UNIVERZA NA PRIMORSKEM
FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA
(DOCTORAL THESIS)

DOLOČENI RAZREDI (HIPER)GRAFOV IN NJIHOVE ALGEBRIČNE LASTNOSTI

(SOME CLASSES OF (HYPER)GRAPHS VIA ALGEBRAIC PROPERTIES)

PAWEŁ PETECKI

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MENTOR: PROF. DR. DRAGAN STEVANOVIĆ
SOMENTORICA: IZR. PROF. DR. KLAVDIJA KUTNAR

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Contents

Abstract

Some Classes of (Hyper)graphs via Algebraic Properties

This thesis contains a number of different topics from graph theory with special emphasis given to algebraic graph theory, gives solutions for some open problems like decomposition of complete hypergraphs into Hamilton cycles and extend results from graph theory to signed graph theory. More precisely, the following open problems are considered in this thesis:

(i) Which hypergraphs can be decomposed into Hamiltonian cycles?

(ii) Determine the full automorphism groups of double generalized Petersen graphs.

(iii) Can sign lollipop graphs be characterized by their Laplacian eigenvalues?

(iv) Is it possible to characterize all sign graphs with small second Laplacian eigenvalue?

Problem (i) is solved for a special decomposition, called cyclic Hamiltonian decomposition. Necessary and sufficient conditions are given for such a decomposition. Problem (ii) is completely solved. It is proven that question (iii) has a positive answer. Problem (iv) is solved for graphs of order greater than 7. All signed graphs whose second largest Laplacian eigenvalue does not exceed 3 are characterized and identified. In particular, it is shown that almost all signed friendship graphs are determined by the spectrum of the Laplacian matrix.

Math. Subj. Class (2010): 05C22, 05C45, 05C50, 05C51, 05C65.

Key words: hypergraph, hamiltonian cycle, decomposition, double generalized Petersen graph, automorphism group, vertex-transitive, sign graph, L-eigenvalue, lollipop graph.

Izvleček

Določeni razredi (hiper)grafov in njihove algebraične lastnosti

Disertacija povezuje različna področja teorije grafov, s posebnim poudarkom na algebraični teoriji grafov, predstavi rešitve določenih odprtih problemov, kot so dekompozicija polnih hipergrafov na hamiltonske cikle, in razširi rezultate teorije grafov na teorijo predznačnih grafov. V disertaciji se še posebej posvetimo sledečim odprtim problemom:

(i) Kateri hipergrafi premorejo dekompozicijo na hamiltonske cikle?

(ii) Kako najti celotno grupo avtomorfizmov dvojno posplošenega Petersenovega grafa?

(iii) Kaj lahko karakteriziramo predznačne lizika grafe glede na njihove L lastne vrednosti?

(iv) Je mogoče karakterizirati vse predznačne grafe z majhno drugo L lastno vrednostjo?

Problem (i) je rešen s posebno dekompozicijo, imenovano ciklična hamiltonska dekompozicija. Podani so potrebi in zadostni pogoji za takšno dekompozicijo. Problem (ii) je popolnoma rešen. Dokazano je, da ima vprašanje (iii) pozitiven odgovor. Problem (iv) je rešen za grafe z redom večjim od 7. Vsi grafi, katerih druga največja Laplaceova lastna vrednost ne presega 3, so karakterizirani in identificirani. Pokazano je, da so skoraj vsi predznačni prijateljski grafi določeni s spektrom pripadajoče Laplaceove matrike.

Math. Subj. Class (2010): 05C22, 05C45, 05C50, 05C51, 05C65.

Ključne besede: hipergraf, hamiltonski cikli, dekompozicija, dvojno posplošeni Petersenov graf, grupa avtomorfizmov, točkovna tranzitivnost, predznačni graf, L lastne vrednosti, lizika graf.

Chapter 1

Introduction

The PhD Thesis deals with graph theory from the algebraic point of view. Two aspects of algebraic graph theory are considered: group theoretic aspect and spectral one. First, it uses the group theory in Hamiltonian decomposition problem and determines full automorphism groups of the so-called double generalized Petersen graphs. Next we study some spectral properties of lollipop graphs and characterize and identify all signed graphs whose second largest Laplacian eigenvalue does not exceed 3.

In 1884 Walecki proved that for odd integers n the complete graphs K_n admit a Hamiltonian decomposition whereas for even integers n the complete graphs K_n admit a decomposition into a perfect matching and Hamiltonian cycles (see [15]). For $k = 3$ Bermond [11] has shown a Hamiltonian decomposition of a complete 3-uniform hypergraph K_n^3 for $n \equiv 2 \pmod 3$ and $n \equiv 4 \pmod 6$. Verrall [46] later completed the solution for the case $n \equiv 1 \pmod 6$, and has proved that for $n \equiv 0 \pmod 3$ $K_n^3 - I$ has a Hamiltonian decomposition, where I is a perfect matching. Kühn and Osthus [38] showed the existence of Berge-type Hamiltonian decompositions for arbitrary n and k. A similar problem was considered for complete graphs by Buratti and Del Fra in [16]. In Chapter 3 we will show sufficient and necessary conditions for the existence of a *cyclic* Hamiltonian decomposition of K_n^k for arbitrary k and n.

The generalized Petersen graphs $\mathrm{GP}(n, k)$, first introduced by Coxeter in [18], are a natural generalization of the well-known Petersen graph. Next step in generalizations of the generalized Petersen graphs are the double generalized Petersen graphs $\mathrm{DP}(n, t)$, first introduced in [57] as examples of vertex-transitive non-Cayley graphs. Chapter 4 aims at obtaining information about how structural properties of double generalized Petersen graphs are linked with the structural properties of generalized Petersen graphs [4, 17]. Hamiltonicity properties, vertex-coloring and edge-coloring of double generalized Petersen graphs will be also considered. In particular, we will show that any $\mathrm{DP}(2n, t)$ has a Hamilton cycle whereas for $\mathrm{DP}(2n+1, t)$ the existence of a Hamilton cycle will be shown only for t being a generator of $\mathbb{Z}_{2n+1}$. Since any $\mathrm{DP}(2n, t)$ is bipartite, two colors suffices for proper vertex-coloring whereas for $\mathrm{DP}(2n + 1, t)$ three colors are needed. Finally, we will show that there are no snarks amongst double generalized Petersen graphs.

In Chapter 5 we consider the spectral characterization problem extended to the adjacency matrix and the Laplacian matrix of signed graphs. We study the spectral determination of signed lollipop graphs, and we show that any signed lollipop graphs is determined by the spectrum of its Laplacian matrix. The second problem from spectral graph theory considered in Chapter 6 is the Laplacian theory of signed graphs, and we focus our attention to the signed graphs whose second largest Laplacian eigenvalue is fairly small. Similar investigations have been considered in the literature, sometimes motivated from applications [35], and several researchers have investigated the structure of (unsigned) graphs with small second largest eigenvalue of some prescribed graph matrix [25, 30, 40, 44]. Recently [9], the authors have put more light on the observation that the spectral theory of signed graphs is a natural generalization of the spectral theory of simple graphs, especially when considering the Laplacian theory of signed graphs. The Laplacian theory of signed graphs generalizes both the spectral theory of Laplacian and signless Laplacian of graphs, and it can give some explanation to phenomena which seem to have an unpredictable behavior, e.g., in [9] the extremal graphs with respect to the magnitude of Laplacian polynomial coefficients. On the other hand, the graphs with small second largest eigenvalue of the Laplacian and signless Laplacian eigenvalues not exceeding 3 have been studied for both Laplacian and signless Laplacian graphs [3, 39, 48], so it is a natural question to revise such results in the wider setting of the Laplacian of signed graphs. It is worth mentioning that, among others, we obtain the (signed) friendship graphs, where the triangles can now be balanced or unbalanced. Finally, we will consider the spectral determination problem in setting of signed graphs (see also [7]). This is usually more complicated for the signed case than it is for the unsigned case. In fact, the Laplacian spectrum of signed graphs cannot distinguish connected signed graphs from disconnected ones. In this PhD Thesis we will characterize and identify all signed graphs whose second largest Laplacian eigenvalue does not exceed 3 and study the spectral determination problem for the signed firefly graphs. In particular, we will show that almost all signed friendship graphs are determined by the spectrum of the Laplacian matrix.

The results of this PhD Thesis are published in the following articles:

- F. Belardo, P. Petecki, Spectral characterizations of signed lollipop graphs, Linear Algebra Appl. 480 (2015) 144–167.
- F. Belardo, P. Petecki, J.F. Wang, On signed graphs whose second largest L-eigenvalue does not exceed 3, accepted in Linear and Multilinear Algebra.
- K. Kutnar, P. Petecki, On automorphisms and structural properties of double generalized Petersen graphs, submitted.
- P. Petecki, On cyclic Hamiltonian decompositions of complete k-uniform hypergraphs, Discrete Math. 325 (2014) 74–76.

Chapter 2

Notations, Definitions and Preliminary Results

Throughout this thesis we are using notion, concepts, definitions, theorems from a wide vary of theories, such as linear algebra, group theory, permutation group theory, combinatorics and group theory (see [26], [42], [45]).

2.1 Graphs

A graph $G = (V, E)$ is a pair of two sets, the set of vertices $V(G)$ and the set of edges $E(G) \subset \binom{V(G)}{2}$. On a drawing vertices are represented by dots and edges by lines connecting dots (see Figure 2.1).

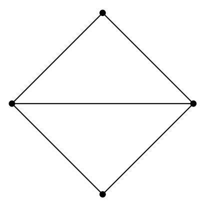

Figure 2.1: An example of a graph.

A vertex v is called *incident* with an edge e when $v \in e$. Two vertices incident with the same edge e are said to be *adjacent*. Two edges are adjacent if they are incident with the same vertex. For a vertex v we define $N(v)$ as the set of the neighbors of v, that is, set of all vertices adjacent to v. The degree $d(v)$ of a vertex v is number of edges incident to it. By $\Delta(G)$ we will denote maximal vertex degree in G. If each vertex of a graph has the same degree d then we call such a graph a d-regular graph. If a graph G is 3-regular we call it *cubic* (see Figure 2.2). For a graph G the bijection $\phi : V(G) \to V(G)$ such that $xy \in E(G) \Leftrightarrow \phi(x)\phi(y) \in E(G)$ is called an automorphism of G. The set of all automorphisms of G together with composition of mappings forms a group called the *automorphism group* $\mathrm{Aut}(G)$ of G. A graph G is said to be *vertex-transitive* and *edge-transitive* if its automorphism group $\mathrm{Aut}(G)$ acts transitively on $V(G)$ and $E(G)$, respectively. A *path* in a graph is a sequence of distinct vertices, where consecutive vertices are adjacent in G. A *cycle* in a graph is a closed path, that is, a path in which the first and the last vertex are adjacent in G. A *Hamilton cycle* in a graph is a

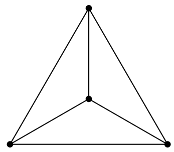

Figure 2.2: The smallest cubic graph.

cycle containing all the vertices of the graph. A subgraph G' of graph G is a graph with $V(G') \subset V(G)$ and $E(G') \subset E(G)$. *Proper coloring* of the vertices (edges) of a graph is a labeling of the vertices of the graph (edges) with colors in such way that any two adjacent vertices (edges) have different colors. As through this thesis no other types of coloring are considered for short we will write *coloring* instead of proper coloring.

2.2 Hypergraphs

A *hypergraph* $H = (V, E)$ is a pair of two sets, set of vertices $V = V(H)$ and set of hyperedges $E = E(H) = \{e_0, e_1, \dots, e_m\}$, where $e_i \subseteq V$. If $|e_i| = k$, for all $i \in \{0, 1, \dots, m\}$, then H is said to be a *k-uniform* hypergraph. The *complete k-uniform hypergraph* on n vertices $V = V(H) = \mathbb{Z}_n = \{0, 1, \dots, n-1\}$ has all k-subsets of $\{0, 1, \dots, n-1\}$ as edges; we denote this hypergraph by K_n^k.

There exist several different notions of Hamilton cycles in hypergraphs, each of which is a valid generalization of the standard notion for ordinary graphs. However, this thesis considers Berge's [10] notion which we recall next. A *Hamilton cycle* in a k-uniform hypergraph H is a sequence

$$(x_0, e_0, x_1, e_1, \dots, x_{n-1}, e_{n-1}, x_0)$$

where $x_0, x_1, \dots, x_{n-1}$ is a list of vertices of H, and $e_0, e_1, \dots, e_{n-1}$ are hyperedges of H such that

(i) $x_i, x_{i+1} \in e_i$, $0 \le i \le n-1$, where indices of the vertices are considered modulo n,

(ii) $e_i \neq e_j$ for $i \neq j$.

Example 2.2.1 *Observe that in K_5^3 the sequence*

$$(0, \{0, 1, 2\}, 1, \{1, 2, 3\}, 2, \{2, 3, 4\}, 3, \{3, 4, 0\}, 4, \{4, 0, 1\}, 0)$$

is a Hamilton cycle.

A hypergraph $H = (V, E)$ is said to be *decomposable* into Hamilton cycles if there exists a family of Hamilton cycles $\mathcal{C} = \{C_1, C_2, \dots, C_h\}$ such that

(i) $E(C_i) \cap E(C_j) = \emptyset$ *for* $i \neq j$

(ii) $\displaystyle\bigcup_{i=1}^{h} E(C_i) = E(H)$.

Example 2.2.2 *For K_5^3 the cycles*

$$C_1 = (0, \{0, 1, 2\}, 1, \{1, 2, 3\}, 2, \{2, 3, 4\}, 3, \{3, 4, 0\}, 4, \{4, 0, 1\}, 0)$$

and

$$C_2 = (0, \{0, 2, 3\}, 2, \{2, 4, 0\}, 4, \{4, 1, 2\}, 1, \{1, 3, 4\}, 3, \{3, 0, 1\}, 0)$$

give its Hamiltonian decomposition.

2.3 Signed graphs

Let $G = (V(G), E(G))$ be a graph of order $n = |V(G)|$ and size $m = |E(G)|$, and let $\sigma : E(G) \to \{+, -\}$ be a mapping defined on the edge set of G. Then $\Gamma = (G, \sigma)$ is a *signed graph* (sometimes called also sigraph). The graph G is its underlying graph, while σ its sign function (or signature). It is common to interpret the signs as the integers $\{1, -1\}$. An edge e is positive (negative) if $\sigma(e) = 1$ (respectively $\sigma(e) = -1$). If $\sigma(e) = 1$ (resp. $\sigma(e) = -1$) for all edges in $E(G)$ then we write $(G, +)$ (respectively $(G, -)$). A cycle of Γ is said to be *balanced*, or positive, if it contains an even number of negative edges, otherwise the cycle is *unbalanced*, or negative. A signed graph is said to be balanced if all its cycles are *balanced*; otherwise, it is unbalanced. By $\sigma(\Gamma)$ we denote the product of signs of all cycles in Γ. Most of the concepts defined for graphs are directly extended to signed graphs. For example, the degree of a vertex v in G (denoted by $\deg(v)$) is also its degree in Γ. So $\Delta(G)$, the maximum (vertex) degree in G, also stands for $\Delta(\Gamma)$, interchangeably.

If some subgraph of the underlying graph is observed, then the sign function for the subgraph is the restriction of the previous one. Thus, if $v \in V(G)$, then $\Gamma - v$ denotes the signed subgraph having $G - v$ as the underlying graph, while its signature is the restriction from $E(G)$ to $E(G - v)$ (note, all edges incident to v are deleted). Similar considerations hold for the disjoint union of signed graphs. If $U \subset V(G)$ then $\Gamma[U]$ denotes the signed induced subgraph of U, while $\Gamma - U = \Gamma[V(G) \setminus U]$. For $\Gamma = (G, \sigma)$ and $U \subset V(G)$, let Γ^U be the signed graph obtained from Γ by reversing the signature of the edges in the cut $[U, V(G) \setminus U]$, namely $\sigma_{\Gamma^U}(e) = -\sigma_\Gamma(e)$ for any edge e between U and $V(G) \setminus U$, and $\sigma_{\Gamma^U}(e) = \sigma_\Gamma(e)$ otherwise. The signed graph Γ^U is said to be (signature) switching equivalent to Γ. In fact, switching equivalent signed graphs can be considered as (switching) isomorphic graphs and their signatures are said to be equivalent. Observe that switching equivalent graphs have the same set of positive cycles (see Figure 2.3).

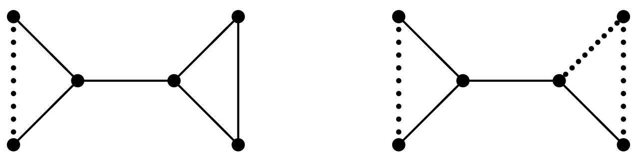

Figure 2.3: A pair of switching equivalent sign graphs.

In the literature, simple graphs are studied by means of the eigenvalues of several matrices associated to graphs. The adjacency matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ whenever vertices i and j are adjacent and $a_{ij} = 0$ otherwise, is one of the most studied together with the Laplacian, or Kirchhoff, matrix $L(G) = D(G) - A(G)$, where $D(G) = \mathrm{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ is the diagonal matrix of vertex degrees. In the last years another graph matrix has attracted the attention of many researchers, the so-called signless Laplacian matrix defined as $Q(G) = A(G) + D(G)$. Matrices can be associated to signed graphs, as well. The adjacency matrix $A(\Gamma) = (a^\sigma_{ij})$ with $a^\sigma_{ij} = \sigma(ij)a_{ij}$ is called the (signed) adjacency matrix and $L(\Gamma) = D(G) - A(\Gamma)$ is the corresponding Laplacian matrix. Both the adjacency and Laplacian matrices are real symmetric matrices, so the eigenvalues are real.

In this thesis we shall consider both the characteristic polynomial of the adjacency matrix and of the Laplacian matrix of a signed graph Γ. Hence to avoid confusion we denote by

$$\phi(\Gamma, x) = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n,$$

the adjacency characteristic polynomial (or A-polynomial) whose roots, namely the adjacency eigenvalues (A-eigenvalues), are denoted by $\lambda_1(\Gamma) \geq \lambda_2(\Gamma) \geq \cdots \geq \lambda_n(\Gamma)$. Similarly, for the Laplacian matrix, we denote by

$$\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n,$$

the Laplacian polynomial (or L-polynomial), and $\mu_1(\Gamma) \geq \mu_2(\Gamma) \geq \cdots \geq \mu_n(\Gamma) \geq 0$ are the Laplacian eigenvalues (L-eigenvalues). Suffix and variables will be omitted if it is clear from the context (so $\phi(\Gamma, x) = \phi(\Gamma)$). A connected signed graph is balanced if and only if $\mu_n = 0$ (see [51]). If Γ is disconnected, then its polynomial is the product of the components polynomials.

