a set of unordered pairs (2-element subsets) of V, called *the edges* of G. If $\{x, y\} \in E$, x and y are called *adjacent* and they are incident with the edge $\{x, y\}$. The *order* of a graph $G = (V, E)$ is $|V|$, the number of its vertices. The *size* of G is $|E|$, the number of its edges. The *degree* (or *valency*) of a vertex $x \in V$, denoted by $d(x)$, is the number of edges incident with it. A *walk* in a graph is an alternating sequence of vertices and edges $x_0e_1x_1e_2x_2 \dots x_{n-1}e_nx_n$ such that $x_0, x_1, \dots, x_n$ are vertices and e_i is an edge connecting x_{i-1} and x_i for each $i, 1 \leq i \leq n$. A *path* is a walk with no repeated vertices. A *tree* is an undirected graph in which any two vertices are connected by exactly one path. The *distance* between two vertices u and v in a connected graph G is denoted by $d_G(u, v)$ and defined as the minimum length (that is, number of edges) of a u, v-path in G. The maximum distance in G is called the *diameter* of G. A *cycle* is a closed walk with no repeated vertices other than the initial one and the final one, i.e., $x_i = x_j$ for $i < j$ if and only if $i = 0$ and $j = n$. The *girth* of a graph is the length of a shortest cycle contained in the graph. A *regular graph* is a graph where each vertex has the same number of neighbours; i.e., every vertex has the same degree or valency. A *bipartite graph* is a graph whose vertices can be divided into two disjoint and independent sets U and V such that every edge connects a vertex in U to one in V. Vertex sets U and V are usually called the partitions of the graph. A *complete graph* is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. Two graphs X and Y are *isomorphic* if there is a bijection ϕ from $V(X)$ to $V(Y)$ such that x is adjacent to y if and only if $\phi(x)$ is adjacent to $\phi(y)$ in Y. We say that ϕ is an *isomorphism* from X to Y. A *subgraph* of a graph X is a graph Y such that $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$. If $V(Y) = V(X)$, we say that Y is a *spanning subgraph* of X. An *induced subgraph* of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in the subset. An *automorphism* of a graph is a graph isomorphism with itself, i.e., a mapping from the vertices of the given graph G back to vertices of G such that the resulting graph is isomorphic to G. The set of automorphisms defines a permutation group known as the graph's *automorphism group.* A *vertex-transitive graph* is a graph whose automorphism groups act transitively upon their vertices. A connected graph G with diameter d is called *distance-regular* if there are constants c_i, a_i, b_i called *intersection numbers*, such that for all $i = 0, 1, \ldots, d$, and all vertices x and y at distance $i = d_G(x, y)$, among the neighbours of y, there are c_i at distance $i-1$ from x, a_i at distance i, and b_i at distance $i+1$. A (v, k, λ, μ)-*strongly regular graph* G is a k-regular graph of order v in which every pair of adjacent vertices belongs to λ triangles and every pair of non-adjacent vertices is connected via μ paths of length 2. A graph of diameter d is said to be *antipodal* if, for any vertices u, v, w such that $d(u, v) = d$ and $d(u, w) = d$, it follows that $d(v, w) = d$ or $v = w$.

2.2 Digraphs

By a *digraph* we mean a structure $G = (V, A)$, where $V(G)$ is a finite set of *vertices*, and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices $u, v \in V(G)$ called *arcs*. The *order* of the digraph G is the number of vertices in G. An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in A(G)$. Similarly, an *out-neighbour* of a vertex v is a vertex w such that $(v, w) \in A(G)$. The *in-degree*, respectively the *out-degree*, of a vertex $v \in V(G)$ is the number of its in-neighbours, respectively out-neighbours. If both the in-degree and the out-degree equal d for every vertex, then the digraph G is called a *diregular digraph* of degree d. A *walk* W of length k in G is an alternating sequence $(v_0 a_1 v_1 a_2 \ldots a_k v_k)$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i. If the arcs $a_1, a_2, \ldots, a_k$ of a walk W are distinct, W is called a *trail*. If the vertices $v_0, v_1, \ldots, v_k$ are also distinct, W is called a *path*. A *cycle* C_k of length k is a closed trail of length $k > 0$ with all vertices distinct (except the first and the last). The *distance* from vertex u to vertex v in G is the length of the shortest directed path from u to v. The *diameter* k of a digraph G is the maximum distance between any two vertices in G. A digraph $G = (V, A)$ is said to be *complete* if both uv and $vu \in A$ for all $u, v \in V$. The *line digraph* of G, denoted by $L(G)$, is the digraph with vertex set $V(L(G)) = \{a_{ij} \mid a_{ij} = (v_i, v_j) \in A\}$, and a vertex a_{ij} is adjacent to a vertex a_{st} in $L(G)$ if and only if $v_j = v_s$ in G. For an integer n, the nth *iterated line digraph* of G is recursively defined as $L^n(G) = L(L^{n-1}(G))$ with $L^0(G) = G$.

2.3 Spectral Graph Theory

The *adjacency matrix*, sometimes also called the *connection matrix*, of a simple labeled graph G is a matrix with rows and columns labeled by graph vertices, with a 1 or 0 in position (u, v) according to whether u and v are adjacent or not. For a simple graph with no self-loops, the adjacency matrix must have 0s on the diagonal. For an undirected graph, the adjacency matrix is symmetric. If G is a graph of diameter d and order n, then for each integer i in the range $0 \le i \le d$, we define the $n \times n$ *distance matrix* $A_i = A_i(G)$ as follows. The rows and columns of A_i correspond to the vertices of G, and the entry in position (u, v) is 1 if the distance $d(u, v)$ between the vertices u and v is i, and zero otherwise. Clearly, $A_0 = I$ and $A_1 = A$ is the usual adjacency matrix of G. An *eigenvalue* of a graph G is an eigenvalue of its adjacency matrix A; i.e., a $\lambda \in \mathbb{R}$ for which there is an *eigenvector* $v \in \mathbb{R}^{|V(G)|}$, $v \neq 0$, such that $Av = \lambda v$. The *multiplicity* $m(\lambda)$ of λ is the dimension of the subspace of $\mathbb{R}^{|V(G)|}$ spanned by all eigenvectors for λ (its eigenspace). The *spectrum* of G is the multiset of eigenvalues λ with their multiplicities $m(\lambda)$, which we denote as follows

$$Spec(G) = Spec(A) = (\lambda)^{m(\lambda)}(\mu)^{m(\mu)} \ldots (w)^{m(w)}.$$

The *trace* of an $n \times n$ square matrix A is defined to be the sum of the elements on the main diagonal, i.e., $trace(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \ldots + a_{nn}$, where a_{ii} denotes the entry on the i-th row and i-th column of A.

2.4 Cage Problem

We use the term (k, g)-*graph* to denote a (finite, simple) k-regular graph of girth g. A (k, g)-*cage* is a smallest k-regular graph of girth g; its order is denoted by $n(k, g)$. The existence of (k, g)-graphs for any degree/girth pair (k, g) with $k \geq 2$ and $g \geq 3$ has been known since the 1960's [30, 71], but the orders $n(k, g)$ have been determined only for very limited sets of parameters (k, g) [33].

The so-called *Moore bound* is a well-known lower bound on the order $n(k, g)$ of k-valent cages of girth g. The precise form of the bound depends on the parity of g:

$$n(k,g) \geq M(k,g) = \begin{cases} 1 + k + k(k-1) + ... + k(k-1)^{(g-3)/2}, & g \text{ odd} \\ 2(1 + (k-1) + ... + (k-1)^{(g-2)/2}), & g \text{ even} \end{cases} \tag{2.1}$$

Graphs whose orders equal the Moore bound are called *Moore graphs* and are very rare. They are known to exist if

$k = 2$ and $g \geq 3$: cycles;

$g = 3$ and $k \geq 2$: complete graphs;

$g = 4$ and $k \geq 2$: complete bipartite graphs;

$g = 5$ and $k = 2, 3, 7$: the 5-cycle, the Petersen graph, the Hoffman-Singleton graph;

$g = 6, 8,$ or 12: symmetric generalized n-gons of order $k - 1$;

with the existence of the $(57, 5)$-graph or order matching the Moore bound still unresolved [5, 25, 33].

Outside the above cases for which the classification of the parameter pairs of the Moore graphs asserts the existence of a graph whose order matches the Moore bound, the obvious lower bound for the order of a (k, g)-cage is the value of the Moore bound plus one, $M(k, g)+1$, when k is even, and the value of the Moore bound plus two, $M(k, g) + 2$, when k is odd (with the second statement following from the fact that odd degree regular graphs must be of even order, and the fact that both Moore bounds are even for odd k).

The difference between the order of a (k, g)-graph and the Moore bound $M(k, g)$ is called the *excess* of the graph, and it is almost universally believed that the excess of the majority of cages is significantly bigger than 2. No unified opinion appears to exist on whether the excess of cages can be arbitrarily large. Inspecting the lists of the best (smallest) known (k,g)-graphs in [33] quickly reveals a significant gap between the orders of the best known graphs and the orders predicted by the Moore bound; with the excess quickly becoming a multiple of the Moore bound. In [17], Brown showed that $n(k,5)$ is never equal to $M(k,5)+1$. For girth 7, Eroh and Schwenk [31] showed the non-existence of k-regular graphs of girth 7 and order $M(k,7)+1$. Note that in this case, the McGee graph achieves the lower bound $M(3,7)+2$, hence is a cage. The case of k-regular cages of odd girth and excess 1 has been completed by Bannai and Ito [6] who have shown (using spectral methods) that no k-regular graphs of order $M(k,g)+1$ exist for any odd $g \geq 5$. For excess 2, Kovács has shown that no graphs of excess 2, girth 5, and odd degree k which is not of the form $\ell^2+\ell+3$ or $\ell^2+\ell-1$, where ℓ is a positive integer, exist [51]. Eroh and Schwenk [31] showed that $n(k,5)$ is not equal to $M(k,5)+2$ for $5 \leq k \leq 11$. Most recent results concerning odd girth and excess 2 are due to Garbe [44]. He showed the non-existence of graphs of excess 2 for parameters $(k,9)$ with $k \equiv 1 \pmod 2$; $(k,13)$ with $k \equiv 0 \pmod 5$, $k \equiv 5 \pmod 7$, $k \equiv 4 \pmod{11}$ and $k \equiv 2,5,10,12 \pmod{13}$; $(k,17)$ with $k \equiv 3 \pmod 5$, $k \equiv 0,2 \pmod 7$, $k \equiv 9 \pmod{11}$ and $k \equiv 6 \pmod{13}$; $(k,21)$ with $k \equiv 7,10 \pmod{11}$; $(k,25)$ with $k \equiv 2,3,4 \pmod 5$, $k \equiv 2,6 \pmod 7$, $k \equiv 2,6,8 \pmod{11}$ and $k \equiv 4,7 \pmod{13}$; $(k,29)$ with $k \equiv 2 \pmod 5$ and $k \equiv 0,8,11 \pmod{13}$. Furthermore, he showed that there are no excess 2 graphs in the families of $(3,2s+1)$-graphs with $s \equiv 0 \pmod 4$, $(5,2s+1)$-graphs with $s \equiv 2 \pmod 4$, $(7,2s+1)$-graphs with $s \equiv 0 \pmod 2$, and $(9,2s+1)$-graphs with $s \equiv 0 \pmod 4$.

All that is known for even girth is summarized in the following two theorems of Biggs and Ito.

Theorem 2.1 ([15]). *Let G be a (k,g)-cage of girth $g=2m \geq 6$ and excess e. If $e \leq k-2$, then e is even and G is bipartite of diameter $m+1$.*

Let $D(k,2)$ be the incidence graph of a symmetric $(v,k,2)$-design.

Theorem 2.2 ([15]). *Let G be a a (k,g)-cage of girth $g=2m \geq 6$ and excess 2. Then $g=6$, G is a double-cover of $D(k,2)$, and k is not congruent to 5 or 7(mod 8).*

2.5 Degree/Diameter problem for undirected graphs

Let $n(k,d)$ denote the largest order of any undirected graph of maximum degree k and diameter d. It is easy to show that the order $|V(\Gamma)|$ of any graph Γ of maximum degree k and diameter d, and therefore also the parameter $n(k,d)$, satisfy the following inequality:

$$|V(\Gamma)| \leq n(k,d) \leq M(k,d) = 1+k+k(k-1)+k(k-1)^2+...+k(k-1)^{d-1}.$$

The above value $M(k,d)$ is called the *Moore bound*. A graph whose order is equal to the Moore bound is called a *Moore graph*; such a graph is necessarily regular of degree k. Moore graphs are proved to be very rare. They are the complete graphs on $k+1$ vertices; the cycles on $2d+1$ vertices; and for diameter 2, the Petersen graph, the Hoffman-Singleton graph, and possibly a graph of degree $k=57$. The difference between the Moore bound

$M(k,d)$ and the order of a specific graph Γ of maximum degree k and diameter d is called the *defect* of Γ, and is denoted by $\delta(\Gamma)$. Thus, if Γ is a largest graph of maximum degree k and diameter d, then $n(k,d) = M(k,d) - \delta(\Gamma)$. It needs to be noted that very little is known about the exact relation between the Moore bounds $M(k,d)$ and the corresponding extremal orders $n(k,d)$. There is a considerable gap between the orders of the largest known/constructed graphs of maximum degree k and diameter d and the corresponding Moore bounds. In particular, it is not even known whether the two parameters are of the same order of magnitude (with computational evidence strongly suggesting that they are not). Thus, the below stated long open *question of Bermond and Bollobás* [11] can be viewed as the natural first attempt at shedding light on the nature of the relation between $M(k,d)$ and $n(k,d)$:

Is it true that for each positive integer c there exist k and d such that the order of the largest graph of maximum degree k and diameter d is at most $M(k,d) - c$?

Moore graphs of degree k and diameter d are well-known to be the only (k,d)-graphs of girth $2d+1$; the girth of any other (i.e., non-Moore) graph of maximum degree k and diameter d is strictly smaller than $2d+1$. It will also prove useful to note that even though it is not known whether the extremal graphs of diameter d, maximal degree k, and of the maximal order $n(k,d)$, are necessarily k-regular, graphs Γ containing vertices of degree smaller than k can be easily shown to satisfy the following stricter upper bound:

$$|V(\Gamma)| \leq M(k,d) - 1 - (k-1) - \ldots - (k-1)^{d-1} = M(k,d) - \frac{(k-1)^d - 1}{k-2}. \qquad (2.2)$$

The *bipartite Moore bound* is the maximum number $B(k,d)$ of vertices in a bipartite graph of maximum degree k and diameter at most d. This bound is due to Biggs [14]:

$$B(2,d) = 2d, \text{ and } B(k,d) = \frac{2(k-1)^d - 2}{k-2}, \text{ if } k > 2, \qquad (2.3)$$

and is smaller than the Moore bound by $(k-1)^d$ (i.e, $M(k,d) - B(k,d) = (k-1)^d$, for $k \geq 3$). Bipartite (k,d)-graphs of order $B(k,d)$ are called *bipartite Moore graphs*. The bipartite Moore bound represents not only an upper bound on the number of vertices of a bipartite graph of maximum degree k and diameter d, but it is also a lower bound on the number of vertices of a regular graph Γ of degree k and girth $g = 2d$.

For degrees 1 or 2, bipartite Moore graphs are K_2 and the $2d$-cycles, respectively. When $k \geq 3$ the possibility of the existence of bipartite Moore graphs was settled by Feit and Higman [35] in 1964 and, independently, by Singleton [72] in 1966. They proved that such graphs exist only if the diameter is $2, 3, 4$ or 6. For $d = 2$ and each $k > 3$ the bipartite Moore graphs of degree k are the complete bipartite graphs of degree k. For $d = 3, 4$ or 6 bipartite Moore graphs of degree k have been constructed only when $k-1$ is a prime power [10]. Furthermore, Singleton [72] proved that the existence of a bipartite Moore graph of diameter 3 is equivalent to the existence of a projective plane of order $k-1$. On the other hand, for $d = 3$, there are values of k with no bipartite Moore graphs. The question of whether or not bipartite Moore graphs of diameter $3, 4$ or 6 exist for other values of k remains open, and represents one of the most famous problems in combinatorics.

2.6 Degree/Diameter problem for directed graphs

Let (d,k)-digraph denote a directed graph of maximum out-degree d and diameter k, and let $n_{d,k}$ be the largest order of a (d,k)-digraph. Let n_i, for $0 \le i \le k$, be the number of vertices at distance i from a distinguished vertex. Then, $n_i \le d^i$, for $0 \le i \le k$. Hence,

$$n_{d,k} = \sum_{i=0}^{k} n_i \le 1 + d + ... + d^{k-1} + d^k = \begin{cases} \frac{d^{k+1}-1}{d-1}, & \text{if } d > 1, \\ k+1, & \text{if d=1.} \end{cases} \tag{2.4}$$

The number on the right-hand side of (2.4), denoted by $M_{d,k}$, is called the *Moore bound* for (d,k)-digraphs. A digraph whose order is equal to this Moore bound is called a *Moore digraph*. It is a well known that $n_{d,k} = M_{d,k}$ only in the trivial cases, when $d = 1$ (directed cycles of length $k+1$) or $k = 1$ (complete digraphs of order $d+1$), see [16] and [68]. A *defect* δ of a given (d,k)-digraph G is the difference between the corresponding Moore bound $M_{d,k}$ and the order of G. Since there are no Moore (d,k)-digraphs for $d \neq 1$ and $k \neq 1$, the problem of the existence of diregular digraphs of degree $d \ge 2$, diameter $k \ge 2$ and the number of vertices $M_{d,k} - \delta$, $\delta \neq 0$, becomes an interesting problem. Such digraphs are called (d,k,δ)*-digraphs*, where δ is the defect from the Moore bound and $d \ge \delta \ge 1$.

Concerning the existence of $(d,k,1)$-digraphs, Fiol, Alegre and Yebra in [42] showed that $(d,2,1)$-digraphs do exist for any degree $d \ge 2$; such family of digraphs are Kautz digraphs $K(d,2)$ (the line digraphs of complete digraphs K_{d+1}) [50]. Conde, Gimbert, González, Miret and Moreno used concepts and techniques from algebraic number theory combined with spectral techniques to prove that $(d,k,1)$-digraphs do not exist for any degree when the diameter is 3 [21]. Later, using a similar method they proved that $(d,k,1)$-digraphs of diameter 4 also do not exist [20]. In [58], Miller and Fris proved that there are no $(2,k,1)$-digraphs for $k \ge 3$. Moreover, Baskoro, Miller, Širáň and Sutton in [9] showed that $(3,k,1)$-digraphs do not exist for $k \ge 3$.

Every $(d,k,1)$-digraph G has the property that for every vertex $u \in G$ there is a unique vertex $v \in G$ such that there are exactly two walks of length $\le k$ from u to v in G, (e.g. [7]). Such a vertex v is called the *repeat* of u, denoted by $r(u)$. If $r(u) = v$, then $r^{-1}(v) = u$. In the case when $r(u) = u$, u is called a *selfrepeat* of G. Motivated by the technique of Bridges and Toueg in [16], Baskoro, Miller, Plesník and Znám used matrix theory (the eigenvalues of the adjacency matrix) to prove that there is no diregular $(d,k,1)$-digraph of degree $d \ge 2$, diameter $k \ge 3$ and with every vertex a selfrepeat, that is, every vertex on a directed cycle C_k. For $k = 2$ and degree $2 \le d \le 12$ they showed that if there is a C_2 then every vertex lies on a C_2 (that is, all vertices are selfrepeats or none is). These results can be found in [8].

Theorem 2.3 ([8]). *For $d \ge 2$ and $k \ge 3$ there is no $(d,k,1)$-digraph with every vertex in C_k.*

For general (d,k)-digraphs of defect 2, the only known result is that $(2,k,2)$-digraphs do not exist for $k \ge 3$. This result was obtained by Miller and Širáň in [59]. For the remaining values of $k \ge 2$ and $d \ge 3$, the question of whether digraphs of defect 2 exist or not remains open.

Chapter 3

On (k, g)-graphs of excess at most 3

The results of this chapter are published in [47]. As we mentioned in Section 2.4 no examples of even girth cages of order $M(k, g) + 1$ have been found. In fact, Theorem 2.1 excludes the possibility of odd excess for the vast majority of even girth cages. Combining Theorem 2.1 with counting cycles, we show that no even girth (k, g)-graphs with $k \geq 3, g \geq 6$, and order $M(k, g) + 1$ exist (Corollary 3.10). Together with the result of Bannai and Ito [5], this means that the only (k, g)-graphs of order $M(k, g) + 1$ can possibly be graphs of girths 3 and 4. Based on these observations, we obtain a complete classification of the parameters (k, g) for which there exists a (k, g)-graph of order $M(k, g) + 1$ (Theorem 3.11). Excess larger than 1 is even harder to deal with. We limit ourselves to excesses 2 and 3, and present formulas for counting cycles of lengths close to the girth for the majority of parameter pairs (k, g). We then attempt to use these exact counts to argue the non-existence of (k, g)-graphs of excess $e \leq 3$ for infinite families of parameters (k, g). Even though the original motivation for our approach is based on a method employed in [46] for the case of small vertex-transitive graphs of given degree and girth, it turns out that an almost identical approach appears already in the 1971 paper of Friedman [43] who employed counting cycles to show that Moore graphs for certain parameter pairs (k, g) cannot exist (the paper appeared prior to the completion of the classification of Moore graphs). When compared to [43], we deal with a wider range of cycle lengths, includes graphs of even girth, which were not considered by Friedman, and considers excesses greater than 0.

Overall, our approach in this chapter is meant to be a comprehensive treatise of the strengths and weaknesses of using counting cycles for determining parameter pairs (k, g) for which (k, g)-graphs of excess in the range $0, 1, 2, 3$ do not exist. For that reason, and for the sake of completeness, we occasionally include results that have been previously proved by other techniques. Another reason for that is that we aim to develop our technique from the very beginning and to demonstrate its versatility.

In addition to determining the order of the smallest (k, g)-graph, one may be interested in determining the *entire spectrum of orders of* (k, g)*-graphs for a given parameter pair* (k, g). The original article on the existence of (k, g)-graphs [30] already contains the observation that given any pair of parameters (k, g), $k, g \geq 3$, a k-regular graph of girth g and order $2m$ exists for every $m \geq 2\sum_{t=1}^{g-2}(k-1)^t$. As odd-degree regular graphs cannot have an odd number of vertices, this result shows that, in the case of odd k, the spectrum of orders of (k, g)-graphs contains all sufficiently large possible orders. As for the remaining even orders in case of even k, a paper of Sachs [71] establishes the existence of such graphs for certain multiples of g, and could be most likely extended to prove the existence of (k, g)-graphs with even k for all (i.e., odd and even) sufficiently large orders. The results presented in this chapter can be viewed as an attempt to determine the other side of the spectrum – the possible orders of (k, g)-graphs that do not differ from the Moore bound by more than 3.

The main conclusion of our investigation is the (surprising?) observation that counting cycles does not exclude the existence of many parameter families even when graphs of only a small excess are considered. The situation is somewhat similar to that of the existence of strongly regular graphs. It is easy to see that a Moore graph of diameter 2 must necessarily be a strongly regular graph. Thus, the existence of the 'unresolved' $(57, 5)$-Moore graph is also a question of the existence of the corresponding strongly regular graph. In [22], Chvátal observed (and provided a precise argument for his observation) that counting cycles cannot exclude the existence of strongly regular graphs whose non-existence could not be established using arguments based on spectral properties of their adjacency matrices. Even though the results obtained in this article strongly suggest the existence of similar results with regard to (k, g)-graphs of orders close to the Moore bound, making such statements more precise would require developing spectral methods for these families of graphs. To the best of our knowledge, nobody has made much progress with respect to spectral theory of graphs of excess larger than 2.

In addition to obtaining a number of results concerning the numbers of cycles of lengths close to the girth for graphs with excess $0 \leq e \leq 3$, we map the situation for each of these excesses and apply the obtained results to exclude as many families of pairs as possible.

3.1 Counting cycles in Moore graphs

The number of cycles of length c passing through a given vertex in a vertex-transitive graph is easily seen to be independent of the choice of the vertex. A similar observation can be made for cycles of length c close to the girth in Moore graphs (we will make this observation more precise), but no such result holds for cages in general. The key observation of the forthcoming section is that even though the numbers of cycles of the same length passing through vertices of a (general) cage may differ from vertex to vertex, these numbers must be the same for all vertices of the Moore graphs. This is, in a way, a curious observation – the vast majority of the known Moore graphs are vertex-transitive – and so the causality is not all that clear in this case. Are the vertex-transitive Moore graphs highly symmetric because the numbers of cycles through each vertex is the same or are these numbers the same because these Moore graphs have to be vertex-transitive? The vertex-transitivity of Moore graphs does not appear to follow from their combinatorial properties and proving that the numbers of small cycles are the same everywhere does not require vertex-transitivity. These connections are particularly interesting with regard to the existence of the $(57, 5)$-Moore graph which has been proved (if it exists) to not be vertex-transitive and to have a small automorphism group [3, 55, 56]. As we will show in this section, if it exists, it has to have the property that the numbers of cycles of length close to 5 must be the same for each of its vertices. Thus, if such a graph exists, it must have an unusual structure: it has to be regular with respect to the number of cycles through each of its vertices, but it cannot be vertex-transitive. For any given vertex $v \in V(G)$ and integer $c \geq 3$, let $\mathbf{c}_G(v,c)$ denote the *number of cycles* of length c in G passing through v. The following lemma is obvious:

Lemma 3.1. *Let G be a graph and $c \geq 3$. The sum*

$$\sum_{v \in V(G)} \mathbf{c}_G(v,c)$$

is divisible by c.

The next lemma illustrates the key calculations used in this chapter. As we already mentioned, the first two results of this lemma have already been proved by Friedman [43]. The third result concerning cycles of length $g+2$ has been stated by Friedman for 3-regular graphs only and was stated in the form $\mathbf{c}_G(v, g+2) = 0$, with the proof omitted. The fourth result is new.

Lemma 3.2. *Let G be (k,g)-Moore graph of* odd girth. *Then the following hold for all $v \in V(G)$:*

1. $\mathbf{c}_G(v,g) = \frac{k}{2}(k-1)^{(g-1)/2}$,
2. $\mathbf{c}_G(v,g+1) = \frac{k(k-2)}{2}(k-1)^{(g-1)/2}$,
3. $\mathbf{c}_G(v,g+2) = \frac{k(k-2)(k-3)}{2}(k-1)^{(g-1)/2}$,
4. $\mathbf{c}_G(v,g+3) = \frac{k(k-2)(k^2-4k+5)}{2}(k-1)^{(g-1)/2}$.

Proof. Let v be an arbitrary vertex of G, and let $N_i(v) = \{u \in V(G) \mid d_G(v,u) = i\}$. The following properties follow easily from the properties of the Moore graphs:

1. $N_i(v) \cap N_j(v) = \emptyset$, for all $0 \leq i \neq j \leq \frac{g-1}{2}$;
2. $|N_0(v)| = 1, |N_1(v)| = k$ and $|N_i(v)| = k(k-1)^{i-1}$, for all $0 \leq i \leq (g-1)/2$;
3. $V(G) = \bigcup_{0 \leq i \leq (g-1)/2} N_i(v)$;
4. each vertex $u \in N_{(g-1)/2}(v)$ is adjacent to exactly one vertex in $N_{(g-3)/2}(v)$ and $(k-1)$ vertices in $N_{(g-1)/2}(v)$ (we will call the edges connecting the vertices from $N_{(g-1)/2}(v)$ *horizontal*).

We call the tree obtained from G by removing the horizontal edges *the Moore tree of* G (as its vertices give us the Moore bound), and note that this tree consists of k branches rooted at v. Each vertex $u \in N_{(g-1)/2}(v)$ is adjacent through a horizontal edge to each of the $(k-1)$ branches distinct from its own.

Consider now a cycle of length g passing through v. The fact that G is a Moore graph implies that any such cycle has to consist of two disjoint paths starting at v of length $(g-1)/2$ connecting v to $u_1, u_2 \in N_{(g-1)/2}(v)$ and one horizontal edge connecting u_1 to u_2.

As the two $(g-1)/2$-paths from v to u_1 and u_2 are uniquely determined by the vertices u_1 and u_2, there is a one-to-one correspondence between the horizontal edges connecting the vertices in $N_{(g-1)/2}(v)$ and the cycles of length g through v. Therefore,

$$\mathbf{c}_G(v,g) = |N_{(g-1)/2}(v)|(k-1)/2 = k(k-1)^{(g-3)/2} \cdot \frac{(k-1)}{2} = \frac{k}{2}(k-1)^{(g-1)/2},$$

as claimed in the first part of our lemma.

Consider next a cycle of length $g+1$ passing through v. It again has to include two vertex-disjoint paths (sharing no other vertex than v) of length $(g-1)/2$ connecting v to a pair of vertices $u_1, u_2 \in N_{(g-1)/2}(v)$ and a 2-path connecting u_1 to u_2. Note that no path of length 2 between two vertices in $N_{(g-1)/2}(v)$ can use other than horizontal edges. Moreover, no such path connects two vertices u_1 and u_2 with the property that the shortest path between v and u_1 and the shortest path between v and u_2 share more than the vertex v: if the two paths shared more than just v, say a vertex w from $N_i(v)$ for some $0 < i \leq (g-1)/2$, this would cause $d_G(w,u_1)$ and $d_G(w,u_2)$ both to be smaller than $(g-1)/2$ and would give rise to a cycle containing w, u_1 and u_2 of length $d_G(w,u_1)+d_G(w,u_2)+2 \leq 2(g-3)/2+2 < g$, a contradiction. Consequently, the number of 2-paths connecting two vertices in $N_{(g-1)/2}(v)$ belonging to different branches with respect to v is equal to $\mathbf{c}_G(v,g+1)$. As the number of such 2-paths starting at a fixed vertex $u \in N_{(g-1)/2}(v)$ is equal to $(k-1)(k-2)$ (there are only $(k-2)$ 'unused' horizontal edges adjacent to the neighbor of u chosen in the first step), we obtain the desired bound

$$\mathbf{c}_G(v,g+1) = |N_{(g-1)/2}(v)|(k-1)(k-2)/2 =$$
$$k(k-1)^{(g-3)/2} \cdot \frac{(k-1)(k-2)}{2} = \frac{k(k-2)}{2}(k-1)^{(g-1)/2}.$$

For any cycle of length $g+2$ passing through v, the two sub-paths of this cycle of length $(g-1)/2$ connecting v to a pair of vertices $u_1, u_2 \in N_{(g-1)/2}(v)$ have to be connected by a path of length 3. It is still the case that any such 3-path has to consist exclusively of horizontal edges: to be a part of a cycle through u_1, any such path has to start with a horizontal edge (there is only one non-horizontal edge adjacent to u_1 and it had already been used to get to u_1): if we followed this initial horizontal edge by a pair of non-horizontal edges finishing in u_2, we would use up our only non-horizontal edge connecting u_2 to v and if we followed this initial horizontal edge by a single non-horizontal edge, we would end up in $N_{(g-3)/2}(v)$ and would have no horizontal edge to complete the path. Thus, distinct cycles of length $g+2$ through v determine 3-paths consisting of horizontal edges. While there are exactly $(k-1)(k-2)(k-2)$ paths of length 3 starting at any $u_1 \in N_{(g-1)/2}(v)$ that consist entirely of horizontal edges, not all such paths give rise to a $(g+2)$-cycle through v: only those of these 3-paths give rise to a $(g+2)$-cycle through v that connect vertices $u_1, u_2 \in N_{(g-1)/2}(v)$ with the property that the shortest path between v and u_1 and the shortest path between v and u_2 share no more than the vertex v. Therefore, when choosing the third horizontal edge to complete a 3-path giving rise a $(g-2)$-cycle through v, we have to avoid using the one horizontal edge terminating at a vertex whose shortest path to v shares more than v with the shortest path connecting v to u_1 (each $u \in N_{(g-1)/2}(v)$ has exactly one such neighbor). This forces the number of horizontal 3-paths starting at u_1 and corresponding to $(g+2)$-cycles through v to be equal to $(k-1)(k-2)(k-3)$ and the total number of such paths to be equal to

$$|N_{(g-1)/2}(v)|(k-1)(k-2)(k-3)/2 =$$
$$k(k-1)^{(g-3)/2} \cdot \frac{(k-1)(k-2)(k-3)}{2} = \frac{k(k-2)(k-3)}{2}(k-1)^{(g-1)/2},$$

and the result follows.

Finally consider a cycle through v of length $g+3$. Unlike the above cases, this time we have two types of cycles to consider. Both types have to consist of two paths of length $(g-1)/2$ sharing no other vertices but v and connecting v to $u_1, u_2 \in N_{(g-1)/2}(v)$, but they differ in the way u_1 and u_2 are connected. The first type is just like the cases considered above: u_1 and u_2 are connected via a 4-path comprised of horizontal edges. However, the number of such paths has to be calculated more carefully than in the previous case and splits into two calculations. While it is still true that for the first edge of the path we can choose any of the $(k-1)$ horizontal edges of u_1, and similarly, we have $(k-2)$ choices for the second and $(k-2)$ choices for the third edge, the number of choices for the fourth edge differs according to the branch of the Moore tree of G to which the other end of the third edge belongs: If the third edge ends in the branch of u_1 (which happens exactly once), we can choose any of its $(k-2)$ remaining horizontal edges to complete the path (as we will always end up in a branch different from the branch of u_1). When choosing any of the horizontal edges not terminating in the branch of u_1 limits our choice of the fourth edge to those not terminating in the branch of u_1. Thus, the number of length $(g+3)$-cycles of this first type is

$$|N_{(g-1)/2}(v)| \cdot \frac{1}{2} \cdot [(k-1)(k-2) \cdot 1 \cdot (k-2) + (k-1)(k-2)(k-3)(k-3)] =$$
$$k(k-1)^{(g-3)/2} \frac{(k-1)(k-2)(k^2-5k+7)}{2} = \frac{k(k-2)(k^2-5k+7)}{2}(k-1)^{(g-1)/2},$$

For the rest of the cycles of length $g+3$, the path connecting u_1 and u_2 does not have to consist entirely of horizontal edges. While it is still the case that the first and fourth edge of the path have to be horizontal (as we use the only non-horizontal edges adjacent to u_1 and u_2 in the paths connecting them to v), the second edge can dip down into $N_{(g-3)/2}(v)$ and the third edge then needs to come back into $N_{(g-1)/2}(v)$. As we only have one non-horizontal edge for the dip-down and $(k-2)$ edges to come back up, and the last (horizontal) edge cannot connect to the branch of u_1 (so we have $(k-2)$ edges to choose from for the last horizontal edge), the total count of the $(g+3)$-cycles of this second type comes to:

$$|N_{(g-1)/2}(v)|(k-1)(k-2)^2/2 =$$
$$k(k-1)^{(g-3)/2} \cdot \frac{(k-1)(k-2)^2}{2} = \frac{k(k-2)^2}{2}(k-1)^{(g-1)/2},$$

Adding the numbers of the two different types of cycles of length $(g+3)$ yields the final claim of the lemma. □

Combining Lemmas 3.1 and 3.2 yields:

Corollary 3.3. *Let G be a (k,g)-Moore graph of* odd girth. *Then the following hold:*

1. $g \mid M(k,g) \cdot \frac{k}{2}(k-1)^{(g-1)/2}$, *for all* $v \in V(G)$;
2. $(g+1) \mid M(k,g) \cdot \frac{k(k-2)}{2}(k-1)^{(g-1)/2}$, *for all* $v \in V(G)$;
3. $(g+2) \mid M(k,g) \cdot \frac{k(k-2)(k-3)}{2}(k-1)^{(g-1)/2}$, *for all* $v \in V(G)$;
4. $(g+3) \mid M(k,g) \cdot \frac{k(k-2)(k^2-4k+5)}{2}(k-1)^{(g-1)/2}$, *for all* $v \in V(G)$.

Even though the parameter pairs (k,g) of the Moore graphs are classified and all but one pairs (k,g) for which (k,g)-Moore graphs exist are known, the main argument of the classification uses linear algebra of adjacency matrices and integral eigenvalues, and no 'purely graph theoretical' proof is known. The corollary we just proved provides an alternate way to exclude specific pairs. Given a pair (k,g), it is easy to calculate the four values listed in the corollary, and then check their divisibility by the corresponding cycle lengths. If either of the four tests fails, no Moore graph with parameters (k,g) exists. Considering, for example, the smallest degrees for which k-regular graphs of girth 5 fail to exist, we obtain the following table:

(k,g)	$\mathbf{c}_G(v,g)$	$\mathbf{c}_G(v,g+1)$	$\mathbf{c}_G(v,g+2)$	$\mathbf{c}_G(v,g+3)$
(4,5)	**306**	612	**612**	**3,060**
(5,5)	1,040	3,120	**6,240**	31,200
(6,5)	2,775	11,100	**33,300**	**188,700**
(8,5)	12,740	76,440	382,200	2,828,280
(9,5)	**23,616**	165,312	991,872	8,265,600
(10,5)	40,905	327,240	2,290,680	21,270,600

where the bold faced numbers fail to meet the divisibility requirements. Out of the first six potential candidates, four are correctly excluded by our test. It is easy to see why this test can never provide us with complete lists of excluded pairs: once $2k$ or $(k+1)$ are divisible by all four cycle lengths $g, g+1, g+2, g+3$, none of the tests can exclude the pair (k,g). The most interesting case is of course the pair $(57,5)$ – the only pair of parameters for which the existence of a Moore graph is unresolved. In agreement with the results of Chvátal [22], all four of the corresponding numbers pass the divisibility tests, and we fail to resolve this last open case.