Finally, it is important to observe that switching equivalent signed graphs will have similar adjacency and Laplacian matrices. In fact, any switching on U can be interpreted as a diagonal matrix $S_U = \text{diag}(s_i)$ having $s_i = 1$ for any $i \in U$ and $s_i = -1$ otherwise. S_U is usually called the state matrix. Hence, $A(\Gamma) = S_U A(\Gamma^U) S_U$ and $L(\Gamma) = S_U L(\Gamma^U) S_U$. Similar effect features with eigenvectors. When we consider a signed graph Γ, from a spectral viewpoint, we are considering its switching isomorphism class $[\Gamma]$.

Chapter 3

Cyclic Hamiltonian Decompositions of Complete k-uniform Hypergraphs

Results of this chapter are published in [43].

3.1 Cyclic Hamiltonian decompositions of K_n^k

Let k and n be positive integers, and let H be a complete k-uniform hypergraph on n vertices K_n^k. Since all k-subsets of $\{0, 1, \ldots, n-1\}$ are edges of H one can easily see that the automorphism group of H is the full symmetric group $\mathrm{Sym}(n)$, and thus $o_1 = (0\,1\,2\,\ldots\,n-1) \in \mathrm{Aut}(H)$. In particular, the group (of rotations) $\mathcal{O} = \langle o_1 \rangle$ generated by o_1 acts on the set of k-edges of H by the rule

$$o_i(e) := o_1^i(e) = o_i(\{v_0, v_1, \ldots, v_{k-1}\}) = \{v_0 + i, v_1 + i, \ldots, v_{k-1} + i\},$$

where the sums are considered modulo n. A Hamiltonian decomposition $\mathcal{C}$ of H is called *cyclic* if for every Hamilton cycle $C \in \mathcal{C}$ there exists a k-edge e of H, and a rotation $o_i \in G$ such that

$$C = (v_0, e, v_1, o_i(e), v_2, o_i^2(e), \ldots, v_{n-1}, o_i^{n-1}(e), v_0).$$

Example 3.1.1 *For K_5^3 the cycles*

$$C_1 = (0, \{0, 1, 2\}, 1, \{1, 2, 3\}, 2, \{2, 3, 4\}, 3, \{3, 4, 0\}, 4, \{4, 0, 1\}, 0)$$

and

$$C_2 = (0, \{0, 2, 3\}, 2, \{2, 4, 0\}, 4, \{4, 1, 2\}, 1, \{1, 3, 4\}, 3, \{3, 0, 1\}, 0)$$

give its cyclic Hamiltonian decomposition.

We can now state the main theorem of this chapter, which gives a necessary and sufficient conditions for the existence of a cyclic Hamiltonian decomposition of K_n^k for arbitrary k and n.

Theorem 3.1.1 *Let k and n be positive integers, and let λ be the smallest non-trivial divisor of n. Then the k-uniform hypergraph K_n^k admits a cyclic Hamiltonian decomposition if and only if n is relatively prime to k and $\lambda k > n$.*

The theorem is proved in the following section. We first prove that for any k-edge e of H its orbit $Orb(e) = \{o_i(e)\colon i = 0, 1, \ldots, n-1\}$ with respect to the action of $\mathcal{O}$ contains exactly n-elements, and then we prove that for every k-edge e the elements of $Orb(e)$ form a Hamilton cycle.

Definition 3.1.2 *The* orbit *of an element x under action of the group $\mathcal{G}$ is the set*

$$Orb_{\mathcal{G}(x)} = \{g(x) \mid g \in \mathcal{G}\}$$

If the group is clear from the context we will write $Orb(x)$ instead of $Orb_{\mathcal{G}(x)}$.

3.2 Sufficient condition

The proof of Theorem 3.1.1 follows from the following two lemmas. In the proof of the first lemma we use orbit counting lemma which says that the number of orbits N of a finite group $\mathcal{G}$ acting on a set X is given by the following formula

$$N = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} |\text{Fix } g|, \tag{3.1}$$

where Fix $g = \{x \in X\colon g(x) = x\}$ is the set of points fixed by $g \in \mathcal{G}$.

Lemma 3.2.1 *Let k and n be positive integers such that k is relative prime to n and $\lambda k > n$, where λ is the smallest non-trivial divisor of n. Then the k-uniform hypergraph K_n^k admits a cyclic Hamiltonian decomposition.*

Proof. Let $H = K_n^k$, $X = E(H)$, $o_1 = (0\,1\,2\, \ldots\, n-1) \in \text{Aut}(H)$, and $\mathcal{O} = \langle o_1 \rangle$. Let i be the smallest positive integer such that Fix $o_i \neq \emptyset$ and let $e \in$ Fix o_i. Suppose that $e = \{v_0, v_1, \ldots, v_{k-1}\}$. Then, without loss of generality, we may assume that $0 \leq v_0 < v_1 < \ldots < v_{k-1} \leq n-1$. Since $e \in$ Fix o_i there exists j such that

$$\begin{array}{rcl} v_0 + i &=& v_j \\ v_1 + i &=& v_{j+1} \\ & \cdots & \\ v_{k-1} + i &=& v_{j+k-1}. \end{array}$$

It follows that

$$\textstyle\sum_{l=0}^{k-1} v_l + i \cdot k = \sum_{l=0}^{k-1} v_l,$$

and consequently $i \cdot k \equiv 0 \pmod{n}$. Since, by assumption n and k are relatively prime, we have $i = 0$, implying that Fix $g \neq \emptyset$ only when $g = Id$. Therefore, by (3.1), we can conclude that

$$N = \frac{1}{|G|} \left(\binom{n}{k} + 0 \cdot (n-1) \right), \text{ implying that } n \cdot N = \binom{n}{k}. \tag{3.2}$$

Since $\mathcal{O} = \{o_1^i\colon i = 0, 1, \ldots, n-1\}$ is of order n, by the Orbit-Stabilizer theorem, every orbit $Orb(e)$ with respect to the action of $\mathcal{O}$ on the set of k-edges X contains at most n elements. Hence (3.2) implies that any orbit $Orb(e)$ is of length n.

To complete the proof we need to indicate an ordering of the k-edges in $Orb(e)$ forming a Hamilton cycle. Let $e = \{v_0, v_1, \ldots, v_{k-1}\}$, $0 \leq v_0 < v_1 < \ldots < v_{k-1} \leq n-1$, and let $d = \min\{v_i - v_{i-1} \pmod{n} : i = 0, 1, \ldots, k-1\}$ (where $v_{-1} = v_{k-1}$). Then clearly $d < \frac{n}{k} < \lambda$, and hence d is relatively prime to n. Let i_0 be such an index that $v_{i_0} - v_{i_0 - 1} = d \pmod{n}$. Observe that $v_{i_0} \in e \cap o_d(e)$, and thus

$$(v_{i_0-1}, e, v_{i_0}, o_d(e), o_d(v_{i_0}), o_d^2(e), o_d^2(v_{i_0}), \ldots, o_d^{n-1}(e), o_d^{n-1}(v_{i_0}))$$

is a Hamilton cycle, as required ($o_d^{n-1}(v_{i_0}) = v_{i_0-1}$). ■

3.3 Necessary condition

Lemma 3.3.1 *Let k and n be positive integers such that the k-uniform hypergraph K_n^k admits a cyclic Hamiltonian decomposition. Then k is relative prime to n and $\lambda k > n$, where λ is the smallest non-trivial divisor of n.*

PROOF. Assume that there is a cyclic Hamiltonian decomposition $\mathcal{C} = \{C_1, C_2, \ldots, C_N\}$ of K_n^k. We will show that if n is not relatively prime to k then the number of orbits of $\mathcal{O} = \langle o_1 \rangle = \{o_1^i : i = 0, 1, \ldots, n-1\}$ in its action on the set of k-edges of K_n^k exceeds N.

Suppose that $\alpha = \gcd(n,k) > 1$, and write

$$\begin{array}{rllll} e = & \{0, & 1, & \ldots, & \frac{k}{\alpha} - 1, \\ & \frac{n}{\alpha}, & \frac{n}{\alpha} + 1, & \ldots, & \frac{n}{\alpha} + \frac{k}{\alpha} - 1, \\ & \vdots & \vdots & & \vdots \\ & \frac{n}{\alpha}(\alpha - 1), & \frac{n}{\alpha}(\alpha - 1) + 1, & \ldots, & \frac{n}{\alpha}(\alpha - 1) + \frac{k}{\alpha} - 1\}. \end{array}$$

Then one can easily see that $e \in \text{Fix } o_{\frac{n}{\alpha}}$, and thus, by (3.1), the number of orbits of the group $\mathcal{O}$ with respect to the action on the set of k-edges of K_n^k is equal to

$$N = \frac{1}{|\mathcal{O}|} \sum_{i=0}^{n-1} |Fix\ o_i| = \frac{1}{|\mathcal{O}|} \left(\binom{n}{k} + \sum_{i=1}^{n-1} |Fix\ o_i| \right) > \frac{1}{|\mathcal{O}|} \binom{n}{k} = \frac{\binom{n}{k}}{n}. \tag{3.3}$$

But on the other hand, since each orbit is supposed to be a Hamilton cycle, we have $nN = \binom{n}{k}$, contradicting (3.3). This shows that $\gcd(n,k) = 1$.

Since by assumption K_n^k admits its cyclic Hamiltonian decomposition $\mathcal{C}$, each k-edge of K_n^k lies on some Hamilton cycle from $\mathcal{C}$. Suppose that $\lambda k \leq n$, where λ is the smallest non-trivial divisor of n, and consider the k-edge $\{0, \lambda, 2\lambda, 3\lambda, \ldots, (k-1)\lambda\}$. Since the difference of any two vertices from this edge is divisible by λ, there is no pair of vertices on this edge at distance d relatively prime to n, implying that this edge does not lie on a Hamilton cycle from $\mathcal{C}$, a contradiction.

■

Chapter 4

On automorphisms and structural properties of double generalized Petersen graphs

Results of this chapter are published in [37].

4.1 Definition of $\mathrm{DP}(n,t)$

The generalized Petersen graphs $\mathrm{GP}(n,k)$, first introduced by Coxeter in [18], are a natural generalization of the well-known Petersen graph (see Figure 4.1).

Definition 4.1.1 *Given an integer $n \geq 3$ and $k \in \mathbb{Z}_n \setminus \{0\}$, $2 \leq 2k < n$, the generalized Petersen graph $\mathrm{GP}(n,k)$ is defined to have vertex set $\{u_i, v_i | \; i \in \mathbb{Z}_n\}$ and edge set the union $\Omega \cup \Sigma \cup I$, where*

$\Omega = \{\{u_i, u_{i+1}\}, | \; i \in \mathbb{Z}_n\}$ *(the outer edges),*

$\Sigma = \{\{u_i, v_i\}, | \; i \in \mathbb{Z}_n\}$ *(the spokes), and*

$I = \{\{v_i, v_{i+k}\}, | \; i \in \mathbb{Z}_n\}$ *(the inner edges).*

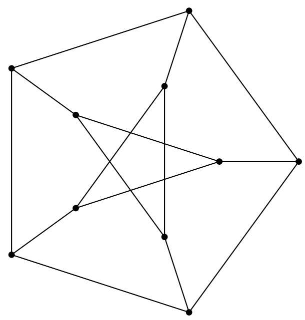

Figure 4.1: The generalized Petersen graph $\mathrm{GP}(5,2)$ (the Petersen graph).

A natural generalization of the generalized Petersen graphs are the double generalized Petersen graphs $\mathrm{DP}(n,t)$, first introduced in [57] as examples of vertex-transitive non-Cayley graphs. They are defined as follows (two examples are given in Figure 4.2).

Definition 4.1.2 *Given an integer $n \geq 3$ and $t \in \mathbb{Z}_n \setminus \{0\}$, $2 \leq 2t < n$, the double generalized Petersen graph* DP(n,t) *is defined to have vertex set $\{x_i, y_i, u_i, v_i |\ i \in \mathbb{Z}_n\}$ and edge set the union $\Omega \cup \Sigma \cup I$, where*

$\Omega = \{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_n\}$ *(the outer edges),*

$\Sigma = \{\{x_i, u_i\}, \{y_i, v_i\} \mid i \in \mathbb{Z}_n\}$ *(the spokes), and*

$I = \{\{u_i, v_{i+t}\}, \{v_i, u_{i+t}\} \mid i \in \mathbb{Z}_n\}$ *(the inner edges).*

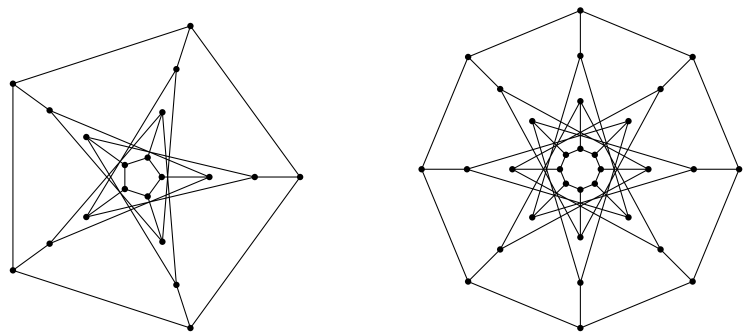

Figure 4.2: The double generalized Petersen graph DP$(5,2)$, which is isomorphic to the generalized Petersen graph GP$(10,2)$ (the dodecahedron), on the left hand-side picture, and the double generalized Petersen graph DP$(8,3)$ on the right hand-side picture.

The motivation for the research in this chapter, resulting in a complete characterized of automorphism groups of double generalized Petersen graphs (see Propositions 4.2.5 and 4.2.6, , Remark 4.2.11 and Corollary 4.2.12), comes from questions post in [57]. The characterization is obtained with a generalization of the method that was used in [28] to obtain a characterization of automorphisms of generalized Petersen graphs.

Aiming at obtaining the information how structural properties of double generalized Petersen graphs are linked with the structural properties of generalized Petersen graphs [4, 17] hamiltonicity properties, vertex-coloring and edge-coloring of double generalized Petersen graphs are also considered. In particular, it is shown that any DP$(2n,t)$ has a Hamilton cycles (see Lemma 4.3.1) while for DP$(2n+1,t)$ the existence of a Hamilton cycles is proven only for t being a generator of $\mathbb{Z}_{2n+1}$ (see Proposition 4.3.2). Any DP$(2n,t)$ is bipartite, thus two colors suffices for proper vertex-coloring while for DP$(2n+1,t)$ three colors are needed (see Lemmas 4.4.1 and 4.4.2). Finally, it is shown that there are no snarks amongst double generalized Petersen graphs (see Lemma 4.4.3).

4.2 Automorphisms of DP(n,t)

In this section, with a generalization of the methods used in [28] to characterize automorphisms of generalized Petersen graphs, we give a complete characterization of automorphism groups of double generalized Petersen graphs, implying the characterization of vertex-transitive double generalized Petersen graphs and a complete classification of edge-transitive double generalized Petersen graphs. Throughout this section let the automorphism group of the double generalized Petersen graph DP(n,t) be denoted by $A(n,t)$, and let its subgroup preserving Σ set-wise be denoted by $B(n,t)$. Define permutations α, β and γ on $V(\mathrm{DP}(n,t))$ by

$$\begin{array}{lll} \alpha: & x_i \mapsto x_{i+1},\ y_i \mapsto y_{i+1},\ u_i \mapsto u_{i+1},\ v_i \mapsto v_{i+1} & \text{(rotation);} \\ \beta: & x_i \mapsto y_i,\ y_i \mapsto x_i,\ u_i \mapsto v_i,\ v_i \mapsto u_i & \text{(flip symmetry);} \\ \gamma: & x_i \mapsto x_{-i},\ y_i \mapsto y_{-i},\ u_i \mapsto u_{-i},\ v_i \mapsto v_{-i} & \text{(reflection);} \end{array}$$

and mappings $\delta, \eta, \psi, \phi\colon V(\mathrm{DP}(n,t)) \to V(\mathrm{DP}(n,t))$ by

$$\begin{array}{rl}
\delta: & x_{2i} \mapsto u_{2it},\ x_{2i+1} \mapsto v_{(2i+1)t},\ y_{2i} \mapsto v_{2it},\ y_{2i+1} \mapsto u_{(2i+1)t}, \\
& u_{2i} \mapsto x_{2it},\ u_{2i+1} \mapsto y_{(2i+1)t},\ v_{2i} \mapsto y_{2it},\ v_{2i+1} \mapsto x_{(2i+1)t}; \\
\eta: & x_{2i} \mapsto x_{2i+k},\ x_{2i+1} \mapsto x_{2i+1+k},\ y_{2i} \mapsto y_{2i},\ y_{2i+1} \mapsto y_{2i+1}, \\
& u_{2i} \mapsto u_{2i+k},\ u_{2i+1} \mapsto u_{2i+1+k},\ v_{2i} \mapsto v_{2i},\ v_{2i+1} \mapsto v_{2i+1}, \text{where } n = 2k; \\
\psi: & x_{2i} \mapsto u_{2it},\ x_{2i+1} \mapsto v_{(2i+1)t},\ y_{2i} \mapsto u_{2it+k},\ y_{2i+1} \mapsto v_{(2i+1)t+k}, \\
& u_{2i} \mapsto x_{2it},\ u_{2i+1} \mapsto y_{(2i+1)t},\ v_{2i} \mapsto x_{2it+k},\ v_{2i+1} \mapsto y_{(2i+1)t+k}, \text{where } n = 2k; \\
\phi: & x_{2i} \mapsto u_{2it},\ x_{2i+1} \mapsto v_{(2i+1)t},\ y_{2i} \mapsto v_{2it+k},\ y_{2i+1} \mapsto u_{(2i+1)t+k}, \\
& u_{2i} \mapsto x_{2it},\ u_{2i+1} \mapsto y_{(2i+1)t},\ v_{2i} \mapsto y_{2it+k},\ v_{2i+1} \mapsto x_{(2i+1)t+k}, \text{where } n = 2k.
\end{array}$$

Observe that the mappings δ, η, ϕ and ψ are not always permutations of $V(\mathrm{DP}(n,t))$. However they are permutations of $V(\mathrm{DP}(n,t))$ as well as automorphisms of $DP(n,t)$ under certain conditions on (n,t), see Proposition 4.2.5 and the following equalities:

- If $n \equiv 0 \pmod 2$ and $t^2 \equiv \pm 1 \pmod n$ (note that t is odd), then

$$\begin{array}{l}
\delta(\{x_{2i}, u_{2i}\}) = \{u_{2it}, x_{2it}\}, \\
\delta(\{x_{2i+1}, u_{2i+1}\}) = \{y_{(2i+1)t}, v_{(2i+1)t}\}, \\
\delta(\{y_{2i}, v_{2i}\}) = \{v_{2it}, y_{2it}\}, \\
\delta(\{y_{2i+1}, v_{2i+1}\}) = \{u_{(2i+1)t}, x_{(2i+1)t}\}, \\
\delta(\{x_{2i}, x_{2i+1}\}) = \{u_{2it}, v_{2it+t}\}, \\
\delta(\{x_{2i+1}, x_{2i+2}\}) = \{v_{2it+t}, u_{2it+2t}\}, \\
\delta(\{y_{2i}, y_{2i+1}\}) = \{v_{2it}, u_{2it+t}\}, \\
\delta(\{y_{2i+1}, y_{2i+2}\}) = \{u_{2it+t}, v_{2it+2t}\}, \\
\delta(\{u_{2i}, v_{2i+t}\}) = \{x_{2it}, x_{(2i+t)t}\} = \{x_{2it}, x_{2it+t^2}\} = \{x_{2it}, x_{2it\pm1}\}, \\
\delta(\{v_{2i}, u_{2i+t}\}) = \{y_{2it}, y_{(2i+t)t}\} = \{y_{2it}, y_{2it+t^2}\} = \{y_{2it}, y_{2it\pm1}\}, \\
\delta(\{u_{2i+1}, v_{2i+1+t}\}) = \{y_{2it+t}, y_{(2i+t+1)t}\} = \{y_{2it+t}, y_{2it+t+t^2}\} = \{y_{2it+t}, y_{2it+t\pm1}\}, \\
\delta(\{v_{2i+1}, u_{2i+1+t}\}) = \{x_{2it+t}, x_{(2i+t+1)t}\} = \{x_{2it+t}, x_{2it+t+t^2}\} = \{x_{2it+t}, x_{2it+t\pm1}\}.
\end{array}$$

- If $n \equiv 0 \pmod 2$ and $4t = n$, then

$$\begin{array}{l}
\eta(\{x_{2i}, u_{2i}\}) = \{x_{2i+2t}, u_{2i+2t}\}, \\
\eta(\{x_{2i+1}, u_{2i+1}\}) = \{x_{2i+1+2t}, u_{2i+1+2t}\}, \\
\eta(\{y_{2i}, v_{2i}\}) = \{y_{2i}, v_{2i}\}, \\
\eta(\{y_{2i+1}, v_{2i+1}\}) = \{y_{2i+1}, v_{2i+1}\}, \\
\eta(\{x_{2i}, x_{2i+1}\}) = \{x_{2i+2t}, x_{2i+2t}\}, \\
\eta(\{x_{2i+1}, x_{2i+2}\}) = \{x_{2i+1+2t}, x_{2i+2+2t}\}, \\
\eta(\{y_{2i}, y_{2i+1}\}) = \{y_{2i}, y_{2i+1}\}, \\
\eta(\{y_{2i+1}, y_{2i+2}\}) = \{y_{2i+1+2t}, y_{2i+2+2t}\}, \\
\eta(\{u_{2i}, v_{2i+t}\}) = \{u_{2i+2t}, x_{2i+t}\}, \\
\eta(\{v_{2i}, u_{2i+t}\}) = \{v_{2i}, u_{2i+t+2t}\} = \{v_{2i}, u_{2i+3t}\}, \\
\eta(\{u_{2i+1}, v_{2i+1+t}\}) = \{u_{2i+1+2t}, v_{2i+1+t}\}, \\
\eta(\{v_{2i+1}, u_{2i+1+t}\}) = \{v_{2i+1}, u_{2i+t+2t}\} = \{v_{2i+1}, u_{2i+3t}\}.
\end{array}$$

- If $n \equiv 2 \pmod 4$ and $t^2 = k \pm 1 \pmod n$ where $2k = n$ (k is odd and t is even), then

$$
\begin{aligned}
&\psi(\{x_{2i}, u_{2i}\}) = \{u_{2it}, x_{2it}\},\\
&\psi(\{x_{2i+1}, u_{2i+1}\}) = \{v_{(2i+1)t}, y_{(2i+1)t}\},\\
&\psi(\{y_{2i}, v_{2i}\}) = \{u_{2it+k}, x_{2it+k}\},\\
&\psi(\{y_{2i+1}, v_{2i+1}\}) = \{v_{(2i+1)t+k}, y_{(2i+1)t+k}\},\\
&\psi(\{x_{2i}, x_{2i+1}\}) = \{u_{2it}, v_{2it+t}\},\\
&\psi(\{x_{2i+1}, x_{2i+2}\}) = \{v_{2it+t}, u_{2it+2t}\},\\
&\psi(\{y_{2i}, y_{2i+1}\}) = \{u_{2it+k}, v_{2it+k+t}\},\\
&\psi(\{y_{2i+1}, y_{2i+2}\}) = \{v_{2it+t+k}, u_{2it+2t+k}\},\\
&\psi(\{u_{2i}, v_{2i+t}\}) = \{x_{2it}, x_{(2i+t)t+k}\} = \{x_{2it}, x_{2it+t^2+k}\} = \{x_{2it}, x_{2it+k+k\pm1}\} = \{x_{2it}, x_{2it\pm1}\},\\
&\psi(\{v_{2i}, u_{2i+t}\}) = \{x_{2it+k}, x_{(2i+t)t}\} = \{x_{2it+k}, x_{2it+t^2}\} = \{x_{2it+k}, x_{2it+k\pm1}\},\\
&\psi(\{u_{2i+1}, v_{2i+1+t}\}) = \{y_{2it+t}, y_{(2i+t+1)t+k}\} = \{y_{2it+t}, y_{2it+t+t^2+k}\} = \{y_{2it+t}, y_{2it+t\pm1}\},\\
&\psi(\{v_{2i+1}, u_{2i+1+t}\}) = \{y_{2it+t+k}, y_{(2i+t+1)t}\} = \{y_{2it+t+k}, y_{2it+t+t^2}\} = \{y_{2it+t+k}, y_{2it+t+k\pm1}\}.
\end{aligned}
$$

- If $n \equiv 0 \pmod 4$ and $t^2 = k \pm 1 \pmod n$ where $2k = n$ (note that in this case k is even and t is odd), then

$$
\begin{aligned}
&\phi(\{x_{2i}, u_{2i}\}) = \{u_{2it}, x_{2it}\},\\
&\phi(\{x_{2i+1}, u_{2i+1}\}) = \{v_{(2i+1)t}, y_{(2i+1)t}\},\\
&\phi(\{y_{2i}, v_{2i}\}) = \{v_{2it+k}, y_{2it+k}\},\\
&\phi(\{y_{2i+1}, v_{2i+1}\}) = \{u_{(2i+1)t+k}, x_{(2i+1)t+k}\},\\
&\phi(\{x_{2i}, x_{2i+1}\}) = \{u_{2it}, v_{2it+t}\},\\
&\phi(\{x_{2i+1}, x_{2i+2}\}) = \{v_{2it+t}, u_{2it+2t}\},\\
&\phi(\{y_{2i}, y_{2i+1}\}) = \{v_{2it+k}, u_{2it+k+t}\},\\
&\phi(\{y_{2i+1}, y_{2i+2}\}) = \{u_{2it+t+k}, v_{2it+2t+k}\},\\
&\phi(\{u_{2i}, v_{2i+t}\}) = \{x_{2it}, x_{(2i+t)t+k}\} = \{x_{2it}, x_{2it+t^2+k}\} = \{x_{2it}, x_{2it+k+k\pm1}\} = \{x_{2it}, x_{2it\pm1}\},\\
&\phi(\{v_{2i}, u_{2i+t}\}) = \{x_{2it+k}, x_{(2i+t)t}\} = \{x_{2it+k}, x_{2it+t^2}\} = \{x_{2it+k}, x_{2it+k\pm1}\},\\
&\phi(\{u_{2i+1}, v_{2i+1+t}\}) = \{y_{2it+t}, y_{(2i+t+1)t+k}\} = \{y_{2it+t}, y_{2it+t+t^2+k}\} = \{y_{2it+t}, y_{2it+t\pm1}\},\\
&\phi(\{v_{2i+1}, u_{2i+1+t}\}) = \{y_{2it+t+k}, y_{(2i+t+1)t}\} = \{y_{2it+t+k}, y_{2it+t+t^2}\} = \{y_{2it+t+k}, y_{2it+t+k\pm1}\}.
\end{aligned}
$$

Following simple computation we can easily see that the following proposition is straightforward.

Proposition 4.2.1 *For all n, t, for which* DP(n,t) *exists, we have $\beta\alpha = \alpha\beta$, $\alpha^{n-1}\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta$, $|\langle\alpha, \beta, \gamma\rangle| = 4n$ and $\langle\alpha, \beta, \gamma\rangle \leq B(n,t)$.* ■

Lemma 4.2.2 *If $\pi \in A(n,t)$ fixes set-wise any of the sets Ω, Σ or I then it either fixes all three of these sets or fixes Σ set-wise and interchanges Ω and I.*

Proof. Since each spoke is incident with two outer edges on one side and two inner edges on the other, it follows that if any outer (inner) edge is mapped onto a spoke, the pair of outer (inner) edges incident to it must be mapped onto one outer and one inner edge. Thus if an automorphism does not preserve the spokes, it cannot preserve any of the sets Ω, Σ or I. ■

In the next proposition the following characterization of vertex-transitive double generalized Petersen graphs given by Feng and Zhou in [58] will be needed.

Proposition 4.2.3 *The graph* DP(n,t) *is vertex-transitive if and only if either $(n,t) = (5,2)$ or $n = 2k$ and $t^2 \equiv \pm 1 \pmod k$.* ■

Definition 4.2.4 *The set of all elements of the group $\mathcal{G}$ acting on a set X fixing an element x is called a* stabilizer *of x in $\mathcal{G}$ and it is denoted $Stab_{\mathcal{G}}(x)$.*

Proposition 4.2.5 *If $(n,t) \neq (4,1)$ then $B(n,t)$ can be characterize as follows:*

(i) If $n \equiv 0 \pmod 2$ and $4t = n$, then $B(n,t) = \langle \alpha, \beta, \gamma, \eta \rangle$.

(ii) If $n \equiv 0 \pmod 2$ and $t^2 \equiv \pm 1 \pmod n$, then $B(n,t) = \langle \alpha, \beta, \gamma, \delta \rangle$.

(iii) If $n \equiv 2 \pmod 4$ and $t^2 \equiv k \pm 1 \pmod n$, where $n = 2k$, then $B(n,t) = \langle \alpha, \beta, \gamma, \psi \rangle$.

(iv) If $n \equiv 0 \pmod 4$ and $t^2 \equiv k \pm 1 \pmod n$, where $n = 2k$, then $B(n,t) = \langle \alpha, \beta, \gamma, \phi \rangle$.

(v) In all other cases (that is, for (n,t) not satisfying any of the conditions in (i) – (iv) above) we have $B(n,t) = \langle \alpha, \beta, \gamma \rangle$.

Proof. Considerations of vertex stabilizers, with the use of Proposition 4.2.3 and the well-known Orbit-Stabilizer Lemma, will enable us to determine the order of $B(n,t)$. The proposition will then follow from the facts that the groups indicated in the statement of the proposition are indeed subgroups of $A(n,t)$ of such orders. Throughout the proof the action of $B(n,t)$ on $V(D(n,t))$ is considered, unless specified otherwise.

Suppose first that $\mathrm{DP}(n,t)$ is not vertex-transitive. Then, by Proposition 4.2.3, n is odd and $(n,t) \neq (5,2)$ or n is even and $t^2 \not\equiv \pm 1 \pmod k$, where $n = 2k$. Observe that $B(n,t)$ has two orbits on $V(D(n,t))$, each of which is of length $2n$ (see Proposition 4.2.1). Let us now consider possibilities for the order of the vertex stabilizer $Stab_{B(n,t)}(x_0)$ of the vertex x_0 under the action of $B(n,t)$. Let $\theta \in Stab_{B(n,t)}(x_0)$. Then $\theta(x_0) = x_0$, and $\theta(N(x_0)) = N(x_0) = \{x_1, x_{n-1}, u_0\}$. Since u_0 is the only neighbor of x_0, adjacent to x_0 via a spoke edge, we have $\theta(u_0) = u_0$, and the other two neighbors of x_0 are either both fixed or they are interchanged by θ. Observe that for $\theta \in Stab_{B(n,t)}(x_0)$ also $\gamma\theta \in Stab_{B(n,t)}(x_0)$, where γ is as defined on page 12. Hence, it suffices to consider the case in which θ fixes $N(x_0)$ point-wise, and consequently also $\{x_i, u_i \mid i \in \mathbb{Z}_n\}$. Now let us consider the possibilities for the action of θ on the vertices v_i. Since $\theta(u_0) = u_0$ the neighbors v_t and v_{n-t} of u_0 are either interchanged by θ or they are both fixed. In the first case, $v_{n-t} = \theta(v_t)$ is adjacent to $u_{2t} = \theta(u_{2t})$, and thus $2t + t = n - t$, implying that $4t = n$. If, however, θ fixes both v_t and v_{n-t} it also fixes y_t and y_{n-t}, and consequently for $4t \neq n$ all of y_i and v_i are fixed, and thus θ is the identity, implying that for $n \neq 4t$ we have $|Stab_{B(n,t)}(x_0)| = 2$, and thus, by Proposition 4.2.1, we have $B(n,t) = \langle \alpha, \beta, \gamma \rangle$. On the other hand, if $n = 4t$ then there are two numerical possibilities for the action of θ on the vertices y_i, but one can easily see that only one of them gives rise to an automorphism of the graph, except when $(n,t) = (4,1)$, in which case both enumerations give rise to automorphisms. Hence, $|Stab(x_0)| = 2$ for $4t \neq n$ and $|Stab(x_0)| = 4$ for $4t = n,\ n \geq 8$. It therefore follows that for $4t \neq n$ we have $|B(n,t)| = 4n,\ \langle \alpha, \beta, \gamma \rangle \leq B(n,t)$, and so, by Proposition 4.2.1, we have $B(n,t) = \langle \alpha, \beta, \gamma \rangle$. Similarly, for $4t = n$ we have $|B(n,t)| = 8n,\ \langle \alpha, \beta, \gamma, \eta \rangle \leq B(n,t),\ \eta\alpha = \alpha\eta,\ \eta\beta = \alpha^{\frac{n}{2}}\beta\eta,\ \eta\gamma = \gamma\eta$, which implies that $|\langle \alpha, \beta, \gamma, \eta \rangle| = 8n$, and thus $B(n,t) = \langle \alpha, \beta, \gamma, \eta \rangle$.

Suppose now that $\mathrm{DP}(n,t)$ is vertex-transitive. Then, by Proposition 4.2.3, either $(n,t) = (5,2)$ or $n = 2k$ and $t^2 \equiv \pm 1 \pmod k$. For $(n,t) = (5,2)$ we have $B(5,2) = \langle \alpha, \beta, \gamma \rangle$. As for the second possibility three subcases need to be considered.

Assume first that $n \equiv 0 \pmod 2$ and $t^2 \equiv \pm 1 \pmod n$. Then the mapping δ, defined on page 12, is an automorphism of $\mathrm{DP}(n,t)$ preserving spokes and interchanging the sets Ω and I, that is, $\delta \in B(n,t)$. As the graph is vertex-transitive the orbit of any vertex is of length $4n$. Using the same arguments as in the none vertex-transitive case one can conclude that the vertex stabilizer in the action of $B(n,t)$ is of size 2 (it is of the size 8 only for $\mathrm{DP}(4,1)$). Hence, by the Orbit-Stabilizer Lemma, we get $|B(n,t)| = 4n \cdot 2 = 8n$. Furthermore, observe that $\delta\alpha = \alpha^t\beta\delta,\ \delta\beta = \beta\delta,\ \delta\gamma = \gamma\delta$ and when $t^2 \equiv -1 \pmod n$ also $\delta^2 = \gamma$. Hence, $|\langle \alpha, \beta, \gamma, \delta \rangle| = 8n$, and thus $B(n,t) = \langle \alpha, \beta, \gamma, \delta \rangle$.

Assume now that $n \equiv 2 \pmod 4$ and $t^2 \equiv k \pm 1 \pmod n$, where $n = 2k$. Then the mapping ψ, defined on page 12, is an automorphism of $\mathrm{DP}(n,t)$ such that $\psi \in B(n,t)$. The orbit of any vertex is of length $4n$. Using the same arguments as in the none vertex-transitive case one can conclude that the vertex stabilizer in the action of $B(n,t)$ is of size 2. Moreover, observe that $\psi\alpha = \alpha^t\beta\psi$, $\psi\beta = \alpha^k\psi$, $\psi\gamma = \gamma\psi$ and when $t^2 \equiv k-1 \pmod n$ also $\psi^2 = \gamma$. Hence, $|\langle \alpha, \beta, \gamma, \psi\rangle| = 8n$, implying that $B(n,t) = \langle \alpha, \beta, \gamma, \psi\rangle$.

Finally, assume that $n \equiv 0 \pmod 4$ and $t^2 \equiv k \pm 1 \pmod n$, where $n = 2k$. Then the mapping ϕ, defined on page 12, is an automorphism of DP(n,t) such that $\phi \in B(n,t)$. The orbit of any vertex is of length $4n$. Again, using the same arguments as in the none vertex-transitive case one can conclude that the vertex stabilizer in the action of $B(n,t)$ is of size 2. Observe that $\phi\alpha = \alpha^t\beta\phi$, $\phi\beta = \alpha^{\frac{n}{2}}\beta\phi$, and $\phi\gamma = \gamma\phi$, and for $t^2 \equiv k-1 \pmod n$ also $\phi^2 = \gamma$. It follows that $|\langle \alpha, \beta, \gamma, \phi\rangle| = 8n$, and thus $B(n,t) = \langle \alpha, \beta, \gamma, \phi\rangle$. ■

Proposition 4.2.6 $B(4,1) = \langle \alpha, \beta, \gamma, \delta, \eta\rangle$.