For the remainder of this section, we develop results similar to the above for Moore graphs of *even* girth. Instead of counting the number of cycles through a fixed vertex, we switch to counting the *number of cycles through a fixed edge.* The reader familiar with the proof of the Moore bound for even girth graphs can visualize the edge to be the one we start developing the Moore tree from (i.e., each terminal vertex of the edge is the root of a tree of depth $(g-2)/2$ and the leaves of these two trees are to be connected by edges that complete g-cycles, see Figure 3.1).

For any given edge $e \in E(G)$ and integer $c \geq 3$, let $\overline{\mathbf{c}}_G(e,c)$ denote the *number of cycles* of length c in G containing e. The proofs of the following lemmas and corollary follow along the same lines as the above proofs and use the facts that the number of edges in a

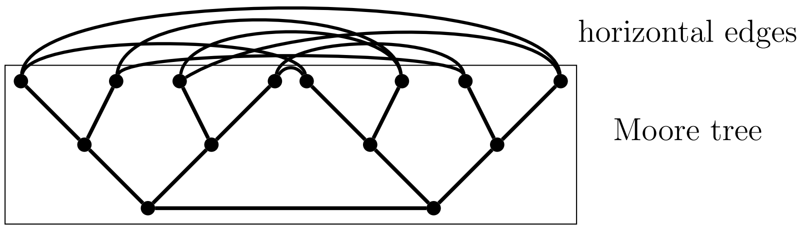

Figure 3.1: The Moore tree and horizontal edges in the $(3, 6)$-cage, Heawood graph.

(k, g)-Moore graph is $\frac{M(k,g)\cdot k}{2}$ and a cycle of length c contains c distinct edges. While it is easy to see that all Moore graphs of even girth are bipartite, and hence contain no cycles of odd length, the non-existence results for cycles of lengths $g+1$ and $g+3$ can be obtained without using this observation.

Lemma 3.4. *Let G be a graph and $c \geq 3$. The sum*

$$\sum_{e \in E(G)} \overline{\mathbf{c}}_G(e, c)$$

is divisible by c.

Lemma 3.5. *Let G be (k, g)-Moore graph of* even girth*. Then the following hold for all $e \in E(G)$:*

1. $\overline{\mathbf{c}}_G(e, g) = (k-1)^{g/2}$,
2. $\overline{\mathbf{c}}_G(e, g+1) = 0$,
3. $\overline{\mathbf{c}}_G(e, g+2) = (k-1)^{g/2}(k-2)^2$,
4. $\overline{\mathbf{c}}_G(e, g+3) = 0$.

In view of the widely believed conjecture that all even girth cages (not just the Moore graphs) must be bipartite (see, for example, [33]), the above lemma together with similar lemmas in the forthcoming sections as well as Theorem 2.1 may be seen supporting this conjecture.

Corollary 3.6. *Let G be a (k, g)-Moore graph of* even girth*. Then the following hold:*

1. $g \mid M(k, g) \cdot \frac{k}{2} \cdot (k-1)^{g/2}$,
2. $(g+2) \mid M(k, g) \cdot \frac{k}{2} \cdot (k-1)^{g/2}(k-2)^2$.

Recall that the existence of Moore graphs of girths $6, 8$ and 12 is equivalent to the existence of projective planes, and symmetric generalized quadrangles and generalized hexagons, respectively. Thus, excluding cages for girths $6, 8$ or 12 would have the potential of proving the non-existence of corresponding generalized polygons whose existence is in question. Unfortunately, not a single pair $(k, 6), (k, 8)$ or $(k, 12)$ is excluded by the above corollary for any $3 \leq k \leq 2000$. On the other hand, the above divisibility criteria exclude for example 800 of the pairs $(k, 10)$, 1143 pairs $(k, 14)$, and all the way to 1809 pairs $(k, 38)$ from the range $3 \leq k \leq 2000$.

We conclude the section with one more observation that might be known as 'folklore'. The reason we decided to mention this result is that we will return with similar arguments in the forthcoming sections as well.

Lemma 3.7. *If $k \geq 3$ and $g \geq 3$ is odd, such that a (k, g)-Moore graph exists, then there is a $(k-1, g)$-graph of order $k(k-1)^{(g-3)/2}$.*

If $k \geq 3$ and $g \geq 4$ is even, such that a (k, g)-Moore graph exists, then there is a $(k-1, g)$-graph of order $2(k-1)^{(g-2)/2}$.

Proof. In both cases, the graph is obtained by considering the graph induced in the corresponding Moore graph by the leaves of any of its Moore trees. □

3.2 Graphs of excess 1

In this section we focus on (k, g)-graphs of orders exceeding the Moore bound by 1. It is easy to check that for odd k, both Moore bounds $M(k, g)$ in (2.1) are even, and thus, $M(k, g) + 1$ is odd, and as such, cannot be the order of a regular graph of odd degree. Hence, no odd degree regular graphs of order $M(k, g) + 1$ exist. Also, based on the result of Bannai and Ito [5], no such graphs exist for odd girth $g \geq 5$, and based on Theorem 2.2 of Biggs and Ito, no such graphs exist for even girth $g \geq 8$ or even girth $g \geq 6$ and $k \geq 4$. The above results together yield the non-existence of (k, g)-graphs of excess 1 for all $g \geq 5$ and $k \geq 3$. In what follows, we employ counting cycles to reprove some of the results concerning graphs of even girth (and to demonstrate our techniques again) and complete the classification of (k, g)-graphs of order $M(k, g) + 1$.

Other than Moore graphs, cages do not necessarily have the property $\mathbf{c}_G(v, c) = \mathbf{c}_G(v', c)$ or $\overline{\mathbf{c}}_G(e, c) = \overline{\mathbf{c}}_G(e', c)$, for all $v, v' \in V(G), e, e' \in E(G)$, and $c \geq 3$. Nevertheless, in case of excess 1, these numbers do have to be equal for all edges of the graph.

Lemma 3.8. *Let $k, g \geq 4$ be even integers, and G be (k, g)-graph of order $M(k, g) + 1$. Then,*

1. $\overline{\mathbf{c}}_G(e, g) = (k-1)^{g/2} - \frac{k}{2}$, *for all $e \in E(G)$,*

2. $\overline{\mathbf{c}}_G(e, g+1) = \frac{k^2}{4}$, *for all $e \in E(G)$.*

Proof. Let $k, g \geq 4$ be even, $|V(G)| = M(k, g) + 1$, and $e = \{u, v\}$ be any edge of G. The key observation to proving these results is to note that all (k, g)-graphs with the above parameters and excess 1 are of the same structure. Namely, they consist of the (k, g)-Moore tree 'rooted' at e and one extra vertex w of distance $\frac{g-2}{2} + 1$ from both u and v that is attached via $\frac{k}{2}$ edges to the subtree attached to u and via the same number of edges to the subtree attached to v. This observation follows easily by understanding the way the Moore tree might be completed into a Moore graph: All the $(k-1)^{g/2}$ edges emanating from the the subtree attached to u must be paired with the $(k-1)^{g/2}$ edges emanating from the the subtree attached to v, and thus, if the number of edges connecting w to the two branches differed between them, we would be left with some 'dangling' edges that could not be attached to anything.

The rest of the proof follows the lines of the proofs from the previous section. The number of g-cycles must be equal to the number of 'horizontal' edges connecting the two branches and all such cycles must avoid the new vertex w. The $(g+1)$-cycles must use the new vertex w, and thus each $(g+1)$-cycle must use one of the $\frac{k}{2}$ left edges followed by one of the $\frac{k}{2}$ right edges attached to w. □

Corollary 3.9. *Let $k, g \geq 4$ be even integers, and G be (k,g)-graph of order $M(k,g)+1$. Then,*

1. $g \mid [M(k,g)+1] \cdot \frac{k}{2} \cdot \left[(k-1)^{g/2} - \frac{k}{2}\right]$,
2. $(g+1) \mid [M(k,g)+1] \cdot \frac{k^3}{8}$.

Once again, the above test cannot possibly exclude all pairs (k,g) for a fixed degree k: Once $\frac{k}{2}$ is divisible by both g and $g+1$, both values satisfy the divisibility requirement and one cannot exclude the possibility $n(k,g) = M(k,g)+1$. Thus, we cannot obtain the same kind of general result as the result of Bannai and Ito for odd g from Corollary 3.9 alone. However, as bipartite graphs do not contain odd length cycles, Lemma 3.8 that asserts the existence of a non-zero number of cycles of odd length $g+1$ together with Theorem 2.1 that asserts the bipartiteness of these graphs yield:

Corollary 3.10. *If $k \geq 4$ and $g \geq 6$ are both even, the order of any (k,g)-graph G that is not a Moore graph is greater than $M(k,g)+1$.*

We close this section with a complete classification of parameter pairs (k,g) for which there exists a (k,g)-graph of order $M(k,g)+1$.

Theorem 3.11. *Let $k \geq 2$ and $g \geq 3$. A (k,g)-graph of order $M(k,g)+1$ exists if and only if $k \geq 4$ is even and $g = 3$.*

Proof. As stated in the introduction for this section, due to parity reasons, if k is odd, no (k,g)-graphs of order $M(k,g)+1$ exist. If $k=2$, $M(2,g)=g$, and any $(2,g)$-graph is a system of disjoint cycles of length at least g, with at least one of the cycles of length exactly g. Therefore, there is no $(2,g)$-graph of order $M(2,g)+1 = g+1$. (If one were willing to allow for the $(g+1)$-cycle to be considered a $(2,g)$-graph, even though it does not contain a cycle of length g, then a $(2,g)$-graph would exist for all $g \geq 3$.)

If $k \geq 4$, the results of Bannai and Ito assert the non-existence of (k,g)- graphs of order $M(k,g)+1$ for all odd girths $g \geq 5$. The above corollary (as well as the results of Biggs and Ito [15]) excludes the existence of such graphs for all even girths $g \geq 6$. Thus, the only possible pairs that might admit the existence of a (k,g)-graph of order $M(k,g)+1$ are the pairs $(k,3)$, $(k,4)$, $k \geq 4$ even.

We first show the existence of such graphs for all pairs $(k,3)$, $k \geq 4$ and even. It is easy to see that, for any odd $n \geq 5$, the graph $K_{n+1} - \frac{n+1}{2}K_2$ (the complete graph minus a perfect matching) is an $(n-1,3)$-graph of order $M(n-1,3)+1$, and hence a $(k,3)$-graph of order $M(k,3)+1$ exists for every even $k \geq 4$.

We conclude our proof by arguing that no $(k,4)$-graphs, $k \geq 4$, of order $M(k,4)+1$ exist. We proceed by contradiction. Let G be a $(k,4)$-graph of order $M(k,4)+1$, $k \geq 4$ even. Let $e=\{u,v\}$ be any edge of G. Then G contains the Moore tree rooted at e containing two subtrees attached to the end-points of e, with the subtrees in this case simply being $(k-1)$-stars (recall that $g=4$). It also contains an extra vertex w of degree k. All the edges adjacent to w must terminate in the leaf sets of the Moore tree (which are simply neighbors of u or v), and since the number of edges terminating in each leaf set must be the same (so that we can balance out the edges between the two leaf sets), $\frac{k}{2}$ edges connect w to one leaf set and $\frac{k}{2}$ to the other. Let $v' \neq v$ be a neighbor of u that is also a neighbor of w. There are still $k-2$ edges connecting v' to the neighbors of v different from u. Since w is adjacent to $\frac{k}{2}$ of them, and $(k-2)+\frac{k}{2} > k-1$ for $k \geq 4$, there must exist a neighbor of v, say u', connected to both v' and w. However, the vertices u', w, v' form a triangle, and G cannot be of girth 4; a contradiction. □

The above classification yields the interesting observation that no Moore (k,g)-graph of girth greater than 3 can be extended into a k-regular graph of the same girth by adding a single vertex.

3.3 Graphs of excess 2

We focus first on graphs of odd girth g. Every vertex u of a (k,g)-graph G of odd girth has to be the root of a Moore tree on $M(k,g)$ vertices. The 'additional' excess vertices must lie outside this Moore tree and must be of distance at least $\frac{g+1}{2}$ from u. Thus, for each vertex u of G, the excess vertices are determined by their distance from u being larger than $\frac{g-1}{2}$, and we will call them the *excess vertices with respect to* u and will denote the set of these vertices by X_u. In case of excess 2, each vertex u of G corresponds to two excess vertices, say $X_u=\{w_1,w_2\}$. We claim that both vertices must be of distance $\frac{g+1}{2}$ from u. If this was not the case, and for example w_1 was of distance larger than $\frac{g+1}{2}$ from u, all of its edges would have to connect w_1 to vertices of distance at least $\frac{g+1}{2}$ from u, but there are at most two such vertices, and w_1 is one of them. Hence, both w_1 and w_2 are adjacent to the leaves of the Moore tree rooted at u, and there are only two possibilities to consider: either w_1 and w_2 are adjacent or they are not. Unlike the case of Moore graphs, G may contain vertices of both kinds: those whose excess vertices are adjacent and those whose are not. The following two examples exhibit several cases of such co-existence (in rather small graphs).

Both examples can be constructed from the Petersen graph by deleting edges and adding two vertices. The first example starts from the Petersen graph viewed as a 6-cycle with the $(3,5)$-Moore tree attached in its center. Choosing any two opposing edges of the 6-cycle, subdividing them by introducing a new vertex into each, and subsequently joining the two new vertices via an edge, results in a $(3,5)$-graph of order 12 (see Figure 3.2 left). The second example is constructed from the Petersen graph by removing every other edge of the outer 6-cycle and attaching two extra vertices to the vertices of degree 2 (see Figure

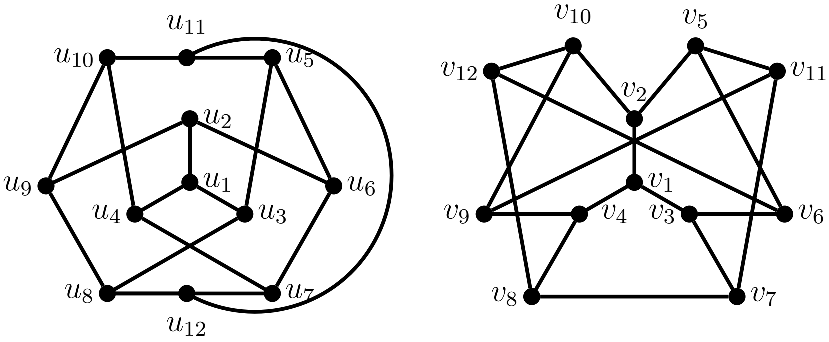

Figure 3.2: The two $(3, 5)$-graphs of order 12

3.2 right). Both graphs are of order $12 = M(3, 5) + 2$, and there exist no other $(3, 5)$-graphs of order 12. This last claim is easy to verify either via a careful consideration of the possibilities or via a computer search [32].

Examining the two examples from Figure 3.2 yields several interesting observations. First, the graph on the left contains both vertices whose excess set consists of two adjacent vertices (the vertices u_1, u_2, u_{11}, u_{12}) as well as vertices whose excess points are of distance 2 (all other vertices). On the other hand, the graph on the right contains vertices whose excess vertices are of distance 3 (the vertices v_1, v_{11}, v_{12}), as well as vertices whose excess sets are formed by adjacent vertices (all other vertices).

The next example of a graph of girth 5 and excess 2 is Robertson's graph of degree 4 and order 19 which is the unique $(4, 5)$-cage [33]. This graph exhibits vertices of all three types. It contains 4 vertices whose excess vertices are of distance 1, 12 vertices with excess vertices of distance 2, and 3 vertices with excess vertices of distance 3.

Based on the above observations, we will call the vertices of G whose excess vertices are of distance 1 *vertices of type $d1$*, vertices whose excess vertices are of distance 2 *vertices of type $d2$*, and all other vertices *vertices of type $d3$*. The number of cycles passing through a vertex u of G depends of its type. We have the following.

Lemma 3.12. *Let G be (k, g)-graph of* odd *girth $g \geq 5$ and excess 2.*

1. *If $u \in V(G)$ is of type $d1$, then*

   (a) $\mathbf{c}_G(u, g) = \frac{k}{2}(k-1)^{\frac{g-1}{2}} - k + 1$;

   (b) $\mathbf{c}_G(u, g+1) = 2\binom{k-1}{2} + (k(k-1)^{\frac{g-3}{2}} - 2(k-1))\binom{k-1}{2} + 2(k-1)\binom{k-2}{2}$.

2. *If $u \in V(G)$ is of type $d2$, then*

   (a) $\mathbf{c}_G(u, g) = \frac{k}{2}(k-1)^{\frac{g-1}{2}} - k$;

   (b) $\mathbf{c}_G(u, g+1) = 2\binom{k}{2} + (k(k-1)^{\frac{g-3}{2}} - 2k + 1)\binom{k-1}{2} + 2(k-1)\binom{k-2}{2} + \binom{k-3}{2}$.

3. *If $u \in V(G)$ is of type $d3$, then*

   (a) $\mathbf{c}_G(u, g) = \frac{k}{2}(k-1)^{\frac{g-1}{2}} - k$;

(b) $\mathbf{c}_G(u,g+1) = 2\binom{k}{2} + (k(k-1)^{\frac{g-3}{2}} - 2k)\binom{k-1}{2} + 2k\binom{k-2}{2}$.

Proof. As we have observed several times already, every g-cycle containing u has to contain a unique horizontal edge connecting two leaves of the Moore tree rooted at u, and each such edge determines its own g-cycle. Hence $\mathbf{c}_G(u,g)$ is equal to the number of horizontal edges. If u is of type $d2$ or $d3$, all edges incident to w_1 and w_2 are adjacent to the leaves of the Moore tree, which diminishes the number of horizontal edges by k (there are $2k$ edges from the leaves of the tree to X_u that replace k of the horizontal edges). Hence, the total number of horizontal edges is $\frac{k}{2}(k-1)^{\frac{g-1}{2}} - k$. By the same kind of argument, if u is of type $d1$, $\mathbf{c}_G(u,g) = \frac{k}{2}(k-1)^{\frac{g-1}{2}} - k + 1$.

Consider next the $(g+1)$-cycles. For all types of vertices, these cycles include one (but not both) or none of the excess vertices. If they include an excess vertex, they also include two edges incident to this vertex. These two edges determine the rest of the cycle uniquely, as their other endpoints are connected to u via unique $\frac{g-1}{2}$-paths. It follows that there are $2\binom{k}{2}$ such cycles for u's of types $d2$ and $d3$, and $2\binom{k-1}{2}$ such cycles for type $d1$. If the $(g+1)$-cycles do not include any of the excess vertices, they must contain two incident horizontal edges. Each pair of incident horizontal edges determines the rest of the $(g+1)$-cycle uniquely, and hence the number of the $(g+1)$-cycles containing two horizontal edges is equal to the number of pairs of incident horizontal edges. Each such pair is determined by its shared 'central' vertex. This observation allows us to count the number of cycles of this second type. If u is of type $d3$, the edges attaching the two excess vertices in X_u to the leaves of the Moore tree rooted at u are never attached to the same leaf (as that would make the distance between the excess vertices equal to 2). Consequently, there are exactly $2k$ leaves incident with $(k-2)$ horizontal edges, each contributing $\binom{k-2}{2}$ pairs of mutually incident pairs of horizontal edges. The remaining $k(k-1)^{\frac{g-3}{2}} - 2k$ leaves are incident with $(k-1)$ horizontal edges and therefore contribute $\binom{k-1}{2}$ pairs of mutually incident pairs of horizontal edges. In case when w_1 and w_2 are attached to the same leaf (i.e., u is of type $d2$), there are $k(k-1)^{\frac{g-3}{2}} - 2k + 1$ leaves adjacent to $(k-1)$ horizontal edges, there are $2k-2$ leaves incident with $(k-2)$ horizontal edges, and there is a single leaf (the one both w_1 and w_2 are adjacent to) with $(k-3)$ horizontal edges. Adding the number of cycles involving excess vertices and cycles including two horizontal edges yields the stated equalities for $\mathbf{c}_G(u,g+1)$ for vertices of types $d2$ and $d3$.

The situation with u of type $d1$ is simpler in that w_1, w_2 cannot be adjacent to the same leaf as that would create a 3-cycle (and we assume $g > 3$). Thus, in this case, the set of leaves consists of $k(k-1)^{\frac{g-3}{2}} - 2(k-1)$ leaves incident with $(k-1)$ horizontal edges and $2(k-1)$ leaves incident with $(k-2)$ horizontal edges. The last claim also follows. □

The following theorem is now fairly obvious.

Theorem 3.13. *Let G be a (k,g)-graph of odd girth $g \geq 5$ and excess 2, let x_1 be the number of vertices of type $d1$, x_2 be the number of vertices of type $d2$, and x_3 be the number of vertices of G of type $d3$. Then the following must be satisfied:*

1. $x_1 + x_2 + x_3 = M(k,g) + 2$;

2. $g \mid x_1 \cdot [\frac{k}{2}(k-1)^{\frac{g-1}{2}} - k + 1] + (x_2 + x_3) \cdot [\frac{k}{2}(k-1)^{\frac{g-1}{2}} - k]$;

3. $(g+1) \mid$

$$x_1 \cdot \left[2\binom{k-1}{2} + (k(k-1)^{\frac{g-3}{2}} - 2(k-1))\binom{k-1}{2} + 2(k-1)\binom{k-2}{2}\right]$$
$$+x_2 \cdot \left[2\binom{k}{2} + (k(k-1)^{\frac{g-3}{2}} - 2k+1)\binom{k-1}{2} + 2(k-1)\binom{k-2}{2} + \binom{k-3}{2}\right]$$
$$+x_3 \cdot \left[2\binom{k}{2} + (k(k-1)^{\frac{g-3}{2}} - 2k)\binom{k-1}{2} + 2k\binom{k-2}{2}\right].$$

In particular, if no non-negative integers x_1, x_2, x_3 satisfying the above arithmetic conditions exist, no (k, g)-graphs of excess 2 exist.

In order to take an advantage of the above theorem, we have tested a number of parameter pairs (k, g) for those that do not allow for any solutions to the above divisibility requirements. Unfortunately, we have found (often many) solutions x_1, x_2, x_3 for each of the parameter pairs considered. This might be due to having three parameters and three divisibility criteria which somehow may always allow for a solution. While an obvious solution to this problem would be to add a criterion for divisibility by $(g+2)$, this approach runs into the problem that it forces further divisions of the edges into different types. This leads to additional restrictions together with additional variables. Instead of taking this path, we have decided to focus on the case of girth 5 where we have already argued that k-regular graphs of girth 5 and order $M(k, g)+2$ must be of diameter 3. After adding the restriction $g = 5$, we obtain the following constraints.

Lemma 3.14. *Let G be a $(k, 5)$-graph with excess 2, let x_1 be the number of vertices of G of type $d1$, x_2 be the number of vertices of type $d2$, and x_3 be the number of vertices of type $d3$. Then*

1. *if $x_1 > 0$, then $x_1 \geq 4$;*
2. *if $x_2 > 0$, then $x_2 \geq 4$;*
3. *x_3 must be divisible by 3.*

In particular, if no non-negative integers x_1, x_2, x_3 satisfying the conditions from Theorem 3.13 and the above conditions exist, no $(k, 5)$-graphs of excess 2 exist.

Proof. Let u be a vertex of type $d1$ and w_1, w_2 be the adjacent excess vertices associated with u. Then w_1 is only adjacent to $(k-1)$ of the k branches of the Moore tree rooted at u, and there exists a neighbor of u of distance greater than 2 from w_1. This vertex thus must be the second vertex from X_{w_1}, which makes w_1 into a type $d1$ vertex. Similarly, w_2 is also a type $d1$ vertex, and hence for any vertex of type $d1$, both of its excess vertices are of type $d1$, which (applied to w_1) yields that at least one of the neighbors of u is also of type $d1$; granting 4 vertices of type $d1$.

If u is of type d_2, the two vertices in $X_u = \{w_1, w_2\}$ share a neighbor, and hence $u \in X_{w_1}$, but $w_2 \notin X_{w_1}$. We claim that the second vertex in X_{w_1} different from u must be of distance 2 from u; which makes w_1 into a type $d2$ vertex. Since all edges incident to w_1 connect

w_1 to leaves of the Moore tree rooted at u, w_1 has to be connected to each branch of this Moore tree (it cannot be connected twice to the same branch as that would create a cycle of length 4), and hence all neighbors of u are of distance 2 from w_1, and therefore the vertex in X_{w_1} different from u is not a neighbor of u, neither it is of distance 3 from u, and hence it is of distance 2 from u; as claimed. By a symmetric argument, w_2 is a type $d2$ vertex as well, and consequently, if u is of type $d2$, so are both of its excess vertices. Applying the same argument to w_1 yields that both excess vertices associated with w_1 are also of type $d2$, and hence we obtain at least one more vertex of type $d2$ different from the vertices u, w_1 and w_2.

Consider finally a vertex u of type $d3$. The two excess vertices w_1, w_2 in X_u which ar of distance 3 one from the other are also the only two vertices of distance 3 from u, and hence the triple u, w_1, w_2 consists of vertices any two of which are of distance 3. Thus, $X_{w_1} = \{u, w_2\}$ and $X_{w_2} = \{u, w_1\}$, and all three vertices are of type $d3$. Any vertex u' of type $d3$ distinct from the vertices u, w_1, w_2 must then come with its own pair of type $d3$ vertices, and hence $X_u \cap X_{u'} = \emptyset$ and the claim $3 \mid x_3$ follows by induction. □

We have added the above conditions to our program looking for parameters $(k, 5)$ not allowing the existence of a $(k, 5)$-graph of order $M(k, 5) + 2$. In case $k = 3$, our program determined that the only possible triples (x_1, x_2, x_3) are $(9, 0, 3)$ and $(8, 4, 0)$; both of which are realized by the two graphs obtained from the Petersen graph. For $k = 4$, there are a number of triples satisfying the criteria, with the triple $(4, 12, 3)$ exhibited by Robertson's graph included. Thus, the restrictions obtained so far do not suffice to exclude impossible triples (x_1, x_2, x_3). This can be also seen from the fact that none of the degrees k, $5 \leq k \leq 11$, excluded by Eroh and Schwenk [31] are excluded by our program. Clearly, further, possibly more complicated, criteria would be needed in order to exclude more degrees k for which there exists no $(k, 5)$-graph of excess 2.

In the second part of this section, we consider graphs of excess 2 and even girth. One such example is the Möbius-Kantor graph of degree 3, girth 6, and order 16. This is once more an example of the situation where a Moore graph of degree 3 and girth 6 (and order smaller by 2) also exists; namely the Heawood graph. Note in addition, that the Möbius-Kantor graph is arc-transitive, and therefore must look the same with respect to every edge. Extending the definition of the excess sets to edges in the natural way, we define X_f to consist of all vertices of X whose distance from *both* end-points of the edge f is greater thant $\frac{g-1}{2}$. The arc-transitivity of the Möbius-Kantor graph yields that all subgraphs induced by the excess vertices of any of its edges are isomorphic to K_2. As we will see in the forthcoming paragraphs, this observation holds for all $(k, 6)$-graphs of order $M(k, 6) + 2$ (whether vertex-transitive or not).

Graphs of excess 2 and even girth are covered by Theorem 2.2. Thus, no excess 2 graphs exist for even girths greater than 6, for $k \equiv 5, 7 \pmod 8$, or for parameters which do not allow for a double cover of an incidence graph of a symmetric $(v, k, 2)$-design. As complete classification of $(v, k, 2)$-designs is not known, we focus in the remaining part of this section on counting cycles in $(k, 6)$-graphs of order $M(k, 6) + 2$. We will only consider those graphs that are bipartite due to Theorem 2.1.

Lemma 3.15. *Let $k \geq 4$. If G is a $(k, 6)$-graph of order $M(k, 6) + 2$, then G is bipartite,* *and*

1. $\overline{\mathbf{c}}_G(e,6) = (k-1)^3 - (k-1)$, *for all* $e \in E(G)$, *and*
2. $\overline{\mathbf{c}}_G(e,8) = (k-1)^3(k-2)^2 - k^3 + 6k^2 - 10k + 5$, *for all* $e \in E(G)$.

Proof. If G is a (k,g)-graph of order $M(k,g)+2$, then G is bipartite by Theorem 2.1. It follows that G cannot contain odd-length cycles, and consequently, none of the two extra vertices can be attached to both branches of the Moore tree rooted at e. The only way to achieve this is for each of the two vertices to be attached to just one branch of the tree. It is not possible, however, for either of the two vertices to be attached just to the leaves of one of the branches: since the branches consist of $(k-1)$ sub-branches, a vertex of degree k would have to be attached twice to the same sub-branch. But that would cause a cycle of length $g-2=4$ and violate the girth of G. It follows that each of the two extra vertices is attached to only $(k-1)$ sub-branches of a different branch of the Moore tree, and therefore the two extra vertices have to be adjacent – connected through an edge. This information is sufficient to guarantee that $\overline{\mathbf{c}}_G(e,6)$ is the same for each edge $e \in E(G)$. In fact, $\overline{\mathbf{c}}_G(e,6) = (k-1)^3 - (k-1)$ as exactly $(k-1)$ horizontal edges are lost due to the connections to the two extra vertices. As for the 8-cycles, they come in two kinds: those that consist of two 2-paths in the different branches of the Moore tree completed via a 3-path of horizontal edges (the same kind as in the Moore graph case) and those that pass through the two extra edges. The number of 8-cycles through e and the two extra edges can be easily seen to be equal to $(k-1)^2$. The number of 8-cycles 'lost' in comparison to the Moore graph is $(k-1)(k-2)^2$, and therefore

$$\overline{\mathbf{c}}_G(e,8) = (k-1)^3(k-2)^2 - (k-1)(k-2)^2 + (k-1)^2.$$

□

We see that the additional information about the potential graphs being necessarily bipartite (due to Theorem 2.1) yields a strong restriction on the structure of the graphs considered, and, in particular, implies the existence of a single type of excess set. This situation differs quite a bit from the case of odd girth considered in the first part of this section. Thus, one might expect the restrictions obtained from Lemma 3.15 to exclude at least some pairs (k,g). This is unfortunately not the case, as we have found no pairs (k,g) that could be excluded using the divisibility criteria of Lemma 3.15. On the other hand, the fact that all excess sets must be of the same structure suggests that vertex-transitive graphs may play an important role in this case.

We conclude the section with an analogue of Lemma 3.7.

Lemma 3.16. *If* $k \geq 3$ *and* $g \geq 3$ *is odd, such that a* (k,g)*-graph* G *of excess* 2 *and having at least one vertex of type* $d1$ *exists, then there is a* $(k-1,g)$*-graph of order* $k(k-1)^{(g-3)/2}+2$.

If $k \geq 3$ *and* $g \geq 6$ *is even, such that a* (k,g)*-graph* G *of excess* 2 *exists, then there is a* $(k-1,g)$*-graph of order* $2(k-1)^{(g-2)/2}+2$.

Proof. If u is a $d1$-type vertex, the desired graph is obtained by taking the subgraph of G induced by the leaves of the Moore tree rooted at u and the two excess vertices in X_u, and removing the edge between the excess vertices.

In the even girth case, the graph is bipartite and all excess pairs are joined by an edge, and so one can take the graph induced in G by the leaves of the Moore tree and the excess vertices of any vertex u of G. The edge connecting the two excess vertices must again be removed. □

3.4 Graphs of excess 3

Since both Moore bounds are even for odd degree k, no odd degree graphs of excess 3 exist. The only interesting cases are those where k is even.

Let k be even and g be odd. This case is not covered by any of the previously mentioned results, and the existence of (k,g)-graphs with even k, odd g, and order $M(k,g)+3$, is wide open. The smallest cage with excess 3 is the $(6,5)$-cage of order 40 obtained by removing the vertices of a Petersen graph from the Hoffman-Singleton graph (which is the unique $(7,5)$-cage). The graph is sometimes known as the Anstee-Robertson graph (it appeared for the first time in Robertson's thesis [70], and was independently discovered by Anstee [2]), but was first considered as a cage by O'Keefe and Wong [65].

If we assume that g is at least 5, only three of the four non-isomorphic graphs of order 3 can appear as subgraphs induced by the three excess vertices associated with any vertex of the graph (the 3-cycle is too short). Thus, the subgraphs induced by the excess vertices are either isomorphic to the graph $3K_1$ containing no edges, the union $K_2 \cup K_1$ containing exactly one edge, or the 2-path $\mathcal{P}_2$ of two edges. This makes for a relatively complicated situation, and we only list a result concerning cycles of length g.

Lemma 3.17. *Let $k \geq 3$ be even, $g \geq 5$ be odd, and G be a (k,g)-graph of order $M(k,g)+3$. Let x_1 denote the number of vertices u of G for which the subgraph induced by X_u is isomorphic to $3K_1$, x_2 denote the number of vertices u of G for which the subgraph induced by X_u is isomorphic to $K_2 \cup K_1$, and x_3 denote the number of vertices u of G for which the subgraph induced by X_u is isomorphic to $\mathcal{P}_2$. Then the following hold:*

1. $x_1 + x_2 + x_3 = M(k,g) + 3,$ *and*

2. *g divides the value*

$$(M(k,g)+3)\frac{k(k-1)^{\frac{g-1}{2}}}{2} - x_1\frac{3k}{2} - x_2(\frac{3k}{2}-1) - x_3(\frac{3k}{2}-2).$$

Proof. We have argued the first claim of the lemma prior to its statement. The second claim follows from counting horizontal edges in G with respect to u, as each horizontal edge corresponds to a unique g-cycle through u. If the subgraph induced by X_u is isomorphic to $3K_1$, the number of horizontal edges decreases by $3 \cdot \frac{k}{2}$, and is therefore equal to $\frac{k}{2}(k-1)^{\frac{g-1}{2}} - \frac{3k}{2}$ (where $\frac{k}{2}(k-1)^{\frac{g-1}{2}}$ would be the number of horizontal edges in a (k,g)-Moore graph). If the subgraph induced by X_u is isomorphic to $K_2 \cup K_1$, this number only decreases by $3 \cdot \frac{k}{2} - 1$, and if the subgraph induced by X_u is isomorphic to $\mathcal{P}_2$, it decreases by $3 \cdot \frac{k}{2} - 2$. The rest of the proof then follows by the usual argument. □

As has unfortunately repeatedly been the case before, the above lemma excludes no small pairs of parameters (k, g).

For the case of even k and even g, Theorem 2.1 contains a great deal of information concerning the structure of (k, g)-graphs. Interestingly, the part of the theorem that states that the excess for these graphs must be even is only stated informally in [15]. For the sake of completeness, we include a quick proof of the claim.

Because the first degree considered by Theorem 2.1 for the case of excess 3 is $k = 5$, and the first girth it applies to is 6, we do not know whether 4-regular graphs of even girth and excess 3 or k-regular graphs with even k and girth 4 and excess 3 necessarily have to be bipartite. All the other cases of even girth and even degree are covered by the next theorem.

Theorem 3.18. *Let $k, g \geq 6$ be both even. Then there exist no (k, g)-graphs of* odd *excess $e \leq k - 2$.*

Proof. Assume throughout that $k, g \geq 6$ are even, $g = 2m$. Applying Theorem 2.1 yields that all (k, g)-graphs of excess $e \leq k - 2$ are bipartite of diameter $m + 1$. Thus, all such graphs consist of a bipartite Moore tree rooted at an edge $\{u, v\}$ and extra vertices $w_1, w_2, \ldots, w_e$ of distance $m+1$ from either u or v. Due to the bipartiteness, u and v belong to different partition sets, and consequently the leaves of the Moore tree divide into two distinct partition sets based on whether they are of distance $m - 1$ from u or v. Also, each of the extra vertices $w_1, w_2, \ldots, w_e$ must belong to one of the partition sets. Since we have an odd number of them, the two partition sets contain different numbers of extra vertices. This means that the two leaf sets of the Moore tree are attached to distinct numbers of vertices from the set $w_1, w_2, \ldots, w_e$. This causes an imbalance contradicting the fact that all the edges emanating from one of the sets of leaves that are not adjacent to the vertices $w_1, w_2, \ldots, w_e$ must be paired with the edges emanating from the other set that are not adjacent to the vertices $w_1, w_2, \ldots, w_e$. The number of excess vertices must be even, and they have to split evenly between the two sets of leafs. $\square$

Corollary 3.19. *If $k = 3$ or $k \geq 5$, and $g \geq 6$ is even, then no (k, g)-graphs of excess 3 exist.*

Chapter 4

Improved lower bounds for the orders of even girth cages

The results of this chapter are published in [48]. Recall, the exact values $n(k,g)$ are not known for the majority of parameter pairs (k,g), and very few lower bounds on $n(k,g)$ exceeding the Moore bound exist. While Theorem 2.1 does not specifically exclude any parameter pairs (k,g), Theorem 2.2 only deals with (k,g)-graphs of excess 2. To our best knowledge, outside some small cases for which $n(k,g)$ has been determined and some few cases where the existence of graphs of excess greater than 2 has been proved by exhaustive computer search, no results excluding parameter pairs for excess larger than 2 for either odd or even g are known (i.e., there are no infinite families of pairs (k,g) for which it has been proven that $n(k,g) > M(k,g)+4$). Thus, results obtained in this chapter, which introduce infinite families of parameter pairs (k,g) for which do not exist any (k,g)-graphs of excess smaller than 5, are the first results of this type. Our arguments rely on Lemma 3.1 and Lemma 3.4.