Proof. Using the same arguments as in the proof of Proposition 4.2.5 one can see that $|Stab(x_0)| = 8$, and thus $|B(4,1)| = 128$. The mapping η, defined on page 12, is an automorphism of DP$(4,1)$ belonging to $B(4,1)$. Since $\eta\alpha = \alpha\eta$, $\eta\beta = \alpha^2\beta\eta$, $\eta\gamma = \gamma\eta$, $\eta\delta = \beta\delta\eta$, we have $|\langle \alpha, \beta, \gamma, \delta, \eta\rangle| = 128$, and therefore $B(4,1) = \langle \alpha, \beta, \gamma, \delta, \eta\rangle$. ■

Lemma 4.2.7 *The following three statements are equivalent:*

(i) DP(n,t) *is edge-transitive.*

(ii) There exists $\pi \in A(n,t)$ *which maps some spoke onto an edge which is not a spoke.*

(iii) $B(n,t)$ *is a proper subgroup of* $A(n,t)$.

Proof. By definition of edge-transitivity statement (i) clearly implies statement (ii). Since $B(n,t)$ consists of automorphisms fixing the set of spoke edges the automorphism π from the statement (ii) cannot be in $B(n,t)$, and therefore (ii) implies (iii).

To complete the proof we need to proof that (iii) implies (i). In order to do this suppose that DP(n,t) is not edge-transitive. Then $A(n,t)$ has at least two orbits on the set of edges of the graph. Since orbits of $B(n,t)$ on the edge set of the graph are subsets of the orbits of $A(n,t)$ on the edge set of the graph, and since $B(n,t)$ has either 2 or 3 orbits on the edge set it follows that $B(n,t)$ and $A(n,t)$ have at least one edge-orbit in common. But then $A(n,t)$ must fix Ω, Σ or I, and therefore, by Lemma 4.2.2, it must fix Σ. But then $A(n,t) = B(n,t)$, and thus $B(n,t)$ is not a proper subgroup of $A(n,t)$, which completes the proof. ■

Following the notation in [28], for a cycle Z in DP(n,t) we let $r(Z)$ be the number of outer edges on Z, $s(Z)$ be the number of spokes on Z, and $t(Z)$ be the number of inner edges on Z. Let $\mathcal{Z}_j$ be the set of j-cycles in DP(n,t), and let

$$R_j = \sum_{Z\in\mathcal{Z}_j} r(Z), \quad S_j = \sum_{Z\in\mathcal{Z}_j} s(Z), \quad \text{and} \quad T_j = \sum_{Z\in\mathcal{Z}_j} t(Z).$$

Lemma 4.2.8 *If* $B(n,t) \neq A(n,t)$*, then* $R_j = S_j = T_j$ *for every* $j \geq 3$.

Proof. Let c, c' and c'' be the number of different j-cycles containing a particular spoke, outer edge and inner edge, respectively. The automorphisms $\alpha, \beta \in A(n,t)$ ensure that c, c' and c'' do not depend, respectively, on the choice of a spoke, outer edge and inner edge. It follows that $R_j = 2nc'$, $S_j = 2nc$ and $T_j = 2nc''$. Since $B(n,t) \neq A(n,t)$, Lemma 4.2.7 implies that DP(n,t) is edge-transitive, and hence $c = c' = c''$, implying that $R_j = S_j = T_j$. ■

Proposition 4.2.9 *$B(n,t) = A(n,t)$ if the ordered pair (n,t) is not one of the following three pairs $(5,2)$, $(10,2)$, and $(10,3)$.*

Proof. Let us consider 8-cycles in $\mathrm{DP}(n,t)$. By Lemma 4.2.8 it suffices to show that for $(n,t) \notin \{(5,2),(10,2),(10,3)\}$ the parameters R_8, S_8 and T_8 are not all equal.

In any double generalized Petersen graph $\mathrm{DP}(n,t)$ there exist 8-cycles

$$(x_i, u_i, v_{i+t}, y_{i+t}, y_{i+t+1}, v_{i+t+1}, u_{i+1}, x_{i+1}, x_i) \text{ and}$$

$$(x_i, u_i, v_{i-t}, y_{i-t}, y_{i-t+1}, v_{i-t+1}, u_{i+1}, x_{i+1}, x_i).$$

If these generic 8-cycles are the only 8-cycles in $\mathrm{DP}(n,t)$ the parameters are

$$R_8 = 2 \cdot 2 \cdot n = 4n,\ S_8 = 2 \cdot 4 \cdot n = 8n,\ T_8 = 2 \cdot 2 \cdot n = 4n,$$

and hence Lemma 4.2.8 implies that $B(n,t) = A(n,t)$. Therefore we only need to consider those double generalized Petersen graphs in which there exist additional 8-cycles. Observe that additional 8-cycles exist in $\mathrm{DP}(n,t)$ only if the parameter t satisfies one of the following conditions:

$$\begin{array}{lll} t = 1, & 2t+2 \equiv 0 \pmod{n}, & 4t-2 \equiv 0 \pmod{n}, \\ t = 2, & 2t+4 \equiv 0 \pmod{n}, & 4t+2 \equiv 0 \pmod{n}. \end{array}$$

In $\mathrm{DP}(n,1)$, $n \notin \{3,4,6,8\}$, there exist three types of 8-cycles, the two generic types and

$$(x_i, u_i, v_{i+1}, y_{i+1}, y_i, v_i, u_{i+1}, x_{i+1}, x_i),$$

implying that the parameters are:

$$\begin{aligned} R_8 &= 3 \cdot 2 \cdot n = 6n, \\ S_8 &= 3 \cdot 4 \cdot n = 12n, \\ T_8 &= 3 \cdot 2 \cdot n = 6n. \end{aligned}$$

In $\mathrm{DP}(3,1)$ there are two additional types of 8-cycles $(x_i, x_{i+1}, u_{i+1}, v_{i+2}, u_i, v_{i+1}, u_{i+2}, x_{i+2}, x_i)$ and $(y_i, y_{i+1}, v_{i+1}, u_{i+2}, v_i, u_{i+1}, v_{i+2}, y_{i+2}, y_i)$, and thus the parameters are

$$\begin{aligned} R_8 &= 3 \cdot 2 \cdot 3 + 2 \cdot 2 \cdot 3 = 10 \cdot 3, \\ S_8 &= 3 \cdot 4 \cdot 3 + 2 \cdot 2 \cdot 3 = 16 \cdot 3, \\ T_8 &= 3 \cdot 2 \cdot 3 + 2 \cdot 4 \cdot 3 = 14 \cdot 3. \end{aligned}$$

In $\mathrm{DP}(4,1)$ there are two additional types of 8-cycles $(x_i, u_i, v_{i-1}, y_{i-1}, y_{i-2}, v_{i-2}, u_{i+1}, x_{i+1}, x_i)$ and $(y_i, v_i, u_{i-1}, x_{i-1}, x_{i-2}, u_{i-2}, v_{i+1}, y_{i+1}, y_i)$, and thus the parameters are

$$\begin{aligned} R_8 &= 3 \cdot 2 \cdot 4 + 2 \cdot 2 \cdot 4 = 10 \cdot 4, \\ S_8 &= 3 \cdot 4 \cdot 4 + 2 \cdot 4 \cdot 4 = 20 \cdot 4, \\ T_8 &= 3 \cdot 2 \cdot 4 + 2 \cdot 2 \cdot 4 = 16 \cdot 4. \end{aligned}$$

In $\mathrm{DP}(6,1)$ there are four additional types of 8-cycles

$$\begin{array}{ll} (x_i, x_{i+1}, u_{i+1}, v_{i+2}, u_{i+3}, v_{i+4}, u_{i+5}, x_{i+5}, x_i), & (y_i, y_{i+1}, v_{i+1}, u_{i+2}, v_{i+3}, u_{i+4}, v_{i+5}, y_{i+5}, y_i), \\ (x_i, u_i, v_{i+1}, u_{i+2}, x_{i+2}, x_{i+3}, x_{i+4}, x_{i+5}, x_i), & (y_i, v_i, u_{i+1}, v_{i+2}, y_{i+2}, y_{i+3}, y_{i+4}, y_{i+5}, y_i), \end{array}$$

and thus the parameters are

$$\begin{aligned} R_8 &= 3 \cdot 2 \cdot 6 + 2 \cdot 2 \cdot 6 + 2 \cdot 4 \cdot 6 = 18 \cdot 6, \\ S_8 &= 3 \cdot 4 \cdot 6 + 2 \cdot 2 \cdot 6 + 2 \cdot 2 \cdot 6 = 20 \cdot 6, \\ T_8 &= 3 \cdot 2 \cdot 6 + 2 \cdot 4 \cdot 6 + 2 \cdot 2 \cdot 6 = 18 \cdot 6. \end{aligned}$$

In DP$(8,1)$ there exist four additional types of 8-cycles

$$\begin{aligned}&(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_0), && (y_0,y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_0),\\ &(u_0,v_1,u_2,v_3,u_4,v_5,u_6,v_7,u_0), && (v_0,u_1,v_2,u_3,v_4,u_5,v_6,u_7,v_0),\end{aligned}$$

and thus the parameters are

$$\begin{aligned}R_8 &= 3\cdot 2\cdot 8+2\cdot 8 = 8\cdot 8,\\ S_8 &= 3\cdot 4\cdot 8 = 12\cdot 8,\\ T_8 &= 3\cdot 2\cdot 8+2\cdot 8 = 8\cdot 8.\end{aligned}$$

In DP$(n,2)$, $n\notin\{5,6,8,10\}$, there exist four types of 8-cycles, the two generic types and

$$\begin{aligned}&(x_i,u_i,v_{i+2},u_{i+4},x_{i+4},x_{i+3},x_{i+2},x_{i+1},x_i),\\ &(y_i,v_i,u_{i+2},v_{i+4},y_{i+4},y_{i+3},y_{i+2},y_{i+1},y_i),\end{aligned}$$

and thus the parameters are

$$\begin{aligned}R_8 &= 2\cdot 4\cdot n+2\cdot 2\cdot n = 12n,\\ S_8 &= 2\cdot 2\cdot n+2\cdot 4\cdot n = 12n,\\ T_8 &= 2\cdot 2\cdot n+2\cdot 2\cdot n = 8n.\end{aligned}$$

In DP$(5,2)$ there are two additional types of 8-cycles $(x_i,u_i,v_{i+2},u_{i+4},v_{i+1},u_{i+3},x_{i+3},x_{i+4},x_i)$ and $(y_i,v_i,u_{i+2},v_{i+4},u_{i+1},v_{i+3},y_{i+3},y_{i+4},y_i)$, thus the parameters are

$$\begin{aligned}R_8 &= 2\cdot 4\cdot 5+2\cdot 2\cdot 5+2\cdot 2\cdot 5 = 16\cdot 5,\\ S_8 &= 2\cdot 2\cdot 5+2\cdot 4\cdot 5+2\cdot 2\cdot 5 = 16\cdot 5,\\ T_8 &= 2\cdot 2\cdot 5+2\cdot 2\cdot 5+2\cdot 4\cdot 5 = 16\cdot 5.\end{aligned}$$

In DP$(6,2)$ there are four additional types of 8-cycles

$$\begin{aligned}&(x_i,u_i,v_{i+2},u_{i+4},v_i,u_{i+2},x_{i+2},x_{i+1},x_i), && (y_i,v_i,u_{i+2},v_{i+4},u_i,v_{i+2},y_{i+2},y_{i+1},y_i),\\ &(x_i,u_i,v_{i-2},y_{i-2},y_{i-3},v_{i-3},u_{i+1},x_{i+1},x_i), && (y_i,v_i,u_{i-2},x_{i-2},x_{i-3},u_{i-3},v_{i+1},y_{i+1},y_i),\end{aligned}$$

and thus the parameters are

$$\begin{aligned}R_8 &= 2\cdot 4\cdot 6+2\cdot 2\cdot 6+2\cdot 2\cdot 6+2\cdot 2\cdot 6 = 20\cdot 6,\\ S_8 &= 2\cdot 2\cdot 6+2\cdot 4\cdot 6+2\cdot 2\cdot 6+2\cdot 4\cdot 6 = 24\cdot 6,\\ T_8 &= 2\cdot 2\cdot 6+2\cdot 2\cdot 6+2\cdot 4\cdot 6+2\cdot 2\cdot 6 = 20\cdot 6.\end{aligned}$$

In DP$(8,2)$ there are two additional types of 8-cycles $(x_i,u_i,v_{i+2},u_{i+4},x_{i+4},x_{i+5},x_{i+6},x_{i+7},x_i)$ and $(y_i,v_i,u_{i+2},v_{i+4},y_{i+4},y_{i+5},y_{i+6},y_{i+7},y_i)$, thus the parameters are

$$\begin{aligned}R_8 &= 2\cdot 4\cdot 8+2\cdot 2\cdot 8+2\cdot 4\cdot 8 = 20\cdot 8,\\ S_8 &= 2\cdot 2\cdot 8+2\cdot 4\cdot 8+2\cdot 2\cdot 8 = 16\cdot 8,\\ T_8 &= 2\cdot 2\cdot 8+2\cdot 2\cdot 8+2\cdot 2\cdot 8 = 12\cdot 8.\end{aligned}$$

In DP$(10,2)$ there are two additional types of 8-cycles $(x_i,u_i,v_{i+2},u_{i+4},v_{i+6},u_{i+8},x_{i+8},x_{i+9},x_i)$ and $(y_i,v_i,u_{i+2},v_{i+4},u_{i+6},v_{i+8},y_{i+8},y_{i+9},y_i)$, thus the parameters are

$$\begin{aligned}R_8 &= 2\cdot 4\cdot 10+2\cdot 2\cdot 10+2\cdot 2\cdot 10 = 16\cdot 10,\\ S_8 &= 2\cdot 2\cdot 10+2\cdot 4\cdot 10+2\cdot 2\cdot 10 = 16\cdot 10,\\ T_8 &= 2\cdot 2\cdot 10+2\cdot 2\cdot 10+2\cdot 4\cdot 10 = 16\cdot 10.\end{aligned}$$

In DP(n,t), where $2t+2\equiv 0 \pmod n$ and $n\geq 10$, there exist four types of 8-cycles, the two generic types, and

$(x_i,u_i,v_{i-t},y_{i-t},y_{i-t-1},v_{i-t-1},u_{i+1},x_{i+1},x_i)$ and $(y_i,v_i,u_{i-t},x_{i-t},x_{i-t-1},u_{i-t-1},v_{i+1},y_{i+1},y_i)$, implying that the parameters are

$$R_8 = 4 \cdot 2 \cdot n = 8n,\ S_8 = 4 \cdot 4 \cdot n = 16n,\ T_8 = 4 \cdot 2 \cdot n = 8n.$$

In DP$(8,3)$ there are four additional 8-cycles

$$\begin{array}{ll}(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_0), & (y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_0), \\ (u_0, v_1, u_2, v_3, u_4, v_5, u_6, v_7, u_0), & (v_0, u_1, v_2, u_3, v_4, u_5, v_6, u_7, v_0),\end{array}$$

and thus the parameters are

$$R_8 = 4 \cdot 2 \cdot 8 + 2 \cdot 8 = 10 \cdot 8,\ S_8 = 4 \cdot 4 \cdot 8 = 16 \cdot 8,\ T_8 = 4 \cdot 2 \cdot 8 + 2 \cdot 8 = 10 \cdot 8.$$

In DP(n,t), where $2t + 4 \equiv 0 \pmod n$ and $n \neq 10$, there exist four types of 8-cycles, the two generic types, and

$$\begin{array}{l}(x_i, u_i, v_{i+t}, u_{i+2t}, x_{i+2t}, x_{i+2t+1}, x_{i+2t+2}, x_{i+2t+3}, x_i), \\ (y_i, v_i, u_{i+t}, v_{i+2t}, y_{i+2t}, y_{i+2t+1}, y_{i+2t+2}, y_{i+2t+3}, y_i),\end{array}$$

giving parameters

$$R_8 = 2 \cdot 2 \cdot n + 2 \cdot 2 \cdot n = 12n, S_8 = 2 \cdot 4 \cdot n + 2 \cdot 2 \cdot n = 12n, T_8 = 2 \cdot 2 \cdot n + 2 \cdot 4 \cdot n = 8n.$$

In DP(n,t), where $4t - 2 \equiv 0 \pmod n$ and $n \neq 10$, there exist four types of 8-cycles, the two generic types, and

$$\begin{array}{l}(x_i, u_i, v_{i+t}, u_{i+2t}, v_{i+3t}, u_{i+4t}, x_{i+4t}, x_{i+1}, x_i), \\ (y_i, v_i, u_{i+t}, v_{i+2t}, u_{i+3t}, v_{i+4t}, y_{i+4t}, y_{i+1}, y_i),\end{array}$$

giving parameters

$$\begin{aligned}R_8 &= 2 \cdot 2 \cdot n + 2 \cdot 4 \cdot n = 8n, \\ S_8 &= 2 \cdot 4 \cdot n + 2 \cdot 2 \cdot n = 12n, \\ T_8 &= 2 \cdot 2 \cdot n + 2 \cdot 2 \cdot n = 12n.\end{aligned}$$

In DP$(10,3)$ there exist six types of 8-cycles (note that for $t = 3$ we have $2t + 4 = 4t - 2$), the two generic types, and

$$\begin{array}{ll}(x_i, u_i, v_{i+3}, u_{i+6}, x_{i+6}, x_{i+7}, x_{i+8}, x_{i+9}, x_i), & (y_i, v_i, u_{i+3}, v_{i+6}, y_{i+6}, y_{i+7}, y_{i+8}, y_{i+9}, y_i), \\ (x_i, u_i, v_{i+3}, u_{i+6}, v_{i+9}, u_{i+2}, x_{i+2}, x_{i+1}, x_i), & (y_i, v_i, u_{i+3}, v_{i+6}, u_{i+9}, v_{i+2}, y_{i+2}, y_{i+1}, y_i),\end{array}$$

and thus the parameters are

$$\begin{aligned}R_8 &= 2 \cdot 2 \cdot 10 + 2 \cdot 4 \cdot 10 + 2 \cdot 2 \cdot 10 = 16 \cdot 10, \\ S_8 &= 2 \cdot 4 \cdot 10 + 2 \cdot 2 \cdot 10 + 2 \cdot 2 \cdot 10 = 16 \cdot 10, \\ T_8 &= 2 \cdot 2 \cdot 10 + 2 \cdot 2 \cdot 10 + 2 \cdot 4 \cdot 10 = 16 \cdot 10.\end{aligned}$$

In DP(n,t), where $4t+2 \equiv 0 \pmod n$ and $n \notin \{3,5,6,10\}$, there exist four types of 8-cycles, the two generic types,

$$\begin{array}{l}(x_i, u_i, v_{i+t}, u_{i+2t}, v_{i+3t}, u_{i+4t}, x_{i+4t}, x_{i+4t+1}, x_i) \text{ and} \\ (y_i, v_i, u_{i+t}, v_{i+2t}, u_{i+3t}, v_{i+4t}, y_{i+4t}, y_{i+4t+1}, y_i),\end{array}$$

and thus the parameters are

$$\begin{aligned}R_8 &= 2 \cdot 2 \cdot n + 2 \cdot 2 \cdot n = 8n, \\ S_8 &= 2 \cdot 4 \cdot n + 2 \cdot 2 \cdot n = 12n, \\ T_8 &= 2 \cdot 2 \cdot n + 2 \cdot 4 \cdot n = 12n.\end{aligned}$$

We can now conclude that in DP(n,t) for $(n,t) \notin \{(5,2),(10,2),(10,3)\}$ either $R_8 \neq S_8$ or $S_8 \neq T_8$ or $T_8 \neq R_8$, and therefore, by Lemma 4.2.8, $B(n,t) = A(n,t)$. ■

Lemma 4.2.7 and Proposition 4.2.9 combined together give a classification of edge-transitive double generalized Petersen graphs. (In fact this also follows from complete classification of cubic symmetric tetracirculants given in [19].)