The remaining argument is based on careful counting of cycles of length g in (potential) (k,g)-graphs of excess 4, and showing that, for certain classes of parameters, the resulting numbers violate the divisibility requirements of Lemma 3.4. In addition to obtaining results concerning graphs of excess 4, we prove that the excess grows without bounds for a meaningful but restricted family of (k,g)-graphs. While this last result does not appear suitable for generalization to all (k,g)-graphs, it should be viewed as further evidence for the Moore bound not being a tight bound in the majority of cases.

4.1 The structure of graphs of even girth and excess 4

In this section, we take on the case of (k,g)-graphs of degree $k \geq 6$, *even* girth $g = 2m \geq 6$, and excess 4. All of these graphs are covered by Theorem 2.1 and are therefore bipartite and of diameter $m+1$. Thanks to these results, the structure of G with respect to any edge $f = \{u,v\} \in E(G)$ can therefore be determined. Let $N_G(u,i)$ denote the i-th neighborhood of the vertex u, i.e., the set of vertices of G whose distance from u in G is equal to i. Since

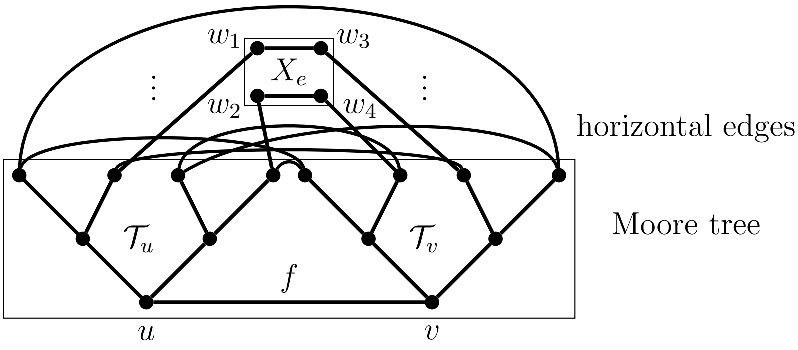

Figure 4.1: The Moore tree and some of the horizontal edges in a potential $(3, 6)$-graph of excess 4

the girth of G is assumed to be equal to g, the set of vertices of G whose distance from u is not larger than $\frac{g-2}{2}$ and whose distance from v is by one larger than their distance from u and the set of of vertices of G whose distance from v is not larger than $\frac{g-2}{2}$ and whose distance from u is by one larger than their distance from v must be disjoint and cannot contain any cycles. Thus, the subgraph of G induced by the first set (determined by u) induces a tree of depth $\frac{g-2}{2}$ rooted at u (we will call it $\mathcal{T}_u$), while the second set induces a tree of depth $\frac{g-2}{2}$ rooted at v (called $\mathcal{T}_v$); with $\mathcal{T}_u$ and $\mathcal{T}_v$ vertex disjoint. The degrees of u or v in their respective trees are equal to $(k-1)$, the degrees of all the non-leave vertices of these trees are equal to k, and all the leaves of these trees are of distance $\frac{g-2}{2}$ from their respective roots. As for the order of these subtrees, they are both of order $1+(k-1)+(k-1)^2+\ldots+(k-1)^{\frac{g-2}{2}}$, with $(k-1)^i$ vertices of distance i from u (or v). We will call the union of $\mathcal{T}_u$ and $\mathcal{T}_v$ together with the edge f the *Moore tree of G rooted at f*; it is the subtree of G that is the basis of the Moore bound for even g. Since G is assumed to be of excess 4, G must contain 4 additional vertices w_1, w_2, w_3, w_4 which do not belong to either $\mathcal{T}_u$ or $\mathcal{T}_v$, and whose distance from both u and v is greater than $\frac{g-2}{2}$. We will call these vertices *the excess vertices with respect to f* and denote this set $X_f = \{w_1, w_2, w_3, w_4\}$. Finally, we shall call the edges not contained in the Moore tree of G *horizontal edges*. The choice of our terminology becomes fairly obvious when consulting Figure 4.1.

We begin with a lemma that restricts the possible ways in which the four excess vertices are attached to the Moore tree.

Lemma 4.1. *Let $k \geq 6$, $g = 2m \geq 6$. Let G be a (k, g)-graph of excess 4, u, v be two adjacent vertices in G, and $X_f = \{w_1, w_2, w_3, w_4\}$ be the four excess vertices with respect to the edge $f = \{u, v\}$. The induced subgraph $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$ (two disjoint copies of K_2) or $\mathcal{P}_3$ (a path of length 3).*

Proof. As shown in Figure 4.1, the graph G consists of a Moore tree rooted at the edge $f = \{u, v\}$ and four excess vertices w_1, w_2, w_3, w_4. Each of these vertices must be attached to at least one of the two subtrees rooted at u or v (for the graph to be of diameter $m+1$), and none can be attached to both, since G is bipartite (and the leaf sets of $\mathcal{T}_u$ and $\mathcal{T}_v$ belong to different bipartite sets). Furthermore, none of the excess vertices can be joined to its corresponding subtree via more than $(k-1)$ edges; this is due to the fact that the excess vertices cannot be joined to any branch of the subtree more than once as multiple attachments would give rise to a cycle shorter than g, and to the fact that the subtrees $\mathcal{T}_u$ and $\mathcal{T}_v$ each consist of exactly $(k-1)$ branches.

The horizontal edges of G are of three kinds. First, there are the horizontal edges directly joining the leaf sets of $\mathcal{T}_u$ and $\mathcal{T}_v$. Second, there are the horizontal edges between the excess vertices w_1, w_2, w_3, w_4 and the leaf sets of $\mathcal{T}_u$ or $\mathcal{T}_v$ (but never simultaneously with both). Finally, there are the horizontal edges between the excess vertices themselves. Note that the number of edges incident with the leaves of $\mathcal{T}_u$ must match the number of edges incident with the leaves of $\mathcal{T}_v$. Thus, in order to balance and pair out the horizontal edges adjacent to the two leaf sets, the number of edges joining the excess vertices to either of the two subtrees must be the same. This easily yields that two of the excess vertices must be attached to one subtree and the other two to the other, and the two pairs belong to different bipartite sets. Without loss of generality, assume that w_1, w_2 are attached to the subtree rooted at u, and w_3, w_4 to the subtree rooted at v (Figure 4.1). Due to bipartedness, w_1 is not adjacent to w_2, and w_3 is not adjacent to w_4. Since the diameter of G is $m+1$, both w_1 and w_2 must be adjacent to at least one of w_3, w_4, and vice versa, both w_3 and w_4 must be adjacent to at least one of w_1, w_2. It follows that the induced subgraph $G[w_1, w_2, w_3, w_4]$ is bipartite, with the two sets consisting of w_1, w_2 and w_3, w_4, and each of its vertices is of degree at least 1. This leaves us with the possibility that all of its vertices are of degree 1, and hence $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$; one vertex in each set is of degree 1 and one is of degree 2, and $G[w_1, w_2, w_3, w_4]$ is isomorphic to $\mathcal{P}_3$; or all of it vertices are of degree 2, in which case $G[w_1, w_2, w_3, w_4]$ is isomorphic to the 4-cycle, which contradicts the assumption that the girth of G is at least 6. $\square$

The number of cycles through any edge of the graph depends now on the form of $G[w_1, w_2, w_3, w_4]$.

Lemma 4.2. *Let $k \geq 6$, $g = 2m \geq 6$. Let G be a (k, g)-graph of excess 4, u, v be two adjacent vertices in G, f be the edge $\{u, v\}$, and w_1, w_2, w_3, w_4 be the four excess vertices with respect to f.*

1. *if $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$, then $\overline{\mathbf{c}}_G(f, g) = (k-1)^m - 2k + 2$;*
2. *if $G[w_1, w_2, w_3, w_4]$ is isomorphic to $\mathcal{P}_3$, then $\overline{\mathbf{c}}_G(f, g) = (k-1)^m - 2k + 3$.*

Proof. Let us assume again that w_1, w_2 are attached to the subtree rooted at u, and w_3, w_4 to the subtree rooted at v.

If $G[w_1, w_2, w_3, w_4]$ is isomorphic to $2K_2$, the number of edges between w_1, w_2 and the corresponding leaves of the Moore tree is $2(k-1)$. Thus, the number of horizontal edges between the two sets of leaves of the Moore tree is equal to $(k-1)^m - 2(k-1)$ (with $(k-1)^m$ being the number of horizontal edges in a (potential) (k, g)-Moore graph). As pointed out before, each horizontal edge corresponds to exactly one g-cycle through f, and no other g-cycles through f exist. Thus, $\overline{\mathbf{c}}_G(f, g) = (k-1)^m - 2(k-1)$.

If $G[w_1, w_2, w_3, w_4]$ is isomorphic to $\mathcal{P}_3$, the number of horizontal edges between the two sets of leaves of the Moore tree is equal to $(k-1)^m - (k-1) - (k-2)$, and the result follows in exactly the same way as above. $\square$

In order to employ the above formulas, we would have to find significant restrictions on the number of edges of one type or the other. On the other hand, it is easy to find arithmetic conditions on k and g that exclude the existence of 'non-mixed' (k,g)-graphs of order $M(k,g)+4$ (by non-mixed we mean graphs that contain only edges for which $G[w_1,w_2,w_3,w_4]$ is isomorphic to $2K_2$ or only edges for which $G[w_1,w_2,w_3,w_4]$ is isomorphic to $\mathcal{P}_3$, but not both). Hence, the situation appears similar to that of the odd-girth graphs of excess 2. Fortunately, this is not the case. In what follows, we show that even-girth graphs of excess 4 and girth larger than 6 cannot be mixed when it comes to counting cycles of length g.

We begin our argument by counting g-cycles passing through vertices. In order to do this, we have to subdivide one of the possibilities considered above for edges (the case $2K_2$). For the first time, this will turn to our advantage.

Let u be a vertex of G incident with an edge $f=\{u,v\}$ for which the subgraph induced by X_f is isomorphic to $2K_2$. Two of the vertices in X_f are then of distance $\frac{g}{2}$ from u (let us denote them w_1,w_2) and two of them are of distance $\frac{g+2}{2}$ from u (say, w_3,w_4). The vertices w_3 and w_4 either share a neighbor (which necessarily has to belong to the set of vertices of distance $\frac{g-2}{2}$ from v), or they do not share a neighbor. It is important to note that if g is assumed to be greater than 4, w_3,w_4 cannot share more than one neighbor as that would lead to a 4-cycle. We say that u is of the *first* $2K_2$ *type* if w_3,w_4 share a neighbor, and we say that u is of the *second* $2K_2$ *type* if they do not. Having defined the types, we can now state the first lemma the proof of which is quite elementary. In analogy with the notation introduced previously for edges, $\mathbf{c}_G(u,g)$ stands for the *number of g-cycles in G rooted at the vertex u.*

Lemma 4.3. *Let $k\geq 6$, $g=2m\geq 6$. Let G be a (k,g)-graph of excess 4 and u be a vertex of G. Then,*

1. *if u is of the first $2K_2$ type, then*

$$\mathbf{c}_G(u,g)=((k-1)^m-2k+2)+((k-1)^{m-1}-2k)\cdot\binom{k-1}{2}+k^3-4k^2+5k-1;$$

2. *if u is of the second $2K_2$ type, then*

$$\mathbf{c}_G(u,g)=((k-1)^m-2k+2)+((k-1)^{m-1}-2k)\cdot\binom{k-1}{2}+k^3-4k^2+5k-2;$$

3. *if u is incident with an edge f whose excess set X_f is isomorphic to $\mathcal{P}_3$, then*

$$\mathbf{c}_G(u,g)=((k-1)^m-2k+3)+((k-1)^{m-1}-2k)\cdot\binom{k-1}{2}+k^3-4k^2+5k-2.$$

Proof. Let u be of the first $2K_2$ type with respect to the edge $f=\{u,v\}$. The g-cycles passing through u come in two kinds. First, there are the $(k-1)^m-2k+2$ g-cycles containing f as claimed in Lemma 4.2. Then there are g-cycles containing u but avoiding f. All of them have to consist of two disjoint $\frac{g-2}{2}$-paths starting at u and connected through a pair of edges attached to vertices of distance $\frac{g}{2}$ from u (the endpoints of the two paths) that share a vertex. The choice of these two final edges completely determines the g-cycles, so we will count the possible pairs of such edges. Both w_1 and w_2 are adjacent to $k-1$ vertices of distance $\frac{g-2}{2}$ from u, which gives us $2\binom{k-1}{2}$ g-cycles through w_1 or w_2. Of the $(k-1)^{\frac{g-2}{2}}$ vertices of distance $\frac{g-2}{2}$ from v, there is one adjacent to both vertices w_3, w_4, there are $2(k-2)$ vertices adjacent to exactly one of the vertices w_3, w_4, and the rest are not adjacent to either w_1 or w_2. It follows that the vertex adjacent to both w_3 and w_4 is incident with $k-3$ horizontal edges, and is therefore contained in $\binom{k-3}{2}$ g-cycles rooted at u. The other $2(k-2)$ vertices give rise to $2\binom{k-2}{2}$ g-cycles, and all the remaining vertices contribute $((k-1)^{\frac{g-2}{2}} - 2k + 3)\binom{k-1}{2}$ g-cycles through u. Adding all these cycles yields

$$\mathbf{c}_G(u, g) =$$

$$((k-1)^m - 2k+2) + ((k-1)^{m-1} - 2k+3) \cdot \binom{k-1}{2} + (2k-4)\binom{k-2}{2} + \binom{k-3}{2} + 2\binom{k-1}{2},$$

which matches the quantity claimed in Case 1.

If u is of the second $2K_2$ type, the situation differs only in a few spots. First, there are the $(k-1)^m - 2k + 2$ g-cycles containing f. The g-cycles not containing f contain either one of the w_1, w_2, and there are $2\binom{k-1}{2}$ of those, or they pass through the $2k-2$ vertices of distance $\frac{g-2}{2}$ from v and adjacent to w_3 or w_4, which contribute $(2k-2)\binom{k-2}{2}$ cycles, or they pass through vertices of distance $\frac{g-2}{2}$ from v adjacent to neither w_3 nor w_4 which finally contribute $((k-1)^{\frac{g-2}{2}} - 2k + 2)\binom{k-1}{2}$ g-cycles through u. Thus,

$$\mathbf{c}_G(u, g) = ((k-1)^m - 2k+2) + ((k-1)^{m-1} - 2k+2) \cdot \binom{k-1}{2} + (2k-2)\binom{k-2}{2} + 2\binom{k-1}{2},$$

and simple arithmetic yields the claim in Case 2.

Assume finally that u is incident with an edge $f = \{u, v\}$ whose excess set X_f is isomorphic to $\mathcal{P}_3$. Without loss of generality we may assume that w_1 is the vertex adjacent to both w_3 and w_4. In a way similar to the argument preceding this proof, the vertices w_3, w_4 cannot share a neighbor among the vertices of distance $\frac{g-2}{2}$ from v: they already share one neighbor, w_1, and the existence of another shared neighbor would cause the existence of a 4-cycle. The counting of cycles through u now follows the usual lines. There are $((k-1)^m - 2k + 3)$ cycles containing f (Lemma 4.2), $\binom{k-2}{2}$ cycles containing w_1, $\binom{k-1}{2}$ cycles containing w_2, $(2k-3)\binom{k-2}{2}$ cycles through the vertices of distance $\frac{g-2}{2}$ from v that are adjacent to w_3 or w_4, and $((k-1)^{m-1} - 2k + 3) \cdot \binom{k-1}{2}$ cycles through the vertices of distance $\frac{g-2}{2}$ from v that are not adjacent to w_3 or w_4:

$$\mathbf{c}_G(u, g) =$$

$$((k-1)^m - 2k+3) + ((k-1)^{m-1} - 2k+3) \cdot \binom{k-1}{2} + (2k-3)\binom{k-2}{2} + \binom{k-2}{2} + \binom{k-1}{2}.$$

□

A simple comparison of the three cases in Lemma 4.3 yields that the first and the third numbers match while the second is by one smaller than the other two. This means that no vertex can be simultaneously incident to edges from the first and second part or the second and third part (since the number of cycles through a vertex has to be unique).

Lemma 4.4. *Let $k \geq 6$, $g = 2m > 6$, and let G be a (k,g)-graph of excess 4. Then, G does not contain edges f for which their corresponding excess set X_f induces a subgraph isomorphic to $\mathcal{P}_3$.*

Proof. Suppose that G satisfies the above conditions, and, by means of contradiction, assume that the excess set X_f induces a subgraph isomorphic to $\mathcal{P}_3$ for some edge f of G. Let us stress right away that we are assuming that $g > 6$ and therefore G does not contain cycles of length 4 or 6. Let $f = \{u,v\}$, $X_f = \{w_1,w_2,w_3,w_4\}$, and the vertices adjacent to u but distinct from v be denoted by $v_1, v_2, \ldots, v_{k-1}$. Also, without loss of generality, assume that w_1 and w_2 are of distance $\frac{g}{2}$ from u and the vertex adjacent to both w_3 and w_4 is the vertex w_2. The number of edges connecting w_2 to the branch of height $\frac{g-2}{2}$ rooted at u is then $k-2$, and therefore w_2 is not attached to one of the sub-branches rooted at the neighbors $v_1, v_2, \ldots, v_{k-1}$ (i.e., w_2 is of distance greater than $\frac{g-2}{2}$ from one of the vertices $v_1, v_2, \ldots, v_{k-1}$). Again without loss of generality, we may assume that this special vertex is the vertex v_1. Let f' be the edge $\{u, v_1\}$. Since the distance of w_2 from v_1 is greater than $\frac{g}{2}$, the excess of f' contains the vertex w_2 together with the vertices w_3, w_4. It follows that the subgraph induced by $X_{f'}$ contains w_2, w_3 and w_4 and since w_2 is adjacent to both w_3 and w_4, the degree of w_2 in the induced subgraph must be 2, and hence the subgraph induced by $f' = \{u, v_1\}$ must be isomorphic to $\mathcal{P}_3$.

Next let f'' be the edge $\{u, v_2\}$. Then both w_1 and w_2 are of distance $\frac{g-2}{2}$ from v_2, and it is easy to see that the excess set of f'' must consist of the vertices w_3, w_4 and two vertices w_5, w_6 belonging to the branch rooted at v, of distance $\frac{g-2}{2}$ from v. We claim that the subgraph induced in G by the set $X_{f''} = \{w_3, w_4, w_5, w_6\}$ cannot be isomorphic to $\mathcal{P}_3$, as this would give rise to a 4-cycle formed by the vertices w_2, w_3, w_5, w_4 or the vertices w_2, w_3, w_6, w_4, depending on whether w_5 or w_6 would be of degree 2 in the induced subgraph. Hence, the subgraph of G induced by $X_{f''}$ is isomorphic to $2K_2$. We further claim that the vertices w_5, w_6 cannot share a neighbor, as if they did share a neighbor, this would give rise to a 6-cycle formed of the vertices w_2, w_3, w_4, w_5, w_6 and the shared neighbor. We conclude that the edge $f'' = \{u, v_2\}$ is of the second $2K_2$ type. This means that u is incident to $f' = \{u, v_1\}$ for which the subgraph induced by $X_{f'}$ is isomorphic to $\mathcal{P}_3$ and to $f'' = \{u, v_2\}$ which of the second $2K_2$ type. However, as pointed out in the discussion preceding this lemma, no vertex of G can be incident with an edge whose excess set induces $\mathcal{P}_3$ and and at the same time with an edge of the second $2K_2$ type. Therefore G cannot contain an edge whose excess set induces $\mathcal{P}_3$. □

4.2 Excluding parameter pairs for even girth and excess 4

Combining Lemma 4.4 with Lemma 4.2 immediately yields:

Lemma 4.5. *Let $k \geq 6$, $g = 2m > 6$, and let G be a (k,g)-graph of excess 4. Then g divides the number*

$$\frac{(M(k,g)+4)\cdot k}{2} \cdot ((k-1)^m - 2k + 2). \tag{4.1}$$

In order to employ this lemma and to exclude some parameter pairs (k,g) for which no (k,g)-graphs of excess 4 exist, we derive a number of simple divisibility results.

Lemma 4.6. *Let $k \geq 6$ and $g = 2m > 6$.*

1. *If $g = 2p$ such that $p > 3$ is prime number and $k \not\equiv 1, 2 \pmod p$, then $M(k,g)+4 \equiv 6 \pmod p$.*

2. *If $g = 4 \cdot 3^s$ such that $s \geq 1$ and k is divisible by 9, then*

   $M(k,g) + 4 \equiv 4 \pmod{3^s}$.

3. *If $g = 2p^2$ such that $p \geq 3$ is a prime number and k is an even number, $k \not\equiv 1, 2 \pmod p$, then $M(k,g) + 4 \equiv 6 \pmod p$.*

4. *If $g = 4p$ such that $p \geq 3$ is a prime number and $k \not\equiv 1, 2 \pmod p$, then $M(k,g)+4 \equiv 2k + 4 \pmod p$.*

5. *If $k \equiv 3 \pmod g$, then $M(k,g) + 4 \equiv 2 \cdot 2^{g/2} + 2 \pmod g$.*

Proof. We proceed case by case.

(1) Let $M(k,g) \equiv r \pmod p$. Since $M(k,g) = 2\left(\frac{(k-1)^{g/2}-1}{k-2}\right)$ and $(k-2,p) = 1$, we get

$$2((k-1)^{g/2} - 1) \equiv r(k-2) \pmod p.$$

Since $(k-1,p) = 1$, Fermat's Little Theorem asserts $(k-1)^p \equiv k-1 \pmod p$. Thus, $(k-2)(r-2) \equiv 0 \pmod p$. Due to the second restriction we have chosen for k, p does not divide $k-2$, and therefore it must divide $r-2$. Hence, $r \equiv 2 \pmod p$, which means that $M(k,g) + 4 \equiv 6 \pmod p$.

(2) Let $M(k,g) \equiv r \pmod{3^s}$. Since $(k-2, 3^s) = 1$. As above, we obtain

$$2((k-1)^{g/2} - 1) \equiv r(k-2) \pmod{3^s}.$$

Since $(k-1, 3^s) = 1$ and the Euler's totient function value $\varphi(3^s) = 2 \cdot 3^{s-1}$, Euler's Theorem yields

$$2((k-1)^{g/2} - 1) \equiv 2((k-1)^{2 \cdot 3^s} - 1) \equiv 2(1-1) \equiv 0 \pmod{3^s}.$$

Thus, $(k-2)r \equiv 0 \pmod{3^s}$, and consequently, $r \equiv 0 \pmod{3^s}$. Therefore $M(k,g) + 4 \equiv 4 \pmod{3^s}$.

(3) Following the same line of argument as above, $2((k-1)^{g/2}-1) \equiv r(k-2) \pmod p$. Since $(2, p^2) = 1$, using the multiplicativity of Euler's function we obtain

$$\varphi(g) = \varphi(2p^2) = \varphi(2) \cdot \varphi(p^2) = p^2(1 - \frac{1}{p}) = p^2 - p.$$

Since $(k-1, g) = 1$, applying Euler's Theorem implies $(k-1)^{\varphi(g)} \equiv (k-1)^{p^2-p} \equiv 1 \pmod g$ i.e. $(k-1)^{p^2} \equiv (k-1)^p \pmod g$. Thus, $r(k-2) \equiv 2((k-1)^p - 1) \pmod g$, and hence, $r(k-2) \equiv 2(k-2) \pmod p$. Since $(k-2, p) = 1$, $r \equiv 2 \pmod p$, and $M(k, g) + 4 \equiv 6 \pmod p$.

(4) Applying Fermat's Little Theorem yields $(k-1)^{g/2} \equiv (k-1)^{2p} \equiv ((k-1)^p)^2 \equiv (k-1)^2 \pmod p$. Therefore, $2((k-1)^{g/2}-1)) \equiv 2(k^2-2k+1-1) \equiv 2k(k-2) \pmod p$. From this follows that $(k-2)(r-2k) \equiv 0 \pmod p$ i.e. $r \equiv 2k \pmod p$, $M(k, g)+4 \equiv 2k+4 \pmod p$.

(5) Since $k-2 \equiv 1 \pmod g$, $M(k, g) \equiv 2((k-1)^{g/2}-1) \equiv 2 \cdot 2^{g/2} - 2 \pmod g$, i.e $M(k, g) + 4 \equiv 2 \cdot 2^{g/2} + 2 \pmod g$.

This completes the proofs for all cases of the lemma. □

We are finally ready to exclude infinite families of parameter pairs.

Theorem 4.7. *Let $k \geq 6$, $g = 2m > 6$. No (k, g)-graphs of excess 4 exist for parameters k, g satisfying at least one of the following conditions:*

(1) *$g = 2p$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod p$;*

(2) *$g = 4 \cdot 3^s$ such that $s \geq 4$, and k is divisible by 9 but not by 3^{s-1};*

(3) *$g = 2p^2$ with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod p$ and even;*

(4) *$g = 4p$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2, 3, p-2 \pmod p$;*

(5) *$g \equiv 0 \pmod{16}$, and $k \equiv 3 \pmod g$.*

Proof. Each of the cases of our proof starts by assuming that there exists a (k, g)-graph G of order $M(k, g) + 4$ whose parameters satisfy the corresponding conditions, after which we derive a contradiction with the divisibility of (4.1) by g from Lemma 4.5.

(1) Lemma 4.6 together with $(2, p) = 1$ yield $\frac{M(k,g)+4}{2} \equiv 3 \pmod p$. Since p divides neither k nor $k-1$, $k((k-1)^p - 2(k-1)) \equiv -k(k-1) \not\equiv 0 \pmod p$. Hence, neither factor of the left side of (9.1) is congruent to 0(mod $2p$), which contradicts (4.1).

(2) Lemma 4.6 forces $\frac{M(k,g)+4}{2} \equiv 2 \pmod{3^s}$. Since $\varphi(3^s) = 2 \cdot 3^{s-1}$, using Euler's Theorem, we obtain $(k-1)^{2 \cdot 3^{s-1}} \equiv 1 \pmod{3^s}$, and consequently, $k((k-1)^{2 \cdot 3^s} - 2(k-1)) \equiv -k(2k-3) \pmod{3^s}$. Since k is not divisible by 3^{s-1}, and since $k \equiv 0 \pmod 9$ yields that $2k-3$ is not divisible by 9, the product $-k(2k-3) \not\equiv 0 \pmod{3^s}$, and we obtain a contradiction with (4.1) again.

(3) The assumptions and Lemma 4.6 imply $\frac{M(k,g)+4}{2} \equiv 3 \pmod{p}$. Since $(k-1,p)=1$, using Euler's Theorem gives us $(k-1)^{p(p-1)} \equiv 1 \pmod{p}$, and therefore $k((k-1)^{p^2} - 2(k-1)) \equiv k((k-1)^p - 2(k-1)) \equiv -k(k-1) \pmod{p}$. Since p divides neither k nor $k-1$, we arrive at the usual contradiction with (4.1).

(4) Since p does not divide $k+2$, $\frac{M(k,g)+4}{2} \equiv k+2 \not\equiv 0 \pmod{p}$ by Lemma 4.6. Since p does not divide $k, k-1$, or $k-3$, we have $k((k-1)^{2p}-2(k-1)) \equiv k((k-1)^2-2k+2) \equiv k(k-1)(k-3) \not\equiv 0 \pmod{p}$. The two congruencies together yield a contradiction with (4.1).

(5) The congruence $k \equiv 3 \pmod{g}$ implies $(k-1)^{g/2} - 2k + 2 \equiv 2^{g/2} - 4 \pmod{g}$, while Lemma 4.6 yields $\frac{(M(k,g)+4)k}{2} \equiv \frac{3(2 \cdot 2^{g/2}+2)}{2} \pmod{g}$. Hence,

$$\frac{(M(k,g)+4)k}{2}((k-1)^{g/2} - 2k + 2) \equiv$$
$$\equiv \frac{3(2 \cdot 2^{g/2}+2)}{2}(2^{g/2} - 4) \equiv 3(2^{g/2}+1)(2^{g/2}-4) \pmod{g}.$$

Using $g \equiv 0 \pmod{16}$ gives us $\frac{g}{2} \geq 8$, and therefore $2^{g/2}$, $2^{g/2+2}$, and 2^g are all congruent to 0 modulo 16, which implies $\frac{(M(k,g)+4)k}{2}((k-1)^{g/2} - 2k+2) \equiv 4 \pmod{16}$, i.e., $\frac{(M(k,g)+4)k}{2}((k-1)^{g/2} - 2k + 2)$ is not congruent to 0 modulo g, and we obtain a contradiction with (4.1).

This completes the proofs. □

The non-existence of (k,g)-graphs of excess 4 with parameters satisfying the conditions of the above theorem does not immediately imply that the excess of a (k,g)-cage must be larger than 4. Nevertheless, combining the above result with the previously known restrictions does imply such conclusion for all of the above parameter pairs. Specifically, as noted in the introduction, there are no Moore graphs of girth 10 or girth greater than 12. Furthermore, Theorem 2.1 claims the non-existence of even girth graphs of excess 1 and degree $k \geq 3$ as well as excess 3 and degree $k \geq 5$. Finally, Theorem 2.2, excludes the possibility of even girth graphs of girth greater than 6 and excess 2. These results, combined with Theorem 4.7 yield the following.

Corollary 4.8. *Let (k,g) be one of the pairs of parameters listed in Theorem 4.7. Then, $n(k,g) \geq M(k,g)+5$, for k even, and $n(k,g) \geq M(k,g)+6$, for k odd.*

Proof. We have proved the corollary prior to its statement for all $g > 12$. The only pair (k,g) covered by Theorem 4.7 that cannot be excluded based on the above arguments is the pair $(3,10)$. However, $n(3,10)$ is known to be equal to 70 (see e.g. [33]), while $M(3,10) = 62$. Hence, the claim is true for the pair $(3,10)$ as well. The case of odd k follows from the fact that the Moore bound for even g is even, and the order of a k-regular graph with odd k must be even. □

4.3 Graphs of even girth and excess larger than 4

It has been observed in the previous chapter that in case of odd degree, even girth, and excess 2, all subgraphs induced by edge excess vertices are isomorphic to K_2. In the previous section, we have proved that in case of even girth greater than 6 and excess 4, all subgraphs induced by the edge excess sets must be isomorphic to $2K_2$. If one was willing to see a pattern in these observations, one might be tempted to try to prove that the edge excess set induced subgraphs of graphs with small excess and large even girth must always be isomorphic to tK_2, for some $t \geq 1$. Graphs of such structure play a prominent role in [15] and are in a way the extreme (k,g)-graphs with odd k and even g and the property that each subgraph X_f induced by the $e = 2t$ excess vertices associated with an edge f contains the minimum necessary number of edges, namely t edges. These are also the graphs that maximize the number of girth cycles through any edge of the graph. In this last section of this chapter, we prove that for any arbitrarily large excess e there exist parameters k and g with the property that the excess of all (k,g)-graphs from our restricted family exceeds e.

Lemma 4.9. *Let $k \geq 6$, $g = 2m \geq 6$, and let G be a (k,g)-graph of even excess $e = 2t \leq k-2$. If f is an edge of G with excess set X_f of size $2t$ and the subgraph induced by X_f in G consists of t copies of K_2, then*

$$\overline{\mathbf{c}}_G(f,g) = (k-1)^m - t(k-1).$$

Proof. The proof is almost identical to that of Lemma 4.2, Part 1. □

Theorem 4.10. *For every $e \geq 1$, there exist parameters k, g, k odd, g even, such that if G is a (k,g)-graph satisfying the property that for every edge f of G the subgraph induced by X_f in G is isomorphic to disjoint copies of K_2's, then G has excess larger than e.*

Proof. Let m be a prime larger than e, and also large enough to admit the existence of an odd k such that $e+2 < k < m$ and $k \equiv 5$ or $7 \pmod 8$. Take $g = 2m$, and assume that G is a (k,g)-graph satisfying the property from our theorem. We claim that the excess of G must be larger than e. To see this, assume to the contrary that the excess of G is $e' \leq e$. Then $e' < k-2$, and Theorem 2.1 asserts that G is bipartite, in which case we know that $e' = 2t'$, for some integer t'. Employing Lemma 4.9 yields $\overline{\mathbf{c}}_G(f,g) = (k-1)^m - t'(k-1)$, for all edges $f \in E(G)$, and therefore g divides $\frac{(M(k,g)+e')\cdot k}{2} \cdot ((k-1)^m - t'(k-1))$. Since $M(k,g) = 2\frac{(k-1)^m-1}{k-2}$, the girth $g = 2m$ of G, and therefore also the prime m, divide the product

$$\frac{(2\frac{(k-1)^m-1}{k-2} + e')\cdot k}{2} \cdot ((k-1)^m - t'(k-1)).$$

We claim, however, that neither of the two factors of this product is divisible by m. We prove our claim separately for each of the factors. Since m is a prime, it follows from Fermat's Little Theorem that $(k-1)^m \equiv k-1 \pmod m$, and therefore

$$\frac{(2\frac{(k-1)^m-1}{k-2} + e')\cdot k}{2} \equiv \frac{(2+e')}{2}\cdot k \pmod m.$$

Since $2 \leq e' + 2 \leq e + 2 < k < m$, $\frac{(2+e')}{2} \equiv (1 + t') \not\equiv 0 \pmod{m}$. Similarly, the choice $e + 2 < k < m$ yields $k \not\equiv 0 \pmod{m}$, and thus neither m nor g divide the first of the factors. Employing Fermat's Little Theorem again, $(k-1)^m - t'(k-1) \equiv (k-1) \cdot (1-t') \pmod{m}$. Note that our choice of $k \equiv 5$ or $7 \pmod{8}$ allows us to use Theorem 2.2 and conclude that $e' \neq 2$, hence $t' \neq 1$, and $(1-t') \not\equiv 0 \pmod{m}$. As $k-1$ is also not divisible by m, the factor $(k-1)^m - t'(k-1)$ is not divisible by m either. Since none of the factors is congruent to 0 modulo m, the product $(M(k,g) + e') \cdot \frac{k}{2} \cdot ((k-1)^m - t'(k-1))$ is not divisible by g, and we obtain a contradiction. The excess of G is therefore bigger than e. □

If one were able to prove that (a sufficient portion) of the (k,g)-graphs whose parameters satisfy the conditions stated at the beginning of the proof of Theorem 4.10 *must* have the structure described in the statement of the theorem, the above result would yield that for each excess $e > 0$, there exist parameters (k,g) with the property that the excess of any (k,g)-graph G exceeds e. The existence of such parameter pairs for arbitrarily large e has already been established for the (much more restricted) family of vertex-transitive (k,g)-graphs [12], but has only been conjectured for the case of general cages. Using as further evidence the excess of the best known (k,g)-graphs listed in the tables of [33], the existence of (k,g)-cages of arbitrarily large excess feels like a foregone conclusion. Nevertheless, any such proof has been elusive so far, and the conjecture, though widely believed, stays frustratingly unproved.

Chapter 5

On bipartite graphs of excess 4

The results of this chapter are published in [36]. Motivated by the result in Theorem 4.7, which was obtained through counting cycles in a hypothetical graph with given parameters and excess 4, in this chapter we address the question of the existence of (k, g)-graphs of excess 4 using spectral properties of their adjacency matrices. The question of the existence of (k, g)-graphs of excess 4 is wide open, and prior to the results given in the previous chapter, no such results were known. The results contained in this chapter extend further our understanding of the structure of the potential graphs of excess 4.