Theorem 4.2.10 *The double generalized Petersen graph* DP(n,t) *is edge-transitive if and only if the ordered pair* (n,t) *is one of the following three pairs* $(5,2)$, $(10,2)$, *or* $(10,3)$.

Proof. By Lemma 4.2.7 a necessary condition for DP(n,t) being edge-transitive is that $B(n,t)$ is a proper subgroup of $A(n,t)$. Therefore, by Proposition 4.2.9, the only candidates for edge-transitive double generalized Petersen graphs are those with

$$(n,t) \in \{(5,2),(10,2),(10,3)\}.$$

If $(n,t) = (5,2)$ then DP(n,t) is isomorphic to the dodecahedron, which is clearly vertex-transitive and edge-transitive (its full automorphism group is of order 120). Since DP$(10,3)$ is isomorphic to DP$(10,2)$ (an isomorphism can be obtained in a natural way by first mapping the 10-cycle on inner edges $(u_0, v_2, u_4, \ldots, v_8, u_0)$ in DP$(10,2)$ onto the 10-cycle on outer edges $(x_0, x_1, x_2, \ldots, x_9, x_0)$ in DP$(10,3)$) we only need to consider the graph DP$(10,2)$. Since

$$\begin{aligned} \mu = {} & (x_1,u_0)(x_2,v_8)(x_3,y_8)(x_4,y_9)(x_5,y_0)(x_6,v_0)(x_7,u_8)(u_1,v_2) \\ & (u_2,u_6)(u_3,y_7)(u_4,v_9)(u_5,y_1)(u_7,v_6)(v_1,v_7)(v_3,y_2)(v_5,y_6) \end{aligned}$$

is an automorphism of DP$(10,2)$, which maps the outer edge $\{x_0,x_1\}$ into the spoke $\{x_0,u_0\}$, Lemma 4.2.2 implies that DP$(10,2)$ is edge-transitive (its automorphism group is of order $|A(10,2)| = 480$ and $A(10,2) = \langle \alpha, \psi, \mu \rangle$, this result was found out with the help of MAGMA [13]). ■

Remark 4.2.11 *Theorem 4.2.10 implies that in Proposition 4.2.9 also implication in the opposite direction holds. In particular, for* DP(n,t) *we have* $B(n,t) = A(n,t)$ *if and only if the ordered pair* (n,t) *is not one of the following three pairs* $(5,2)$, $(10,2)$, *and* $(10,3)$.

Corollary 4.2.12 *The automorphism group* $A(n,t)$ *of the double generalized Petersen graph* DP(n,t) *is characterized as follows:*

(i) *If* $n \equiv 0 \pmod 2$, $4t = n$ *and* $(n,t) \neq (4,1)$, *then* $A(n,t) = \langle \alpha, \beta, \gamma, \eta \rangle$.

(ii) $A(4,1) = \langle \alpha, \beta, \gamma, \delta, \eta \rangle$.

(iii) *If* $n \equiv 0 \pmod 2$, $t^2 \equiv \pm 1 \pmod n$ *and* $(n,t) \neq (10,3)$, *then* $A(n,t) = \langle \alpha, \beta, \gamma, \delta \rangle$.

(iv) $A(10,3) = \langle \alpha, \delta, \lambda \rangle$, *where*

$$\begin{aligned} \lambda = {} & (x_1,u_0)(x_2,v_3)(x_3,y_3)(x_4,y_4)(x_5,y_5)(x_6,v_5)(x_7,u_8)(u_1,v_7) \\ & (u_2,u_6)(u_3,y_2)(u_4,v_4)(u_5,y_6)(u_7,v_1)(v_0,y_1)(v_2,v_6)(v_8,y_7). \end{aligned}$$

(v) *If* $n \equiv 2 \pmod 4$, $t^2 \equiv k \pm 1 \pmod n$, *where* $n = 2k$ *and* $(n,t) \neq (10,2)$, *then* $A(n,t) = \langle \alpha, \beta, \gamma, \psi \rangle$.

(vi) $A(10,2) = \langle \alpha, \psi, \mu \rangle$.

(vii) *If* $n \equiv 0 \pmod 4$ *and* $t^2 \equiv k \pm 1 \pmod n$, *where* $n = 2k$, *then* $A(n,t) = \langle \alpha, \beta, \gamma, \phi \rangle$.

(viii) $A(5,2)$ *is the automorphism group of the dodecahedron.*

(ix) *In all cases different from the eight mentioned above we have* $A(n,t) = \langle \alpha, \beta, \gamma \rangle$.

4.3 Hamilton cycles in $\mathrm{DP}(n,t)$

In this section Hamilton cycles in double generalized Petersen graphs are considered. The first proposition shows the existence of a Hamilton cycle in any double generalized Petersen graph of order $0 \pmod 8$ whereas the second proposition shows the existence of a Hamilton cycle only in a particular subclass of double generalized Petersen graphs of order $4 \pmod 8$. (Observe that any double generalized Petersen graph is either of order $0 \pmod 8$ or $4 \pmod 8$.)

Proposition 4.3.1 *Every double generalized Petersen graph* $\mathrm{DP}(2n,t)$ *admits a Hamilton cycle.*

PROOF. Observe that for different $i \in \mathbb{Z}_n$ the 8-paths

$$u_{2i} \to x_{2i} \to x_{2i+1} \to u_{2i+1} \to v_{2i+1-t} \to y_{2i+1-t} \to y_{2i+2-t} \to v_{2i+2-t}$$

have different vertex sets. Joining all n of them in a natural way therefore gives a Hamilton cycle in $\mathrm{DP}(2n,t)$. ■

Proposition 4.3.2 *Every double generalized Petersen graph* $\mathrm{DP}(2n+1,t)$*, where* $\mathbb{Z}_{2n+1} = \langle t \rangle$*, admits a Hamilton cycle.*

PROOF. The inner edges form a Hamilton cycle in the subgraph of $DP(2n+1,t)$ induced on the set $V' = \{u_i, v_i \mid i \in \mathbb{Z}_{2n+1}\}$. This cycle can be extended to a Hamilton cycle in $\mathrm{DP}(2n+1,t)$ in the following way: first remove the two edges $\{u_{2n}, v_{t-1}\}$ and $\{u_0, v_t\}$ from this Hamilton cycle on V', and then replace these two edges with the following two paths

$$u_0 \to x_0 \to x_1 \to \ldots \to x_{2n} \to u_{2n}, \text{ and } v_t \to y_t \to y_{t+1} \to \ldots \to y_{t-1} \to v_{t-1}.$$

■

A detailed computer-assisted search shows that the double generalized Petersen graphs $\mathrm{DP}(2n+1,t)$ for $n \leq 15$ admit Hamilton cycles also for t not being a generator of $\mathbb{Z}_{2n+1}$. Based on Propositions 4.3.1 and 4.3.2 and these computer-assisted results we post the following conjecture.

Conjecture 4.3.3 *All* $\mathrm{DP}(n,t)$ *are admit Hamilton cycle.*

4.4 Colorings of $\mathrm{DP}(n,t)$

A *snark* is a connected, cyclically 4-edge-connected cubic graph with girth at least 5 which is not 3-edge-colorable. While examples of snarks were initially scarce - the Petersen graph being the first known example - infinite families of snarks are now known to exist (see for instance [34]). As double generalized Petersen graphs are cubic graphs it is interesting to consider whether there exist snarks amongst them. However, we first consider their vertex coloring properties. The first lemma shows that for double generalized Petersen graphs of order $0 \pmod 8$ two colors suffices, and the second lemma shows that for graphs of order $4 \pmod 8$ three colors are needed.

Lemma 4.4.1 $\chi(\mathrm{DP}(2n,t)) = 2.$

PROOF. One can easily see that $c\colon V(\mathrm{DP}(2n,t)) \to \mathbb{N}$ defined by

$$c(x_{2i}) = c(u_{2i+1}) = c(v_{2i}) = c(y_{2i+1}) = 1, c(x_{2i+1}) = c(u_{2i}) = c(v_{2i+1}) = c(y_{2i}) = 2,$$

where $i \in \{0, 1, \ldots, n-1\}$, if t is even, and by

$$c(x_{2i}) = c(u_{2i+1}) = c(v_{2i+1}) = c(y_{2i}) = 1, c(x_{2i+1}) = c(u_{2i}) = c(v_{2i}) = c(y_{2i+1}) = 2,$$

where $i \in \{0, 1, \ldots, n-1\}$, if t is odd, is a well defined coloring function on $V(\mathrm{DP}(2n,t))$. ■

Lemma 4.4.2 $\chi\,(\mathrm{DP}(2n+1,t)) = 3$.

PROOF. Observe that $c\colon V(\mathrm{DP}(2n,t)) \to \mathbb{N}$ defined by

$$c(x_0) = c(u_{\pm i}) = c(y_{2i}) = 1, c(x_{2i}) = c(u_0) = c(y_{2i-1}) = 2, c(x_{2i-1}) = c(v_{\pm i}) = c(y_0) = 3,$$

where $i \in \{1, 2, \ldots, n\}$, is a well defined coloring function on $V(\mathrm{DP}(2n+1,t))$. ■

The next lemma shows that there are no snarks amongst the double generalized Petersen graphs.

Lemma 4.4.3 $\chi'\,(\mathrm{DP}(n,t)) = 3$.

PROOF. Since $\mathrm{DP}(n,t)$ is a cubic graph it is clear that at least 3 colors are needed for its proper edge-coloring. However, it will be shown that three colors suffice. To do so four cases depending on the parity and divisibility by 3 need to be considered.

CASE 1. $n = 2k$.

Then we can use colors 1 and 2 to color outer edges, color 3 for spokes and colors 1 and 2 for inner edges (since all cycles have even length).

CASE 2. $n = 2k+1,\ 3 \mid n,\ 3 \mid t$.

It can be easily checked that function $c\colon E(\mathrm{DP}(n,t)) \to \mathbb{N}$ defined by

$$\begin{aligned}
&c(\{x_{3i}, u_{3i}\}) = c(\{v_{3i}, y_{3i}\}) = c(\{x_{3i+1}, x_{3i+2}\}) = c(\{y_{3i+1}, y_{3i+2}\}) = 1,\\
&c(\{x_{3i+1}, u_{3i+1}\}) = c(\{v_{3i+1}, y_{3i+1}\}) = c(\{x_{3i+2}, x_{3i+3}\}) = c(\{y_{3i+2}, y_{3i+3}\}) = 2,\\
&c(\{x_{3i+2}, u_{3i+2}\}) = c(\{v_{3i+2}, y_{3i+2}\}) = c(\{x_{3i}, x_{3i+1}\}) = c(\{y_{3i}, y_{3i+1}\}) = 3
\end{aligned}$$

is a proper coloring of edges (not all of them). Remaining inner edges form even cycles each adjacent with spokes only in one color, so they can be colored using two colors different from adjacent spokes, which gives us a proper 3-edge-coloring of $\mathrm{DP}(n,t)$.

CASE 3. $n = 2k+1,\ 3 \mid n, 3 \not\mid t$.

In this case the same function can be used as in Case 2. The only difference is that in this case the colors of inner edges are uniquely determined.

CASE 4. $n = 2k+1,\ 3 \not\mid n$.

First color the spoke $\{x_0, u_0\}$ with color 1, the spoke $\{x_1, u_1\}$ with color 2, and all other spokes $\{x_i, u_i\}$ with color 3. These induce 3-coloring of outer edges $\{x_i, x_{i+1}\}$. On inner edges there are two cycles needing special attention, one containing the vertex u_0 and one containing the vertex u_1. For the first one use temporary colors $c^*(\{u_{2ti}, v_{(2i+1)t}\}) = 2$ and $c^*(\{v_{(2i+1)t}, u_{(2i+2)t}\}) = 1$. Next, change the color of the edge $\{u_0, v_{-t}\}$ from 1 to 3, and give color 1 to the lower spoke $\{v_{-t}, y_{-t}\}$. Then give temporary colors $c^*(\{u_{2ti+1}, v_{(2i+1)t+1}\}) = 1$ and $c^*(\{v_{(2i+1)t+1}, u_{(2i+2)t+1}\}) = 2$ to the second special cycle on inner edges. Change the color of the edge $\{u_1, v_{-t+1}\}$ from 2 to 3, and give color 2 to the lower spoke $\{v_{-t+1}, y_{-t+1}\}$. All other cycles on inner edges are of even length, therefore you can use colors 1 and 2 to color them. Finally, color with color 3 all spokes $\{v_i, y_i\}$ except the two mention above. Now, the outer edges $\{y_i, y_{i+1}\}$ can be colored with one of the three colors, which is uniquely determined for each of them. This shows that there are no snarks amongst double generalized Petersen graphs. ∎

Chapter 5

Spectral Characterizations of Signed Lollipop Graphs

Results of this chapter are published in [7].

5.1 Signed graphs - basic properties and known results

In this section we recall some basic results which will be useful for the study of signed graphs from a spectral viewpoint. More details on these results can be found in [9].

We first recall a formula useful to compute the coefficients of Laplacian polynomial of signed graphs. We need first to introduce some additional notation. A signed TU-subgraph H of a signed graph Γ is a subgraph whose components are trees or unbalanced unicyclic graphs. If H is a signed TU-subgraph, then $H = \bigcup_{i=1}^{t} T_i \bigcup_{j=1}^{c} U_j$, where, if any, the T_i's are trees and the U_j's are unbalanced unicyclic graphs. The weight of the signed TU-subgraph H is defined as $w(H) = 4^c \prod_{i=1}^{t} |T_i|$. For the special case $\Gamma = (G, -)$, we get the formula for the signless Laplacian of simple graphs, where instead of signed TU-subgraphs we have the TU-subgraphs, namely subgraphs whose components are trees or odd unicyclic graphs [24].

Theorem 5.1.1 *[9, 20] Let Γ be a signed graph and $\psi(\Gamma, x) = x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n$ be the Laplacian polynomial of Γ. Then we have*

$$b_i = (-1)^i \sum_{H \in \mathcal{H}_i} w(H), \quad i = 1, 2, \ldots, n, \tag{5.1}$$

where $\mathcal{H}_i$ denotes the set of the signed TU-subgraphs of Γ built on i edges.

From the above formula it is (again) evident that the L-polynomial is invariant under switching isomorphisms, since switching preserves the sign of the cycles. Furthermore, it is important to observe that the signature is relevant only on the edges that are not bridges, hence we will always consider the all-positive signature for trees. In the sequel signed trees and unsigned trees will be considered as the same object. For the same reason, the edges which do not lie on some cycle are not relevant for the signature and they will be always considered as positive. Another straight consequence of the above formula is described in the following corollary.

Corollary 5.1.2 *Let (G, σ) and (G, σ') be two signed graphs, on the same underlying graph G. Let $\psi(G, \sigma) = \sum_{i=1}^{n} b_i x^{n-i}$ and $\psi(G, \sigma') = \sum_{i=1}^{n} b'_i x^{n-i}$. If the girth of G is g then $b_i = b'_i$ for $i = 0, 1, \ldots, g-1$.*

Now, we recall some useful formulas, given in [9], which relate the Laplacian polynomial of a signed graph to the adjacency polynomials of its opportunely defined signed subdivision graph and signed line graph. In order to do so, we need to introduce a special oriented vertex-edge incidence matrix B_η of a signed graph $\Gamma = (G, \sigma)$ with n vertices and m edges. Assign any random orientation η on the positive edges of Γ. Then, the $n \times m$ matrix $B_\eta = (b^\eta_{ij})$ is defined as

$$b^\eta_{ij} = \begin{cases} +1 & \text{if } v_i \text{ is incident } e_j \text{ and } \sigma(e_j) = -1, \\ +1 & \text{if } v_i \text{ is the head of } e_j \text{ and } \sigma(e_j) = 1, \\ -1 & \text{if } v_i \text{ is the tail of } e_j \text{ and } \sigma(e_j) = 1, \\ 0 & \text{if } e_j \text{ is not incident } v_i. \end{cases}$$

It is not difficult to see that $L(\Gamma) = B_\eta B_\eta^\top$, which implies that $L(\Gamma)$ is a positive semidefinite matrix. From the above matrix we define two signed graphs, one of order $n + m$ and the other of order m, corresponding to the signed subdivision graph and the signed line graph, respectively. Recall that the subdivision of a simple graph G is the graph $\mathcal{S}(G)$ obtained from G by inserting in each edge a vertex of degree 2. In fact, $\mathcal{S}(G)$ is a graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent if and only if they are incident in G. Now, let us assign an orientation η to the positive edges and consider the corresponding incidence matrix $B_\eta = (b_{ij})$. The signed subdivision graph, associated to B_η, is the signed graph $\mathcal{S}(\Gamma_\eta) = (\mathcal{S}(G), \sigma^{\mathcal{S}}_\eta)$, where

$$\sigma^{\mathcal{S}}_\eta(v_i e_j) = b^\eta_{ij}$$

It is worth to observe that a signed subdivision graph is balanced if and only if each cycle in the signed root graph contains an even number of positive edges.