5.1 Necessary conditions for the existence of graphs of even girth and excess 4

Let us assume that $k \geq 6$, $g = 2d \geq 6$ and G is a (k, g)-graph of excess 4 and order n. Due to the result of Biggs stated in Theorem 2.1, the restriction of the parameters k, g given above allows us to conclude that G is a bipartite graph with diameter $d+1$. For each integer i in the range $0 \leq i \leq d+1$, we consider the distance matrices A_i. Clearly, $A_0 = I, A_1 = A$, the usual adjacency matrix of G. The last non-zero matrix is the matrix A_{d+1}, which we denote by E and refer to it as the *excess matrix*, that is, E is the adjacency matrix of the graph with the same vertex set V as G such that two vertices of V are adjacent if and only if they are at distance $d + 1$. We call this graph the *excess graph* of G and we denote it $G(E)$. If J is the all-ones matrix, the sum of the *i-distance matrices* A_i, for $0 \leq i \leq d$, with the matrix E satisfies the identity

$$\sum_{i=0}^{d} A_i + E = J.$$

Let us define the following polynomials:

$$F_0(x) = 1,\ F_1(x) = x,\ F_2(x) = x^2 - k;$$

$$G_0(x) = 1,\ G_1(x) = x + 1;$$

$$H_{-2}(x) = -\tfrac{1}{k-1},\ H_{-1}(x) = 0,\ H_0(x) = 1,\ H_1(x) = x;$$

$$P_{i+1}(x) = xP_i(x) - (k-1)P_{i-1}(x) \text{ for } \begin{cases} i \geq 2, & \text{if } P_i = F_i, \\ i \geq 1, & \text{if } P_i = G_i, \\ i \geq 1, & \text{if } P_i = H_i. \end{cases} \tag{5.1}$$

In [72], Singleton gives many relationships between these polynomials. We use two of them. Given any $i \geq 0$,

$$G_i(x) = \sum_{j=0}^{i} F_j(x), \tag{5.2}$$

$$G_{i+1}(x) + (k-1)G_i(x) = (x+k)H_i(x). \tag{5.3}$$

The above defined polynomials have a close connection to the properties of a graph G. Namely, for $t < g$, the element $(F_t(A))_{x,y}$ counts the number of paths of length t joining vertices x and y of G. It follows from (5.2) that $G_t(A)$ counts the number of paths of length at most t joining pairs of vertices in G. All of the preceding claims can be found in Delorme, Jørgensen, Miller and Villavicencio [27].

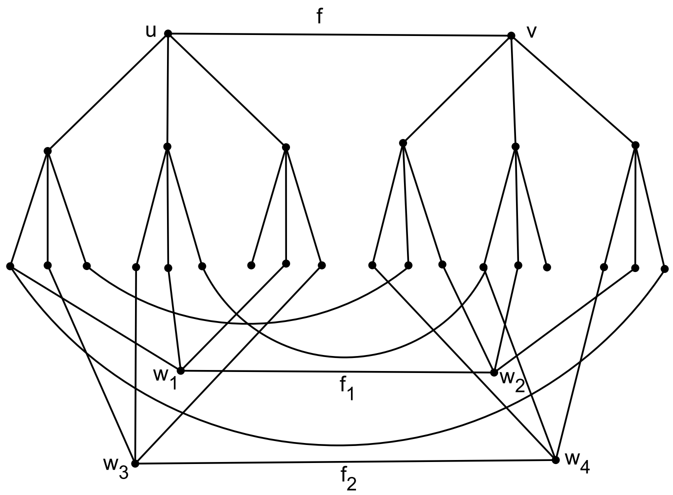

Figure 5.1: The Moore tree and some of the horizontal edges in a potential (4, 6)-graph of excess 4

The next lemma is based on the structure of G described in Lemma 4.1.

Lemma 5.1. *Let $k \geq 6$ and $g = 2d \geq 6$, and let G be a (k,g)-graph of excess 4. If A is the adjacency matrix of G and E is the excess matrix of G, then*

$$F_d(A) = kA_d - AE.$$

Proof. Let $f = \{u, v\}$ be a base edge of the Moore tree and let $f_1 = \{w_1, w_2\}, f_2 = \{w_3, w_4\}$ be the edges of the subgraph induced by X_f. Also, let us assume that $d(u, w_1) = d(u, w_3) =$

d and $d(u,w_2) = d(u,w_4) = d+1$. We consider the case when $G[w_1,w_2,w_3,w_4]$ is isomorphic to $2K_2$, in which case the excess vertices do not share a common neighbour. The other cases when $G[w_1,w_2,w_3,w_4]$ is isomorphic to $2K_2$ and the excess vertices share a common neighbour or the subgraph induced by the excess vertices contains $\mathcal{P}_3$ are analogous. Since there are $k-1$ paths of length d from u to w_1 and w_3, by the definition of $F_i(x)$, we have $(F_d(A))_{u,w_1} = (F_d(A))_{u,w_3} = k-1$. Considering the vertices at distance d from u, there are also the $(k-1)^{d-1}$ leaves of the subtree rooted at v. For $2(k-1)$ of these vertices, there exist $k-1$ paths of length d from u to them. Namely, they are the vertices adjacent to w_2 or w_4. For all the other leaves, there are k paths between them and u. Thus, $(F_d(A))_{u,s} = 0$ if $d(u,s) \neq d$, $(F_d(A))_{u,s} = k$ if s is a leaf of a branch rooted at v and not adjacent to w_2 and w_4, and $(F_d(A))_{u,s} = k-1$ if s is w_1, w_3 or a leaf of a branch rooted at v and adjacent to w_2 or w_4. This yields the matrix kA_d, such that $(kA_d)_{u,s} = k$ if $d(u,s) = d$ and $(kA_d)_{u,s} = 0$ if $d(u,s) \neq d$. Now, let s be a vertex of G such that $d(u,s) = d$ and s is adjacent to w_2 or w_4. If $s = w_1$ or $s = w_3$, then it is easy to see that $(AE)_{u,s} = 1$. On the other hand, since s is adjacent to the subtree rooted at u through $k-2$ different horizontal edges, it follows that, between the $k-1$ branches of the subtree rooted at u, there exists one sub-branch that is not adjacent to s through a horizontal edge. Let s_1 be the root of that sub-branch. Then, $d(s,s_1) = d+1$ and $d(u,s_1) = 1$, which implies $(A)_{u,s_1} = 1$ and $(E)_{s_1,s} = 1$. Let s_2 be the other vertex at distance $d+1$ from s. Because all neighbours of u, except s_1, are at distance smaller than $d+1$ from s, we have $(A)_{u,s_2} = 0$ and $(E)_{s_2,s} = 1$. Thus $(AE)_{u,s} = 1$. If s is a vertex of G such that $d(u,s) = d$ and s is not adjacent to w_2 or w_4, then the distance between s and the neighbours of u is $d-1$. In this case, $(AE)_{u,s} = 0$. If $d(u,s) \neq d$, then the distance between s and the neighbours of u is different from $d+1$, and therefore $(AE)_{u,s} = 0$. The required identity follows from summing up the above conclusions. □

Lemma 5.2. *Let $k \geq 6$ and $g = 2d \geq 6$, and let G be a (k,g)-graph of excess 4. If A is the adjacency matrix of G, E is the excess matrix of G and J is the all-ones matrix, then*

$$kJ = (A + kI)(H_{d-1}(A) + E).$$

Proof. By the definition of the polynomials $G_i(x)$ and using the fact that G has diameter $d+1$, we conclude $J = G_{d-1}(A) + A_d + E$. The relation (5.2), setting $i = d$, asserts $G_d(A) = G_{d-1}(A) + F_d(A)$. Substituting this identity in (5.3), where we fix $i = d-1$, we get $kG_{d-1}(A) + F_d(A) = (A+kI)H_{d-1}(A)$. Due to Lemma 5.1 the last identity is equivalent to $kG_{d-1}(A) + kA_d + kE = (A+kI)(H_{d-1}(A) + E)$. From $kJ = kG_{d-1}(A) + kA_d + kE$ follows $kJ = (A + kI)(H_{d-1}(A) + E)$. □

The next theorem gives a relationship between the eigenvalues of the matrices A and E (this result is an analogue of Theorem 3.1 in Delorme, Jørgensen, Miller and Villavicencio [27]).

Theorem 5.3. *If $\mu(\neq \pm k)$ is an eigenvalue of A, then*

$$H_{d-1}(\mu) = -\lambda,$$

where λ is an eigenvalue of E.

Proof. Let us suppose that μ is an eigenvalue of A. Since G is a k-regular graph, the all-ones matrix J is a polynomial in A. This implies that any eigenvector of A is also an eigenvector of J. From $kJ = (A + kI)(H_{d-1}(A) + E)$ and since $H_{d-1}(A)$ is also a polynomial in A, we have that E is a polynomial in A, and consequently, every eigenvector of A is an eigenvector of E. Therefore, the eigenvalues of kJ are of the form $(\mu + k)(H_{d-1}(\mu) + \lambda)$. As is well known, the eigenvalues of kJ are kn (with multiplicity 1) and 0 (with multiplicity $n - 1$). The eigenvalue kn corresponds to $\mu = k$, and so all the remaining eigenvalues, except for $-k$, satisfy the above equation. □

Since the eigenvalues of a disjoint union of cycles are known, we are now in a position to determine the spectrum of A.

Lemma 5.4. *Let $k \geq 6$ and $g = 2d \geq 6$, and let G be a (k, g)-graph of excess 4. If A and E are, respectively the adjacency matrix and the excess matrix of G, then:*

(1) The matrix E is the adjacency matrix of a graph $G(E)$, consisting of a disjoint union of c cycles C_i of length l_i with $1 \leq i \leq c$. Moreover, if d is odd and V_1 and V_2 are the two partition sets of the bipartite graph G, then every cycle in $G(E)$ is completely contained either in V_1 or V_2.

(2) The spectrum of A consists of:

(2.1) $\pm k$, $c-2$ solutions of $H_{d-1}(x) = -2$, and one solution of each equation $H_{d-1}(x) = -2\cos(\frac{2\pi j}{l_i})$, for $j = 1, ..., l_i - 1, 1 \leq i \leq c$ and d odd.

(2.2) $\pm k$, $c - 1$ solutions of $H_{d-1}(x) = -2$, and one solution of each equation (except one) $H_{d-1}(x) = -2\cos(\frac{2\pi j}{l_i})$, for $j = 1, ..., l_i - 1, 1 \leq i \leq c$ and d even.

Proof. (1) Our proof is analogous to that of Kovács [51] for girth 5, and Garbe's proof [44] for odd girth $g = 2k + 1 > 5$. Let $f = \{u, v\}$ be a base edge of a bipartite Moore tree of G. Lemma 4.1 asserts that there exist exactly two vertices of G at distance $d + 1$ from u. Namely, they are the excess vertices adjacent to the leaves of the subtree rooted at v. The excess matrix E is the adjacency matrix for the graph $G(E)$ with same vertex set V as G such that two vertices of $G(E)$ are adjacent if and only if they are at distance $d + 1$. Because, for each vertex $u \in V(G)$, there are exactly two vertices at distance $d + 1$ from u, every component of $G(E)$ is a cycle. Let c be the number of these cycles and let l_i, for $i = 1, ..., c$, be the lengths of these cycles ordered in an arbitrary manner. Moreover, if d is an odd number, any two vertices of G at distance $d + 1$ lie in the same partite set. Therefore, any connected component of $G(E)$ is entirely contained either in V_1 or V_2.
(2) The eigenvalues of an n-cycle are known and are equal to $2\cos(\frac{2\pi j}{n})$, for $j = 0, ..., n - 1$. Therefore the eigenvalues of $G(E)$ are $2\cos(\frac{2\pi j}{l_i})$, for $j = 0, 1, ..., l_i - 1$ and $1 \leq i \leq c$, (see Garbe [44]). Since G is a k-regular bipartite graph, it has (among others) the eigenvalues k and $-k$. Let V_1 and V_2 be the partition sets of G. Hence, the eigenvector of A corresponding to k consists of the all-ones vector j, and the eigenvector corresponding to $-k$ is the vector j' with values 1 on V_1 and values -1 on V_2. If d is an odd number, then two vertices of $G(E)$ are adjacent if and only if they are in the same partite set. Therefore $E \cdot j' = 2j'$, which implies that from the set of c solutions on $H_{d-1}(x) = -2$, we need to subtract two

multiplicities for the eigenvalues k and $-k$. If d is an even number, then two vertices of $G(E)$ are adjacent if and only if they are in different partite sets. Thus $E \cdot j' = -2j'$. In this case, from the set of c solutions on $H_{d-1}(x) = -2$, we need to subtract one multiplicity for the eigenvalue k and from the set of all solutions on $H_{d-1}(x) = 2$, we need to subtract one multiplicity for the eigenvalue $-k$. □

Lemma 5.5. *Let $k \geq 6$ and $g = 2d \geq 6$ and let G be a (k, g)-graph of excess 4. Let c be the number of cycles of $G(E)$ and c_2 be the number of cycles of even length. Then:*

(1) If $H_{d-1}(x) - 2$ is irreducible over $\mathbb{Q}[x]$, then $d - 1$ divides $c - 1$ or $c - 2$.

(2) If $H_{d-1}(x) + 2$ is irreducible over $\mathbb{Q}[x]$, then $d - 1$ divides $c_2 - 1$ or c_2.

Proof. (1) Combining Theorem 5.3 and Lemma 5.4 (2), we obtain that $H_{d-1}(x) - 2$ is an irreducible factor of the characteristic polynomial of A. Realizing that all the roots of an irreducible factor of a characteristic polynomial of a given rational symmetric matrix have the same multiplicities, (see Kovács [51]), from Lemma 5.4 (2) we have the following: If d is an even number, then the $d - 1$ roots of $H_{d-1}(x) - 2$ have multiplicity $\frac{c-1}{d-1}$, which has to be a positive integer. If d is odd, then the $d - 1$ roots have multiplicity $\frac{c-2}{d-1}$.
(2) This proof follows the same reasoning as (1). □

We can base the testing of irreducibility of $H_{d-1}(x) \pm 2$ on the well known Eisenstein's criterion that asserts for a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ and a prime p that divides a_i for all $0 \leq i < n$, does not divide a_n and p^2 does not divide a_0. Now we are ready for the main result of this section.

Theorem 5.6. *Let $k(\geq 7)$ be an odd number and let $g = 2d \geq 8$. Let c be the number of cycles of $G(E)$ and c_2 be the number of cycles with even length. If there exists a (k, g)-graph of excess 4, then:*

(1) If d is an odd number, then $d - 1$ divides $c - 2$ and c_2.

(2) If d is an even number, then $d - 1$ divides $c - 1$ and $c_2 - 1$.

Proof. According to Lemma 5.5, it is enough to prove that the polynomials $H_{d-1}(x) - 2$ and $H_{d-1}(x) + 2$ are irreducible. We prove, using induction on $d \geq 4$, that $H_{d-1}(x) = x^{d-1} + (k-1)P_{d-3}(x)$, where $P_{d-3}(x)$ is an integer polynomial of degree $d-3$. We calculate $H_3(x) = x^3 - 2(k-1)x$. Let us suppose that the above formula holds for $H_{d-2}(x)$ and $H_{d-3}(x)$. That yields

$$\begin{aligned} H_{d-1}(x) = x(x^{d-2} + (k-1)P_{d-4}(x)) - (k-1)(x^{d-3} + (k-1)P_{d-5}(x)) = \\ = x^{d-1} + (k-1)P_{d-3}(x). \end{aligned}$$

Therefore, $H_{d-1}(x) \pm 2 = x^{d-1} + (k-1)P_{d-3}(x) \pm 2$. By the induction hypothesis, it follows that $H_{d-1}(0) = (-1)^{\frac{d-1}{2}}(k-1)^{\frac{d-1}{2}}$ for an odd d, and $H_{d-1}(0) = 0$ for an even d. Hence, for an odd $d(\geq 5)$ $|(-1)^{\frac{d-1}{2}}(k-1)^{\frac{d-1}{2}} \pm 2|$ is not divisible by 2^2, and clearly for an even $d(\geq 4)$, ± 2 is not divisible by 2^2. Since $k-1$ is even, it follows that every coefficient on $H_{d-1}(x) \pm 2$ except for the coefficient 1 of x^{d-1} is divisible by 2. Thus, the conditions of the Eisenstein's criterion are satisfied, and $H_{d-1}(x) \pm 2$ is irreducible. □

5.2 The non-existence of bipartite graphs of cyclic or bicyclic excess

In this section we deal with the same family of graphs considered in the previous section. Again, let $k \geq 6$ and $g = 2d \geq 6$, and let G be a (k, g)-graph of excess 4 and order n. Clearly, n is an even number. We proved that the excess graph $G(E)$ consists of a disjoint union of c cycles C_i, for $1 \leq i \leq c$. If $c = 1$ and $G(E)$ consists of an n-cycle, then G is of *cyclic excess* 4, and if $c = 2$ and $G(E)$ consists of a disjoint union of two cycles, then G is of *bicyclic excess* 4. These are the graphs we study in this section. Note that there is no graph G with cyclic excess 4 if d is an odd number; in this case, we showed that each cycle of $G(E)$ is completely contained either in V_1 or V_2.

Let d be an even number and let L_n be an n-cycle formed by the vertices of $G(E)$. If A' is the adjacency matrix of L_n, its characteristic polynomial $\chi(L_n, x)$ satisfies $\chi(L_n, x) = (x-2)(x+2)(R_n(x))^2$, where R_n is a monic polynomial of degree $\frac{n}{2} - 1$. Consider the factorization $x^n - 1 = \prod_{l|n} \Phi_l(x)$, where $\Phi_l(x)$ denotes the *l-th cyclotomic polynomial*. In the following paragraph, we summarize the properties of cyclotomic polynomials as listed in Delorme and Villavicencio [28].
The cyclotomic polynomial $\Phi_l(x)$ has integral coefficients, it is irreducible over $\mathbb{Q}[x]$, and it is *self-reciprocal* ($x^{\phi(l)}\Phi_l(1/x) = \Phi_l(x)$). From the irreducibility and the self-reciprocity of $\Phi_l(x)$ follows that the degree of $\Phi_l(x)$ is even for $l \geq 2$.
Thus, we obtain the following factorization of $R_n(x)$: $R_n(x) = \prod_{3 \leq l|n} f_l(x)$, where f_l is an integer polynomial of degree $\frac{\phi(l)}{2}$ satisfying $x^{\phi(l)/2} f_l(x + 1/x) = \Phi_l(x)$. Also, f_l is irreducible over $\mathbb{Q}[x]$, $f_3(x) = x+1, f_4(x) = x, f_5(x) = x^2 + x - 1$ and $f_6(x) = x - 1$. Substituting $y = -H_{d-1}(x)$ into $\frac{\chi(L_n,y)}{(y-2)}$, we obtain a polynomial $F(x)$ of degree $(n-1)(d-1)$, which satisfies $F(A)u = 0$ for each eigenvector u of A orthogonal to the all -one vector. Then, $F_{l,k,d-1}(x) = f_l(-H_{d-1}(x))$ yields

$$F(x) = (-H_{d-1}(x) + 2) \prod_{3 \leq l|n} (F_{l,k,d-1}(x))^2.$$

Lemma 5.7. *Let $g = 2d > 6$, and $l \geq 3$ be a divisor of n. If there is a (k, g)-graph with cyclic excess 4 and order n, then $F_{l,k,d-1}(x)$ must be reducible over $Q[x]$.*

Proof. The degree of $F_{l,k,d-1}(x)$ is equal to $(d-1)\frac{\phi(l)}{2}$. If $F_{l,k,d-1}(x)$ is irreducible over $\mathbb{Q}[x]$, then all its roots must be eigenvalues of A. Employing Observation 3.1. from Delorme and Villavicencio [28], we conclude that there are at most $\phi(l)$ roots of $F_{l,k,d-1}(x)$ that are eigenvalues of A. Thus $(d-1)\frac{\phi(l)}{2}{=}\phi(l)$, that is, $d = 3$. This contradicts the assumption that $2d > 6$. □

Note that $deg(F_{l,k,d-1}(x)) = d - 1$ if and only if $\phi(l) = 2$, that is, if and only if $l \in \{3, 4, 6\}$.

Lemma 5.8. *Let $k \geq 6$ and $g = 2d > 6$, and let n be the order of a (k, g)-graph with cyclic excess 4.*

(1) If $n \equiv 1 \pmod 3$, then $H_{d-1}(x) - 1$ must be reducible over $\mathbb{Q}[x]$.

(2) *If* $n \equiv 0 \pmod 4$, *then* $H_{d-1}(x)$ *must be reducible over* $\mathbb{Q}[x]$.

(3) *If* $n \equiv 0 \pmod 6$, *then* $H_{d-1}(x) + 1$ *must be reducible over* $\mathbb{Q}[x]$.

Proof. It follows directly from Lemma 5.7, with the additional assumptions $f_3(x) = x + 1, f_4(x) = x$ and $f_6(x) = x - 1$. □

If $n \equiv 0 \pmod 4$, then using the formula for the order of G, $d-1$ must be odd. On the other hand, since $H_1(x) = x, H_3(x) = x^3 - 2(k-1)x$ and $H_{d-1}(x) = xH_{d-2}(x) - (k-1)H_{d-3}(x)$, we see that if $d-1$ is an odd number, then x divides $H_{d-1}(x)$, which implies that $H_{d-1}(x)$ is reducible. Therefore, (2) holds.
The irreducibility of the polynomials $H_{d-1}(x) - 1$ over $\mathbb{Q}[x]$ is examined in Delorme, Jørgensen, Miller and Villavicencio [27], where it is analytically proven that these polynomials are irreducible for $d \in \{4, 6, 8\}$; and the paper contains a conjecture that if $d \geq 10$, then $H_{d-1}(x) - 1$ is irreducible. From the irreducibility of $H_{d-1}(x) - 1$, we obtain the main non-existence result of our paper.

Theorem 5.9. *If k and g satisfy one of the following conditions, there exists no (k, g)-graph of cyclic excess* 4:

(1) $k \equiv 1, 2 \pmod 3$ *and* $g = 8$.

(2) $k \equiv 1 \pmod 3$ *and* $g = 12$.

(3) $k \equiv 1 \pmod 3$ *and* $g = 16$.

Proof. Because the order of the graphs is equal to

$$4 + 2\left(1 + (k-1) + \cdots + (k-1)^{(g-2)/2}\right),$$

we conclude $n \equiv 0 \pmod 3$. Since the polynomial $H_{d-1}(x) - 1$ is known to be irreducible for $d \in \{4, 6, 8\}$, we get a contradiction to (1) from Lemma 5.8. □

Remark 5.10. Since d is an even number, Theorem 5.6 asserts that $d-1$ divides $c-1$ and $c_2 - 1$. This claim is satisfied because $c = c_2 = 1$.

Next, let us consider graphs of bicyclic excess 4. In this case, we can assume an arbitrary (even or odd) d, as this case does not depend on the parity of d. So, let $G(E)$ be a graph consisting of a disjoint union of two cycles C_1 and C_2. If d is an odd number, then the vertex sets of the cycles C_1 and C_2 correspond to the partite sets V_1 and V_2, respectively.
If $n \equiv 0 \pmod 4$, d is even, each edge of $C(E)$ has endpoints in V_1 and V_2. Therefore, each of the cycles has even length, that is, $c_2 = 2$. Furthermore, $k-1$ must be odd. Unfortunately, this will not help us in excluding any family of pairs (k, g) for which G does not exist. In fact, for an odd $d-1$ and an odd $k-1$, we cannot conclude the irreducibility of $H_{d-1}(x)+2$, thus, we cannot employ Lemma 5.5.
If $n \equiv 2 \pmod 4$ and d is odd, then the lengths of C_1 and C_2 are equal to $\frac{n}{2}$ (clearly, $n = 2s + 1$ is odd). Therefore $c_2 = 0$, and $d-1$ divides $c-2$ and c_2.

The main result about the non-existence of graphs G with bicyclic excess 4 is given in the following theorem.

Theorem 5.11. *If $k(\geq 7)$ is odd and $g = 2d \geq 8$, where d is an even integer, then there exists no (k, g)-graph with bicyclic excess 4.*

Proof. We have $c = 2$. Theorem 5.6 implies that $d - 1$ divides $c - 1$, which is a contradiction. $\square$

Chapter 6

On the non-existence of antipodal cages of even girth

The results of this chapter are published in [37]. In this chapter we investigate in the existence of cages of even girth and small excess having antipodal property. Recall, a graph of diameter d is said to be *antipodal* if, for any vertices u, v, w such that $d(u,v) = d$ and $d(u,w) = d$, it follows that $d(v,w) = d$ or $v = w$, (see e.g. [14]). The n-dimensional cubes Q_n are trivially antipodal graphs. These graphs are bipartite and have the antipodal property, since every vertex of Q_n has a unique vertex at maximum distance from it. Also, for $n \geq 2$, the complete bipartite graph $K_{n,n}$ is antipodal. Here the antipodal partition is the same as the bipartition. The dodecahedron is an example of trivially antipodal, but not bipartite graph. Examples of graphs which are non-trivially antipodal and not bipartite are the complete tripartite graphs $K_{n,n,n}$, which have diameter 2, and the line graph of the Petersen graph, which has diameter 3. Further potential antipodal cages of even girth may exist among the $(k,6)$-cages of excess $e \leq k-2$. Any graph having the property that for each its edge the excess set induces a subgraph with just $\frac{e}{2}$ edges is antipodal. Moreover, each such graph is a λ-fold cover of a graph $D(k,\lambda)$, with $\lambda = \frac{e}{2}+1$, see Theorem C in [15]. The cages described in Theorem 2.2 belong to this family of antipodal graphs because their excess sets induce subgraphs with 1 edge. Finally, let us mention one more known antipodal cage of even girth, the unique $(7,6)$-cage; a graph discovered by O'Keefe and Wong [66]. This graph is the unique 3-fold cover with girth 6 of the incidence graph of the points and planes of $PG(3,2)$. It is also a bipartite and antipodal cage of order 90 and excess 4, [33].
The problem of the existence of antipodal regular graphs of odd girth was considered by Bannai and Ito. Using the same approach they used to prove the non-existence of the regular graphs with excess 1 and girth $2d+1 > 5$, they proved the following result.

Theorem 6.1 (Bannai and Ito [6]). *For $d \geq 3$, there exist no antipodal regular graphs with diameter $d+1$ and girth $2d+1$.*

Motivated by Theorem 6.1, in this chapter we address the question of the existence of the antipodal (k,g)-cages of even girth and excess $e \leq k-2$. Employing the methodology used in [6], [15] and [67], we prove the non-existence of antipodal (k, g)-cages of excess e, for $k \geq e + 2 \geq 6$ and $g = 2d \geq 8$; Theorem 6.12.

6.1 On (k, g)-cages of even girth and excess $e \leq k - 2$

Let k, g, d and e be positive integers such that $k \geq e + 2$ and $g = 2d \geq 6$. Let G be a (k, g)-cage of excess e; Theorem 2.1 asserts that e is even and G is a bipartite graph of diameter $d + 1$. Since G is a bipartite graph, it contains no odd cycle; consequently there exists no edge between the excess vertices of the same partite set. Moreover, in order to balance the Moore tree of G and to pair out the horizontal edges of G, it is easy to observe that half (that is, $\frac{e}{2}$) of the excess vertices belong to the first, and the other half to the second partite set of G. It implies that for each vertex of $V(G)$ there exist exactly $\frac{e}{2}$ vertices at distance $d + 1$ from it (see, for example, Figure 6.1).
In order to study the spectral properties of the (k, g)-cage G, we use the polynomials G_i, F_i and H_i introduced in Chapter 5.

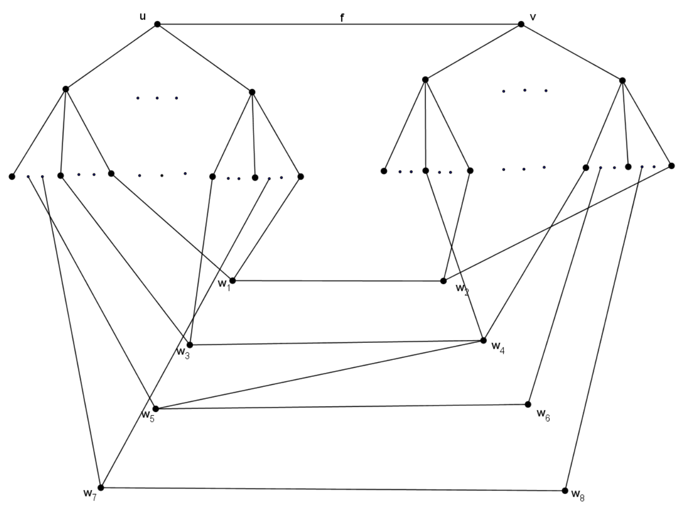

Figure 6.1: The Moore tree and some of the horizontal edges in a potential (k, 6)-graph of excess 8

The next lemma is a generalization of Lemma 5.1 from the previous chapter, where it was used to investigate the properties of cages of even girth and excess 4.

Lemma 6.2. *Let $k \geq e + 2$ and $g = 2d \geq 6$, and let G be a (k, g)-cage of excess e. If A is the adjacency matrix of G, then*

$$F_d(A) = kA_d - AA_{d+1}.$$

Proof. Let $f = \{u, v\}$ be a base edge of the Moore tree for G and let $X_f = \{w_1, w_2, \ldots, w_e\}$ be the excess set with respect to f. Also, let us assume that $d(u, w_1) = d(u, w_3) = \ldots = d(u, w_{e-1}) = d$ and $d(u, w_2) = d(u, w_4) = \ldots = d(u, w_e) = d + 1$. Let l_i be the number of edges between w_i and the leaves of $\mathcal{T}_u$ and $\mathcal{T}_v$, where $1 \leq i \leq e$. We consider the case when the excess vertices do not share common neighbour among the leaves of $\mathcal{T}_u$ and $\mathcal{T}_v$. The opposite case can be proved in a similar way. By the definition of $F_i(x)$, we have $(F_d(A))_{u,w_i} = l_i$, for each odd i, $1 \leq i \leq e-1$. Considering the vertices at distance d from u, there are the $(k-1)^{d-1}$ leaves of $\mathcal{T}_v$. For $l_2 + l_4 + \ldots + l_e$ of these vertices, there exist $k-1$ paths of length d from u to them. Namely, they are the vertices adjacent to $w_2, w_4, \ldots, w_{e-2}$ or w_e. For all the other leaves, there are k paths between them and u. Thus, $(F_d(A))_{u,s} = 0$ if $d(u,s) \neq d$, $(F_d(A))_{u,s} = k$ if s is a leaf of $\mathcal{T}_v$ and not adjacent to $w_2, w_4, \ldots, w_e$, $(F_d(A))_{u,s} = k-1$ if s is a leaf of $\mathcal{T}_v$ and adjacent to one of $w_2, w_4, \ldots, w_e$, and $(F_d(A))_{u,w_i} = l_i$, for each odd i; $1 \leq i \leq e-1$. For the matrix kA_d we have $(kA_d)_{u,s} = k$ if $d(u,s) = d$ and $(kA_d)_{u,s} = 0$ if $d(u,s) \neq d$. Now, let s be a vertex of G such that $d(u,s) = d$ and s is adjacent to one of $w_2, w_4, \ldots, w_e$. If s is a vertex among the vertices $w_1, w_3, \ldots, w_{e-1}$, then it is easy to see that $(AA_{d+1})_{u,s} = k - l_i$. On the other hand, since s is adjacent to $\mathcal{T}_u$ through $k-2$ different horizontal edges, it follows that, between the $k-1$ branches of $\mathcal{T}_u$, there exists one sub-branch that is not adjacent to s through a horizontal edge. Let s_1 be the root of that sub-branch. Then, $d(s, s_1) = d+1$ and $d(u, s_1) = 1$, which implies $(A)_{u,s_1} = 1$ and $(A_{d+1})_{s_1,s} = 1$. Let s_i, $2 \leq i \leq \frac{e}{2}$ be the remaining vertices at distance $d+1$ from s. Because all neighbours of u, except s_1, are at distance smaller than $d+1$ from s, we have $(A)_{u,s_i} = 0$ and $(A_{d+1})_{s_i,s} = 1$, for $2 \leq i \leq \frac{e}{2}$. Thus $(AA_{d+1})_{u,s} = 1$. If s is a vertex of G such that $d(u,s) = d$ and s is not adjacent to $w_2, w_4, \ldots, w_e$, then the distance between s and the neighbours of u is $d-1$. In this case, $(AA_{d+1})_{u,s} = 0$. If $d(u,s) \neq d$, then the distance between s and the neighbours of u is different from $d+1$, and therefore $(AA_{d+1})_{u,s} = 0$. The required identity follows from summing up the above conclusions. □

Based on the previous lemma and the properties of the polynomials G_i, H_i and F_i, we obtain the next two results. Theorem 6.4 is the main result in this section; it gives a relationship between the eigenvalues of the matrices A and A_{d+1}. We omit the proofs since they are analogous to the proofs of Lemma 5.2 and Theorem 5.3 from Chapter 5.

Lemma 6.3. *Let $k \geq e+2 \geq 4$ and $g = 2d \geq 6$, and let G be a (k,g)-cage of excess e. If A is the adjacency matrix of G and J is the all-ones matrix, then*

$$kJ = (A + kI)(H_{d-1}(A) + A_{d+1}).$$

Theorem 6.4. *If $\theta(\neq \pm k)$ is an eigenvalue of A, then*

$$H_{d-1}(\theta) = -\lambda,$$

where λ is an eigenvalue of A_{d+1}.

6.2 Spectral analysis of antipodal cages of even girth and small excess

In this section, we study the spectral properties of antipodal (k,g)-cages of even girth $g = 2d \geq 6$ and excess at most $k-2$. Let G be such graph, A be its adjacency matrix and let n be its order. Recall that G is a bipartite graph of diameter $d+1$. Let V_1 and V_2 be the partition sets of G. If d is an even number, then any two vertices of $V(G)$ at distance $d+1$ belong to different partite sets. Clearly, this case is not possible in case of antipodal bipartite graphs. Therefore, for the rest of the chapter we assume d odd. Since for each vertex $u \in V(G)$ there exist exactly $\frac{e}{2}$ vertices at the diameter distance $d+1$ (they are the excess vertices of the same partite set), we observe that the distance matrix A_{d+1} of G is an adjacency matrix of a disjoint union of $K_{\frac{e}{2}+1}$-complete graphs. Let c be the number of such complete graphs. Obviously $c = \frac{2n}{e+2}$. The spectrum of the disjoint union of c complete graphs of order $\frac{e}{2}+1$ is known and determined by $\{(\frac{e}{2})^c, (-1)^{n-c}\}$ (see Prop. 6 in [26]). Applying this result in Theorem 6.4, we are in a position to determine the spectrum of A.

Theorem 6.5. *If $\theta(\neq \pm k)$ is an eigenvalue of A, then*

$$H_{d-1}(\theta) - \epsilon = 0, \tag{6.1}$$

where $\epsilon = -\frac{e}{2}, 1$.

The roots of $H_{d-1}(x)$ are equal to $2\sqrt{k-1}\cos\frac{i\pi}{d}$ for $i = 1, \ldots, d-1$, (see [72]). Therefore we assume $x = -2\sqrt{k-1}\cos\phi, 0 < \phi < \pi$. Let $s = \sqrt{k-1}$. Then we have

$$H_{d-1}(x) = (-s)^{d-1}\frac{\sin d\phi}{\sin\phi}.$$

Putting $\phi = \frac{i\pi-\alpha}{d}$, as suggested in [6] and [15], we transform the equation (6.1) as follows

$$\sin\alpha - \eta_i s^{-d+1}\sin\left(\frac{i\pi-\alpha}{d}\right) = 0, \tag{6.2}$$

where $\eta_i = \epsilon(-1)^{d+i}$. The following result follows similarly as Lemma 3.3 from [15] and Lemma 2.2 from [67].

Lemma 6.6. *The equation (6.1) has $d-1$ distinct roots $\theta_1 < \theta_2 < \ldots < \theta_{d-1}$, with $\theta_i = -2s\cos\phi_i, (0 < \phi_i < \pi)$. If we set $\phi_i = \frac{i\pi-\alpha_i}{d}$ then*

$$0 < \alpha_i < min\{s^{-d+1}\phi_i, s^{-d+1}(\pi-\phi_i)\} \text{ if } \eta_i = 1;$$

$$max\{-s^{-d+1}\phi_i, -s^{-d+1}(\pi-\phi_i)\} < \alpha_i < 0 \text{ if } \eta_i = -1;$$

$$0 < \alpha_i < min\{\tfrac{e}{2}s^{-d+1}\phi_i, \tfrac{e}{2}s^{-d+1}(\pi-\phi_i)\} \text{ if } \eta_i = \tfrac{e}{2};$$

$$max\{-\tfrac{e}{2}s^{-d+1}\phi_i, -\tfrac{e}{2}s^{-d+1}(\pi-\phi_i)\} < \alpha_i < 0, \text{ if } \eta_i = -\tfrac{e}{2}.$$

From the bounds of α_i we derive the bounds of ϕ_i as follows.

$$\tfrac{i\pi}{d+s^{-d+1}} < \phi_i < \tfrac{i\pi}{d} \text{ if } \eta_i = 1;$$

$$\tfrac{i\pi}{d} < \phi_i < \tfrac{i\pi}{d-s^{-d+1}} \text{ if } \eta_i = -1;$$

$$\tfrac{i\pi}{d+\frac{e}{2}s^{-d+1}} < \phi_i < \tfrac{i\pi}{d} \text{ if } \eta_i = \tfrac{e}{2};$$

$$\tfrac{i\pi}{d} < \phi_i < \tfrac{i\pi}{d-\frac{e}{2}s^{-d+1}} \text{ if } \eta_i = -\tfrac{e}{2}.$$

We claim that $tr(A^q) = n(B_d^q)_{0,0}$ for $q = 0, 1, \ldots, 2d-1$, where

$$B_D = \begin{pmatrix} 0 & 1 & & & & & & \\ k & 0 & 1 & & & & 0 & \\ & k-1 & 0 & 1 & & & & \\ & & k-1 & 0 & 1 & & & \\ & & & & \ddots & \ddots & & \\ & & & & \ddots & & & \\ & 0 & & & & k-1 & 0 & k \\ & & & & & & k-1 & 0 \end{pmatrix}$$

is the $(D+1)\times(D+1)$ intersection matrix of a Moore bipartite graph of degree k, diameter D and of girth $2D$, (see [67]). If $q < g(G)$, the number of closed walks of length q that start from a fixed vertex u is independent of the vertex u and the excess. Furthermore, the entry $(B^q_{\lceil \frac{g(G)}{2} \rceil})_{0,0}$ gives this number, where $(B_i^q)_{0,0}$ is the $(0,0)$-entry of B_i^q, (see [45]). The number of closed walks of length q in G is given by $tr(A^q)$. Since G is a bipartite graph, it follows that G contains no closed walk of odd length. Thus, $tr(A^q) = n(B_d^q)_{(0,0)}$ for $q = 1, 3, \ldots, 2d-3, 2d-1$. Moreover, since the girth of G is $2d$ we obtain $tr(A^q) = n(B_d^q)_{(0,0)}$ for $q = 0, 1, \ldots, 2d-1$.

Theorem 6.7. *Let θ be a root of $H_{d-1}(x) - \epsilon$. The multiplicity $m(\theta)$ of θ in G, $\theta \neq \pm k$, is given by*

$$m(\theta) = \frac{nek(k-1)H_{d-2}(\theta)}{2\epsilon(2\epsilon + \frac{e}{2} - 1)H'_{d-1}(\theta)(k^2 - \theta^2)}. \tag{6.3}$$

Proof. In order to compute the multiplicity of an eigenvalue θ of G, we employ the approach from [6], [15] and [67]. Let $\xi(x) = (x^2 - k^2)(H_{d-1}(x) + \frac{e}{2})(H_{d-1}(x) - 1)$ and $\xi_\theta(x) = \frac{\xi(x)}{x-\theta}$. Since $\xi(A) = 0$, it follows $m(\theta) = \frac{tr(\xi_\theta(A))}{\xi_\theta(\theta)}$.
As $deg(H_{d-1}(x)) = d-1$ we obtain that $deg(\xi_\theta(x)) = 2d-1$. Therefore, let us assume $\xi_\theta(x) = x^{2d-1} + a_{2d-2}x^{2d-2} + \ldots + a_1 x + a_0$. Hence,

$$tr(\xi_\theta(A)) = tr(A^{2d-1}) + a_{2d-2}tr(A^{2d-2}) + \ldots + a_1 tr(A) + a_0 tr(I_n).$$

Since $tr(A^q) = n(B_d^q)_{0,0}$ for $0 \leq q \leq 2d-1$, we have

$$tr(\xi_\theta(A)) = n(\xi_\theta(B_d))_{0,0}.$$

The polynomial $(x^2 - k^2)H_{d-1}(x)$ is a minimal polynomial of B_d, (see [72]). That yields

$$\xi_\theta(B_d) = -\frac{e}{2}\frac{B_d^2 - k^2 I_n}{B_d - \theta I_n}.$$

Setting $L_{i+1}(x) = \frac{x^2-k^2}{x-\theta}(H_i(x) - H_i(\theta))$ for $i = 0, \ldots, d-1$, we get

$$L_d(B_d) = -H_{d-1}(\theta)\frac{B_d^2 - k^2 I_n}{B_d - \theta I_n} = -\epsilon\frac{B_d^2 - k^2 I_n}{B_d - \theta I_n}.$$

Therefore, $\xi_\theta(B_d) = \frac{e}{2\epsilon}L_d(B_d)$.
Calculating the derivative of $(x-\theta)\xi_\theta(x)$, that is, $((x-\theta)\xi_\theta(x))' = ((x^2-k^2)(H_{d-1}(x) + \frac{e}{2})(H_{d-1}(x)-1))'$, we have $\xi_\theta(\theta) = (2\epsilon + \frac{e}{2} - 1)H'_{d-1}(\theta)(\theta^2 - k^2)$. Thus

$$m(\theta) = \frac{ne}{2\epsilon(2\epsilon + \frac{e}{2} - 1)} \frac{(L_d(B_d))_{0,0}}{H'_{d-1}(\theta)(\theta^2 - k^2)}.$$

In [67] was proven that $(L_d(B_d))_{0,0} = -k(k-1)H_{d-2}(\theta)$. Substituting this identity in the previous expression we obtain

$$m(\theta) = \frac{nek(k-1)}{2\epsilon(2\epsilon + \frac{e}{2} - 1)} \frac{H_{d-2}(\theta)}{H'_{d-1}(\theta)(k^2 - \theta^2)}.$$

□

6.2.1 Multiplicities as functions of $\cos\phi$

Let θ be a root of $H_{d-1}(x) - \epsilon$ and let $\theta = -2s\cos\phi, 0 < \phi < \pi$. We express the multiplicity of θ, $m(\theta)$, as a function of $\cos\phi$. For that purpose we define the following functions $f(z), g_1(z), g_2(z)$ and $g_3(z)$.

$$f(z) = \frac{4s^2(1-z^2)}{k^2 - 4s^2z^2};$$

$$g_1(z) = \frac{k(k-1)(\sqrt{1 - s^{-2d+2}(1-z^2)} + s^{-d+1}z)}{d\sqrt{1 - s^{-2d+2}(1-z^2)} + s^{-d+1}z};$$

$$g_2(z) = \frac{k(k-1)(\sqrt{1 - \frac{e^2}{4}s^{-2d+2}(1-z^2)} - \frac{e}{2}s^{-d+1}z)}{d\sqrt{1 - \frac{e^2}{4}s^{-2d+2}(1-z^2)} - \frac{e}{2}s^{-d+1}z};$$

$$g_3(z) = \frac{k(k-1)(\sqrt{1 - \frac{e^2}{4}s^{-2d+2}(1-z^2)} + \frac{e}{2}s^{-d+1}z)}{d\sqrt{1 - \frac{e^2}{4}s^{-2d+2}(1-z^2)} + \frac{e}{2}s^{-d+1}z}.$$

Lemma 6.8. *For either value of ϵ, if we set $\theta_i = -2s\cos\phi_i$ for $1 \le i \le d-1$, then*

$$m(\theta_i) = \frac{ne}{4s^2(\frac{e}{2}+1)} f(\cos\phi_i) g_1(\eta_i \cos\phi_i), \text{ if } \epsilon = 1;$$

$$m(\theta_i) = \frac{n}{2s^2(\frac{e}{2}+1)} f(\cos\phi_i) g_2(\cos\phi_i), \text{ if } \epsilon = -\tfrac{e}{2} \text{ and } i \text{ is odd};$$

$$m(\theta_i) = \frac{n}{2s^2(\frac{e}{2}+1)} f(\cos\phi_i) g_3(\cos\phi_i), \text{ if } \epsilon = -\tfrac{e}{2} \text{ and } i \text{ is even}.$$

Proof. The derivative of $H_{d-1}(x)$ is computed in [67]. We have

$$H'_{d-1}(\theta_i) = \frac{(-s)^{d-1}(-1)^i}{2s\sin^2\phi_i}(d\cos\alpha_i + \eta_i s^{-d+1}\cos\phi_i).$$

Substituting $H_{d-2}(\theta_i) = (-s)^{d-2}(-1)^{i+1}\frac{\sin(\phi_i+\alpha_i)}{\sin\phi_i}$ and $H'_{d-1}(\theta_i)$ in (6.3), we obtain

$$m(\theta_i) = \frac{nek(k-1)\sin\phi_i\sin(\phi_i+\alpha_i)}{\epsilon(2\epsilon+\frac{e}{2}-1)(k^2-\theta_i^2)(d\cos\alpha_i+\eta_i s^{-d+1}\cos\phi_i)}.$$

The equation (6.2) yields $\sin(\phi_i+\alpha_i) = \sin\phi_i(\cos\alpha_i + \eta_i s^{-d+1}\cos\phi_i)$. Hence

$$m(\theta_i) = \frac{ne\sin^2\phi_i}{\epsilon(2\epsilon+\frac{e}{2}-1)(k^2-\theta_i^2)}\frac{k(k-1)(\cos\alpha_i+\eta_i s^{-d+1}\cos\phi_i)}{(d\cos\alpha_i+\eta_i s^{-d+1}\cos\phi_i)}.$$

By equation (6.2) and Lemma 6.6, as $k, d \geq 3$, it follows that, if $\eta_i = 1$ or $\eta_i = \frac{e}{2}$, then $0 < \alpha_i < \frac{\pi}{2}$. Similarly, if $\eta_i = -1$ or $\eta_i = -\frac{e}{2}$, then $-\frac{\pi}{2} < \alpha_i < 0$. Therefore $\cos\alpha_i > 0$, and thus, $\cos\alpha_i = \sqrt{1-\eta_i^2 s^{-2d+2}(1-\cos^2\phi_i)}$. It implies

$$m(\theta_i) = \frac{ne}{4s^2\epsilon(2\epsilon+\frac{e}{2}-1)}\frac{4s^2(1-\cos^2\phi_i)}{k^2-4s^2\cos^2\phi_i}\frac{k(k-1)(\sqrt{1-\eta_i^2 s^{-2d+2}(1-\cos^2\phi_i)}+\eta_i s^{-d+1}\cos\phi_i)}{(d\sqrt{1-\eta_i^2 s^{-2d+2}(1-\cos^2\phi_i)}+\eta_i s^{-d+1}\cos\phi_i)}.$$

Using the formulas for f, g_1, g_2 and g_3 we get the desired result. □

The following two lemmas concern the monotonicity of f, g_1, g_2 and g_3. The first lemma is given in [15] and [67] (Lemma 3.5 and Lemma 4.1).

Lemma 6.9. *For $k \geq 3$ and $|z| < 1$ the function $f(z)$ is even and concave down.*

Lemma 6.10. *For $k \geq 3$, $d \geq 3$ and $|z| < 1$, the monotonicities of $g_1(z), g_2(z)$ and $g_3(z)$ behave as follows.*

(1) *$g_1(z)$ is monotonically increasing;*

(2) *$g_2(z)$ is monotonically decreasing;*

(3) *$g_3(z)$ is monotonically increasing;*

(1) It suffices to prove that $g_1'(z)$ is positive on the interval $(-1, 1)$. We have

$$g_1'(z) = \frac{k(k-1)(d-1)s^{-d+1}(1-s^{-2d+2})}{\sqrt{1+s^{-2d+2}(-1+z^2)}(d\sqrt{1+s^{-2d+2}(-1+z^2)}+s^{-d+1}z)^2} > 0.$$

(2) In this case we prove that $g_2'(z)$ is negative on the interval $(-1, 1)$.

$$g_2'(z) = \frac{-\frac{e}{2}s^{-d+1}k(k-1)(d-1)(1-\frac{e^2}{4}s^{-2d+2})}{\sqrt{1+\frac{e^2}{4}s^{-2d+2}(-1+z^2)}(d\sqrt{1+\frac{e^2}{4}s^{-2d+2}(-1+z^2)}-\frac{e}{2}s^{-d+1}z)^2}.$$

Since $k, d \geq 3$ and $k \geq e+2$, we easily conclude that $\frac{e^2}{4}s^{-2d+2} < 1$ and $|\frac{e^2}{4}s^{-2d+2}(-1+z^2)| < 1$.

(3) It follows from the same reasoning as (2).

6.3 Main result

Let $\lambda_1 < \lambda_2 < \ldots < \lambda_{d-1}$ be the roots of $H_{d-1}(x) + \frac{e}{2}$, and let $\mu_1 < \mu_2 < \ldots < \mu_{d-1}$ be the roots of $H_{d-1}(x) - 1$.

Lemma 6.11. *Let $\lambda_1, \ldots, \lambda_{d-1}$ and $\mu_1, \ldots, \mu_{d-1}$ be defined as above.*

(1) *If $k \geq 3$ and $d \geq 3$ is an odd number, then $m(\lambda_i) = m(\lambda_{d-i})$ and $m(\mu_i) = m(\mu_{d-i})$, for $1 \leq i \leq d-1$;*

(2) *If $k \geq 6$ and $d \geq 5$ is an odd number, then $m(\lambda_1) < m(\lambda_i)$ and $m(\mu_1) < m(\mu_i)$, for $2 \leq i \leq d-2$.*

(1) If d is odd $H_{d-1}(-x) = H_{d-1}(x)$. Therefore θ is a root of $H_{d-1}(x) - \epsilon$, if and only if, $-\theta$ is a root of $H_{d-1}(x) - \epsilon$, (see [1]). Then $\lambda_i + \lambda_{d-i} = \mu_i + \mu_{d-i} = 0$. By checking (6.3) and using $H_{d-2}(-x) = -H_{d-2}(x)$, we obtain $m(\lambda_i) = m(\lambda_{d-i})$ and $m(\mu_i) = m(\mu_{d-i})$ for each $1 \leq i \leq d-1$.

(2) Since μ_i is a root of $H_{d-1}(x) - 1$, we have $\epsilon = 1$. According to Lemma 6.6 let us set $\mu_i = -2s \cos\phi_i$, for $1 \leq i \leq d-1$. In this case $\eta_i = \epsilon(-1)^{d+i} = (-1)^{i+1}$. Since $-\mu_2 = \mu_{d-2}$ we obtain $-\cos\phi_2 = \cos\phi_{d-2}$. Now, for $3 \leq i \leq d-3$, we have $-\cos\phi_2 = \cos\phi_{d-2} < \cos\phi_i < \cos\phi_2$. Since f is even and concave down function we have

$$f(\cos\phi_2) < f(\cos\phi_i) \text{ for } 3 \leq i \leq d-3.$$

The inequality $\cos\phi_i < |\cos\phi_2|$ and the fact that $g_1(z)$ is a monotonically increasing function yield $g_1(\eta_2 \cos\phi_2) = g_1(-\cos\phi_2) < g_1(\pm\cos\phi_i)$.
Therefore, for $3 \leq i \leq d-3$, we conclude

$$m(\mu_2) = \frac{ne}{4s^2(\frac{e}{2}+1)} f(\cos\phi_2) g_1(\eta_2 \cos\phi_2) < \frac{ne}{4s^2(\frac{e}{2}+1)} f(\cos\phi_i) g_1(\pm\cos\phi_i) = m(\mu_i).$$

Next, we will show that $m(\mu_1) < m(\mu_2)$. Since $\cos\phi_2 < -\cos\phi_{d-1}$, we use the following result given in [67],

$$\frac{f(\cos\phi_2)}{f(\cos\phi_{d-1})} > 1 + \frac{2(s^{d-1}-1)}{s^{d-1}+1} \frac{k^2-4s^2}{4k^2-s^2}.$$

In order to prove that $m(\mu_1) < m(\mu_2)$ we will prove that $f(\cos\phi_{d-1}) g_1(\eta_{d-1}\cos\phi_{d-1}) < f(\cos\phi_2) g_1(\eta_2 \cos\phi_2)$. Since $\eta_2 = \eta_{d-1} = -1$ and g_1 is increasing function we have

$$\frac{g_1(\eta_{d-1}\cos\phi_{d-1})}{g_1(\eta_2\cos\phi_2)} < \frac{g_1(1)}{g_1(-1)} = \frac{(1+s^{-d+1})(d-s^{-d+1})}{(1-s^{-d+1})(d+s^{-d+1})} < 1 + \frac{2s^{-d+1}}{1-s^{-d+1}}.$$

Therefore, it is enough to prove $\frac{s^{d-1}-1}{s^{d-1}+1}\frac{k^2-4s^2}{4k^2-s^2} > \frac{s^{-d+1}}{1-s^{-d+1}}$, that is, $\frac{(s^{d-1}-1)^2}{s^{d-1}+1}\frac{k^2-4s^2}{4k^2-s^2} > 1$. We easily conclude that if $k \geq 6$ and $d \geq 5$, then $\frac{(s^{d-1}-1)^2}{s^{d-1}+1} > 10$ and $\frac{k^2-4s^2}{4k^2-s^2} > \frac{1}{10}$.

We proceed similarly when λ_i is a root of $H_{d-1}(x) + \frac{e}{2}$. In this case $\epsilon = -\frac{e}{2}$ and $\eta_i = \frac{e}{2}(-1)^i$. Again let $\lambda_i = -2s\cos\phi_i$, for $1 \leq i \leq d-1$. Following the same

reasoning as above we have $f(\cos\phi_1) < f(\cos\phi_i)$ for $2 \le i \le d-2$.
Now, let i be an odd number such that $3 \le i \le d-2$. For such i we note $\eta_i = -\frac{e}{2} < 0$. Sine $g_2(z)$ is a monotonically decreasing and $\cos\phi_i < \cos\phi_1$, we have $g_2(\cos\phi_1) < g_2(\cos\phi_i)$. Thus, for odd i such that $3 \le i \le d-2$ we have

$$m(\lambda_1) = \frac{n}{2s^2(\frac{e}{2}+1)} f(\cos\phi_1) g_2(\cos\phi_1) < \frac{n}{2s^2(\frac{e}{2}+1)} f(\cos\phi_i) g_2(\cos\phi_i) = m(\lambda_i).$$

Since $m(\lambda_i) = m(\lambda_{d-i})$ occurs $m(\lambda_1) < m(\lambda_i)$ for each $2 \le i \le d-2$.

Based on Lemma 6.11, we are ready to give the main result in this chapter.

Theorem 6.12. *If $k \ge e+2 \ge 6$ and $g = 2d \ge 8$, then there exist no antipodal (k,g)-cages of excess e.*

Proof. We assume that $d \ge 5$ since we already concluded the non-existence of antipodal (k,g)-cages of excess $e \le k-2$ and girth $g = 2d$ where d is an even number. Since $m(\mu_1) < m(\mu_i)$ for $2 \le i \le d-2$ we obtain that μ_1 and $\mu_{d-1} = -\mu_1$ are either conjugate quadratic irrationals or integers. Therefore, μ_1^2 is an integer. Analogously, λ_1^2 is an integer. Hence $\mu_1^2 - \lambda_1^2$ is an integer number. By Lemma 6.6 we have

$$-2s\cos\frac{\pi}{d+s^{-d+1}} < \mu_1 < -2s\cos\frac{\pi}{d}$$
$$-2s\cos\frac{\pi}{d} < \lambda_1 < -2s\cos\frac{\pi}{d-\frac{e}{2}s^{-d+1}}.$$

Then, as $\mu_1^2 > 4s^2\cos^2\frac{\pi}{d}$ and $\lambda_1^2 < 4s^2\cos^2\frac{\pi}{d}$, we have that $\mu_1^2 - \lambda_1^2 > 0$.
Now we will prove that $\mu_1^2 - \lambda_1^2 < 1$. As $\mu_1^2 < 4s^2\cos^2\frac{\pi}{d+s^{-d+1}}$ and $\lambda_1^2 > 4s^2\cos^2\frac{\pi}{d-\frac{e}{2}s^{-d+1}}$, we have that

$$\mu_1^2-\lambda_1^2 < 4s^2\left(\cos^2\frac{\pi}{d+s^{-d+1}} - \cos^2\frac{\pi}{d-\frac{e}{2}s^{-d+1}}\right) = 4s^2\left(\sin^2\frac{\pi}{d-\frac{e}{2}s^{-d+1}} - \sin^2\frac{\pi}{d+s^{-d+1}}\right) =$$
$$= 4s^2\left(\sin\frac{\pi}{d-\frac{e}{2}s^{-d+1}} - \sin\frac{\pi}{d+s^{-d+1}}\right)\left(\sin\frac{\pi}{d-\frac{e}{2}s^{-d+1}} + \sin\frac{\pi}{d+s^{-d+1}}\right) =$$
$$= 16s^2\sin\left(\frac{\pi}{2(d-\frac{e}{2}s^{-d+1})} - \frac{\pi}{2(d+s^{-d+1})}\right)\cos\left(\frac{\pi}{2(d-\frac{e}{2}s^{-d+1})} + \frac{\pi}{2(d+s^{-d+1})}\right)\cdot$$
$$\cdot\sin\left(\frac{\pi}{2(d-\frac{e}{2}s^{-d+1})} + \frac{\pi}{2(d+s^{-d+1})}\right)\cos\left(\frac{\pi}{2(d-\frac{e}{2}s^{-d+1})} - \frac{\pi}{2(d+s^{-d+1})}\right) <$$
$$< 4s^2\pi^2\left(\frac{1}{(d-\frac{e}{2}s^{-d+1})^2} - \frac{1}{(d+s^{-d+1})^2}\right) = \frac{4\pi^2(\frac{e}{2}+1)s^{-d+3}(2d+(1-\frac{e}{2}1)s^{-d+1})}{(d-\frac{e}{2}s^{-d+1})^2(d+s^{-d+1})^2}.$$

Since $(d-\frac{e}{2}s^{-d+1})^2 > (d-1)^2 > 2d+1 > 2d+(1-\frac{e}{2})s^{-d+1}$, it is suffices to prove that $d+s^{-d+1} > 2\pi\sqrt{\frac{e}{2}+1}s^{\frac{-d+3}{2}}$. Using $k \ge e+2 \ge 6$ and $d \ge 5$, we obtain

$$s^{\frac{d-3}{2}}(d+s^{-d+1}) > \sqrt{k-1}d \ge \sqrt{e+1}d > 2\pi\sqrt{\frac{e}{2}+1}.$$

Therefore, $\mu_1^2 - \lambda_1^2$ is an integer number such that $0 < \mu_1^2 - \lambda_1^2 < 1$, which is impossible. $\square$

Chapter 7

On the excess of the vertex-transitive graphs of given degree and girth

The results of this chapter are published in [39]. Many of the cages as well as the smallest known (k, g)-graphs turn out to be *vertex-transitive* [33]. The reason for such frequent occurrence among the smallest (k, g)-graphs is not well understood, but one of the reasons might lie in the fact that vertex-transitive graphs are locally isomorphic around each vertex, and hence each of their vertices lies on cycles of the same lengths. This seems to be a feature shared by the extreme (k, g)-graphs as well. Based on this observation, considering a restricted version of the original cage problem and looking for smallest vertex-transitive (k, g)-graphs (which we shall refer to as *vertex-transitive cages*) and their corresponding orders $vt(k, g)$ will most likely lead to improvements in our understanding of both the general Cage Problem and the structure of vertex-transitive graphs. Obviously, $vt(k, g) \geq n(k, g)$.

The existence of vertex-transitive (k, g)-graphs for any pair $k, g \geq 3$ has been established for example in [46]. In the case of general cages, the question of whether there exists a universal bound on the excess is still open. On the other hand, in the more specialized case of vertex-transitive cages, this question has been answered in negative, and the excess of vertex-transitive (k, g)-cages can be arbitrarily large. This result is due to Biggs:

Theorem 7.1 ([12])**.** *For each odd integer $k \geq 3$, there is an infinite sequence of values of g such that the excess e of any vertex-transitive graph of degree k and girth g satisfies $e > \dfrac{g}{k}$.*

In the PhD thesis, we show that Biggs' result [12] holds not only for infinitely many g's, but, in fact, holds for almost all g's for any given $k \geq 4$. More specifically, we show that for any given excess e and degree $k \geq 4$, the set of g's for which $vt(k, g) - M(k, g) < e$ is of asymptotic density 0 (when compared to the set of all girths $g \geq 3$). The main technique used here depends on counting cycles in graphs whose orders are close to the Moore bound (this technique was introduced in Chapter 3). Our counting techniques rely on the following fairly obvious lemma.

Lemma 7.2 ([34])**.** *If G is a vertex-transitive graph and $n \geq 3$ is a positive integer, then the following hold:*

(1) $\boldsymbol{c}_G(x,n) = \boldsymbol{c}_G(y,n)$, *for all* $x, y \in V(G)$;

(2) n *divides the product* $\boldsymbol{c}_G(x,n) \cdot |V(G)|$, *for all* $x \in V(G)$.

In addition to obtaining the above stated density results, we address the question of the magnitude of the excess. As the odd- and even-girth cases differ is several important characteristics, we derive our results separately for odd- and even-girth regular graphs.

7.1 The excess of vertex-transitive graphs of odd girth

In this section we prove that for any fixed pair $k \geq 4, e \geq 1$, the excess for (k,g)-vertex-transitive cages exceeds e for almost all odd girths g. Our arguments are analogous to those used in [34] in which the authors show similar density results in the so-called Degree/Diameter Problem.

We begin by presenting an upper and lower bound on the number of g-cycles containing a fixed vertex v in a general (i.e., not necessarily vertex-transitive) (k,g)-graph.

Lemma 7.3. *Let G be a (k,g)-graph of degree $k \geq 3$, odd girth g, excess $e(G) \geq 1$, and let v be an arbitrary vertex of G. The number $\boldsymbol{c}_G(v,g)$ of g-cycles containing v satisfies the following lower and upper bounds:*

$$\frac{k(k-1)^{(g-1)/2}}{2} - \frac{ek}{2} \leq \boldsymbol{c}_G(v,g) \leq \frac{k(k-1)^{(g-1)/2}}{2}. \tag{7.1}$$

Proof. If G were a Moore graph, the number of g-cycles through a fixed vertex v would satisfy the identity $\mathbf{c}_G(v,g) = \frac{1}{2}k(k-1)^{(g-1)/2}$, proved in [43]. In order to prove the bounds for graphs G whose orders exceeds the Moore bound $M(k,g)$, i.e., graphs with excess $e(G) \geq 1$, we use the following notation introduced in Chapter 3. Let v be an arbitrary vertex of G, and let $N_G^i(v)$ denote the set of vertices in G whose distance from v is equal to i. Let $g = 2t+1$, and let $\mathcal{T}_v^G$ be the subgraph of G whose vertices belong to the union $\bigcup_{i=0}^t N_G^i(v)$ and whose edges are the edges of the subgraph of G induced by the subset $\bigcup_{i=0}^t N_G^i(v)$ *minus* the edges with both ends belonging to the last layer $N_G^t(v)$. Since G contains no cycles shorter than g, it is easy to see that $\mathcal{T}_v^G$ is a tree of order $M(k,g)$; we will refer to this tree as the *Moore tree of G with respect to v*. In addition, we will call the edges connecting the vertices from $N_G^t(v)$ (and thus excluded from $\mathcal{T}_v^G$) *edges horizontal with respect to v*. The key observation of our argument relates the number of g-cycles through v with the number of edges horizontal with respect to v. More specifically, since $\mathcal{T}_v^G$ is a tree of depth t rooted at v, any $g = (2t+1)$-cycle through v must consist of two edge-disjoint paths of length t connecting v to vertices u, w in $N_G^t(v)$ and a single horizontal edge connecting u and w. Consequently, $\mathbf{c}_G(v,g)$ is equal to the number of edges horizontal with respect to v. This yields the upper bound $\mathbf{c}_G(v,g) \leq \dfrac{k(k-1)^{(g-1)/2}}{2}$, where the right side of the inequality is the maximal possible number of horizontal edges – the number of vertices in $N_G^t(v)$ multiplied by $(k-1)$ and divided by 2 (each edge is counted twice this way). Let X_v now be the *excess set of G* with respect to v, i.e., the set of e vertices of G that do not belong to $\mathcal{T}_v^G$. Note that the only vertices from $\mathcal{T}_v^G$ the vertices from X_v

might be connected to are the vertices in $N_G^t(v)$. Since the e vertices in X_v are of degree k, the maximum number of edges between X_v and $N_G^t(v)$ is ek. The edges horizontal with respect to v are the edges emanating from the vertices in $N_G^t(v)$ that do not connect to the vertices in $N_G^{t-1}(v)$ nor to the vertices in X_v. It follows that the number of horizontal edges with respect to v is at least $\frac{k(k-1)^{(g-1)/2}}{2} - \frac{ek}{2}$, which yields the desired lower bound on $\mathbf{c}_G(v,g)$. □

The next lemma follows from Lemmas 7.3 and 7.2:

Lemma 7.4. *Let $k, g \geq 3$ be integers, and g be odd. If p is a prime divisor of g, and $e \geq 1$ has the property that none of the integers in the interval*

$$\mathcal{I} = [k(k-1)^{(g-1)/2} - ke, k(k-1)^{(g-1)/2} + ke] \tag{7.2}$$

is divisible by p, then the excess of any vertex-transitive (k,g)-graph is greater than e.

Proof. Suppose that k, g, p and e satisfy the assumptions of the lemma, and assume, by means of contradiction, that the order $vt(k,g)$ of a vertex-transitive (k,g)-cage does not exceed $M(k,g)+e$. Thus, $M(k,g) \leq vt(k,g) \leq M(k,g)+e$, which translates into

$$vt(k,g) \in \{\frac{k(k-1)^{(g-1)/2}-2}{k-2}, \frac{k(k-1)^{(g-1)/2}-2}{k-2}+1, \ldots, \frac{k(k-1)^{(g-1)/2}-2}{k-2}+e\}. \tag{7.3}$$

If G is a vertex-transitive (k,g)-graph, and p is a prime divisor of g, using Lemma 7.2 yields

$$p \mid |V(G)| \cdot \mathbf{c}_G(v,g),$$

which implies that p divides at least one of the factors $|V(G)|$ or $\mathbf{c}_G(v,g)$. Due to Lemma 7.3,

$$\mathbf{c}_G(v,g) \in \{\frac{k(k-1)^{(g-1)/2}}{2} - \frac{ek}{2}, \ldots, \frac{k(k-1)^{(g-1)/2}}{2}\},$$

and since p is necessarily odd, if p divides one of the above numbers, it also divides at least one of the numbers in

$$\mathcal{I}_1 = \{k(k-1)^{(g-1)/2} - ek, \ldots, k(k-1)^{(g-1)/2}\}.$$

Similarly, if p divides one of the numbers in the set defined in (7.3), it also divides one of the numbers in

$$\mathcal{I}_2 = \{k(k-1)^{(g-1)/2} - 2, \ldots, k(k-1)^{(g-1)/2} - 2 + e(k-2)\}.$$

Note, however, that $\mathcal{I}_1 \cup \mathcal{I}_2 \subseteq \mathcal{I}$, and therefore, if p were to divide $|V(G)| \cdot \mathbf{c}_G(v,g)$, it would have to divide at least one number in $\mathcal{I}$. Since we assume that p divides none of the integers in $\mathcal{I}$, we obtain a contradiction with the assumption that the excess of a vertex-transitive (k,g)-cage is at most e. □

Remark: Since the union of the sets $\mathcal{I}_1$ and $\mathcal{I}_2$ is a proper subset of $\mathcal{I}$, we could use $\mathcal{I}_1 \cup \mathcal{I}_2$ in place of $\mathcal{I}$ in the statement of Lemma 7.4. However, this would complicate but not strengthen our forthcoming results, hence we choose to use $\mathcal{I}$ instead.

Next, using ideas from number theory, we will prove that for any fixed $k \geq 4$ and $e \geq 1$, the density of the set of odd girths g for which the excess of any vertex-transitive (k, g)-graph is less than e is 0 (in the set of all odd $g \geq 3$). This approach comes originally from [34], and our proof follows a similar line of arguments.

Let $\mathcal{A}$ be a set of positive integers. For any $n > 1$, let $\mathcal{A}(n) = |\mathcal{A} \cap [1, n]|$, the number of elements in $\mathcal{A}$ that do not exceed n. In what follows, we will use the following densities.

Lower asymptotic density of $\mathcal{A}$:

$$\underline{d}(\mathcal{A}) = \liminf_{n\to\infty} \frac{\mathcal{A}(n)}{n},$$

Upper asymptotic density of $\mathcal{A}$:

$$\overline{d}(\mathcal{A}) = \limsup_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

Clearly, $0 \leq \underline{d}(\mathcal{A}) \leq \overline{d}(\mathcal{A}) \leq 1$. If, in addition, $\underline{d}(\mathcal{A}) = \overline{d}(\mathcal{A})$, we say that $\mathcal{A}$ has the asymptotic density

$$d(\mathcal{A}) = \lim_{n\to\infty} \frac{\mathcal{A}(n)}{n}.$$

Given integers a, b, c, d, q such that $a \neq 0, c \neq 0$ and $q > 2$, let $\mathcal{A}(a, b, c, d, q)$ denote the set of integers

$$\{n \in \mathbb{N} \mid n \text{ is odd, and } n \mid (aq^{(n-1)/2} + b)(cq^{(n-1)/2} + d)\}.$$

The next result was proved in [34], Lemma 4.3.

Lemma 7.5 ([34])**.** *Let a, b, c, d and q be integers such that $a \neq 0, c \neq 0$, and $q > 2$. Then,*

$$d(\mathcal{A}(a, b, c, d, q)) = 0.$$

We now have all the necessary ingredients to prove the main theorem of this section.

Theorem 7.6. *Let $k \geq 4$ and $e \geq 1$ be fixed integers. The asymptotic density of the set of all odd g for which there exists a vertex-transitive (k, g)-graph with excess not exceeding e is* 0.

Proof. Lemma 7.2 asserts for all vertex-transitive (k, g)-graphs G that g divides $\mathbf{c}_G(v, g) \cdot |V(G)|$, and therefore

$$g \mid ((k-2)|V(G)|) \cdot (2\mathbf{c}_G(v, g)).$$

Since both factors $(k-2) \cdot |V(G)|$ and $2 \cdot \mathbf{c}_G(v, g)$ belong to the interval $\mathcal{I}$ defined in (7.2), it follows that a vertex-transitive (k, g)-graph may exist only if at least one of the numbers in the set

$$\mathcal{I}^2 = \{(k(k-1)^{(g-1)/2} + i) \cdot (k(k-1)^{(g-1)/2} + j) \ \mid -ke \leq i, j \leq ke\}$$

is divisible by g. Obviously, each of the numbers contained in $\mathcal{I}^2$ has the form of the product in the definition of the set $\mathcal{A}(a,b,c,d,q)$ when substituting $a=k$, $b=i$, $c=k$, $d=j$ and $q=k-1$. Thus, if some g satisfies $g \mid (k(k-1)^{(g-1)/2}+i)\cdot(k(k-1)^{(g-1)/2}+j)$ for some i and j, then g belongs to $\mathcal{A}(k,i,k,j,k-1)$. Therefore, the set of all odd girths g for which there exists a vertex-transitive (k,g)-graph whose excess does not exceed e is a subset of the set

$$\bigcup_{-ke\leq i,j\leq ke} \mathcal{A}(k,i,k,j,k-1).$$

By Lemma 7.5, all sets $\mathcal{A}(k,i,k,j,k-1)$ are of density 0, and hence, the above union is a finite union of sets of asymptotic density 0, and is therefore a set of asymptotic density 0 itself. Being a subset of a set of asymptotic density 0, the set of all odd girths g for which there exists a vertex-transitive (k,g)-graph with excess not exceeding e must also be of asymptotic density 0, as claimed. □

7.2 The excess of vertex-transitive graphs of even girth

The essence of arguments used in this section is close to those of the previous section. Nevertheless, the details are different enough to justify considering the even girth case separately. We often omit the details.

Lemma 7.7. *Let G be a (k,g)-graph of degree $k\geq 3$, even girth $g\geq 4$, excess $e\geq 1$, and let v be an arbitrary vertex of G. The number $\mathbf{c}_G(v,g)$ of g-cycles containing v satisfies the following lower and upper bounds:*

$$\frac{k(k-1)^{g/2}}{2} - \frac{ke(k^2-3k+5)}{4} \leq \mathbf{c}_G(v,g) \leq \frac{k(k-1)^{g/2}}{2}. \tag{7.4}$$

Proof. The 'usual proof' of the Moore bound for a (k,g)-graph G, g even, considers a pair of trees rooted at two endpoints of a fixed edge of the graph. However, to prove the upper bound, we simply consider the tree of depth $\frac{g-2}{2}$ rooted at v, denoted by $\mathcal{T}_v^G$ again, and consisting of k-branches of height $\frac{g-2}{2}$. All non-root and non-leaf vertices of $\mathcal{T}_v^G$ are of degree k, and the total number of vertices of $\mathcal{T}_v^G$ is

$$1+k+k(k-1)+\ldots+k(k-1)^{\frac{g-4}{2}}.$$

All g-cycles of G containing v consist of two $\frac{g-2}{2}$-paths starting at v, having no other shared vertices and whose endpoints are connected to a single vertex at distance $\frac{g}{2}$ from v. It is easy to see that the minimum number of vertices at distance $\frac{g}{2}$ from v in G is $\frac{k(k-1)^{\frac{g-2}{2}}}{k} = (k-1)^{\frac{g-2}{2}}$; which happens when all the distance $\frac{g}{2}$ vertices are joined to the leaves of $\mathcal{T}_v^G$ via all of their adjacent edges. It is also easy to see that this is the situation in which G contains the maximum number of g-cycles through v with each vertex at distance $\frac{g}{2}$ from v giving rise to $\binom{k}{2}$ g-cycles. This yields the upper bound on $\mathbf{c}_G(v,g)$ $(k-1)^{\frac{g-2}{2}}\cdot\binom{k}{2}$ asserted in our lemma.