Next we define the signed line graph associated to B_η. The signed line graph of $\Gamma = (G, \sigma)$ is the signed graph $(\mathcal{L}(G), \sigma^{\mathcal{L}}_\eta)$, where $\mathcal{L}(G)$ is the (usual) line graph and

$$\sigma^{\mathcal{L}}_\eta(e_i e_j) = \begin{cases} b^\eta_{ki} b^\eta_{kj} & \text{if } e_i \text{ is incident } e_j \text{ at } v_k; \\ 0 & \text{otherwise.} \end{cases}$$

Note that both $\mathcal{S}(\Gamma_\eta)$ and $\mathcal{L}(\Gamma_\eta)$ depends on the chosen edge orientation η, but it is not difficult to see that a different orientation η' gives rise to a, respectively, switching equivalent signed subdivision graph and signed line graph. For example, reverting the orientation of some (positive) edge corresponds to having the value -1 in the state matrix entry related to the vertex subdividing the edge. Hence, $\mathcal{S}(\Gamma_\eta)$ and $\mathcal{L}(\Gamma_\eta)$ are uniquely defined up to switching isomorphisms, and for this reason the index η will be not anymore specified. For further details, the interested reader is referred to [9]. The following result holds

Theorem 5.1.3 ([9]) *Let Γ be a signed graph of order n and size m, and $\phi(\Gamma)$ and $\psi(\Gamma)$ its adjacency and Laplacian polynomials, respectively. Then*

(i) $\phi(\mathcal{L}(\Gamma), x) = (x + 2)^{m-n} \psi(\Gamma, x + 2)$,

(ii) $\phi(\mathcal{S}(\Gamma), x) = x^{m-n} \psi(\Gamma, x^2)$.

Remark 5.1.4 *In [9] the authors gave an interpretation of the oriented incidence matrix B_η in terms of bi-oriented graphs, here we have considered a slightly different (but equivalent) interpretation that is analogous to the Laplacian theory of mixed graphs, for which the edges can be either oriented or unoriented. It is clear that the Laplacian theory of mixed graphs is the same as that of signed graphs, e.g., [27, 56]. Finally, it is necessary to observe that in the literature we have also different definitions of signed line graphs for a signed graph (see, for example, [2, 52]).*

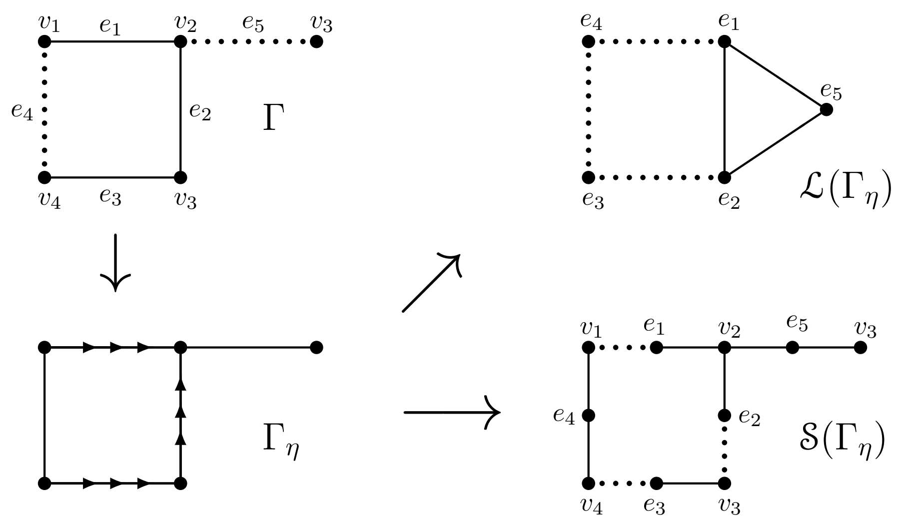

Figure 5.1: A signed graph and the corresponding signed subdivision and line graphs.

An example of subdivision and line graphs of a signed graph are depicted in Fig 5.1, where positive edges are bold lines, while negative edges are dotted lines.

The following result is the interlacing theorem in the edge variant. It can be deduced from the ordinary vertex variant interlacing theorem for the adjacency matrix combined with Theorem 5.1.3 (ii).

Theorem 5.1.5 *Let $\Gamma = (G, \sigma)$ be a signed graph and $\Gamma - e$ be the signed graph obtained from Γ by deleting the edge e. Then*

$$\mu_1(\Gamma) \geq \mu_1(\Gamma - e) \geq \mu_2(\Gamma) \geq \mu_2(\Gamma - e) \geq \cdots \geq \mu_n(\Gamma) \geq \mu_n(\Gamma - e).$$

From the above theorem, we can characterize the signed graphs whose Laplacian spectral radius does not exceed 4. Recall that the signatures of trees are omitted. Also, for the sake of readability, for signed unicyclic graphs, the signature denoted by $\bar{\sigma}$ means that the unique cycle is unbalanced. Note that signed unicyclic graphs have just two non-switching equivalent signatures: the all-positive edges, denoted by $\sigma = +$, and the unique cycle is unbalanced, denoted by $\bar{\sigma}$. Under the above notation we have the following results (cf. also [27]).

Lemma 5.1.6 *Let $\Gamma = (C_{2n}, \bar{\sigma})$ be the unbalanced cycle on $2n$ vertices. Then $\mu_1(C_{2n}, \bar{\sigma}) < 4$.*

Proof. In view of Corollary 5.1.2, $\psi(C_{2n}, +)$ and $\psi(C_{2n}, \bar{\sigma})$ have all coefficients but one equal. In fact, it is not difficult to see that $\psi((C_{2n}, +), x) - \psi((C_{2n}, \bar{\sigma}), x) = -4 < 0$ for every $x \in \mathbb{R}$. Since the spectral radius of $\psi(C_{2n}, +)$ is 4, then $\psi((C_{2n}, \bar{\sigma}), x) > 0$ for all $x \geq 4$. The latter implies that the corresponding spectral radius is less than 4. ■

Theorem 5.1.7 *Let μ be the largest eigenvalue, or spectral radius, of the Laplacian of a connected signed graph $\Gamma = (G, \sigma)$. The following statements hold:*

(i) $\mu(\Gamma) = 0$ *iff* $\Gamma = K_1$*;*

(ii) $\mu(\Gamma) = 2$ *iff* $\Gamma = K_2$*;*

iii) $\mu(\Gamma) = 3$ *iff* $\Gamma \in \{P_3, (K_3, +)\}$*;*

(iv) $3 < \mu(\Gamma) < 4$ *iff* $\Gamma \in \{P_n\ (n \geq 4), (C_{2n}, \bar{\sigma}), (C_{2n+1}, +)\ (n \geq 2)\}$*;*

(v) $\mu(\Gamma) = 4$ *iff* $\Gamma \in \{(C_{2n}, +), (C_{2n+1}, \bar{\sigma})\ (n \geq 2), K_{1,3}, (K_{1,3}^+, +), (K_4^-, +), (K_4, +)\}$.

where $K_{1,3}^+$ *(*K_4^-*) is obtained from* $K_{1,3}$ *(resp.,* K_4*) by adding (resp., deleting) an edge.*

Proof. Most of the above values of the Laplacian spectral radii of signed graphs can be deduced from the ordinary (signless) Laplacian theory of simple graphs (e.g., [49]). Let us denote by $\varphi(G)$ the characteristic polynomial of the signless Laplacian $D(G) + A(G)$, and by $\kappa(G)$ the corresponding spectral radius.

Clearly, $\Delta(\Gamma) < 4$, otherwise $K_{1,4}$ appears and $\mu(\Gamma) \geq 5$ by Theorem 5.1.5. Items (i), (ii) and (iii) are trivial and they can be easily verified.

Regarding the graphs in Items (iv) and (v), we have the following considerations. From the Laplacian theory of unsigned graph we get that $\mu(P_n) < 4$, $\mu(C_{2n+1}, +) < 4$, $\mu(K_{1,3}) = \mu(K_{1,3}^+, +) = \mu(K_4^-, +) = \mu(K_4, +) = 4$. From the signless Laplacian theory of graphs we get that $\psi(C_{2n}, +\sigma) = \varphi(C_{2n})$ with $\kappa(C_{2n}) = 4$, $\psi(C_{2n+1}, \bar{\sigma}) = \varphi(C_{2n+1})$ with $\kappa(C_{2n+1}) = 4$, and $\psi(K_{1,3}^+, \bar{\sigma}) = \varphi(K_{1,3}^+)$ with $\kappa(K_{1,3}^+) > 4$.

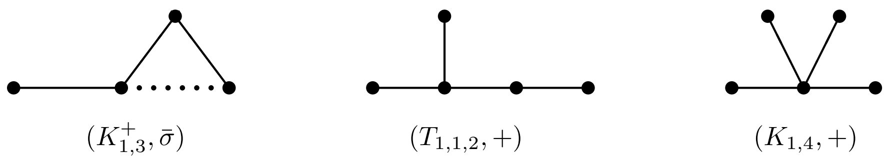

Figure 5.2: Forbidden subgraphs for $\mu_1 \leq 4$.

We now consider those spectra which cannot be deduced from the theory of unsigned graphs. One graph is $(C_{2n}, \bar{\sigma})$, for which we have that $\mu(C_{2n}, \bar{\sigma}) < 4$ by Lemma 5.1.6. Any other (connected) graph different from the previous ones will contain a vertex of degree 3, and at least one of the two following graphs: $(K_{1,3}^+, \bar{\sigma})$ or the tree $T_{1,1,2}$ (see Fig 5.2) which, according to Theorem 5.1.5, lead to signed graphs with spectral radius greater than 4. ■

In general, we can give the following upper bound for the largest Laplacian eigenvalue of a signed graph. Other similar bounds can be found in [36].

Lemma 5.1.8 *Let* $\Gamma = (G, \sigma)$ *be a signed graph with* Δ_1 *and* Δ_2 *being the first and second largest vertex degrees in* G*, and let* $\mu(\Gamma)$ *be it s Laplacian spectral radius. Then* $\mu(\Gamma) \leq \Delta_1 + \Delta_2$*, with equality if and only if* $\Gamma = K_{1,n}$ *or* $\Gamma = (K_n, -)$*.*

Proof. For any given matrix A, let $|A|$ be the absolute value matrix whose entries are obtained from A by replacing each entry with the corresponding absolute value. Recall that the largest eigenvalue of a square matrix A is less than or equal to the largest eigenvalue of $|A|$ (due to the Perron-Frobenius theorem). Hence, we have that $\mu(\Gamma) = \mu(BB^\top) \leq \mu(|BB^\top|) = \mu(G, -)$, namely the largest L-eigenvalue of a signed graph is bounded by the largest L-eigenvalue of the corresponding all-negative edges signed graph. On the other hand, $L(G,-)$ is the signless Laplacian of the underlying simple graph G, for which the inequality $\mu(G,-) \leq \Delta_1 + \Delta_2$ holds (e.g., Theorem 4.2 in [23]). The equality holds if and only if G is either the n-star $K_{1,n}$ or the complete graph K_n. This completes the proof. ∎

We conclude this section with two formulas (see Theorems 3.2 and 3.4 in [6]) useful for the computation of the A-polynomial of any weighted non oriented graph.

Theorem 5.1.9 *Let $A = (a_{ij})$ be the adjacency matrix of a weighted graph G. Let $v \in G$ be any vertex. Then we have*

$$\phi(G,x) = (x - a_{vv})\phi(G-v,x) - \sum_{u\sim v} a_{uv}^2 \phi(G-u-v,x) - 2\sum_{C\in\mathcal{C}_v} \omega(C)\phi(G\setminus V(C),x),$$

$$\phi(G,x) = \phi(G-uv,x) - a_{uv}^2\phi(G-u-v,x) - 2\sum_{C\in\mathcal{C}_{uv}} \omega(C)\phi(G\setminus V(C),x),$$

where $\mathcal{C}_a$ is the set of cycles passing through a and $\omega(C) = \prod_{uw\in C} a_{uw}$.

The formulas in Theorem 5.1.9 have a natural use in the context of the adjacency matrix. However, they can be used for the Laplacian of signed graphs by mapping the Laplacian matrix of a signed graph as the adjacency matrix of a weighted multigraph. In fact, by doing so, any positive edge becomes a negative edge and viceversa, while the vertices degrees are expressed as weighted loops. The weight of a n-cycle C will be $+1$ if the cycle contains an even number of positive edges, and -1 if it contains an odd number of positive edges, that is $(-1)^n\sigma(C)$.

5.2 Spectral determination of signed graphs

In this section we give some results which will be useful for the study of spectral determination of signed graphs. This problem was, possibly, first introduced by Acharya in [1] in the context of the adjacency matrix.

Definition 5.2.1 *We say that a signed graph $\Gamma = (G,\sigma)$ is determined by the spectrum, or the eigenvalues, of its matrix $M(\Gamma)$ if and only if any other signed graph $\Lambda = (H,\sigma')$ such that $M(\Lambda)$ has the same spectrum of $M(\Gamma)$ implies that Γ and Λ are two switching isomorphic graphs. In the latter case, we say that Γ is determined by the spectrum of the matrix M, or a DMS graph for short. If Λ is not switching isomorphic to Γ, we say that the two graphs are M-cospectral, or, equivalently, Λ is a M-cospectral mate of Γ.*

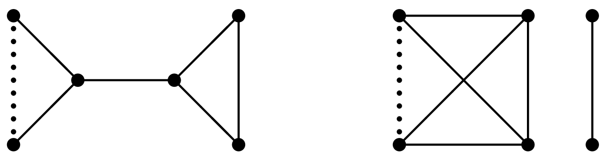

Figure 5.3: A pair of A-cospectral signed graphs.

As shown in Fig 5.3, there are pairs of cospectral signed graphs. However, cospectral mates share the spectral invariants. Let us first consider some spectral invariants which can be deduced from the powers of the matrices of signed graphs. For this purpose, we need to introduce some additional notation. A walk of length k in a signed graph Γ is a sequence $v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_{k+1}$ of vertices $v_1, v_2, \ldots, v_{k+1}$ and edges $e_1, e_2, \ldots, e_k$ such that $v_i \neq v_{i+1}$ for each $i = 1, 2, \ldots, k$; a walk is said to be positive if it contains an even number of positive edges, otherwise it is said to be negative. Let $w^{+}_{v_i v_j}(k)$ (resp. $w^{-}_{v_i v_j}(k)$) denote the number of positive (resp., negative) walks of length k from the vertex v_i to the vertex v_j. Finally let t^{+}_{Γ} (resp., t^{-}_{Γ}) denote the number of balanced (resp., unbalanced) triangles in Γ (the suffix is omitted if clear from the context). The following fact is well known (see, for example, [52]):

Lemma 5.2.2 *Let Γ be a signed graph and A its adjacency matrix. Then the (i,j)-entry of the matrix A^k is $w^{+}_{v_i v_j}(k) - w^{-}_{v_i v_j}(k)$.*

From the above lemma we immediately deduce the following corollary.

Corollary 5.2.3 *Let Γ be a signed graph, A its adjacency matrix, D the diagonal matrix of vertex degrees and t^+ (resp., t^-) the number of balanced (resp., unbalanced) triangles.*
Then $\mathrm{tr}\,(A^2) = \mathrm{tr}(D)$, and $\mathrm{tr}\,(A^3) = 6(t^+ - t^-)$.

Let $T_k = \sum_{i=1}^{n} \mu_i^k$ $(k = 0, 1, 2, \ldots)$ be the k-th spectral moment for the Laplacian spectrum of a signed graph Γ.

Theorem 5.2.4 *Let $\Gamma = (G, \sigma)$ be a signed graph with n vertices, m edges, t^+ balanced triangles, t^- unbalanced triangles, and degree sequence $(d_1, d_2, \ldots, d_n)$. We have*

$$T_0 = n, \quad T_1 = \sum_{i=1}^{n} d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^{n} d_i^2, \quad T_3 = 6\,(\,t^- - t^+) + 3\sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i^3.$$

Proof. Recall that $\mathrm{tr}\,MN = \mathrm{tr}\,NM$ for any two feasible matrices M and N. The formulas for T_0 and T_1 are obvious. The formula for T_2 follows from $\mathrm{tr}\,L^2 = \mathrm{tr}\,(D-A)^2 = \mathrm{tr}\,D^2 + \mathrm{tr}\,A^2$, since $\mathrm{tr}\,AD = \mathrm{tr}\,DA = 0$ and, by Corollary 5.2.3, we have $\mathrm{tr}\,A^2 = \mathrm{tr}\,D = 2m$. Finally, $T_3 = \mathrm{tr}\,(D-A)^3 = \mathrm{tr}\,D^3 + 3\mathrm{tr}\,A^2 D - 3\mathrm{tr}\,AD^2 - \mathrm{tr}\,A^3$. Since $\mathrm{tr}\,AD^2 = 0$, and, by Corollary 5.2.3, $\mathrm{tr}\,(A^3) = 6(t^+ - t^-)$ we get the assertion.

■

It is well-known that the multiplicity of the eigenvalue 0 counts the number of balanced components (see, for example, [51]). The below result synthesizes the considerations so far made.

Theorem 5.2.5 *Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \sigma')$ be two L-cospectral signed graphs. Then,*

(i) Γ and Λ have the same number of vertices and edges;

(ii) Γ and Λ have the same number of balanced components;

(iii) Γ and Λ have the same Laplacian spectral moments;

(iv) Γ and Λ have the same sum of squares of degrees, $\sum_{i=1}^{n} d_G(v_i)^2 = \sum_{i=1}^{n} d_H(v_i)^2$;

(v) $6(t^-_{\Gamma} - t^+_{\Gamma}) + \sum_{i=1}^{n} d_G(v_i)^3 = 6(t^-_{\Lambda} - t^+_{\Lambda}) + \sum_{i=1}^{n} d_H(v_i)^3$.

The following theorem can be useful in those situations in which a signed graph Γ and its signed subdivision graph $\mathcal{S}(\Gamma)$ maintain the same structure (e.g., lollipop graphs).