To prove the lower bound, we go back to considering the 'usual' Moore tree rooted at an edge $\{v,u\}$ with $k-1$ branches of depth $\frac{g-2}{2}$ rooted at v and $k-1$ branches of depth

$\frac{g-2}{2}$ rooted at u. In this view, we have three types of g-cycles passing through v (which we will refer to as Type 1., Type 2., or Type 3. cycles):

1. the cycles that avoid u, consist of two edge-disjoint paths of length $\frac{g-2}{2}$ starting at v with the endpoints of the two paths adjacent via horizontal edges to a vertex in the tree rooted at u;

2. the cycles that avoid u, consist of two edge-disjoint paths of length $\frac{g-2}{2}$ started at v with the endpoints of the two paths adjacent to a vertex belonging to the excess set $X_{v,u}$;

3. and the cycles that contain u and consist of a $\frac{g}{2}$-path started at v and containing u, a $\frac{g-2}{2}$-path started at v not containing u and one horizontal edge connecting the endpoints of these two paths.

Let $X_{v,u}$ (in parallel with the X_v used in the previous section) denote the e excess vertices with respect to the vertices v and u, and let l denote the number of edges between the vertices in $X_{v,u}$. The existence of these l edges implies the non-existence of some horizontal edges, more precisely, there are only $(k-1)^{\frac{g}{2}} - (\dfrac{ke}{2} - l)$ horizontal edges in G. There are several cases we need to consider.

Suppose first, that no two excess vertices share a common neighbor among the leaves of the part of the Moore tree rooted at u. Let a_i, $1 \le i \le e$, be the number of edges from the i-th excess vertex to the leaves of the branch rooted at v. Then, the number of Type 2. g-cycles through v is $\binom{a_1}{2} + \ldots + \binom{a_e}{2}$. The number of Type 1. g-cycles is $((k-1)^{(g-2)/2} - (\dfrac{ke}{2} - l))\binom{k-1}{2} + (\dfrac{ke}{2} - l))\binom{k-2}{2}$, since the leaves of the subtree rooted at u divide into those that are incident with $k-1$ horizontal edges and those incident to $k-2$ horizontal edges. Finally, the number of Type 3. g-cycles is the number of horizontal edges, $(k-1)^{g/2} - (\dfrac{ke}{2} - l)$. In summary,

$$\mathbf{c}_G(v,g) = ((k-1)^{g/2} - (\frac{ke}{2} - l)) + ((k-1)^{(g-2)/2} - (\frac{ke}{2} - l))\binom{k-1}{2} + (\frac{ke}{2} - l))\binom{k-2}{2} +$$

$$+\binom{a_1}{2} + \ldots + \binom{a_e}{2}.$$

Next, let us suppose there exist excess vertices which share a common neighbor among the leaves of the subtree rooted at u. Let s be the number of leaves of the subtree rooted at u which are adjacent to the excess vertices. Then $s < \dfrac{ke}{2} - l$ and $s = s_1 + s_2$, where s_1 denotes the number of vertices adjacent just to one excess vertex and s_2 denotes the number of vertices adjacent to at least two excess vertices. In this case,

$$\mathbf{c}_G(v,g) \ge ((k-1)^{g/2} - (\frac{ke}{2} - l)) + ((k-1)^{(g-2)/2} - s)\binom{k-1}{2} + s_1\binom{k-2}{2}$$

$$+\binom{a_1}{2} + \ldots + \binom{a_e}{2}.$$

Thus, whether there exist vertices which share a common neighbor among the leaves of the subtree rooted at u or not, the parameter $\mathbf{c}_G(v,g)$ satisfies:

$$\mathbf{c}_G(v,g) \geq ((k-1)^{g/2} - (\frac{ke}{2} - l)) + ((k-1)^{(g-2)/2} - (\frac{ke}{2} - l))\binom{k-1}{2} + \binom{a_1}{2} + \ldots + \binom{a_e}{2}.$$

Since the number of edges from the excess vertices to the subtree rooted at v and the number of edges from the excess vertices to the subtree rooted at u must be equal (in order to balance the number of horizontal edges started at the two subtrees), $a_1 + a_2 + \ldots. + a_e = \frac{ke}{2} - l$. Let $a_{i_1}, \ldots, a_{i_t}$, $t \leq e$, be the non-zero a_i's. Applying the inequality between the quadratic and the arithmetic mean to the positive integers $a_{i_1}, \ldots, a_{i_t}$ yields:

$$\binom{a_1}{2} + \ldots + \binom{a_e}{2} = \binom{a_{i_1}}{2} + \ldots + \binom{a_{i_t}}{2} = \frac{a_{i_1}^2 + \ldots + a_{i_t}^2}{2} - \frac{a_{i_1} + a_{i_2} + \ldots + a_{i_t}}{2} \geq$$

$$\frac{(a_{i_1} + a_{i_2} + \ldots + a_{i_t})^2}{2t} - \frac{a_{i_1} + a_{i_2} + \ldots + a_{i_t}}{2} \geq \frac{(\frac{ke}{2} - l)^2}{2e} - \frac{(\frac{ke}{2} - l)}{2}.$$

All of the above yield the lower bound

$$\mathbf{c}_G(v,g) \geq \frac{k(k-1)^{g/2}}{2} + \frac{(\frac{ke}{2} - l)^2}{2e} - (\frac{ke}{2} - l)(\frac{k^2 - 3k + 5}{2}) >$$

$$> \frac{k(k-1)^{g/2}}{2} - \frac{ke(k^2 - 3k + 5)}{4}.$$

□

The next lemma follows from Lemmas 7.7 and 7.2.

Lemma 7.8. *Let $k \geq 4$, $g \geq 4$ even, $e \geq 1$, and let G be a vertex-transitive (k,g)-graph. If p is a prime divisor of g that does not divide any of the integers in the interval*

$$\mathcal{J} = [2k(k-1)^{g/2} - ke(k^2 - 3k + 5), 2k(k-1)^{g/2} + k(k-2)e - 2k],$$

then the excess of G is greater than e.

Proof. Let G be a vertex-transitive (k,g)-graph, and p be a prime divisor of g. Lemma 7.2 yields

$$p \mid |V(G)| \cdot \mathbf{c}_G(v,g),$$

which implies in turn that p divides at least one of the factors $|V(G)|$ or $\mathbf{c}_G(v,g)$. Due to Lemma 7.7,

$$\mathbf{c}_G(v,g) \in \{\frac{k(k-1)^{g/2}}{2} - \frac{ke(k^2 - 3k + 5)}{4}, \ldots, \frac{k(k-1)^{g/2}}{2}\}.$$

If p divides one of the above numbers, it also divides at least one of the numbers in

$$\mathcal{J}_1 = \{2k(k-1)^{g/2} - ke(k^2 - 3k + 5), \ldots, 2k(k-1)^{g/2}\}.$$

The order of any vertex-transitive (k, g)-graph G of even girth g and excess not exceeding e satisfies
$M(k, g) \leq |V(G)| \leq M(k, g) + e$, i.e.,

$$|V(G)| \in \{\frac{2(k-1)^{g/2}-2}{k-2}, \frac{2(k-1)^{g/2}-2}{k-2}+1, \ldots, \frac{2(k-1)^{g/2}-2}{k-2}+e\}.$$

Similarly, if p divides one of the above numbers, it also divides one of the numbers in

$$\mathcal{J}_2 = \{2k(k-1)^{g/2}-2k, 2k(k-1)^{g/2}+k^2-4k, \ldots, 2k(k-1)^{g/2}-2k+ke(k-2)\}.$$

As the integers in $\mathcal{J}_1$ and $\mathcal{J}_2$ all appear in $\mathcal{J}$, any prime divisor of at least one of the integers in these intervals must also divide at least one integer in $\mathcal{J}$. Therefore, p divides none of the numbers in $\mathcal{J}$, and the excess of every vertex-transitive (k, g)-graph must be greater than e. □

In the case of even girth, given integers a, b, c, d, q such that $a \neq 0, c \neq 0$, and $q > 2$, we need to consider the set of even n such that

$$\mathcal{B}(a, b, c, d, q) = \{n \in \mathbb{N} \mid n \text{ is even and } n \mid (aq^{\frac{n}{2}}+b)(cq^{\frac{n}{2}}+d)\}.$$

For any prime p and any set of positive integers $\mathcal{B}$, let $\mathcal{B}_p = \{n \in \mathcal{B} \mid p \parallel n\}$, where $p \parallel n$ indicates that p divides n, but p^2 does not. We will employ the following two theorems from Number Theory. The first one is the contrapositive of Corollary 3 of Niven [64].

Theorem 7.9 ([64]). *Let $\{p_i\}_{i=1}^{\infty}$ be a set of primes such that $\sum_{i=1}^{\infty} 1/p_i = +\infty$. If $\mathcal{B}$ is a set of positive integers such that $\sum \overline{d}(\mathcal{B}_{p_i}) < +\infty$, then $d(\mathcal{B}) = 0$.*

The second result is a theorem of Erdős [29].

Theorem 7.10 ([29]). *Let $\{a_i\}_{i=1}^{\infty}$ be a set of integers such that $a_i \nmid a_j$, unless $i = j$. Then $\sum_{i=1}^{\infty} \frac{1}{a_i \log a_i}$ converges.*

Lemma 7.11. *Let a, b, c, d and q be integers such that $a \neq 0, c \neq 0$, and $q > 2$. Then,*

$$d(\mathcal{B}(a, b, c, d, q)) = 0.$$

Proof. Let $p > 2$ be a fixed prime such that $p \nmid acq$ and let $n \in \mathcal{B} = \mathcal{B}(a, b, c, d, q)$ be a multiple of p. Then, $p \mid (aq^{\frac{n}{2}}+b)(cq^{\frac{n}{2}}+d)$, and thus, $p \mid aq^{\frac{n}{2}}+b$ or $p \mid cq^{\frac{n}{2}}+d$. Since $p \nmid acq$, it follows that $(a, p) = (c, p) = 1$. Thus, there exist integers a^{-1} and c^{-1} such that $aa^{-1} \equiv 1 \pmod p$ and $cc^{-1} \equiv 1 \pmod p$. By $(a^{-1}, p) = (c^{-1}, p) = 1$ it follows that $q^{\frac{n}{2}} \equiv -a^{-1}b \pmod p$ or $q^{\frac{n}{2}} \equiv -c^{-1}d \pmod p$. Since $p \mid n$ and n is an even number, $n = pr$, where r is even. Since $(p, q) = 1$, Fermat's little theorem yields $q^{\frac{n}{2}} \equiv (q^p)^{\frac{r}{2}} \equiv q^{\frac{r}{2}} \pmod p$. Hence,

$$q^{\frac{r}{2}} \equiv -a^{-1}b \pmod p \quad \text{or} \quad q^{\frac{r}{2}} \equiv -c^{-1}d \pmod p.$$

Recalling $(p, q) = 1$ again, let k denote the smallest positive integer satisfying $q^k \equiv 1 \pmod p$. This yields $p < q^k$, and therefore $k > \dfrac{\log p}{\log q}$. As shown above, there are only two possible residue classes modulo k that $\frac{r}{2}$ might belong to. Thus, the asymptotic density of the multiples rp for which r satisfies the above conditions within the set of all multiples of p is at most $\frac{2}{k}$. The asymptotic density of the set of multiples of p within the set of all positive integers is clearly equal to $\frac{1}{p}$, and therefore

$$\overline{d}(\mathcal{B}_p) \;<\; \frac{1}{p}\cdot\frac{2}{k} \;\leq\; \frac{2\log q}{p\log p} \;.$$

Thus, due to Erdős's Theorem 7.10,

$$\sum_{p>acq}\overline{d}(\mathcal{B}_p) < \sum_{p>acq}\frac{2\log q}{p\log p} = 2\log q\cdot\sum_{p>acq}\frac{1}{p\log p} < +\infty \;.$$

Finally, since $\sum_{p>acq}\frac{1}{p}$ diverges and $\sum_{p>acq}\overline{d}(\mathcal{B}_p) < +\infty$, applying Niven's Theorem 7.9 yields $d(\mathcal{B}(a,b,c,d,q)) = 0$. □

We can now present the main theorem of this section:

Theorem 7.12. *Let $k \geq 4$ and $e \geq 1$ be fixed. The asymptotic density of the set of all even g for which there exists a vertex-transitive (k,g)-graph with excess not exceeding e is 0.*

Proof. Let $k \geq 4$, $e \geq 1$, and let G be a vertex-transitive (k,g)-graph of excess at most e. If $g \mid |V(G)|\cdot\mathbf{c}_G(v,g)$, then $g \mid ((k-2)k\cdot|V(G)|)\cdot(4\mathbf{c}_G(v,g))$. Since both $(k-2)k\cdot|V(G)|$ and $4\cdot\mathbf{c}_G(v,g)$ belong to $\mathcal{J}$, a (k,g)-vertex-transitive graph may exist only if at least one of the numbers in the set

$$\mathcal{J}^2 = \{(2k(k-1)^{g/2}+i)\cdot(2k(k-1)^{g/2}+j) \mid -ke(k^2-3k+5) \leq i,j \leq k(k-2)e-2k\}$$

is divisible by g. Each of the numbers in $\mathcal{J}^2$ takes the form of a product from $\mathcal{B}(a,b,c,d,q)$, where $a = 2k$, $b = i$, $c = 2k$, $d = j$ and $q = k-1$. Thus, if some g satisfies $g \mid (2k(k-1)^{g/2}+i)\cdot(2k(k-1)^{g/2}+j)$ for some i and j, then g belongs to $\mathcal{B}(2k,i,2k,j,k-1)$. Therefore, the set of all even g's for which there exists a vertex-transitive (k,g)-graph with excess not exceeding e is a subset of the set

$$\bigcup_{-ke(k^2-3k+5)\leq i,j\leq k(k-2)e-2k}\mathcal{B}(2k,i,2k,j,k-1).$$

This is a finite union of sets of asymptotic density 0 (satisfying the conditions of Lemma 7.11). As a subset of a set of asymptotic density 0, the set of all even g's for which there exists a vertex-transitive (k,g)-graph with excess not exceeding e must also be of asymptotic density 0. □

7.3 A lower bound on the growth of the excess in the odd girth case

As mentioned in the introduction of this chapter, the existence of vertex-transitive cages whose orders exceed the Moore bound by an arbitrarily large value has already been established by Biggs. In the previous sections, we have shown that such cages constitute the 'majority' of all cages. In this section, we want to derive an analogue to the second part of Biggs' Theorem 7.1, namely to the part where the theorem asserts that the resulting excesses satisfy $e > \frac{g}{k}$. We begin with a lemma whose proof relies on basic number theory.

Lemma 7.13. *Let $e \geq 1$ be fixed, r be an odd integer, and k be large enough to satisfy the inequality $(k-1)^{(r-1)/2} > e$. If p is a prime larger than $2k(k-1)^{(r-1)/2}$ and $g = rp$, then the excess of any vertex-transitive (k, g)-graph is greater than e.*

Proof. The assumption $g = rp$ yields $\frac{g-1}{2} = \frac{rp-1}{2} = \frac{r(p-1)}{2} + \frac{r-1}{2}$. Since p is a prime, employing Euler's criterion yields:

$$(k-1)^{(g-1)/2} \equiv ((k-1)^{(p-1)/2})^r \cdot (k-1)^{(r-1)/2} \equiv$$
$$\equiv \left(\frac{k-1}{p}\right)^r \cdot (k-1)^{(r-1)/2} \equiv \left(\frac{k-1}{p}\right) \cdot (k-1)^{(r-1)/2} \pmod p,$$

where $\left(\frac{k-1}{p}\right)$ is the *Legendre symbol.* The interval $\{k(k-1)^{(g-1)/2} + i \mid -ke \leq i \leq ke\}$, is equivalent modulo p to

$$\{\left(\frac{k-1}{p}\right) k(k-1)^{(r-1)/2} + i \mid -ke \leq i \leq ke\}. \tag{7.5}$$

First, consider the case $\left(\frac{k-1}{p}\right) = 1$. In this case we have

$$\left(\frac{k-1}{p}\right) k(k-1)^{(r-1)/2} + i = k(k-1)^{(r-1)/2} + i \leq k(k-1)^{(r-1)/2} + ke < 2k(k-1)^{(r-1)/2} < p,$$

and $\left(\frac{k-1}{p}\right) k(k-1)^{(r-1)/2} + i = k(k-1)^{(r-1)/2} + i \geq k(k-1)^{(r-1)/2} - ke > 0.$

Thus, $0 < \left(\frac{k-1}{p}\right) k(k-1)^{(r-1)/2} + i < p$, for all $-ke \leq i \leq ke$. Using identical argument in the case when $\left(\frac{k-1}{p}\right) = -1$ yields $-p < \left(\frac{k-1}{p}\right) k(k-1)^{(r-1)/2} + i < 0$. The above inequalities imply that the interval defined in (7.5) does not contain 0, which means that the interval $\{k(k-1)^{(g-1)/2} + i \mid -ke \leq i \leq ke\}$ contains no multiples of p, and therefore, by Lemma 7.4, the excess of any vertex-transitive (k, g)-graph is necessarily greater than e. □

The following corollary of Lemma 7.13 provides us with a partial solution to a conjecture stated in [46] which predicts that the order of a vertex-transitive graph of degree k and odd girth g which is not a Moore graph should always be at least $M(k, g) + k - 1$.

Corollary 7.14. *Let $k \geq 3$ be an integer, $r \geq 3$ be an odd integer, and p be a prime number such that $p > 2k(k-1)^{(r-1)/2}$. If If $g = rp$, and G is a vertex-transitive graph of degree k and girth g, then $|V(G)| \geq M(k,g) + k$.*

Proof. The result follows from Lemma 7.13 and the easy observation that $(k-1)^{(r-1)/2} \geq e$, for all $r \geq 3$ and $e \leq k-1$. Hence, the excess of any vertex-transitive (k,g)-graph exceeds $e \geq k-1$, and therefore $vt(k,g) \geq M(k,g) + k$. □

Next we prove a technical lemma that will then allow us to improve on Biggs' Theorem 7.1.

Lemma 7.15. *For every $k \geq 3$ there exists a constant C_k such that for each $e \geq 1$ one can find an odd girth $g < C_k e \log(e+1)$ with the property that the excess of any vertex-transitive (k,g)-graph exceeds e.*

Proof. Let us fix $k \geq 3$ and $e \geq 1$, and let r be the smallest odd integer satisfying $(k-1)^{(r-1)/2} > e$, or equivalently $\dfrac{r-1}{2} > \log_{k-1}(e)$. Applying Bertrand's Postulate [49], we are assured of the existence of a prime p such that

$$2k(k-1)^{(r-1)/2} < p < 4k(k-1)^{(r-1)/2}.$$

The lower bound implies that $p > 2ke$, while the upper bound yields

$$p < 4k(k-1)^{(r-1)/2} \leq 4k(k-1)^{\lceil \log_{k-1}(e) \rceil + 1} < 4k(k-1)^{(\log_{k-1}(e)+1)+1} = 4k(k-1)^2 e.$$

Since p and r satisfy the conditions from Lemma 7.13, choosing $g = rp$ yields that any vertex-transitive (k,g)-graph has excess greater than e. In addition,

$$g = rp < (2(\log_{k-1}(e) + 2) + 1)4k(k-1)^2 e = (2\log_{k-1}(e) + 5)4k(k-1)^2 e < C_k \log(e+1)e.$$

□

We are ready to prove:

Theorem 7.16. *For any $k \geq 3$, there exists an infinite sequence of odd girths $\{g_i\}_{i=1}^{\infty}$, such that the excess of any vertex-transitive (k, g_i)-graph is greater than $g_i^{1/(1+o(1))}$.*

Proof. Since C_k from the previous lemma is a constant with respect to k, $\lim_{e\to\infty} \dfrac{\log(C_k \log(e+1))}{\log(e)} = 0$. Therefore, $\dfrac{\log(C_k \log(e+1))}{\log(e)} = o(1)$, or $C_k \log(e+1) = e^{o(1)}$, and hence, applying Lemma 7.15, we obtain $e > g_i^{1/(1+o(1))}$. □

7.4 A lower bound on the growth of the excess in the even girth case

In the end, we obtain results parallel to those in the previous section; this time for even girth. Once again, this is a generalization of Biggs' Theorem 7.1.

Lemma 7.17. *Let G be a vertex-transitive graph of degree $k \geq 3$ and girth g, and let $e \geq 1$. Let $s \geq 3$ be a natural number such that $2(k-1)^s > e(k^2-3k+5)$ and let p be a prime number satisfying the inequality $p > 2k(k-1)^s + k(k-2)e - 2k$. If $g = 2ps$, then $|V(G)| > M(k,g) + e$.*

Proof. Due to Lemma 7.8, it is enough to prove that the integers in the interval $\mathcal{J}$ are not divisible by p. Using Fermat's theorem, we obtain: $(k-1)^{g/2} \equiv ((k-1)^p)^s \equiv (k-1)^s \pmod p$. Let $2k(k-1)^{g/2}+i$ be a number from the interval $\mathcal{J}$. Clearly, $-ke(k^2-3k+5) \leq i \leq k(k-2)e - 2k$. As argued above, $2k(k-1)^{g/2} + i \equiv 2k(k-1)^s + i \pmod p$ and $2k(k-1)^s + i \leq 2k(k-1)^s + k(k-2)e - 2k < p$. On the other hand, $2k(k-1)^s + i \geq 2k(k-1)^s - ke(k^2-3k+5) > 0$. Thus, p divides none of the integers in $\mathcal{J}$, and it follows that the excess of any vertex-transitive (k,g)-graph whose parameters satisfy the requirements of our lemma is greater than e. $\square$

Lemma 7.18. *Let $k \geq 3$ and $e \geq 1$ be fixed. Then there exists a constant C_k and an even girth $g < C_k \log(e+1)e$, such that the excess of any vertex-transitive (k,g)-graph is greater than e.*

Proof. Let s be the smallest integer with the property $2(k-1)^s > e(k^2-3k+5)$. Then, $s > \log_{k-1}\left(\frac{e(k^2-3k+5)}{2}\right)$ or $s = \lceil \log_{k-1}\left(\frac{e(k^2-3k+5)}{2}\right)\rceil$. Using Bertrands Postulate [49], we can deduce that there exists a prime p such that

$$2k(k-1)^s + k(k-2)e - 2k < p < 4k(k-1)^s + 2k(k-2)e - 4k.$$

Thus, using the above estimate,

$$p < 4k(k-1)^s + 2k(k-2)e - 4k < 4k(k-1)^{\log_{k-1}\left(\frac{e(k^2-3k+5)}{2}\right)+1} + 2k(k-2)e - 4k <$$

$$< (2k(k-1)(k^2-3k+5) + 2k(k-2))e.$$

Since p and s satisfy the conditions from Lemma 7.17, if we take $g = 2ps$, then any vertex-transitive (k,g)-cage has excess greater than e. Therefore,

$$g = 2ps <$$

$$< (4k(k-1)(k^2-3k+5)+4k(k-2))\frac{e}{\log(k-1)} \cdot \left(\log(e) + \log\left(\frac{k^2-3k+5}{2}\right) + \log(k-1)\right) <$$

$$< C_k \log(e+1)e.$$

$\square$

Our last theorem follows directly from the previous lemma.

Theorem 7.19. *For any $k \geq 3$, there exists an infinity sequence of even girths $\{g_i\}_{i=1}^{\infty}$ such that the excess of any vertex-transitive (k,g_i)-graph is greater than $g_i^{1/(1+o(1))}$.*

Proof. Since $\lim_{e\to\infty} \dfrac{\log(C_k \log(e+1))}{\log(e)} = 0$, then $\dfrac{\log(C_k \log(e+1))}{\log(e)} = o(1)$, or

$$e^{o(1)} = C_k \log(e+1).$$

Using Lemma 7.18, we conclude that $e > g_i^{1/(1+o(1))}$. $\square$

Chapter 8

A connection between a question of Bermond and Bollobás and Ramanujan graphs

The results of this chapter are published in [40]. The question of Bermond and Bollobás has already been answered in positive for the more specialized families of vertex-transitive and Cayley graphs [34]. If we let $vt(k,d)$ denote the largest order of a vertex-transitive (k,d)-graph, and $C(k,d)$ denote the largest order of a Cayley (k,d)-graph, then

$$C(k,d) \leq vt(k,d) \leq n(k,d).$$

Exoo et al. proved in [34] that for any fixed $k \geq 3$ and $c \geq 2$ there exists a set $\mathcal{S}$ of natural numbers of positive density such that $vt(k,d) \leq M(k,d) - c$, for all $d \in \mathcal{S}$. The same holds for Cayley graphs as well. We list the result for the sake of completeness.

Theorem 8.1 ([34])**.** *Let $k \geq 3$ and $c \geq 2$. Let r be an odd integer, and let p be a prime such that $p > 2k(k-1)^{(r-1)/2} > 8k(k-1)^2c^2$. If $2d+1 = rp$, then any vertex-transitive (k,d)-graph has defect greater than c.*

In this chapter, we will not extend the results of [34]. Instead, we present a connection between the question of Bermond and Bollobás and the class of Ramanujan graphs defined as follows. Let Γ be a connected k-regular graph, and let $\lambda(\Gamma)$ denote the largest absolute value of an eigenvalue of Γ distinct from k or $-k$. We say that Γ is a *Ramanujan graph* if $\lambda(\Gamma) \leq 2\sqrt{k-1}$ [54]. The significance of the value $2\sqrt{k-1}$ with regard to $\lambda(\Gamma)$ has been established by Alon and Boppana [1] who proved that $\lambda(\Gamma)$ is not much smaller than $2\sqrt{k-1}$ for the majority of k-regular graphs. More precisely, if we let $X_{n,k}$ denote a k-regular graph on n vertices, then:

Theorem 8.2 ([54])**.** $\liminf_{n\to\infty} \lambda(X_{n,k}) \geq 2\sqrt{k-1}$.

Ramanujan graphs of degree k and arbitrarily large order are known to exist when $k-1$ is a prime [54] or a prime power [61]. In the breakthrough paper [57], the authors proved the existence of infinitely many bipartite Ramanujan graphs of every degree $k \geq 2$. However, the bipartedness of their graphs is deeply imbedded in their construction method, and the existence of arbitrary large non-bipartite Ramanujan graphs for degrees k for which $k-1$ is neither a prime nor a prime number is a well-known open problem; the first open degree being $k = 7$ [62].

In the main result of this chapter, Theorem 8.3, we prove that a negative answer to the question of Bermond and Bollobás would imply the existence of arbitrarily large non-bipartite Ramanujan graphs for any (prime power or not) degree k, and conversely, the non-existence of arbitrarily large non-bipartite Ramanujan graphs for some fixed degree k would yield a positive answer to the question of Bermond and Bollobás.

8.1 Spectral analysis

For many of our arguments, we rely on the techniques of spectral analysis applied to graphs extremal with respect to the Degree/Diameter Problem. We begin with a brief review of the basic facts as listed in [60].

Let Γ be a connected (k,d)-graph of order n and δ and let $\mathbf{A}$ be its adjacency matrix. Recall, the polynomials $G_{k,i}(x)$ for all $x \in \mathbb{R}$ are defined as follows:

$$\begin{cases} G_{k,0}(x) = 1 \\ G_{k,1}(x) = x + 1 \\ G_{k,i+1}(x) = xG_{k,i}(x) - (k-1)G_{k,i-1}(x) \text{ for } i \geq 1. \end{cases} \tag{8.1}$$

Moreover, the entry $(G_{k,i}(\mathbf{A}))_{\alpha,\beta}$ counts the number of paths of length at most i joining the vertices α and β in Γ. Regular graphs with defect δ and order n satisfy the matrix equation

$$G_{k,d}(\mathbf{A}) = \mathbf{J}_n + \mathbf{B},$$

where $\mathbf{B}$ is a non-negative integer matrix with the row and column sums equal to δ. The matrix $\mathbf{B}$ is called the *defect matrix* (see [60]).

Next, we follow the line of argument that originally appeared in [13]. Since Γ is regular and connected, the all-ones matrix $\mathbf{J}_n$ is a polynomial of $\mathbf{A}$, say, $J_n(\mathbf{A})$. From now on, we adopt the convention that matrices will be denoted by upper-case bold-face characters while their corresponding polynomials will be denoted by the same character but not bold-faced. Thus, $\mathbf{B} = B(\mathbf{A}) = G_{k,d}(\mathbf{A}) - J_n(\mathbf{A})$, and $\mathbf{J}_n = J_n(\mathbf{A}) = G_{k,d}(\mathbf{A}) - B(\mathbf{A})$. It follows that if λ is an eigenvalue of $\mathbf{A}$, then $G_{k,d}(\lambda) - B(\lambda) = J_n(\lambda)$ is an eigenvalue of $\mathbf{J}_n$. Substituting the value k for λ yields the eigenvalue n of $\mathbf{J}_n$, $G_{k,d}(k) - B(k) = n$. An easy calculation yields that $G_{k,d}(k) = M(k,d)$, and therefore $B(k) = M(k,d) - n = \delta$ is an eigenvalue of $\mathbf{B}$. Since each row and column of $\mathbf{B}$ sums up to δ, every eigenvalue of $\mathbf{B}$ has value at most δ. If $\lambda \neq k$ is another eigenvalue of $\mathbf{A}$, then $G_{k,d}(\lambda) - B(\lambda)$ must be the zero eigenvalue of $\mathbf{J}_n$. Therefore, $G_{k,d}(\lambda) - B(\lambda) = 0$, and since $|B(\lambda)| \leq \delta$, we obtain $|G_{k,d}(\lambda)| \leq \delta$. Thus, the value $|G_{k,d}(\lambda)|$ is a lower bound for the defect $\delta(\Gamma)$. In summary, if Γ is a graph of diameter d, degree k, and order $M(k,d) - \delta$, then every eigenvalue $\lambda \neq k$ of Γ satisfies

$$|G_{k,d}(\lambda)| \leq \delta. \tag{8.2}$$

Since $\mathbf{A}$ is symmetric, all eigenvalues of $\mathbf{A}$ are real. Let $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1}$ be the eigenvalues of Γ and let λ be the eigenvalue with the second largest absolute value. It is well known from *Perron-Frobenius theory* (e.g., [53]), that $\lambda_0 = k$ and k is of multiplicity one if and only if Γ is connected. Moreover, if Γ is non-bipartite, then $\lambda_{n-1} > -k$. Therefore, if Γ is a connected and non-bipartite graph, $\lambda = \max\{\lambda_1, |\lambda_{n-1}|\}$.

8.2 Main result

The following theorem is the main result in this chapter.

Theorem 8.3. *Let $c \geq 1$ and $k \geq 3$ be fixed integers. Then there exists an even $D_{c,k}$ such that any graph Γ of maximum degree k, even diameter $d \geq D_{c,k}$, and order greater than $M(k,d) - c$, is a non-bipartite k-regular Ramanujan graph with $\lambda(\Gamma) < 2\sqrt{k-1}$. If $k > c$, all these Ramanujan graphs must be of girth $2d$ or $2d-1$.*

Proof. Let $c \geq 1$ and $k \geq 3$. As argued in the introduction, Moore and bipartite Moore graphs exist only for very limited diameters d. Therefore, assuming that $d > 6$ yields the non-existence of Moore or bipartite Moore (k,d)-graphs (and hence the non-existence of (k,d)-graphs of girth $2d+1$). In addition, due to (2.2), taking $d > \log_{k-1}(c(k-2)+1)$ makes the order of any *non-regular* graph of maximum degree k and diameter d smaller than $M(k,d) - c$. Similarly, due to (2.3), taking $d > \log_{k-1}(c)$ makes the order of any *bipartite* (k,d)-graph smaller than $M(k,d) - c$. Thus, any (k,d)-graph Γ of diameter $d > \max\{6, \log_{k-1}(c(k-2)+1)\}$ and of order greater than $M(k,d) - c$ must be k-regular, non-bipartite, and of girth smaller than $2d+1$. We claim that there exists an integer $D_{c,k} > \max\{6, \log_{k-1}(c(k-2)+1)\}$ such that every (k,d)-graph Γ of even diameter $d > D_{c,k}$ and of order greater than $M(k,d) - c$ is in fact a non-bipartite Ramanujan graph.

We proceed by contradiction. Let us assume that Γ is a (k,d)-graph of even diameter $d > \max\{6, \log_{k-1}(c(k-2)+1)\}$, of order greater than $M(k,d) - c$, and satisfying the inequality $\lambda(\Gamma) \geq 2\sqrt{k-1}$. Since Γ is k-regular and non-bipartite, $\lambda(\Gamma) \in (-k, -2\sqrt{k-1}] \cup [2\sqrt{k-1}, k)$. To obtain the desired contradiction, we will show that $|G_{k,d}(x)| > c$, for all $x \in (-k, -2\sqrt{k-1}] \cup [2\sqrt{k-1}, k)$ and all sufficiently large even diameters d. This, combined with the inequality (8.2) will imply that the defect of (k,d)-graphs from this class is larger than c.

To achieve our goal, we derive an explicit formula for $G_{k,d}(x)$. Fixing the variable x makes the last equation of (8.1) into a second order linear homogeneous recurrence equation for $G_{k,d}(x)$ with respect to the parameter d, subject to the initial conditions $G_{k,0}(x) = 1$ and $G_{k,1}(x) = x+1$. We will only calculate the values of $G_{k,d}(x)$ for $x \in (-k, -2\sqrt{k-1}] \cup [2\sqrt{k-1}, k)$, and thus we only need to consider the recurrence relation in the case when the roots of the corresponding second degree polynomial equation $t^2 - xt + (k-1) = 0$ are real; with a double-root when $x = \pm 2\sqrt{k-1}$. Solving this recurrence equation for a fixed $x \in (-k, -2\sqrt{k-1}) \cup (2\sqrt{k-1}, k)$, we obtain the explicit formula

$$G_{k,d}(x) = \frac{x+2+\sqrt{x^2-4k+4}}{2\sqrt{x^2-4k+4}} \left(\frac{x+\sqrt{x^2-4k+4}}{2}\right)^d -$$

$$-\frac{x+2-\sqrt{x^2-4k+4}}{2\sqrt{x^2-4k+4}} \left(\frac{x-\sqrt{x^2-4k+4}}{2}\right)^d.$$

In the case when $x = \pm 2\sqrt{k-1}$, the second degree polynomial equation has a double root and we obtain:

$$G_{k,d}(2\sqrt{k-1}) = (d+1)\sqrt{k-1}^{\,d} + d\sqrt{k-1}^{\,d-1},$$

$$G_{k,d}(-2\sqrt{k-1}) = (-1)^d((d+1)\sqrt{k-1}^{\,d} - d\sqrt{k-1}^{\,d-1}).$$

It is easy to see from (8.1) that the function $G_{k,d}(x)$ is a polynomial of degree d in x, and thus differentiable. Calculating the derivative of $G_{k,d}(x)$, for $x \in (-k, -2\sqrt{k-1}) \cup (2\sqrt{k-1}, k)$, we obtain

$$G'_{k,d}(x) = \frac{d(x-\sqrt{x^2-4k+4})^d(x+2-\sqrt{x^2-4k+4})}{2^{d+1}(x^2-4k+4)} + \frac{(x-\sqrt{x^2-4k+4})^{d+1}}{2^{d+1}(x^2-4k+4)} +$$

$$+\frac{x(x-\sqrt{x^2-4k+4})^d(x+2-\sqrt{x^2-4k+4})}{2^{d+1}(x^2-4k+4)^{\frac{3}{2}}} + \frac{d(x+\sqrt{x^2-4k+4})^d(x+2+\sqrt{x^2-4k+4})}{2^{d+1}(x^2-4k+4)} +$$

$$+\frac{(x+\sqrt{x^2-4k+4})^{d+1}}{2^{d+1}(x^2-4k+4)} - \frac{x(x+\sqrt{x^2-4k+4})^d(x+2+\sqrt{x^2-4k+4})}{2^{d+1}(x^2-4k+4)^{\frac{3}{2}}}.$$

We will now use the above formulas to show that for sufficiently large even d's the polynomials $G_{k,d}(x)$ are positive on both intervals $(-k, -2\sqrt{k-1})$ and $(2\sqrt{k-1}, k)$ and they are decreasing on the interval $(-k, -2\sqrt{k-1})$ and increasing on the interval $(2\sqrt{k-1}, k)$.

The assumption $x \in (2\sqrt{k-1}, k)$ yields the inequalities $x^2-4k+4 > 0$, $x > \sqrt{x^2-4k+4}$, and $k > x$. Thus, under this assumption, $G_{k,d}(x) > 0$, for all $x \in (2\sqrt{k-1}, k)$ (as well as $G_{k,d}(2\sqrt{k-1}) > 0$) and all d. Similarly, the first five terms of $G'_{k,d}(x)$ are positive numbers, while it is easy to see that there exists a positive integer D_1 such that the inequality

$$\frac{d(x+\sqrt{x^2-4k+4})^d(x+2+\sqrt{x^2-4k+4})}{2^{d+1}(x^2-4k+4)} > \frac{x(x+\sqrt{x^2-4k+4})^d(x+2+\sqrt{x^2-4k+4})}{2^{d+1}(x^2-4k+4)^{\frac{3}{2}}},$$

holds for all $x \in (2\sqrt{k-1}, k)$ and all $d \geq D_1$. Hence, $G'_{k,d}(x) > 0$, for all $x \in (2\sqrt{k-1}, k)$ and all $d \geq D_1$.

Since $G_{k,d}(x)$ is continuous on $[2\sqrt{k-1}, k]$ (being a polynomial), and differentiable on $(2\sqrt{k-1}, k)$, this means that $G_{k,d}(x)$ is positive and increasing on $[2\sqrt{k-1}, k]$, and therefore assumes its minimum value at $2\sqrt{k-1}$. This provides us with a lower bound on the defect of Γ in this case:

$$\delta(\Gamma) \geq |G_{k,d}(\lambda_1)| = G_{k,d}(\lambda_1) \geq G_{k,d}(2\sqrt{k-1}) = (d+1)\sqrt{k-1}^{\,d} + d\sqrt{k-1}^{\,d-1}.$$

It is easy to see the existence of a positive integer D_2 such that

$$\delta(\Gamma) \geq (d+1)\sqrt{k-1}^{\,d} + d\sqrt{k-1}^{\,d-1} > c,$$

for all $d \geq D_2$.

Next, let us suppose that $x \in (-k, -2\sqrt{k-1})$. This assumption implies the inequalities $0 > x+2+\sqrt{x^2-4k+4} > x+2-\sqrt{x^2-4k+4}$, and if d is an *even* number, $(x -$

$\sqrt{x^2-4k+4})^d > (x+\sqrt{x^2-4k+4})^d > 0$. It follows that the functions $G_{k,d}(x)$ are positive on $(-k,-2\sqrt{k-1})$ for even d's. Using the above inequalities again, we can also deduce that $G'_{k,d}(x)$ are negative on $(-k,-2\sqrt{k-1})$. Thus, if d is an even number, the functions $G_{k,d}(x)$ are positive and decreasing on $[-k,-2\sqrt{k-1}]$. This implies the bound

$$\delta(\Gamma) \geq |G_{k,d}(\lambda_{n-1})| = G_{k,d}(\lambda_{n-1}) \geq G_{k,d}(-2\sqrt{k-1}) = (-1)^d((d+1)\sqrt{k-1}^d - d\sqrt{k-1}^{d-1})$$

as well as the existence of a positive integer D_3 such that

$$\delta(\Gamma) \geq (-1)^d((d+1)\sqrt{k-1}^d - d\sqrt{k-1}^{d-1}) > c,$$

for all *even* $d \geq D_3$.

The above arguments yield that any bipartite as well as any non-Ramanujan non-bipartite k-regular graph Γ of even diameter $d > D_{c,k} = \max\{6, \log_{k-1}(c(k-2)+1), D_1, D_2, D_3\}$ and $\lambda(\Gamma) \geq 2\sqrt{k-1}$ has defect larger than c. That proves the first part of our theorem.

To prove the last part, let us assume that the girth $g(\Gamma)$ of Γ is at most $2d-2$, i.e., $3 \leq g(\Gamma) \leq 2d-2$, and assume that $b \in V(\Gamma)$ lies on a g-cycle. Let

$$N_\Gamma(b,i) = \{v \mid v \in V(\Gamma), d_\Gamma(b,v) = i\}, \text{ for } 0 \leq i \leq d.$$

It is easy to see that $|N_\Gamma(b,0)| = 1, |N_\Gamma(b,1)| = k$, and $|N_\Gamma(b,i)| \leq k(k-1)^{i-1}$ for $2 \leq i \leq d-2$. Since b lies on a g-cycle, where $g \leq 2d-2$, we obtain $|N_\Gamma(b,d-1)| \leq k(k-1)^{d-2}-1$. Hence $|N_\Gamma(b,d)| \leq k(k-1)^{d-1}-(k-1)$. This implies the inequality

$$\delta(\Gamma) = M(k,d) - |V(\Gamma)| = (1+k+k(k-1)+\ldots+k(k-1)^{d-1}) - (|N_\Gamma(b,0)| + |N_\Gamma(b,1)| +$$

$$+\ldots+|N_\Gamma(b,d-1)| + |N_\Gamma(b,d)|) \geq k > c.$$

□

8.3 Concluding remarks

Remark 8.4. If the answer to the question of Bollobás were negative (i.e., if there existed a positive integer c such that $M(k,d) - n(k,d) \leq c$, for all $k \geq 3, d \geq 2$), there would have to exist a (k,d)-graph Γ of order $|V(\Gamma)| \geq M(k,d) - c$ for every pair k,d. Due to Theorem 8.3, for any fixed degree $k \geq 3$ and sufficiently large even diameter d, the graph Γ would have to be a non-bipartite k-regular Ramanujan graph. This would yield the existence of infinitely many non-bipartite k-regular Ramanujan graphs for every $k \geq 3$.

Remark 8.5. As the reader might have noticed, the extremal values $(d+1)\sqrt{k-1}^d + d\sqrt{k-1}^{d-1}$ and $(-1)^d((d+1)\sqrt{k-1}^d - d\sqrt{k-1}^{d-1})$ can not just be made larger than c by choosing a sufficiently large d, but can be made arbitrarily large. Because of the continuity of the functions $G_{k,d}$ this implies that the values $G_{k,d}(x)$, for sufficiently large d's, are not only larger than c on the intervals $(-k,-2\sqrt{k-1}] \cup [2\sqrt{k-1},k)$, but there exist positive $\delta_{k,d}$'s such that $G_{k,d}(x) > c$ for all $x \in (-k,-2\sqrt{k-1}+\delta_{k,d}] \cup [2\sqrt{k-1}-\delta_{k,d},k)$. This, in combination with Theorem 8.2 of Alon and Boppana, appears to suggest a positive answer to the question of Bermond and Bollobás. The key to proving the positive answer would however require finding a meaningful relation between the rate of the convergency in Theorem 8.2 and the rate of decrease of the values $\delta_{k,d}$ (with respect to increase in d). The definition of the derivative yields the rough estimate $\delta_{k,d} \approx \frac{G_{k,d}(2\sqrt{k-1})-c}{G'_{k,d}(2\sqrt{k-1})}$. For the rate of convergency in Theorem 8.2, Alon in [63] has shown that if Γ is k-regular and $diam(\Gamma) \geq 2b+2 \geq 4$, for some natural b, then

$$\lambda(\Gamma) \geq 2\sqrt{k-1} - \frac{2\sqrt{k-1}-1}{b}.$$

A further improvement of this result is due to Solé [73, 52], who proved that if Γ is a (k,d)-graph of girth g, then

$$\lambda(\Gamma) \geq 2\sqrt{k-1}(1 - \frac{2\pi^2}{g^2} + O(\frac{1}{g^4})).$$

Under the assumption $k > c$, the second part of Theorem 8.3 yields the inequality $g \geq 2d-1$, and hence the above result of Solé yields the lower bound

$$\lambda(\Gamma) \geq 2\sqrt{k-1}(1 - \frac{2\pi^2}{g^2}) \geq 2\sqrt{k-1}(1 - \frac{2\pi^2}{(2d-1)^2}).$$

Thus, if one were able to show that

$$2\sqrt{k-1} \cdot \frac{\pi^2}{2d^2} \;\leq\; \delta_{k,d}$$

for sufficiently large d's, the arguments included in our proof of Theorem 8.3 would yields the proof for the positive answer to the question of Bermond and Bollobás. We were however unable to prove such inequality. Instead, we conclude this remark with the following theorem that follows from the above discussion.

Theorem 8.6. *Let $c > 0$, $k > c$, and $d > \max\{6, \log_{k-1}(c(k-2)+1)\}$ be positive integers. If $G_{k,d}(x) > c$ for all $x \in (-k, -2\sqrt{k-1} + \frac{2\pi^2}{(2d-1)^2}] \cup [2\sqrt{k-1} - \frac{2\pi^2}{(2d-1)^2}, k)$, then $n(k,d) < M(k,d) - c$.*

Remark 8.7. The calculations included in the proof of Theorem 8.3 allow for estimating the defect $\delta(\Gamma)$ of any k-regular graph Γ of sufficiently large even diameter d and girth at least $2d-1$. First, the order of a k-regular graph Γ of girth at least $2d-1$ is known to satisfy the inequality [33]:

$$|V(\Gamma)| \geq M(k, d-1).$$

This yields an upper bound on the defect of Γ:

$$\delta(\Gamma) = M(k,d) - |V(\Gamma)| \leq \frac{k(k-1)^d - 2}{k-2} - \frac{k(k-1)^{d-1} - 2}{k-2} = k(k-1)^{d-1}.$$

Using the lower bound on the defect $\delta(\Gamma)$ of any k-regular graph of sufficiently large even diameter d and of girth at least $2d-1$ derived in the proof of Theorem 8.3, we conclude that $\delta(\Gamma)$ for such graphs belongs to the interval $[(d+1)\sqrt{k-1}^d - d\sqrt{k-1}^{d-1}, k(k-1)^{d-1}]$.

This observation is related to the concept of a generalized Moore graph [18]: A k-regular graph Γ of diameter d and girth at least $2d-1$ is called a *generalized Moore graph*. As argued above, the defect of any generalized Moore graph is bounded from above by $k(k-1)^{d-1}$. For example, both Moore graphs and bipartite Moore graphs are generalized Moore graphs, with the Moore graphs having the defect 0 and the bipartite Moore graphs having the defect $(k-1)^d$. It has been conjectured that the diameter of a generalized Moore graph cannot exceed 6 (the conjecture can be found, for example, in the notes for the talk delivered by L.K. Jørgensen in Bandung, 2012). Our lower bounds on the defects of non-bipartite generalized Moore graphs do not seem to contribute to the resolution of this conjecture.

Chapter 9

On degree and diameter monotonicity of digraphs

The results of this chapter are published in [38]. Let (d,k)-digraph denote a directed graph of maximum out-degree d and diameter k, and let $n_{d,k}$ be the largest order of a (d,k)-digraph. Let n_i, for $0 \le i \le k$, be the number of vertices at distance i from a distinguished vertex. Then, $n_i \le d^i$, for $0 \le i \le k$. Hence,

$$n_{d,k} = \sum_{i=0}^{k} n_i \le 1 + d + \ldots + d^{k-1} + d^k = \begin{cases} \frac{d^{k+1}-1}{d-1}, & \text{if } d > 1, \\ k+1, & \text{if d=1.} \end{cases} \tag{9.1}$$

The number on the right-hand side of (9.1), denoted by $M_{d,k}$, is called the *Moore bound* for (d,k)-digraphs. In this chapter we give a positive answer to the question concerning the degree/diameter problem of digraphs asked in [60];

is $n_{d,k}$ monotonic in d and k?

Employing *Kautz digraphs* we show $n_{d,k}$ is strictly monotonic increasing in k and in d. These graphs are diregular digraphs of large order. Moreover, they are iterated line digraphs of complete digraphs. The Kautz digraph of degree d and diameter k has order $d^k + d^{k-1}$, (see [60] and [50]).

Our proof is based on a relatively simple idea. The unknown optimal digraphs must have at least as many vertices as the Kautz digraph, but no more than the Moore digraph. Using the numbers $d^k + d^{k-1}$ and $d^k + d^{k-1} + \ldots + d + 1$ as a lower and an upper bound of $n_{d,k}$, respectively, we give an elementary proof of our claim.

Diameter monotonicity

Theorem 9.1. *For each $k, d \ge 1$ holds $n_{d,k+1} > n_{d,k}$.*

Proof. If $d = 1$, then $n_{d,k} = n_{1,k} = M_{1,k} = k + 1$. It implies

$$n_{d,k+1} = n_{1,k+1} = k + 2 > k + 1 = n_{1,k} = n_{d,k}.$$

If $d > 1$, then we have

$$n_{d,k+1} \geq d^{k+1} + d^k > d^{k+1} > \frac{d^{k+1} - 1}{d - 1} \geq n_{d,k}.$$

$\square$

Degree monotonicity

Theorem 9.2. *For each $k, d \geq 1$ holds $n_{d+1,k} > n_{d,k}$.*

Proof. If $d = 1$, then we get

$$n_{d+1,k} = n_{2,k} \geq 2^k + 2^{k-1} > k + 1 = n_{1,k} = n_{d,k}.$$

If $d > 1$, then we have

$$n_{d+1,k} \geq (d+1)^k + (d+1)^{k-1} = (d+2)(d+1)^{k-1} \geq \frac{d^2}{d-1}(d+1)^{k-1} \geq$$

$$\geq \frac{d^2}{d-1} d^{k-1} > \frac{d^{k+1} - 1}{d - 1} \geq n_{d,k}.$$

$\square$

Chapter 10

On the non-existence of families of (d, k, δ)-digraphs containing only selfrepeats

The results of this chapter are published in [41]. To address the question of the existence of (d, k, δ)-digraphs with $\delta \geq 2$, we will generalize the concept of selfrepeat vertices as follows. For a fixed vertex $u \in V(G)$, let $R(u)$ denote the set of vertices $v \in V(G)$ for which there are at least two walks of length $\leq k$ connecting u to v. In general, for $\delta > 1$, $|R(u)|$ can be larger than 1. In the case when $R(u) = u$, we say again that u is a *selfrepeat*. It is interesting to observe that the all vertices of the $(2, 2, 1)$-digraph, as well as of the $(d, 2, d)$-digraphs for $d = 2, 3, 7$ and possibly $d = 57$, are selfrepeats [7]. Moreover, in the unpublished paper [7], Baskoro and Garminia proved the non-existence of (d, k, δ)-digraphs containing only selfrepeats among the $(d, 2, 2)$-digraphs with $d \geq 3$, the $(d, 2, \delta)$-digraphs with $\delta = 4, 5$ or 6 and $d \geq \delta$, and the $(d, 3, 4)$-digraphs with $d \geq 4$.

10.1 Spectral analysis of (d, k, δ)-digraphs with all selfrepeat vertices

We consider a diregular digraph of degree d, diameter k with number of vertices

$$n = d^k + d^{k-1} + \ldots + d + 1 - \delta. \tag{10.1}$$

As we stated in Section 2.6, we shall call such a digraph a (d, k, δ)-*digraph*. Similarly as for the $(d, k, 1)$-digraphs, we can easily see that hold the following propositions for the (d, k, δ)-digraphs, with $d \geq \delta \geq 2$. These propositions are stated in [7].

Proposition 10.1. *For any two vertices u, v of a (d, k, δ)-digraph G there is at most one walk of length l $(< k)$ from u to v. Moreover, G contains no cycle C_l of length l $(< k)$.*

Proposition 10.2. *For every vertex u of a (d, k, δ)-digraph there exists a non-empty set $S \subseteq V(G)$ such that for each $v \in S$ there are at least two walks of length $\leq k$ from u to v.*

The set S is called the *repeat-set* of u, and it is denoted by $R(u) = S$. Recall that, if $R(u) = \{u\}$, then we say that u is a *selfrepeat*. As argued before, graphs consisting of selfrepeats only are interesting and we focus on these graphs from now on.
Let $d \geq \delta \geq 1$, let G be a (d, k, δ)-digraph containing only selfrepeat vertices and let A be its adjacency matrix. Because of Proposition 10.1, in such a digraph, any two distinct vertices are connected by a unique directed path of length at most $k-1$, that is, there are no directed cycles of length less than k in G. Since each vertex of G is a selfrepeat, any two distinct vertices at distance k are connected by a unique directed path of length k. Thus, among the d in-neighbours of each vertex of G, exactly δ are at distance $k-1$ from it. It implies that each vertex of G lies on exactly δ directed cycles of length k, C_k. Using this argument, we observe that the adjacency matrix A fulfills the following matrix equation

$$A^k + A^{k-1} + \ldots + A + I = J + \delta I \tag{10.2}$$

where I is the $n \times n$ identity matrix and J is the $n \times n$ matrix whose entries are all 1's. It is well known that the eigenvalues of J are n (with multiplicity 1) and 0 (with multiplicity $n-1$). Therefore the eigenvalues of A are d (this corresponds to n) and the roots of

$$x^k + x^{k-1} + \ldots + x + 1 - \delta = 0. \tag{10.3}$$

Let $\xi_A(x) = x^k + x^{k-1} + \ldots + x + 1 - \delta$. In the following lemma we analyze the roots of $\xi_A(x)$ and their multiplicities.

Lemma 10.3. *Let $k \geq 3$ and $\delta \geq 1$. The polynomial $\xi_A(x) = x^k + x^{k-1} + \ldots + x + 1 - \delta$ has at most one negative real root, one positive real root and at least $k-2$ complex roots. Moreover, all of the roots of $\xi_A(x)$ are simple.*

Proof. In the proof we use Descartes' rule of signs: the number of positive roots of a single-variable polynomial with real coefficients is equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number; similarly, the number of negative roots is the number of sign changes after multiplying the coefficients of odd-power terms by -1, or fewer than it by an even number [23]. According to Descartes' rule of signs, the polynomial $\xi_A(x)$ has exactly one positive real root. If θ is a positive real root of $\xi_A(x)$ with multiplicity greater than 1, then θ also is a root of its derivative $kx^{k-1} + (k-1)x^{k-2} + \ldots + 2x + 1$, which is impossible. Thus, the unique positive real root of the polynomial $\xi_A(x)$ is simple. If $\delta \neq k+1$, then the equation (10.3) has the same roots as the equation

$$x^{k+1} - \delta x + \delta - 1 = 0, \tag{10.4}$$

except for the extra root of (10.4) $x = 1$; if $\delta = k+1$, then $x = 1$ is a root of (10.3) with multiplicity 1 and a root of (10.4) with multiplicity 2. Using Descartes' rule of signs we have that the equation $x^{k+1} - \delta x + \delta - 1 = 0$ has at most two positive real roots (one of them is $x = 1$) and at most one negative real root. If we suppose that θ is a root of (10.4) with multiplicity greater than 1, we deduce that θ also satisfies the first derivative of (10.4), that is, $(k+1)x^k - \delta$. Combining (10.4) and the identity $(k+1)x^k - \delta = 0$ we obtain $\theta = \frac{(\delta-1)(k+1)}{\delta k} \geq 0$, which yields that there exist no negative real root nor complex roots of $\xi_A(x)$ with multiplicity greater than 1. $\square$

To calculate the multiplicities of the eigenvalues of G we use the following lemma stated in [35] by Feit and Higman.

Lemma 10.4 ([35]). *Let θ be a simple root of the polynomial $f(x)$, and put $f_\theta(x) = \frac{f(x)}{x-\theta}$. If M is a matrix satisfying $f(M) = 0$ then $\frac{trace(f_\theta(M))}{f_\theta(\theta)}$ is the multiplicity of θ as a characteristic root of M, and so is rational.*

Since all the roots of $\xi_A(x)$ are simple we can apply the result of Feit and Higman to any of its roots. In the following lemma we give a formula for calculating multiplicities of the eigenvalues of a (d,k,δ)-digraph G.

Lemma 10.5. *Let $d \geq \delta \geq 1$ and let G be a (d,k,δ)-digraph of order n containing only selfrepeat vertices. If θ is an eigenvalue of G different to d and 1, then its multiplicity $m(\theta)$ satisfies the following*

$$m(\theta) = \frac{n(\delta + d\theta^k - d\delta)(\theta - 1)}{((k+1)\theta^k - \delta)(\theta - d)}. \tag{10.5}$$

Proof. We analyze the multiplicities of the eigenvalues of A using Lemma 10.4. Since the spectrum of A consists of d and the roots of $\xi_A(x)$, which are simple roots of $\xi_A(x)$, we conclude that $(x-d)\xi_A(x)$ is the minimal polynomial of A. It is well-known that every square matrix over a commutative ring satisfies its own minimal equation. We can consider $f(x) = (x-d)\xi_A(x) = (x-d)(x^k + x^{k-1} + \ldots + x + 1 - \delta)$, and thus, $f_\theta(x) = \frac{(x-d)(x^k+x^{k-1}+\ldots+x+1-\delta)}{x-\theta}$. By Lemma 10.4, the multiplicity of θ equals

$$m(\theta) = \frac{trace(f_\theta(A))}{f_\theta(\theta)}. \tag{10.6}$$

Since $f_\theta(x)$ is a monic polynomial of degree k we can set $f_\theta(x) = x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0$. Now, comparing the coefficients in front of x^i, for $0 \leq i \leq k$, on both sides of the identity

$$(x-\theta)(x^k + a_{k-1}x^{k-1} + \ldots + a_1 x + a_0) = (x-d)(x^k + x^{k-1} + \ldots + x + 1 - \delta)$$

we obtain the following system

$$\begin{cases} a_{k-1} - \theta = 1 - d \\ a_{k-2} - a_{k-1}\theta = 1 - d \\ \quad \cdots \\ a_{i-1} - a_i\theta = 1 - d \\ \quad \cdots \\ a_1 - a_2\theta = 1 - d \\ a_0 - a_1\theta = 1 - \delta - d \\ -a_0\theta = d(\delta - 1) \end{cases} \tag{10.7}$$

Solving (10.7) we find $a_0 = \theta^k + (1-d)(\theta^{k-1} + \theta^{k-2} + \ldots + \theta + 1) - \delta$ and $a_i = \theta^{k-i} + (1-d)(\theta^{k-i-1} + \theta^{k-i-2} + \ldots + \theta + 1)$, for $1 \leq i \leq k-1$. Replacing the obtained values for a_i in $f_\theta(x)$ we get

$$f_\theta(x) = x^k + (\theta+1-d)x^{k-1} + (\theta^2 + (1-d)(\theta+1))x^{k-2} + \ldots + \theta^k + (1-d)(\theta^{k-1} + \theta^{k-2} + \ldots + \theta + 1) - \delta.$$

Substituting θ for x yields

$$f_\theta(\theta) = (k+1)\theta^k + (1-d)(k\theta^{k-1} + (k-1)\theta^{k-2} + \ldots + 2\theta + 1) - \delta.$$

Using $\theta^k + \theta^{k-1} + \ldots + \theta + 1 - \delta = 0$, we obtain

$$f_\theta(\theta) = (k+1)\theta^k + (1-d)(k\theta^{k-1} + (k-1)\theta^{k-2} + \ldots + 2\theta + 1) - \delta = ((k+1)\theta^k - \delta)(\frac{\theta - d}{\theta - 1}). \quad (10.8)$$

Since the (d,k,δ)-digraphs do not contain directed cycles of length less than k, (see Propos. 10.1), the traces of the matrices A^i, with $1 \le i \le k-1$, are all equal to 0, $trace(A^i) = 0$. Hence, using additivity of traces, we have

$$trace(f_\theta(A)) = trace(A^k + (\theta+1-d)A^{k-1} + (\theta^2 + (1-d)(\theta+1))A^{k-2} + \ldots +$$

$$+(\theta^k + (1-d)(\theta^{k-1} + \theta^{k-2} + \ldots + \theta + 1) - \delta)I_n) = trace(A^k) + (\theta^k + (1-d)(\theta^{k-1} + \theta^{k-2} + \ldots + 1) - \delta)n.$$

As we showed previously, each vertex of a (d,k,δ)-digraph containing only selfrepeats lies on exactly δ directed cycles of length k. It implies

$$trace(A^k) = \delta n. \quad (10.9)$$

By $\theta^k + \theta^{k-1} + \ldots + \theta + 1 - \delta = 0$ it follows

$$n(\theta^k + (1-d)(\theta^{k-1} + \theta^{k-2} + \ldots + 1) - \delta) = nd(\theta^k - \delta). \quad (10.10)$$

In the end, substituting the results from (10.8), (10.9) and (10.10) in (10.6) we derive the formula for the multiplicity of θ,

$$m(\theta) = \frac{trace(A^k) + (\theta^k + (1-d)(\theta^{k-1} + \theta^{k-2} + \ldots + 1) - \delta)n}{(k+1)\theta^k + (1-d)(k\theta^{k-1} + (k-1)\theta^{k-2} + \ldots + 2\theta + 1) - \delta} =$$

$$= \frac{n(\delta + d\theta^k - d\delta)(\theta - 1)}{((k+1)\theta^k - \delta)(\theta - d)}.$$

□

Taking advantage of Lemma 10.5 we are in a position to give the main result in this chapter.

Theorem 10.6. *Let $d \ge \delta \ge k+1 \ge 4$. Then there exist no (d,k,δ)-digraphs containing only selfrepeat vertices.*

Proof. Let G be a (d,k,δ)-digraph containing only selfrepeat vertices. Recall that the spectrum of G consists of d and of the roots of the polynomial $\xi_A(x) = x^k + x^{k-1} + \ldots + x + 1 - \delta$. From Lemma 10.3 we easily see that for $k \ge 4$ and $\delta \ge 1$ the polynomial $\xi_A(x)$ has at least two complex roots. If $k = 3$, then by Descartes' rule it follows that $x^4 - \delta x + \delta - 1$ has no negative real root. Therefore, the polynomial $x^3 + x^2 + x + 1 - \delta$ has one positive real root and two complex roots.
In order to prove the non-existence of G, we fix a complex eigenvalue with negative real

part, (in Remark 10.7 below we prove the existence of such eigenvalue), and we show that its multiplicity is again a complex number, which will leads to a contradiction.

Let θ be a complex eigenvalue of G with negative real part and let $|\theta| = r$, that is, $\theta = r(\cos\phi + i\sin\phi)$ and $\cos\phi < 0$. Comparing the imaginary and the real parts of both sides of $\theta^{k+1} = \theta\delta - \delta + 1$ we obtain $\frac{\sin(k+1)\phi}{\sin\phi} = \frac{\delta}{r^k}$ and $r^{k+1}\cos(k+1)\phi = r\delta\cos\phi - \delta + 1$, respectively. These two relations yield

$$\frac{\sin k\phi}{\sin\phi} = \frac{\sin(k+1)\phi\cos\phi - \sin\phi\cos(k+1)\phi}{\sin\phi} = \frac{\delta-1}{r^{k+1}}. \tag{10.11}$$

By $\theta^{k+1} = \theta\delta - \delta + 1$ it follows $(\delta + d\theta^k - d\delta)(\theta - 1) = -d\theta^k + \delta\theta + d - \delta$ and $((k+1)\theta^k - \delta)(\theta - d) = -d(k+1)\theta^k + k\delta\theta + (k+1)(1-\delta) + d\delta$.
Next, we substitute the trigonometric form of θ into the formula (10.5), and then we multiply its numerator and denominator by the conjugate of the denominator of $m(\theta)$. We obtain

$$m(\theta) = n\cdot$$

$$\cdot\frac{-dr^k\cos k\phi + \delta r\cos\phi + d - \delta + i(-dr^k\sin k\phi + \delta r\sin\phi)}{-d(k+1)r^k\cos k\phi + k\delta r\cos\phi + (k+1)(1-\delta) + d\delta + i(-d(k+1)r^k\sin k\phi + k\delta r\sin\phi)}\cdot$$

$$\cdot\frac{-d(k+1)r^k\cos k\phi + k\delta r\cos\phi + (k+1)(1-\delta) + d\delta - i(-d(k+1)r^k\sin k\phi + k\delta r\sin\phi)}{-d(k+1)r^k\cos k\phi + k\delta r\cos\phi + (k+1)(1-\delta) + d\delta - i(-d(k+1)r^k\sin k\phi + k\delta r\sin\phi)}.$$

After performing the required multiplications we are able to determine the imaginary part of $m(\theta)$. The fact that $m(\theta)$ is a natural number makes the imaginary part equals to 0, which leads to the identity

$$d\delta r^{k+1}\sin(k-1)\phi + dr^k((k+1)(d-1) - d\delta)\sin k\phi + \delta r((\delta-d)k + (k+1)(1-\delta) + d\delta)\sin\phi = 0. \tag{10.12}$$

Since $\frac{\delta}{r^k} = \frac{\sin(k+1)\phi}{\sin\phi} = \frac{\delta-1}{r^{k+1}}\cos\phi + \cos k\phi$, we have $\cos k\phi = \frac{\delta}{r^k} - \frac{\delta-1}{r^{k+1}}\cos\phi$. Thus

$$\frac{\sin(k-1)\phi}{\sin\phi} = \frac{\sin k\phi}{\sin\phi}\cos\phi - \cos k\phi = \frac{\delta-1}{r^{k+1}}\cos\phi - \cos k\phi = \frac{2(\delta-1)}{r^{k+1}}\cos\phi - \frac{\delta}{r^k}.$$

Substituting $\sin(k-1)\phi = \frac{2(\delta-1)}{r^{k+1}}\sin\phi\cos\phi - \frac{\delta}{r^k}\sin\phi$ and $\sin k\phi = \frac{\delta-1}{r^{k+1}}\sin\phi$ in (10.12), and dividing by $\sin\phi$, we have

$$2d\delta(\delta-1)\cos\phi = \frac{d(\delta-1)}{r}((k+1)(1-d) + d\delta) + \delta r((d-\delta)k + (k+1)(\delta-1)). \tag{10.13}$$

If $\delta \geq k+1$, it can be easily seen that the right side of (10.13) is a positive number. On the other hand, we have chosen θ to be a complex eigenvalue of G whose real part is negative, ($\cos\phi < 0$). This provides us the required contradiction. ☐

Remark 10.7. Let $\delta \geq k+1 \geq 4$. If $\delta > k+1$, then $\xi_A(1) = k+1-\delta < 0$. Since $\xi_A(d) = n > 0$ and $\xi_A(1) < 0$, we note that the unique positive root of $\xi_A(x)$ belongs to the interval $(1,d)$, and we denote it by θ_1. Descartes' rule asserts that the polynomial $\xi_A(x)$ has no negative real root when k is an odd number. We will prove the existence of a complex root of $\xi_A(x)$

with negative real part when $\xi_A(x)$ has a negative real root θ_2; in such case k must be an even number. The case when $\xi_A(x)$ has no negative root can be handled similarly.
Let $\theta_j = p_j + q_j i$, with $3 \leq j \leq k$, be the complex roots of $\xi_A(x)$. By way of contradiction we assume that $p_j \geq 0$, for all $3 \leq j \leq k$. Using the fact that the complex roots come in conjugate pairs and applying Vieta's formulas to the polynomial $\xi_A(x)$, we obtain

$$-1 = \theta_1 + \theta_2 + \ldots + \theta_k = \theta_1 + \theta_2 + (p_3 + \ldots + p_k) \geq \theta_1 + \theta_2.$$

On the other hand, since k is an even number and θ_1 is a positive, using the inequality $(1+\theta_1)^t > (1+\theta_1)^{t-1} + \theta_1^t + \theta_1^{t-1}$, for $2 \leq t \leq k$, we can deduce

$$\xi_A(-1-\theta_1) = (-1-\theta_1)^k + (-1-\theta_1)^{k-1} + \ldots + (-1-\theta_1) + 1 - \delta =$$

$$= (1+\theta_1)^k - (1+\theta_1)^{k-1} + \ldots + (1+\theta_1)^2 - (1+\theta_1) + 1 - \delta > \theta_1^k + \theta_1^{k-1} + \ldots + 1 - \delta = 0.$$

Since $\xi_A(0) < 0$ and $\xi_A(-1-\theta_1) > 0$, it follows that the negative root of $\xi_A(x)$ belongs to the interval $(-1-\theta_1, 0)$, that is, $-1-\theta_1 < \theta_2 < 0$. Thus $\theta_1 + \theta_2 > -1$, which is in contradiction to $\theta_1 + \theta_2 \leq -1$. For $\delta = k+1$ we proceed similarly.

Remark 10.8. Using the proof of Theorem 10.6 we can re-prove the non-existence of the $(d, k, 1)$-digraphs containing only selfrepeats, with $k \geq 3$ and $d \geq 2$; the result given in Theorem 2.3. Since $\delta = 1$, the eigenvalues of this family of digraphs are d (with multiplicity 1) and the roots of $x^k + x^{k-1} + \ldots + x = 0$, which consist of 0 and the k-th roots of unity, $e^{2\pi ji/k}$, with $1 \leq j < k$. Note that $\theta = e^{2\pi ji/k}$ implies $r = |\theta| = 1$, for each θ. Substituting $\delta = r = 1$ in (10.13) we have $0 = (d-1)k$, which is impossible.

The problem of the non-existence of (d, k, δ)-digraphs with only selfrepeats, for specific parameters (d, k, δ), can be answered directly employing the formulas (10.5) or (10.13). For each triplet (d, k, δ) with $d \geq \delta \geq 2$ and $k \geq 3$ we can determine the order and the eigenvalues of the (d, k, δ)-digraphs containing only selfrepeats. Then, using (10.5) we can calculate the corresponding multiplicities. The non-existence of the considered digraphs can be concluded once we obtain non-integer or negative integer multiplicity. Similarly, the non-existence of (d, k, δ)-digraphs containing only selprepeats can be secured if the right side of (10.13) is greater than $2d\delta(\delta-1)$ or smaller than $-2d\delta(\delta-1)$. Based on the results obtained using the above formulas in *WolframAlpha*, we establish the following conjecture.

Conjecture 10.9. Let $d \geq \delta \geq 2$ and $k \geq 3$. There exist no (d, k, δ)-digraphs containing only selfrepeat vertices.

10.2 Alternate proof of the nonexistence of Moore digraphs

Plesník and Znám in [68] and later Bridges and Toueg in [16] showed that Moore digraphs exist only in the trivial cases, when $d = 1$ (directed cycle C_{k+1}) or when $k = 1$ (complete symmetric digraph K_{d+1}). In this section we give an alternate proof of this result based on the method presented in this chapter. In order to relate the multiplicity result in Section

10.1 with the multiplicities of the eigenvalues of the Moore digraphs, we verify Lemma 10.5 in the case of Moore digraphs, putting $\delta = 0$ in formula (10.5).

First, the (d, k)-Moore digraphs can be viewed as $(d, k, 0)$-digraphs ((d, k)-digraphs with defect $\delta = 0$). If A is the adjacency matrix of a (d, k)-Moore digraph, then we have the following relation:

$$A^k + A^{k-1} + \ldots + A + I = J.$$

This matrix equation yields that the spectrum of A consists of d and the roots of

$$x^k + x^{k-1} + \ldots + x + 1 = 0, \tag{10.14}$$

which are the $(k + 1)$-st roots of unity. It is a well-known fact that the roots of unity of (10.14) are simple roots. Clearly, the equation (10.14) has the same roots as the equation (10.3) when $\delta = 0$.

Since any (d, k)-Moore digraph does not contain cycles of length shorter than $k + 1$, $trace(A^j) = 0$, for each $1 \leq j \leq k$ (see [16]). In comparison, for the (d, k, δ)-digraphs containing only selfrepeat vertices we showed that there exists no cycle of length shorter than k, which implies $trace(A^j) = 0$, for each $1 \leq j \leq k - 1$, while $trace(A^k) = \delta n$. We note that if $\delta = 0$, then $trace(A^k) = 0$.

Now, setting $\delta = 0$ in formula (10.5), we obtain a formula for the multiplicities of the non-trivial eigenvalues of the Moore digraphs:

Corollary 10.10. *Let θ, different to d, be an eigenvalue of a (d, k)-Moore digraph of order n. Then its multiplicity $m(\theta)$ satisfies the identity*

$$m(\theta) = \frac{nd}{k+1} \cdot \frac{\theta - 1}{\theta - d}. \tag{10.15}$$

The only real roots of unity are 1 and -1. Therefore the equation (10.14) has at least $k-1$ complex roots. Thus, for $k > 1$ we can fix a complex eigenvalue θ of A. Using formula (10.15) we conclude that if $d > 1$, then $m(\theta)$ is a complex number (since $\frac{\theta-1}{\theta-d}$ is complex), which is impossible.

Finally, we briefly analyze the remaining two possibilities, $d = 1$ or $k = 1$, determining the spectrum of the corresponding digraphs.

- If $d = 1$, then $m(\theta) = \frac{n}{k+1} = \frac{k+1}{k+1} = 1$. Therefore for the spectrum of the $(1, k)$-Moore digraphs (they are the directed C_{k+1} cycles), we have

  $$Spec(C_{k+1}) = \{1^{(1)}, \theta_1^{(1)}, \ldots, \theta_{k-1}^{(1)}, \theta_k^{(1)}\},$$

  where θ_i is the i-th root of the equation (10.14).

- If $k = 1$, then $n = d + 1$ and $\theta = \theta_1 = -1$. Thus, $m(\theta) = \frac{(d+1)d}{2} \cdot \frac{-2}{-1-d} = d$. For the spectrum of the $(d, 1)$-Moore digraphs (they are the directed complete digraphs K_{d+1}) we have

  $$Spec(K_{d+1}) = \{d^{(1)}, -1^{(d)}\}.$$

Chapter 11

Conclusions

A number of research problems from extremal graph theory are solved, in particular: improvement of the lower bounds on the order of cages of even girth, existence of antipodal cages of even girth and small excess, existence of vertex-transitive graphs of given degree and girth. Moreover, in the thesis we answered the open question about the monotonicity of the function $n_{d,k}$ (the largest possible order of the digraphs with maximum out-degree d and diameter k) in d and in k, and we give a close connection between two well-known open problems from extremal graph theory, the problem posed by Bermond in Bollobás and the problem which address the existence of the non-bipartite Ramanujan graphs for any degree k.