Theorem 5.2.6 *Let $\Gamma = (G, \sigma)$ be a signed graph of order n and size m, and $\mathcal{S}(\Gamma)$ the subdivision graph of Γ.*

(i) *The signed graphs* Γ *and* Λ *are* L*-cospectral iff* $\mathcal{S}(\Gamma)$ *and* $\mathcal{S}(\Lambda)$ *are* A*-cospectral;*

(ii) *Let* Γ *be a signed graph and* $\mathcal{S}(\Gamma)$ *a DAS-graph. Then* Γ *is a DLS-graph;*

(iii) *Let* Γ *be a DLS-graph. If any graph* A*-cospectral to* $\mathcal{S}(\Gamma)$ *is a subdividion of some graph, then* $\mathcal{S}(\Gamma)$ *is a DAS-graph.*

Proof. (i) Since Γ and Λ are L-cospectral, then $\psi(\Gamma, x) = \psi(\Lambda, x)$, and Γ and Λ have the same order and size which implies that $m(\Gamma) - n(\Gamma) = m(\Lambda) - n(\Lambda)$. Thus,

$$x^{m(\Gamma)-n(\Gamma)}\psi(\Gamma, x^2) = x^{m(\Lambda)-n(\Lambda)}\psi(\Lambda, x^2),$$

which implies by Lemma 5.1.3 (i) that $\phi(\mathcal{S}(\Gamma), x) = \phi(\mathcal{S}(\Lambda), x)$. This ends the necessity.

Conversely, since $\mathcal{S}(\Gamma)$ and $\mathcal{S}(\Lambda)$ are A-cospectral, then

$$\phi(\mathcal{S}(\Gamma), x) = \phi(\mathcal{S}(\Lambda), x), \quad n(\mathcal{S}(\Gamma)) = n(\mathcal{S}(\Lambda)), \quad m(\mathcal{S}(\Gamma)) = m(\mathcal{S}(\Lambda)).$$

Note that

$$m(\mathcal{S}(\Gamma)) = 2m(\Gamma), \quad m(\mathcal{S}(\Lambda)) = 2m(\Lambda), \quad n(\mathcal{S}(\Gamma)) = m(\Gamma) + n(\Gamma), \quad n(\mathcal{S}(\Lambda)) = m(\Lambda) + n(\Lambda).$$

From the above equalities, we obtain that $m(\Gamma) = m(\Lambda)$ and $n(\Gamma) = n(\Lambda)$, and so

$$(\sqrt{x})^{n(\Gamma)-m(\Gamma)}\phi(\mathcal{S}(\Gamma), \sqrt{x}) = (\sqrt{x})^{n(\Lambda)-m(\Lambda)}\phi(\mathcal{S}(\Lambda), \sqrt{x}),$$

which shows from that $\psi(\Gamma, x) = \psi(\Lambda, x)$.

(ii) Assume that $\psi(\Lambda, x) = \psi(\Gamma, x)$. Then by (i) we get $\phi(\mathcal{S}(\Lambda), x) = \phi(\mathcal{S}(\Gamma), x)$. Since $\mathcal{S}(\Gamma)$ is a DAS-graph, then $\mathcal{S}(\Lambda)$ is switching isomorphic to $\mathcal{S}(\Gamma)$, that implies Λ being switching isomorphic to Γ.

(iii) Assume that Λ and Λ' are two signed graphs such that $\Lambda = \mathcal{S}(\Lambda')$ and $\phi(\lambda, x) = \phi(\mathcal{S}(\Lambda'), x) = \phi(\mathcal{S}(\Gamma), x)$, which implies from (i) that $\psi(\Lambda', x) = \psi(\Gamma, x)$. Since Γ is a DLS-graph, then Λ' is switching isomorphic to Γ, and so $\Lambda = \mathcal{S}(\Lambda')$ is switching isomorphic to $\mathcal{S}(\Gamma)$ which shows that $\mathcal{S}(\Gamma)$ is indeed a DAS-graph. ■

In view of Theorem 5.2.6 the following problem naturally arises: under which conditions a signed graph $\Gamma = (G, \sigma)$ can be seen as a signed subdivision graph. The answer is the same as that for unsigned graphs, and the signature can be easily deduced.

Theorem 5.2.7 *A signed graph* $\Gamma = (G, \sigma)$ *is the signed subdivision graph of* Λ *if and only if the following items hold:*

(i) G *is bipartite;*

(ii) *One of the two color classes, say* S_2*, consists of exactly* $m(G)/2$ *vertices of degree 2;*

(iii) G *does not contain* C_4 *as its subgraph.*

Then Λ *is obtained from* Γ *by replacing each vertex from* S_2 *with an edge. The signature of the edge will be: a) positive, if the two deleted edges were of different sign; b) negative, if both deleted edges had the same sign.*

5.3 Spectral determination of signed lollipop graphs

In this section we study the Laplacian spectral determination of signed lollipop graphs. Another spectral determination problem is considered in [8] for the signed graphs whose second largest L-eigenvalue does not exceed 3, in which signed friendship graphs are included. A lollipop graph is the coalescence between a cycle and a path for which the end vertex of the path is identified with a vertex from the cycle. By $L_{g,n}$ we denote the lollipop graph whose girth is g and the order is n. Since the lollipop is a unicyclic graph, then it admits only two different non-equivalent signatures: the all positive edges $\sigma = +$, and $\bar{\sigma}$ for which the unique cycle is unbalanced. In Fig 5.4 we depicted an example of signed lollipop graph.

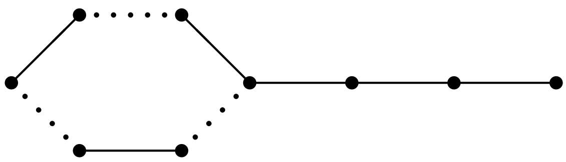

Figure 5.4: The signed lollipop graph $(L_{6,9}, \bar{\sigma})$.

In the literature, the spectral determination of (unsigned) lollipop graphs, and related graphs, has been already considered in the papers [14, 32, 31, 55]. Here we continue such investigations by extending the problem to the wider settings of signed graphs. Clearly, the main results from the above cited papers must be taken into account. We restate such results in terms of signed graphs.

Theorem 5.3.1 *Let $(L_{g,n}, \sigma)$ be a signed lollipop graph of order n and girth g. We have:*

- $(L_{g,n}, +)$ *has no A-cospectral mates with only positive edges [14];*
- $(L_{g,n}, +)$ *has no L-cospectral mates with only positive edges [32];*
- $(L_{g,n}, -)$ *has no L-cospectral mates with only negative edges [33, 55].*

Now we spectrally characterize the signed lollipop graphs and extend the result of Theorem 5.3.1 to all signed lollipop graphs. The following lemma gives two bounds on the first and second largest eigenvalue of any signed lollipop graph.

Lemma 5.3.2 *Let $(L_{g,n}, \sigma)$ be a lollipop graph. Then we have $4 < \mu_1(L_{g,n}, \sigma) < 5$ and $\mu_2(L_{g,n}, \sigma) < 4$.*

Proof. The upper bound for $\mu_1(L_{g,n}, \sigma)$ comes from the fact that largest and second largest vertex degrees of $(L_{g,n}, \sigma)$ are 3 and 2, and from Lemma 5.1.8 we obtain $\mu_1(L_{g,n}, \sigma) < 5$. The lower bound for for $\mu_1(L_{g,n}, \sigma)$ comes from $K_{1,3}$ being a subgraph (interlacing theorem). Finally, from the interlacing theorem applied to the edge in the cycle incident with the vertex of degree 3, we obtain the path P_n. Hence, in view of Theorem 5.1.7 (iii), we have $\mu_1(L_{g,n}, \sigma) \geq 4 > \mu_1(P_n) \geq \mu_2(L_{g,n}, \sigma)$. Finally, it is also easy to see that 4 cannot be an eigenvalue of $(L_{g,n}, \sigma)$ (see for example, Lemma 5.3.15). ∎

Lemma 5.3.3 *Let $\Gamma = (G, \sigma)$ be L-cospectral with $(L_{g,n}, \sigma)$, then Γ has the same degree sequence of $(L_{g,n}, \sigma)$.*

Proof. Let $\Gamma = (G, \sigma)$ be L-cospectral with $(L_{g,n}, \sigma)$. Since $\mu_1(L_{g,n}, \sigma) < 5$, then Γ cannot have vertices whose degree is greater than 3, otherwise $K_{1,4}$ appears as a subgraph of Γ and $\mu_1(\Gamma) \geq 5$. Let n_i be the number of vertices whose degree is i, where $0 \leq i \leq 3$. From Theorem 5.2.4 (i) and (iv) we deduce the following linear system of equations:

$$\begin{cases} n_0 + n_1 + n_2 + n_3 &= n, \\ n_1 + 2n_2 + 3n_3 &= 2n, \\ n_1 + 4n_2 + 9n_3 &= 4n + 2. \end{cases}$$

whose unique (acceptable) solution is indeed $n_0 = 0$, $n_1 = 1$, $n_2 = n - 2$ and $n_3 = 1$. Hence the underlying graph of Γ consists of a lollipop graph with possibly one or more cycles as connected components. ∎

Now we have restricted the structure of a tentative L-cospectral mate of a signed lollipop graph. Let us denote by Γ a signed graph cospectral with $(L_{g,n}, \sigma)$. We have proved that Γ is a signed lollipop graph with possibly one or more cycles as components. However, Γ can not have any kind of cycles as a component. In fact, $(C_{2r+1}, \bar{\sigma})$ and $(C_{2r}, +)$ are not acceptable since 4 would appear as an eigenvalue. Also, the eigenvalue 0 appears at most once, so Γ can have no more than one balanced cycle as a component.

The following lemma lists the spectra of signed cycles and paths (see [22] for the balanced ones, those unbalanced can be deduced from the balanced by Theorem 5.1.3), and it will be useful to the reader. For the sake of readability, the suffices and the polynomials variables will be omitted if clear from the context.

Lemma 5.3.4 *Let P_n and C_n be the path and the cycle on n vertices, respectively. Let $\mathrm{Spec}_M(\Gamma)$ denote the multiset of eigenvalues of $M(\Gamma)$.*

$$\begin{aligned}
\mathrm{Spec}_A\,(C_n, +) &= \{2\cos\frac{2k}{n}\pi,\ k = 0, 1, \ldots, n-1\}; \\
\mathrm{Spec}_A\,(C_n, \bar{\sigma}) &= \{2\cos\frac{2k+1}{n}\pi,\ k = 0, 1, \ldots, n-1\}; \\
\mathrm{Spec}_A\,(P_n) &= \{2\cos\frac{k}{n+1}\pi,\ k = 1, 2, \ldots, n\}; \\
\mathrm{Spec}_L\,(C_{2n}, +) &= \{2 + 2\cos\frac{2k}{2n}\pi,\ k = 0, 1, \ldots, 2n-1\}; \\
\mathrm{Spec}_L\,(C_{2n+1}, +) &= \{2 + 2\cos\frac{2k+1}{2n+1}\pi,\ k = 0, 1, \ldots, 2n\}; \\
\mathrm{Spec}_L\,(C_{2n}, \bar{\sigma}) &= \{2 + 2\cos\frac{2k+1}{2n}\pi,\ k = 0, 1, \ldots, 2n-1\}; \\
\mathrm{Spec}_L\,(C_{2n+1}, \bar{\sigma}) &= \{2 + 2\cos\frac{2k}{2n+1}\pi,\ k = 0, 1, \ldots, 2n\}; \\
\mathrm{Spec}_L\,(P_n) &= \{2 + 2\cos\frac{k}{n}\pi,\ k = 1, 2, \ldots, n\}.
\end{aligned}$$

Remark 5.3.5 *In view of the above lemma we get that $(C_{2n}, +)$ is L-cospectral with $(C_n, +) \cup (C_n, \bar{\sigma})$, and in view of Lemma 5.1.3 (i) the same applies w.r.t. the adjacency spectra. Moreover, the L-spectrum of $(C_{2n+1}, +)$ (resp., $(C_{2n+1}, \bar{\sigma})$) contains the L-spectrum of $(C_d, +)$ (resp., $(C_d, \bar{\sigma})$) for any d divisor of $2n + 1$. The L-spectrum of $(C_{2n}, \bar{\sigma})$ contains the L-spectrum of $(C_d, \bar{\sigma})$ provided that $\frac{2n}{d}$ is an odd number. For example, the L-spectrum $(C_{120}, \bar{\sigma})$ contains the L-spectrum of $(C_d, \bar{\sigma})$ only for $d \in \{8, 24, 40\}$. Similarly, the L-spectrum of $(C_{2n,+})$ contains the L-spectrum of $(C_d, +)$ for all divisors d of $2n$, while it also contains the L-spectrum of $(C_d, \bar{\sigma})$ when $\frac{2n}{d}$ is an even number.*

The lemma below stems from the above observations.

Lemma 5.3.6 *Let $(C_{2n},+)$ be a even balanced cycle and let $2n = 2^{t+1}r$, where t and r are positive integer and r is odd. If $r \geq 3$, then $(C_{2^{t+1}r},+)$ is L-cospectral with $(C_{2^s r},+)\bigcup_{i=s}^{t}(C_{2^i r},\bar{\sigma})$, with $0 \leq s \leq t$. If $r = 1$ then $(C_{2^{t+1}},+)$ is L-cospectral with $(C_{2^s},+)\bigcup_{i=s}^{t}(C_{2^i},\bar{\sigma})$, with $2 \leq s \leq t$.*

Let $\mu(d) = 2+2\cos\frac{\pi}{d}$; for d odd $\mu(d)$ is $\mu_1(C_d,+)$, while for d even it is $\mu_1(C_d,\bar{\sigma})$. The observations of Remark 5.3.5 play a crucial role in the following theorem. Let $\mathrm{GCD}(a,b)$ be the greatest common divisor between the integers a and b. Also, let $[c(n),\sigma]$ be the set of the L-eigenvalues of multiplicity two of the cycle (C_n,σ).

Theorem 5.3.7 *The signed lollipop graph $(L_{g,n},\sigma) = \Lambda$ has simple L-eigenvalues if $\mathrm{GCD}(g,n) = 1$. If $\mathrm{GCD}(g,n) = d \geq 2$, then we have the following possibilities*

- *if g is odd, then the eigenvalues of Λ of multiplicity two are those of $[c(d),\sigma]$;*
- *if g is even, $\frac{d}{g}$ odd (resp., even), and $\sigma = +$, then the eigenvalues of Λ of multiplicity two are those of $[c(d),+]$ (resp., $[c(2d),+]$);*
- *if g is even and $\sigma = \bar{\sigma}$, then for $\frac{g}{d}$ odd the eigenvalues of Λ of multiplicity two are those of $[c(d),\bar{\sigma}]$, while for $\frac{g}{d}$ even, Λ has just simple eigenvalues.*

Proof. Recall that the L-eigenvalues of signed cycles, other than 0 and 4, have multiplicity two.

First, note that P_n is an edge-deleted subgraph of $(L_{g,n},\sigma)$, hence in view of fact the P_n has only simple eigenvalues, by interlacing theorem each L-eigenvalue has at most multiplicity two (and it must be an L-eigenvalue for P_n). Similarly, for $(C_g,\sigma)\cup P_{n-g}$ we have:

$$\begin{aligned}\mu_1(L_{g,n},\sigma) &\geq \mu_1((C_g,\sigma)\cup P_{n-g}) \geq \mu_2(L_{g,n},\sigma) \geq \mu_2((C_g,\sigma)\cup P_{n-g})\\ &\geq \cdots \geq \mu_n(L_{g,n},\sigma) \geq \mu_n((C_g,\sigma)\cup P_{n-g}).\end{aligned}$$

Let μ be, if any, an L-eigenvalue of multiplicity two, then μ is an L-eigenvalue of $(C_g,\sigma)\cup P_{n-g}$. Let us consider the subdivision graph $\mathcal{S}(L_{g,n},\sigma) = (L_{2g,2n},\sigma')$. By applying Theorem 5.1.9 at the hanging path edge that is incident to the vertex of degree 3, we have:

$$\phi(L_{2g,2n},\sigma') = \phi(C_{2g},\sigma')\phi(P_{2n-2g}) - \phi(P_{2g-1})\phi(P_{2n-2g-1}). \tag{5.2}$$

Let $\lambda = \sqrt{\mu}$, in view of Theorem 5.1.3, λ is an A-eigenvalue of multiplicity two, as well. From μ being a L-eigenvalue of $(C_g,\sigma)\cup P_{n-g}$, we deduce that λ is an A-eigenvalue of $(C_{2g},\sigma')\cup P_{2n-2g-1}$. Since λ is a root of $\phi(C_{2g},\sigma')$ or a root of $\phi(P_{2n-2g-1})$, then in (5.2) we have that λ is of multiplicity two if and only if λ is a root of both $\phi(C_{2g},\sigma')$ and $\phi(P_{2n-2g-1})$ (note, if $\lambda \neq 4$ is an A-eigenvalue of (C_{2g},σ'), then it is an A-eigenvalue of P_{2g-1}). The latter implies that μ is an L-eigenvalue of both (C_g,σ) and P_{n-g}. Clearly, if $d = GCD(g,n-g) = 1$, then such a μ cannot exist and the L-eigenvalues of Λ have multiplicity 1. So let $d \geq 2$ in the sequel.

Assume first that g is odd, then also d is odd. Since d divides both g and $n-g$, we have that $[c(g),\sigma]\cap \mathrm{Spec}_L(P_{n-g}) = [c(d),\sigma]$.

Assume next that g is even and $\sigma = +$. Since g is even then also the L-eigenvalues of $(C_r,\bar{\sigma})$ appear in $\mathrm{Spec}_L(C_{g,n},+)$ for any divisor r of g such that $\frac{g}{r}$ is even (note, r is a proper divisor). Hence, if $\frac{g}{d}$ is even, then d divides both g and $n-g$, and we have that $[c(g),+]\cap \mathrm{Spec}_L(P_{n-g}) = [c(2d),+]$. If instead we have $\frac{g}{d}$ odd, then we get $[c(g),+]\cap \mathrm{Spec}_L(P_{n-g}) = [c(d),+]$. In particular, if $g = d$ or $g = 2d$, then the eigenvalues of multiplicity two are those of $[c(g),+]$.

Finally, assume that g is even and $\sigma = \bar{\sigma}$, then $[c(g), \bar{\sigma}] \cap \mathrm{Spec}_L(P_{n-g})$ is non-empty if and only if $\frac{g}{d}$ is odd, and in the latter case we get $[c(d), \bar{\sigma}]$; if $\frac{g}{d}$ is even, then $\frac{g}{r}$ is even for all the divisors r of d, hence $[c(g), \bar{\sigma}] \cap \mathrm{Spec}_L(P_{n-g}) = \emptyset$.