The basic tools used in this research range from combinatorial methods in graph theory combined with number theory and the linear algebra considerations in matrix theory. Finding the multiplicities of the eigenvalues of graphs, counting g-cycles in certain graphs combining with number theory and using the power of the inequalities are the main techniques employed in the thesis.

The results of this PhD Thesis represent a significant contribution to a number of long standing open problems in graph theory.

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Chapter 12

Povzetek v slovenskem jeziku

12.1 Teoretična izhodišča

Topologijo omrežja (na primer telekomunikacijskega, multiprocesorskega ali lokalnega računalniškega omrežja) po navadi modeliramo z grafom, katerega vozlišča predstavljajo 'vozlišča omrežja' (postaje oz. procesorje), neusmerjene ali usmerjene povezave grafa pa predstavljajo povezave med danimi vozlišči. Pri načrtovanju takšnega omrežja moramo upoštevati številne značilnosti grafov. Najpogostešji takšni značilnosti sta omejitev stopnje vozlišč ter omejitev premera grafa. Interpretacija teh dveh parametrov v omrežju je očitna: stopnja vozlišča je število povezav, ki jih ima to vozlišče z drugimi vozlišči, premer pa predstavlja maksimalno število povezav, ki jih mora neko sporočilo prepotovati na poti med poljubnim parom vozlišč. Kakšno je največje število vozlišč v omrežju z omejeno stopnjo vozlišč in omejenim premerom? Če so povezave med vozlišči neusmerjene, potem imamo opravka z naslednjim problemom v teoriji grafov:

- *Problem stopnje in premera:* Za dani naravni števili k in d poiščite največje možno število vozlišč $n(k,d)$, za katero obstaja graf z $n(k,d)$ vozlišči, ki ima največjo stopnjo k in premer d.

Če besedo 'stopnja' zamenjamo z 'izhodna stopnja', dobimo različico problema za usmerjene grafe. Izhodna stopnja vozlišča v usmerjenemu grafu je število usmerjenih povezav z začetkom v danem vozlišču. Tako dobimo naslednji problem:

- *Problem stopnje in premera v usmerjenemu grafu:* Za dani naravni števili d in k poiščite največje možno število vozlišč $n_{d,k}$, za katero obstaja usmerjeni graf z $n_{d,k}$ vozlišči, ki ima največjo izhodno stopnjo d in premer k.

Problem kletke, ki je znan tudi kot problem stopnje in ožine, je tesno povezan s problemom stopnje in premera.

- *Problem kletke (Problem stopnje in ožine):* Za dani naravni števili k in g poiščite najmanjše število vozlišč $n(k,g)$ za katero obstaja graf stopnje k in ožine g.

V doktorski disertaciji smo se osredotočili na odprta vprašanja in probleme, ki se nanašajo na problem kletke ter na problem stopnje in premera v neusmerjenih in usmerjenih grafih. Rezultati doktorske disertacije so objavljeni v naslednjih člankih.

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12.1.1 Problem kletke

Grafu ožine g, ki je k-regularen, pravimo (k, g)-*graf*. Z izrazom (k, g)-*kletka* pa poimenujemo (k, g)-graf, ki ima minimalno število vozlišč. Red takega grafa označimo z $n(k, g)$. Že okoli leta 1960 je bilo znano, da za vsak par (k, g) obstaja neskončno mnogo (k, g)-grafov [30, 71]. Kljub temu je red $n(k, g)$ za (k, g)-kletke znan samo za nekatere posebne pare števil k in g [33].

Moore-ova meja $M(k, g)$ je naravna spodnje meja za red (k, g)-grafa (in potemtakem tudi za red (k, g)-kletke). Njena vrednost je odvisna od parnosti števila g:

$$n(k,g) \geq M(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2}, & \text{če je} g \text{ lih,} \\ 2\left(1 + (k-1) + \cdots + (k-1)^{(g-2)/2}\right), & \text{če je} g \text{ sod.} \end{cases}$$

Znano je, da red velike večine (k, g)-kletk presega Mooreovo mejo (glej npr. [33]). Natančna zveza med številoma $M(k, g)$ in $n(k, g)$ je eno od pomembnečjih odprtih vprašanj v teoriji kletk. Grafi, katerih red je enak Moore-ovi meji, se imenujejo *Moore-ovi grafi*. Poleg tega je znano, da Mooreov graf obstaja če in samo če je $k = 2$ in $g \geq 3$, $g = 3$ in $k \geq 2$, $g = 4$ in $k \geq 2$, $g = 5$ in $k \in \{2, 3, 7\}$ ali pa je $g \in \{6, 8, 12\}$, pri čemer obstaja posplošeni n-kotnik reda $k - 1$ [5, 25, 33]. Obstoj $(57, 5)$-Mooreovega grafa je še vedno odprto vprašanje.

Razlika med redom n nekega (k,g)-grafa G in Moore-ovo mejo $M(k,g)$, ki jo označimo z e, se imenuje *presežek* grafa G, tj. $e = n - M(k,g)$. Izračun presežka (k,g)-kletke je ekvivalenten izračunu števila $n(k,g)$.

Poglavitni del disertacije je študij kletk s sodo ožino in majhnim presežkom. Rezultati o kletkah s sodo ožino, ki so bili znani do sedaj, so povzeti v naslednjih dveh izrekih.

Izrek 12.1 ([15])**.** *Naj bo G (k,g)-kletka z ožino $g = 2m \geq 6$ in presežkom e. Če je $e \leq k-2$, potem je e sodo števil, graf G pa je dvodelen s premerom $m+1$.*

V naslednjem izreku naj $D(k,2)$ označuje incidenčni graf simetričnega $(v,k,2)$-načrta.

Izrek 12.2 ([15])**.** *Naj bo G (k,g)-kletka z ožino $g = 2m \geq 6$ in presežkom 2. Potem je $g = 6$, G je dvojni krov grafa $D(k,2)$, število k pa ni kongruentno 5 ali 7 $(\mathrm{mod}\ 8)$.*

Kolikor nam je znano – razen za določene majhne primere grafov, za katere je bilo število $n(k,g)$ ugotovljeno, in za majhno število grafov s presežkom večjim od 2, katerih obstoj je bil dokazan z izčrpnim preiskovanjem z računalnikom – ni nobenih znanih parov parametrov (k,g), kjer bi bil presežek večji od 2 za bodisi sodo bodisi liho število g (tj. ni znana nobena neskončna družina parov števil (k,g), za katere bi bilo dokazano, da je $n(k,g) > M(k,g)+4$). V disertaciji smo vpeljali neskončne družine parov parametrov (k,g), za katere ne obstaja noben (k,g)-graf s presežkom manjšim od 5. Ti rezultati so prvi rezultati takšnega tipa. Naši argumenti temeljijo na precej očitni lemi, ki jo bomo sedaj predstavili. Naj bo G k-regularen graf ožine g. Za poljubno povezavo $f \in E(G)$ in celo število $c \geq 3$ naj $\overline{\mathbf{c}}_G(f,c)$ označuje *število ciklov* dolžine c v grafu G, ki vsebujejo povezavo f.

Lemma 12.3. *Naj bo G graf in naj bo $c \geq 3$. Število*

$$\sum_{f \in E(G)} \overline{\mathbf{c}}_G(f,c)$$

je deljivo s c.

Ključen argument v dokazu zgoraj opisanega rezultata temelji na skrbnem štetju ciklov dolžine c v (potencialnem) (k,g)-grafu presežka 4.

Predpostavimo, da je $k \geq 6$, $g = 2d \geq 6$ in da je G (k,g)-graf, ki ima presežek 4 in red n. Zaradi Biggsovega rezultata zapisanega v Izreku 12.1, nam restrikcija parametrov k in g, ki je podana zgoraj, omogoča sklep, da je G dvodelen graf s premerom $d+1$. Za vsako celo število i, za katero velja $0 \leq i \leq d+1$, definiramo $n \times n$ matriko $A_i = A_i(G)$ na naslednji način. Vrstice in stolpci matrike A_i ustrezajo vozliščem grafa G. Vrednost na položaju (u,v) je 1, če je razdalja $d(u,v)$ med vozliščema u in v enaka i, sicer pa je vrednost na položaju (u,v) enaka 0. Očitno je $A_0 = I$ in $A_1 = A$, kjer A označuje običajno matriko sosednosti grafa G.

Zadnja neničelna matrika je matrika A_{d+1}, ki jo bomo označili z E in jo poimenovali *matrika presežnosti.* Z drugimi besedami, E je matrika sosednosti grafa, ki ima isto množico vozlišč V kot graf G, dve vozlišči iz V pa sta sosednji natanko tedaj, ko sta na razdalji $d+1$ v grafu G. Tako dobljen graf imenujemo *graf presežnosti* grafa G in ga označimo z $G(E)$.

Če J označuje matriko, ki ima na vsakem položaju vrednost 1, potem *razdaljne matrike* A_i, za $0 \le i \le d$, in matrika E zadoščajo enakosti

$$\sum_{i=0}^{d} A_i + E = J.$$

Definirajmo sedaj naslednje polinome:

$$F_0(x) = 1,\ F_1(x) = x,\ F_2(x) = x^2 - k;$$

$$G_0(x) = 1,\ G_1(x) = x + 1;$$

$$H_{-2}(x) = -\tfrac{1}{k-1},\ H_{-1}(x) = 0,\ H_0(x) = 1,\ H_1(x) = x;$$

$$P_{i+1}(x) = xP_i(x) - (k-1)P_{i-1}(x) \text{ za } \begin{cases} i \ge 2, & \text{če je } P_i = F_i, \\ i \ge 1, & \text{če je } P_i = G_i, \\ i \ge 1, & \text{če je } P_i = H_i. \end{cases}$$

Singleton je dokazal veliko povezav med temi polinomi [72]. Uporabili smo dve izmed teh povezav. Za vsak $i \ge 0$ velja, da je

$$G_i(x) = \sum_{j=0}^{i} F_j(x)$$

in

$$G_{i+1}(x) + (k-1)G_i(x) = (x+k)H_i(x).$$

Zgoraj definirani polinomi so tesno povezani z lastnostmi danega grafa. Za $t < g$ namreč element $(F_t(A))_{x,y}$ predstavlja število poti dolžine t med vozliščema x in y v grafu G. Podobno, $G_t(A)$ predstavlja število poti dolžine največ t med dvema vozliščema. Z uporabo zgornjih polinomov in z uporabo spektralne analize smo izpeljali potrebne pogoje za obstoj dvodelnih (k, g)-grafov, ki imajo presežek 4. Pravimo, da ima graf G *ciklični presežek*, če je graf presežnosti $G(E)$ cikel dolžine n. Za graf G pravimo, da ima *biciklični presežek*, če je $G(E)$ disjunktna unija dveh ciklov. Opozorimo, da ni znanih nobenih (k, g)-grafov s cikličnim presežkom 2, medtem ko lahko primere grafov z bicikličnim presežkom 2 najdemo med $(3, 5)$-grafi presežka 2 [28]. V doktorski disertaciji smo dokazali neobstoj (k, g)-grafov s cikličnim in bicikličnim presežkom 4 za neskončne razrede parametrov (k, g).

Izkazalo se je, da je veliko znanih kletk *vozliščno-tranzitivnih* [33], tj. grupa avtomorfizmov grafa deluje tranzitivno na množici vozlišč. Zdi se, da imajo to lastnost tudi ekstremni (k, g)-grafi. Študij vozliščno-tranzitivnih kletk bo najverjetneje privedel do izboljšav rezultatov glede splošnega problema kletk, kot tudi izboljšal naše razumevanje vozliščno-tranzitivnih grafov. Pri splošnih kletkah je vprašanje obstoja splošne meje za presežek še vedno odprto. Po drugi strani pa je v bolj specializiranem primeru vozliščno-tranzitivnih kletk odgovor na to vprašanje negativen. Presežek vozliščno-tranzitivnih (k, g)-kletk je lahko poljubno velik. Ta rezultat je dokazal Biggs:

Izrek 12.4 ([12])**.** *Za vsako liho število $k \geq 3$ obstaja neskončno zaporedje vrednosti g, tako da za presežek e vsakega vozliščno-tranzitivnega grafa stopnje k in ožine g velja $e > \frac{g}{k}$.*

V disertaciji smo pokazali, da Biggsov rezultat [12] ne drži le za neskončno mnogo vrednosti g, ampak v resnici velja za skoraj vse vrednosti g pri poljubnem $k \geq 4$.

Pravimo, da je graf premera d *antipoden*, če za poljubna vozlišča u, v in w, za katera je $d(u, v) = d$ in $d(u, w) = d$, velja, da je $d(v, w) = d$ ali pa $v = w$ (glej npr. [14]). Lepi primeri antipodnih grafov so n-dimenzionalne kocke Q_n. To so dvodelni grafi, kjer za vsako vozlišče v obstaja natanko eno vozlišče, ki je na maksimalni razdalji od v. Zaradi te lastnosti so vse n-dimenzionalne kocke antipodne. Poleg tega so polni dvodelni grafi $K_{n,n}$, kjer je $n \geq 1$, prav tako antipodni. Pri teh grafih je antipodno razbitje isto kot dvodelno razbitje. Skelet dodekaedra je primer trivialno antipodnega grafa, ki ni dvodelen. Primeri grafov, ki so netrivialno antipodni in niso dvodelni, so polni tridelni grafi $K_{n,n,n}$, ki imajo premer 2, in graf povezav Petersenovega grafa, ki ima premer 3. Možno je, da obstajajo tudi primeri antipodnih kletk sode ožine med $(k, 6)$-kletkami s presežkom $e \leq k - 2$. Antipoden je tudi vsak graf z lastnostjo, da za vsako povezavo tega grafa presežna množica inducira podgraf, ki ima le $\frac{e}{2}$ povezav. Poleg tega je vsak tak graf λ-listni krov indicenčnega grafa nekega simetričnega (v, k, λ)-načrta, ki ga označimo z $D(k, \lambda)$ in za katerega velja, da je $\lambda = \frac{e}{2} + 1$ [15]. Kletke, o katerih govori Izrek 12.2, pripadajo tej družini antipodnih grafov, saj njihove presežne množice inducirajo podgrafe z 1 povezavo. Enolično določena $(7, 6)$-kletka je tudi primer antipodne kletke; ta graf sta odkrila O'Keefe in Wong [66]. Problem obstoja antipodnih regularnih grafov z liho ožino sta obravnavala Bannai in Ito. Dokazala sta naslednji izrek.

Izrek 12.5 ([6])**.** *Za $d \geq 3$ ne obstaja noben antipoden regularen graf z premerom $d + 1$ in ožino $2d + 1$.*

V disertaciji smo se ukvarjali z vprašanjem obstoja antipodnih (k, g)-kletk s sodo ožino in presežkom $e \leq k - 2$. Dokazali smo njihov neobstoj za $k \geq e + 2 \geq 6$ in $g = 2d \geq 8$.

12.1.2 Problem stopnje in premera za neusmerjene grafe

Z izrazom (k, d)-*graf* poimenujemo vsak k-regularen graf Γ z diametrom d. Naj $n(k, d)$ označuje največje možno število vozlišč kateregakoli neusmerjenega grafa maksimalne stopnje k in diametra d. Parameter $n(k, d)$ zadošča sledeči neenakosti:

$$|V(\Gamma)| \leq n(k, d) \leq M(k, d) = 1 + k + k(k - 1) + k(k - 1)^2 + ... + k(k - 1)^{d-1}.$$

Zgoraj opisana meja $M(k, d)$ se imenuje *Mooreova meja*. Graf, katerega število vozlišč je enako Mooreovi meji, imenujemo *Mooreov graf*. Takšen graf je vedno k-regularen. Mooreovi grafi so dokazano zelo redki. Primeri takšnih grafov so popolni grafi s $k + 1$ vozlišči in cikli na $2d + 1$ vozliščih. Nadaljnji primeri takšnih grafov, ki imajo premer enak 2, so Petersenov graf, Hoffman-Singletonov graf in mogoče graf stopnje $k = 57$. Za $k > 2$ in $d > 2$ Mooreovi grafi ne obstajajo [60].

Razliko med Mooreovo mejo $M(k, d)$ in številom vozlišč grafa Γ z maksimalno stopnjo k in premerom d imenujemo *defekt* grafa Γ ter ga označimo z $\delta(\Gamma)$. Če je Γ največji graf maksimalne stopnje k in premera d, potem je $n(k, d) = M(k, d) - \delta(\Gamma)$. Naj omenimo, da je, podobno kot v primeru kletk, zelo malo znanega o točni povezavi med Mooreovo mejo $M(k, d)$ in pripadajočim maksimalnim številom vozlišč $n(k, d)$. Čeprav obstaja med številom vozlišč največjih znanih/konstruiranih grafov maksimalne stopnje k in premera d ter pripadajočo Mooreovo mejo kar velik razkorak, občutno boljše teoretične meje niso znane. Znano ni niti, če sta parametra enakega velikostnega reda (pri čemer računalniški rezultati močno kažejo na to, da nista). Tako lahko na spodaj opisano *vprašanje Bermonda in Bollobása* [11], ki je že dolgo odprto, gledamo kot na naravni prvi korak k razumevanju narave odnosa med $M(k, d)$ in $n(k, d)$.

Ali je res, da za vsako pozitivno celo število c obstajata takšna k in d, da je število vozlišč največjega grafa maksimalne stopnje k in premera d največ $M(k, d) - c$?

V doktorski disertaciji predstavimo povezavo med tem vprašanjem in vrednostjo $2\sqrt{k-1}$ katero je tudi kljucno v definiciji Ramanujanovih grafov definiranih kot k-regularni grafi katerih druge največje lastne vrednosti (v absolutni vrendosti) kvečjemu $2\sqrt{k-1}$.

12.1.3 Problem stopnje in premera za usmerjene grafe

Usmerjeni graf ali *digraf* je struktura $G = (V, A)$, kjer je $V(G)$ končna množica *vozlišč*, $A(G)$ pa je množica urejenih parov (u, v) paroma različnih vozlišč $u, v \in V(G)$, ki jih imenujemo *loki*. *Red* digrafa G je število vozlišč digrafa G.

Vhodni sosed vozlišča v v digrafu G je takšno vozlišče u, da je $(u, v) \in A(G)$. Podobno je *izhodni sosed* vozlišča v takšno vozlišče w, da je $(v, w) \in A(G)$. *Vhodna stopnja* (oz. *izhodna stopnja*) vozlišča $v \in V(G)$ je število njegovih vhodnih (oz. izhodnih) sosedov. Če je vhodna in izhodna stopnja vsakega vozlišča enaki d, imenujemo digraf G *diregularen digraf* stopnje d.

Sprehod W dolžine k v digrafu G je alternirajoče zaporedje $(v_0 a_1 v_1 a_2 \ldots a_k v_k)$ takšnih vozlišč in lokov digrafa G, da je $a_i = (v_{i-1}, v_i)$ za vsak i. Če so loki $a_1, a_2, \ldots, a_k$ sprehoda W paroma različni, imenujemo W *sled*. Če so vsa vozlišča $v_0, v_1, \ldots, v_k$ paroma različna, imenujemo W *pot*. *Cikel* C_k dolžine k je zaprta sled dolžine $k > 0$, kjer so vsa vozlišča (razen prvega in zadnjega) paroma različna. *Razdalja* od vozlišča u do vozlišča v v digrafu G je dolžina najkrajše usmerjene poti od u do v. *Premer* k digrafa G je največja možna razdalja med pari vozlišč digrafa G. Usmerjenemu grafu maksimalne vhodne stopnje d in premera k pravimo (d, k)-digraf. Naj bo $n_{d,k}$ največji možni red (d, k)-digrafa. Naj bo $n_i(x)$, za $0 \le i \le k$, število vozlišč na razdalji i od danega vozlišča x. Potem je $n_i(x) \le d^i$ za $0 \le i \le k$. Sledi

$$n_{d,k} = \sum_{i=0}^{k} n_i(x) \le 1 + d + ... + d^{k-1} + d^k = \begin{cases} \frac{d^{k+1}-1}{d-1}, & \text{če je } d > 1, \\ k+1, & \text{če je } d = 1. \end{cases} \tag{12.1}$$

Število na desni strani enačbe (12.1) označimo z $M_{d,k}$ in ga imenujemo *Mooreova meja* za (d, k)-digrafe. Digraf, katerega red je enak Mooreovi meji, imenujemo *Mooreov digraf*. Dobro znano je, da enakost $n_{d,k} = M_{d,k}$ drži samo v trivialnem primeru, ko je $d = 1$ (usmerjen cikel dolžine $k + 1$) ali $k = 1$ (poln digraf reda $d + 1$), glej [16] in [68]. *Defekt* δ danega (d, k)-digrafa G je razlika med pripadajočo Mooreovo mejo $M_{d,k}$ in redom digrafa

G. Ker za $d \neq 1$ in $k \neq 1$ Mooreovi (d,k)-digrafi ne obstajajo, postane problem obstoja diregularnih digrafov stopnje $d \geq 2$, premera $k \geq 2$ in s številom vozlišč $M_{d,k} - \delta$, $\delta \neq 0$, zanimivo vprašanje. Takšne digrafe imenujemo (d,k,δ)-*digrafi*, kjer je δ defekt in $d \geq \delta \geq 1$.

V doktorski disertaciji smo pozitivno odgovorili na vprašanje, ki se nanaša na problem stopnje in premera digrafov in je bilo zastavljeno v [60]:

Ali je zaporedje $n_{d,k}$ monotono glede na d in k?

Vsak $(d,k,1)$-digraf G ima lastnost, da za vsako vozlišče $u \in G$ obstaja enolično določeno vozlišče $v \in G$, tako da obstajata natanko dve poti dolžine kvečjemu k od u do v v G (glej [7]). Takšno vozlišče v imenujemo *ponovitev* vozlišča u in ga označimo z $r(u)$. Če je $r(u) = v$, potem je $r^{-1}(v) = u$. Vozlišče u, za katero velja $r(u) = u$, imenujemo *samoponovitev* digrafa G. Baskoro, Miller, Plesník in Znám [8] so uporabili spektralno metodo (lastne vrednosti matrike sosednosti) za dokaz neobstoja diregularnih $(d,k,1)$-digrafov stopnje $d \geq 2$, premera $k \geq 3$ in z lastnostjo, da je vsako vozlišče samoponovitev (tj. da vsako vozlišče leži na usmerjenem ciklu C_k). Za $k = 2$ in stopnjo $2 \leq d \leq 12$ so pokazali, da če obstaja C_2, potem vsako vozlišče leži na C_2 (torej je samoponovitev bodisi vsako vozlišče bodisi nobeno).

Izrek 12.6 ([8])**.** *Za $d \geq 2$ in $k \geq 3$ ne obstaja noben $(d,k,1)$-digraf z vsemi vozlišči na C_k.*

S posplošitvijo koncepta samoponovitve bomo naslovili vprašanje obstoja (d,k,δ)-digrafov z $\delta \geq 2$ na naslednji način. Za fiksno vozlišče $u \in V(G)$ naj $R(u)$ označuje množico vozlišč $v \in V(G)$, za katere obstajata vsaj dva sprehoda dolžine kvečjemu k, ki povezujeta u in v. Za $\delta > 1$ je lahko $|R(u)|$ v splošnem primeru večja od 1. Če je $R(u) = u$, pravimo, da je u *samoponovitev.* Zanimivo je, da so vsa vozlišča v $(2,2,1)$-digrafu in $(d,2,d)$-digrafih za $d \in \{2,3,7\}$ ter mogoče tudi $d = 57$ samoponovitve [7]. Poleg tega sta v neobjavljenem članku [7] Baskoro in Garminia dokazala neobstoj $(d,2,2)$-digrafa z $d \geq 3$, $(d,2,\delta)$-digrafa z $\delta = 4,5$ ali 6 in $d \geq \delta$, ter neobstoj $(d,3,4)$-digrafa z $d \geq 4$, ki vsebuje samo samoponovitve.

V disertaciji smo te rezultate razširili z dokazom neobstoja (d,k,δ)-digrafa, ki vsebuje samo vozlišča, ki so samoponovitve, za $d \geq \delta \geq k+1 \geq 4$. Poleg tega smo z uporabo podobnih tehnik ponovno dokazali neobstoj $(d,k,1)$-digrafa, ki vsebuje le samoponovitve, za $k \geq 3$ in $d \geq 2$, ter neobstoj (d,k)-Mooreovih digrafov za $k \geq 2$ in $d \geq 2$.

12.2 Rezultati

V disertaciji smo obravnavanih 8 povezanih tem iz ekstremalne teorije grafov, ki so predstavljene v prejšnjem razdelku. V naslednjih podrazdelkih bomo predstavili najpomembnejša znanstvene rezultate za vsako izmed njih.

12.2.1 Izboljšanje spodnje meje za red kletk sode ožine

Poiskali smo neskončne družine parametrov (k,g), kjer je $g > 6$ sodo število. Za te grafe smo pokazali, da je presežek kateregakoli k-regularnega grafa ožine g večji od 4. Tako smo dobili izboljšano spodnjo mejo za red k-regularnih grafov ožine g z najmanjšim možnim redom. Te grafi se imenujejo (k,g)-kletke. Za omejeno družino (k,g)-grafov, ki imajo dodatno strukturno lastnost in dovolj velika parametra k in g, smo pokazali, da je presežek teh k-regularnih grafov ožine g lahko poljubno velik. V disertaciji smo dokazali naslednja izreka:

Izrek 12.7. *Naj bo $k \geq 6$ in $g = 2m > 6$. Potem ne obstaja (k,g)-graf s presežkom 4 za parametra k, g, ki bi ustrezal vsaj enemu izmed pogojev:*

(1) $g = 2p$, *kjer je* $p \geq 5$ *praštevilo, in* $k \not\equiv 0, 1, 2 \pmod p$*;*

(2) $g = 4 \cdot 3^s$, *kjer je* $s \geq 4$, k *pa je deljiv z* 9 *in ni deljiv s* 3^{s-1}*;*

(3) $g = 2p^2$, *kjer je* $p \geq 5$ *praštevilo,* k *pa je tako sodo število, da* $k \not\equiv 0, 1, 2 \pmod p$*;*

(4) $g = 4p$, *kjer je* $p \geq 5$ *praštevilo, in* $k \not\equiv 0, 1, 2, 3, p-2 \pmod p$*;*

(5) $g \equiv 0 \pmod{16}$ *in* $k \equiv 3 \pmod g$.

Izrek 12.8. *Za vsak $e \geq 1$ obstajata parametra k, g, kjer je k liho in g sodo število, tako da velja: Če je G (k,g)-graf, v katerem je za vsako povezavo f podgraf induciran z X_f izomorfen disjunktni uniji kopij grafa K_2, potem ima G presežek večji od e.*

12.2.2 Dvodelne kletke s presežkom 4

Zanimala sta nas struktura in lastnosti dvodelnih kletk s presežkom 4. Dokazali smo naslednji izrek.

Izrek 12.9. *Naj bo $k \geq 7$ liho število in naj bo $g = 2d \geq 8$. Naj bo c število vseh ciklov grafa $G(E)$ in naj bo c_2 število ciklov sode dolžine. Če obstaja (k,g)-graf s presežkom 4, potem velja:*

(1) Če je d liho število, potem $d-1$ deli $c-2$ in c_2.

(2) Če je d sodo število, potem $d-1$ deli $c-1$ in c_2-1.

Dodatno smo se osredotočili še na obstoj dveh potencialnih družin dvodelnih grafov s presežkom 4. To so dvodelni grafi s cikličnim in dvodelni grafi z dvocikličnim presežkom 4. Dokazali smo še dva izreka.

Izrek 12.10. *Če k in g zadoščata enemu izmed spodnjih pogojev, potem ne obstaja (k,g)-graf s cikličnim presežkom 4:*

(1) $k \equiv 1, 2 \pmod 3$ *in* $g = 8$*;*

(2) $k \equiv 1 \pmod 3$ *in* $g = 12$*;*

(3) $k \equiv 1 \pmod 3$ *in* $g = 16$.

Izrek 12.11. *Če je $k \geq 7$ liho število in $g = 2d \geq 8$, kjer je d sodo naravno število, potem ne obstaja (k,g)-graf z dvocikličnim presežkom 4.*

12.2.3 Neobstoj antipodnih kletk sode ožine

Biggs in Ito sta dokazala, da je vsaka (k, g)-kletka sode ožine $g = 2d \geq 6$, kjer je $e \leq k-2$, dvodelen graf s premerom $d+1$. V disertaciji smo se ukvarjali s spektralnimi lastnostmi teh grafov. Podali smo povezavo med lastnimi vrednostmi matrike sosednosti A in razdaljne matrike A_{d+1} teh grafov ter s tem dokazali naslednji izrek.

Izrek 12.12. *Če je θ $(\neq \pm k)$ lastna vrednost matrike A in je $H_{d-1}(x)$ Dicksonov polinom druge vrste s parametrom $k-1$ in stopnjo $d-1$, potem je*

$$H_{d-1}(\theta) = -\lambda,$$

kjer je λ lastna vrednost matrike A_{d+1}.

Naj bo G kletka sode ožine z antipodno lastnostjo in presežkom e največ $k-2$. Z uporabo prejšnjega izreka in upoštevanjem specifične strukture kletk sode ožine z antipodno lasnostjo smo dokazali naslednja izreka.

Izrek 12.13. *Naj bo θ ničla polinoma $H_{d-1}(x) - \epsilon$, kjer je $\epsilon = 1$ ali $\epsilon = -\frac{e}{2}$. Večkratnost $m(\theta)$ ničle θ v G, $\theta \neq \pm k$, je*

$$m(\theta) = \frac{nek(k-1)H_{d-2}(\theta)}{2\epsilon(2\epsilon + \frac{e}{2} - 1)H'_{d-1}(\theta)(k^2 - \theta^2)}.$$

Izrek 12.14. *Če je $k \geq e+2 \geq 6$ in $g = 2d \geq 8$, potem ne obstajajo antipodne (k, g)-kletke s presežkom e.*

12.2.4 Presežek vozliščno-tranzitivnih grafov dane stopnje in ožine

V disertacij smo obravnavali dobro znani problem kletk na posebnem razredu vozliščno-tranzitivnih grafov. Ukvarjali smo se z iskanjem najmanjšega vozliščno-tranzitivnega k-regularnega grafa z ožino g. Biggsov rezultat smo razširili in dokazali, da je za vsak dan presežek e in vsako dano stopnjo $k \geq 3$, asimptotska gostota množice tistih števil g, za katere obstaja vozliščno-tranzitivna (k, g)-kletka s presežkom kvečjemu e, enaka 0. Ti rezultati so podani z naslednjimi štirimi izreki.

Izrek 12.15. *Naj bosta $k \geq 4$ in $e \geq 1$ celi števili. Asimptotska gostota množice vseh lihih števil g, za katere obstaja vozliščno-tranzitiven (k, g)-graf z presežkom kvečjemu e, je 0.*

Izrek 12.16. *Naj bosta $k \geq 4$ in $e \geq 1$ celi števili. Asimptotska gostota množice vseh sodih števil g, za katere obstaja vozliščno-tranzitiven (k, g)-graf s presežkom kvečjemu e, je 0.*

Izrek 12.17. *Za vsak $k \geq 3$ obstaja neskončno zaporedje lihih števil $\{g_i\}_{i=1}^{\infty}$, tako da je presežek kateregakoli vozliščno-tranzitivnega (k, g)-grafa večji od $g_i^{1/(1+o(1))}$.*

Izrek 12.18. *Za vsak $k \geq 3$ obstaja neskončno zaporedje sodih števil $\{g_i\}_{i=1}^{\infty}$, tako da je presežek kateregakoli vozliščno-tranzitivnega (k, g)-grafa večji od $g_i^{1/(1+o(1))}$.*

12.2.5 Povezava med vprašanjem Bermonda in Bollobása in Ramanujan-ovimi grafi

V [11] sta Bermond in Bollobás postavila naslednjo vprašanje:

Naj bo $c > 0$ pozitivno celo število. Ali obstajata števili k in d, tako da velja $n(k,d) \leq M(k,d) - c$?

V disertacij smo pokazali, da bi negativen odgovor na vprašanje Bermonda in Bollobása impliciral obstoj poljubno velikega ne-dvodelnega Ramanujan-ovega grafa za katerokoli stopnjo k. Pokazali smo tudi obrat: neobstoj poljubno velikega ne-dvodelnega Ramanujan-ovega grafa za neko fiksno stopnjo k, bi impliciral pozitiven odgovor na vprašanje Bermonda in Bollobása. Dokazali smo naslednja dva izreka.

Izrek 12.19. *Naj bosta $c \geq 1$ in $k \geq 3$ celi števili. Potem obstaja sodo štefilo $D_{c,k}$, tako da je vsak graf Γ maksimalne stopnje k, sodega premera $d \geq D_{c,k}$, in velikosti večje od $M(k,d) - c$, ne-dvodelen k-regularen Ramanujan-ov graf z $\lambda(\Gamma) < 2\sqrt{k-1}$. Če je $k > c$, potem imajo vsi takšni Ramanujan-ovi grafi ožino $2d$ ali $2d - 1$.*

Izrek 12.20. *Naj bodo $c > 0$, $k > c$, in $d > \max\{6, \log_{k-1}(c(k-2)+1)\}$ pozitivna cela števila. Če je $G_{k,d}(x) > c$ za vsak $x \in (-k, -2\sqrt{k-1} + \frac{2\pi^2}{(2d-1)^2}] \cup [2\sqrt{k-1} - \frac{2\pi^2}{(2d-1)^2}, k)$, potem velja $n(k,d) < M(k,d) - c$.*

12.2.6 Monotonost stopnje in premera pri digrafih

V disertacij smo podali pozitiven odgovor na spodnje vprašanje, ki se nanaša na problem stopnje in premera pri digrafih. To vprašanje je bilo prvič omenjeno v [60].

Ali je $n_{d,k}$ monotono glede na d in k?

Kot odgovor na zgornje vprašanje smo dokazali naslednja dva izreka.

Izrek 12.21. *Za vsa števila $k, d \geq 1$ velja $n_{d,k+1} > n_{d,k}$.*

Izrek 12.22. *Za vsa števila $k, d \geq 1$ velja $n_{d+1,k} > n_{d,k}$.*

12.2.7 Neobstoj družin (d,k,δ)-digrafov, ki vsebujejo le samoponovitve

V doktorski disertaciji smo izpeljali formulo za izračun večkratnosti lastnih vrednosti (d,k,δ)-digrafov, ki vsebujejo samo samoponovitve.

Izrek 12.23. *Naj bo $d \geq \delta \geq 1$ in naj bo G tak (d,k,δ)-digraf reda n, ki vsebuje samo samoponovitve. Če je θ lastna vrednost digrafa G, ki je različna od d in 1, potem njena večkratnost $m(\theta)$ zadošča naslednji enakosti:*

$$m(\theta) = \frac{n(\delta + d\theta^k - d\delta)(\theta - 1)}{((k+1)\theta^k - \delta)(\theta - d)}.$$

Dokazali smo neobstoj družine takšnih digrafov in sicer tako, da smo pokazali, da večkratnost $m(\theta)$ ni celo število.

Izrek 12.24. *Naj bo $d \geq \delta \geq k+1 \geq 4$. Potem ne obstaja noben (d, k, δ)-digraf, ki vsebuje samo samoponovitve.*

Metodologija

Orodja, ki smo jih uporabili pri predstavljenih raziskavah, vključujejo orodja kombinatoričnih metod v teoriji grafov, kakor tudi orodja teorije števil, linearne algebre in matričnega računa.

Združili smo kombinatorične metode preštevanja g-ciklov v (k, g)-grafih in teorijo števil, da smo prišli do potrebnih pogojev, ki morajo veljati za parametra k in g. Ta pristop smo uporabili za izboljšanje spodnje meje za red kletk sode ožine (Poglavje 4) in pri oceni presežka vozliščno-tranzitivnih grafov dane stopnje in ožine (Poglavje 7). Podobne metode so uporabljene v člankih [43] in [46].

Pri dokazu neobstoja družin dvodelnih grafov s presežkom 4 (s cikličnim in dvocikličnim presežkom 4), opisanih v Poglavje 5, neobstoja antipodnih kletk s sodo ožino in majhnim presežkom (Poglavje 6) in neobstoja (d, k, δ)-digrafov, ki vsebujejo samo samoponovitve (Poglavje 10), smo združili moč kombinatoričnih metod z močjo spektralne analize. Konkretneje, izpeljali smo formule za večkratnost pripadajočih lastnih vrednosti grafa in nato izpeljali relacije med njimi, ki so logično neizpolnljive. Ta pristop je v literaturi že znan in uporabljen v mnogih obstoječih člankih, na pimer v [5, 6, 13, 15] in [67].

Da bi vzpostavili povezavo med vprašanjem Bermonda in Bollobása in Ramanujan-ovimi grafi (Poglavje 8) smo združili spektralne metode z realno analizo.

Za dokazovanje rezultatov o monotonosti stopnje in premera pri digrafih (Poglavje 9) smo uporabljali neenakosti.

Kazalo

Stvarno kazalo

Declaration

I declare that this thesis does not contain any materials previously published or written by another person except where due reference is made in the text.

Slobodan Filipovski