This completes the proof. ■

Corollary 5.3.8 *Let Γ be a signed graph L-cospectral with $(L_{g,n}, \sigma)$, with $GCD(g, n) = d \geq 3$. If (C_r, σ) is a component of Γ, then r divides d.*

Proof. Since (C_r, σ) is a component of Γ, then $\mu(r)$ is in the spectrum of Λ with multiplicity two. Hence, according to Theorem 5.3.7, $\mu(r)$ is in the spectrum of (C_g, σ) and of P_{n-g}. Consequently, r divides both g and $n-g$. The latter implies that r divides d as well. ■

Corollary 5.3.9 *Let Γ be a signed graph L-cospectral with $(L_{g,n}, \sigma)$. If $GCD(g, n) = d \leq 2$, then Γ is connected.*

Proof. Recall that in view of Lemma 5.3.3, any L-cospectral mate of a signed lollipop graph consists of a lollipop graph with possibly one or more cycles as components. From Theorem 5.3.7, we deduce that $(L_{g,n}, \sigma)$ has simple eigenvalues when $\mathrm{GCD}(g, n) = 1$, consequently any L-cospectral mate Γ cannot have cycles as components, as cycles carry eigenvalues of multiplicity two. If $\mathrm{GCD}(g, n) = 2$ there could be eigenvalues of multiplicity two but they belongs to, say, degenerate cycle (C_2, σ), which are not allowed. Hence, when $\mathrm{GCD}(g, n) \leq 2$, Γ must be connected. ■

Lemma 5.3.10 *Let $\psi(\Gamma, x) = \sum_{i=0}^{n} (-1)^n \, b_i(\Gamma) \, x^{n-1}$. Then we have:*

$$b_n(C_n, +) = 0, \quad b_n(C_n, \bar{\sigma}) = 4, \quad b_n(L_{n,g}, +) = 0, \quad b_n(L_{n,g}, \bar{\sigma}) = 4,$$

$$b_{n-1}(C_n, \sigma) = n^2, \quad b_{n-1}(L_{n,g}, +) = gn, \quad b_{n-1}(L_{n,g}, \bar{\sigma}) = gn + 2(n-g)(n-g+1).$$

Proof. The proof is a straightforward application of Theorem 5.1.1. ■

Theorem 5.3.11 *Let Γ be a L-cospectral mate of $(L_{g,n}, \bar{\sigma})$. Then Γ is connected.*

Proof. By Lemma 5.3.3, Γ is a disjoint union of a signed lollipop graph with possibly one or more signed cycles. In view of Theorem 5.2.5, since $(L_{g,n}, \bar{\sigma})$ is unbalanced then Γ cannot have any balanced component, which implies that Γ can possibly have just unbalanced cycles as components. However if Γ consists of $t \geq 2$ components, all of them unicyclic and unbalanced, then $b_n(\Gamma) = 4^t > 4 = b_n(L_{g,n}, \bar{\sigma})$, that is a contradiction. Hence, Γ is a connected graph. ■

Theorem 5.3.12 *Let Γ be a L-cospectral mate of $\Lambda = (L_{g,n}, +)$, with $d = GCD(g, n)$ an odd number. If $g \neq d$ and $n \neq 4d$, then Γ is connected.*

Proof. If $GCD(g, n) = d \leq 2$ the assertion is obviously true, so let $GCD(g, n) = d \geq 3$ for the remainder of the proof. If Γ is disconnected then Γ has one or more cycle as components. Let Λ' be the lollipop component of Γ

Assume first that Γ has an unbalanced cycle component, say, $(C_s, \bar{\sigma})$ with s even. If so, $\mu(s)$ is in the spectrum of Λ with multiplicity two, and s divides g and $n - g$, and consequently also d. But d is odd, so s cannot divide d, and Γ cannot have unbalanced cycles as component. So Γ has a positive cycle and Λ' is an unbalanced component. Assume next that $(C_r, +)$ is the positive cycle of Γ, then $\mu(r)$ is an eigenvalue of Γ with multiplicity two and the same applies to Λ. Similarly to above, r must divide d, so the only possibility is that $d = kr$ with k odd.

Let $k > 1$. Hence, Λ contains the eigenvalue $\mu(kr)$, which cannot belong to the component $(C_r, +)$. The latter implies that $\mu(kr)$ is in Λ', but Λ' has an unbalanced cycle $(C_{g'}, \bar{\sigma})$ as subgraph and $\mu(kr)$ cannot be an eigenvalue of $(C_{g'}, \bar{\sigma})$. So it is $k = 1$ and $r = d$.

By Lemma 5.3.10, we have that $b_{n-1}(\Gamma) = 4d^2 = gn = b_{n-1}(\Lambda)$. Since d divides both g and n, the latter equality implies that either $g = d$ and $n = 4d$, or $g = 2d$ and $n = 2d$, or $g = 4d$ and $n = d$. Clearly, it is $g < n$ and the only acceptable values are $g = d$ and $n = 4d$. Also, Λ' has order $n' = n - r = 3d$. The latter special case requires additional investigation, so we will consider it separately in a subsequent lemma. In all other cases, Γ must be a connected signed graph, hence it reduces to the lollipop component Λ'.

This completes the proof. ■

For the case d even, we need a more involved analysis due to the fact that $(C_{2n}, +)$ is not a DLS graphs. Hence we need more lemmas and results to show that Γ cannot be a disconnected graph.

Let B_n be the matrix of order n obtained from $L(P_{n+1})$ by deleting the row and column corresponding to some end-vertex of P_{n+1}. Let H_n be the matrix of order n obtained from $L(P_{n+2})$ by deleting the rows and columns corresponding to both the end vertices of P_{n+2} respectively. Both matrices represents augmented paths so their spectrum is not depending on the signature of the edges. The first two of the following equalities were given by Guo in [29], the third is proved in [50].

Lemma 5.3.13 *Let P_n be the path of order n and H_n, B_n defined as above. Then*

(i) $x\,\psi(B_n) = \psi(P_{n+1}) + \psi(P_n)$,

(ii) $\psi(P_n) = x\,\psi(H_{n-1})$,

(iii) $\psi(P_n) = (x-2)\psi(P_{n-1}) - \psi(P_{n-2})$.

We now express the L-polynomial of signed cycles and signed lollipop graphs in terms of the polynomials of paths. For a signed unicyclic graph Γ of girth g, let $\varsigma(\Gamma) = (-1)^{g+1}\sigma(\Gamma)$.

Lemma 5.3.14 *We have the following equalities*

$$\psi(C_n, \sigma) = \frac{\psi(P_{n+1})}{x} - \frac{\psi(P_{n-1})}{x} + 2\varsigma(C_n, \sigma),$$

and

$$\begin{aligned}\psi(L_{g,n}\bar{\sigma}) \;&=\; \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma(\Lambda)\Big] \\ &\quad - \frac{1}{x^2}(\psi(P_{n-g}) + \psi(P_{n-g-1}))\psi(P_g).\end{aligned}$$

Proof. The results can be easily obtained by iterated use of Theorem 5.1.9, combined with Lemma 5.3.13

In fact, for $\psi(C_n, \sigma)$, in view of Theorem 5.1.9, we have the following decomposition whose result depends on the parity of n and the value $\sigma(C_n, \sigma)$:

$$\begin{aligned}\psi(C_n, \sigma) \;&=\; \psi(H_n) - \psi(H_{n-2}) - 2(-1)^n \sigma(C_n, \sigma) \\ &=\; \frac{P_{n+1}}{x} - \frac{P_{n-1}}{x} + 2\varsigma(C_n, \sigma).\end{aligned}$$

A similar computation holds for $\Lambda = (L_{g,n}, \sigma)$:

$$\begin{aligned}
\psi(\Lambda) &= (x-3)\psi(H_{g-1})\psi(B_{n-g}) - 2\psi(H_{g-2})\psi(B_{n-g}) - \psi(H_{g-1})\psi(B_{n-g-1}) \\
&\quad -2(-1)^g\sigma(\Lambda)\psi(B_{n-g}) \\
&= \psi(B_{n-g})\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma(\Lambda)\Big] - \frac{1}{x}\psi(B_{n-g-1})\psi(P_g) \\
&= \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma(\Lambda)\Big] \\
&\quad -\frac{1}{x^2}(\psi(P_{n-g}) + \psi(P_{n-g-1}))\psi(P_g).
\end{aligned}$$

This completes the proof. ■

Lemma 5.3.15 *We have*

$\psi(P_n, 4) = 4n$; $\psi((C_{2n}, +), 4) = \psi((C_{2n+1}, \bar{\sigma}), 4) = 0$; $\psi((C_{2n+1}, +), 4) = \psi((C_{2n}, \bar{\sigma}), 4) = 4$;

$$\psi((L_{g,n}, +), 4) = \begin{cases} -4g(n-g), & g \text{ is even,} \\ -4[g(n-g) - (2n-2g+1)], & g \text{ is odd;} \end{cases}$$

$$\psi((L_{g,n}\bar{\sigma}), 4) = \begin{cases} -4[g(n-g) - (2n-2g+1)], & g \text{ is even,} \\ -4g(n-g), & g \text{ is odd.} \end{cases}$$

Proof.
The results can be easily obtained by Lemma 5.3.14. In fact, $\psi(P_1, 4) = 4$ and by induction $\psi(P_n, 4) = (4-2)\psi(P_{n-1}, 4) - \psi(P_{n-2}, 4) = 2(4n-4) - (4n-8) = 4n$.

$$\begin{aligned}
\psi((C_n, \sigma), 4) &= \frac{\psi(P_{n+1}, 4)}{4} - \frac{\psi(P_{n-1}, 4)}{4} + 2\varsigma(C_n, \sigma) \\
&= n + 1 - n + 1 + 2\varsigma(C_n, \sigma) = 2 + 2\varsigma(C_n, \sigma).
\end{aligned}$$

A similar computation holds for $\Lambda = (L_{g,n}, \sigma)$:

$$\begin{aligned}
\psi(\Lambda, 4) &= \frac{1}{4}(\psi(P_{n-g+1}, 4) + \psi(P_{n-g}, 4))\Big[\frac{4-3}{4}\psi(P_g, 4) - \frac{2}{4}\psi(P_{g-1}, 4) + 2\varsigma(\Lambda)\Big] \\
&\quad -\frac{1}{4^2}(\psi(P_{n-g}, 4) + \psi(P_{n-g-1}, 4))\psi(P_g, 4) \\
&= -4[g(n-g) + (2 + 2\varsigma(\Lambda))(2n-2g+1)].
\end{aligned}$$

This completes the proof. ■

Lemma 5.3.16 *Let $(L_{g,n}, \sigma) = \Lambda$ and $(L_{g',n'}, \sigma') = \Lambda'$ be two signed lollipop graphs such that $2n' \leq n$. Then $\mu_2(\Lambda) > \mu_2(\Lambda')$.*

Proof. Since the L-eigenvalues of Λ are interlaced by those of P_n, we have that $\mu_2(\Lambda) \geq \mu_2(P_n) \geq \mu_1(P_{n'}) \geq \mu_2(\Lambda')$. We next prove that it is $\mu_1(P_n) \neq \mu_2(\Lambda)$, so that the last inequality is strict.

Assume first that either g odd and $\sigma = +$, or g even and $\sigma = \bar{\sigma}$, then by (5.2) it is $\mu(P_n) > \mu_1((C_g, \sigma) \cup P_{n-g}) \geq \mu_2(\Lambda)$, that is true for $n > g$ (in the latter settings).

Consider next the case when either g is even and $\sigma = +$, or g odd and $\sigma = \bar{\sigma}$; in both cases the subdivision graph of $(L_{g,n}, \sigma)$ is $(L_{2g,2n}, +)$, since they both contain an even number of positive edges in the cycle. We show that $(L_{2g,2n}, +)$ does not have $\lambda_1(P_{2n-1}) =$

$\sqrt{\mu_1(P_n)} = \underline{\lambda}$ as an A-eigenvalue, which implies that $(L_{g,n}, \sigma)$ can not have $\mu_1(P_n)$ as an L-eigenvalue. Apply Theorem 5.1.9 to one vertex in the cycle of degree 2 adjacent to the vertex of degree 3. We have:

$$\phi((L_{2g,2n}, +), x) = x\phi(P_{2n-1}) - \phi(P_{2n-2}, x) - \phi(P_{2g-2}, x)[2 + \phi(P_{2n-2g}, x)],$$

from which we deduce that

$$\phi((L_{2g,2n}, +), \underline{\lambda}) = -\phi(P_{2n-2}, \underline{\lambda}) - \phi(P_{2g-2}, \underline{\lambda})[2 + \phi(P_{2n-2g}, \underline{\lambda})] < 0.$$

Consequently, $\mu_2(\Lambda) \geq \mu_1(P_{n'}) > \mu_2(\Lambda')$. This completes the proof. ■

Lemma 5.3.17 *Let Γ be a disconnected L-cospectral mate of $\Lambda = (L_{g,n}, +)$. If either $n \neq 2g$, or $n = 2g$ with g even, then $\mu_2(\Gamma)$ is not an eigenvalue of the cycle components of Γ.*

PROOF. Since Γ is disconnected than Γ has at least one cycle as component. Let $GCD(g, n) = d$, if $d \leq 2$, Γ must be connected, so we consider $d \geq 3$. Recall that d divides both g and $n - g$, hence $n = kd$ with $k \geq 2$. According to Corollary 5.3.8, if (C_r, σ) is a cycle of Γ, then r divides d; recall that $\mu_1(C_r, \sigma) < 4$. So r is at most $\frac{n}{2}$, and the latter equality is possible if and only if $g = n - g = d$, that is $n = 2g$. Assume that $n \neq 2g$, then $r < \frac{n}{2}$. Consequently, by interlacing, we have $\mu_2(\Gamma) = \mu_2(\Lambda) \geq \mu_2(P_n) > \mu_1(C_r, \sigma)$.

To complete the proof we need to consider the case $n = 2g$ and g even. So assume that Γ has one cycle of order g. Clearly, $\mu_1(C_g, +) = 4$, so we need to consider only $(C_g, \bar{\sigma})$. Observe that $\mu_1(C_g, \bar{\sigma}) = \mu_1 P_g = \mu_2(P_n)$. We will use a similar strategy to the one used in Lemma 5.3.16, in fact we will show that $\mu_1(P_g)$ is not an L-eigenvalue of $(L_{g,2g}, +)$ by showing that $\lambda_1(P_{2g-1})$ is not an A-eigenvalue of $(L_{2g,4g}, +)$. Let us use Theorem 5.1.9 at the vertex of degree 3, we then obtain

$$\phi(L_{2g,4g}, +) = x\phi(P_{2g})\phi(P_{2g-1}) - 2\phi(P_{2g})\phi(P_{2g-2}) - \phi^2(P_{2g-1}) - 2\phi(P_{2g}).$$

By computing the polynomials in $\lambda_1(P_{2g-1}) = \underline{\lambda}$ we have

$$\phi((L_{2g,4g}, +), \underline{\lambda}) = -2\phi(P_{2g}, \underline{\lambda})[\phi(P_{2g-2}, \underline{\lambda}) + 1] > 0,$$

since $\lambda_2(P_{2g}) < \underline{\lambda} < \lambda_1(P_{2g})$ and $\lambda_1(P_{2g-2}) < \underline{\lambda}$. The latter shows that indeed $\lambda_2(L_{g,2g}, +) > \mu_2(P_n) = \mu_1(P_g) = \mu_1(C_g, \bar{\sigma})$. This completes the proof. ■

We can finally prove the result below.

Theorem 5.3.18 *Let Γ be a L-cospectral mate of $\Lambda = (L_{g,n}, +)$, with $d = GCD(g, n)$ an even number. Then Γ is connected.*

PROOF. Since Λ is a balanced lollipop whose $GCD(g, n) = d$ is even, then by Theorem 5.3.7 the eigenvalues of multiplicity two for Λ are those of $[c(2k), +]$, for some number k equal to either d or $2d$. The even number $2k$ can be written in the form $2^{t+1}r$, where r is a positive odd number. We give the proof for $r \geq 3$, the case $r = 1$ can be solved similarly. By Lemma 5.3.6 we have that $(C_{2^{t+1}r}, +)$ is cospectral with $(C_{2^s r}, +) \bigcup_{i=s}^{t} (C_{2^i r}, \bar{\sigma})$ for any $0 \leq s \leq t$.

Let Γ be a disconnected tentative cospectral mate of Λ, and denote by $\Lambda' = (L_{g',n'}, \sigma)$ the lollipop component of Γ. In the sequel we show that Γ should be one of the two following signed graphs:

(i) $\Gamma = \Lambda' \cup (C_r, +) \bigcup_{i=0}^{t} (C_{2^i r}, \bar{\sigma})$;

(ii) $\Gamma = \Lambda' \bigcup_{i=s}^{t} (C_{2^i r}, \bar{\sigma})$.

From Γ being disconnected, we have that Γ has at least one cycle.

Assume that there is a balanced cycle $(C_q, +)$ among its components, then Λ' is unbalanced. The value q divides g and $n-g$ (see Corollary 5.3.8), but then it divides r as well, due to r being the greatest odd factor of $GCD(g, n-g)$. Evidently, $q = r$ otherwise if $q < r$, $\mu(r)$ cannot be an eigenvalue of $(C_q, +)$ or of Λ', and thus of Γ, while it appears in Λ. Also, Λ' must contain the eigenvalues of $[c(r), \bar{\sigma}]$ with the same multiplicity, since these eigenvalues cannot appear in some cycle component, and the latter implies that $GCD(g', n') = kr$, with k odd. However it is $k = 1$, otherwise Λ' contains the eigenvalues of multiplicity two of a longer odd unbalanced cycle whose eigenvalues do not appear in Λ. In addition g' is odd, otherwise g' is even, $\frac{g'}{r}$ is also even, and the eigenvalues of $[c(r), \bar{\sigma}]$ cannot appear in Λ' with multiplicity two. Now, since $GCD(g', n')$ is odd, then Λ' can not have the eigenvalues of unbalanced even cycles, necessary to complete the spectrum of $(C_{2^t r}, +)$, the latter implies that Γ must have an unbalanced even cycle for each necessary even multiple of r. Consequently, Γ is of type (i).

Assume next that Γ has not any balanced cycle as a component. In this case Λ' is balanced and it contains the eigenvalues of both $[c(r), +]$ and $[c(r), \bar{\sigma}]$ with multiplicity two, which implies Λ' has the eigenvalues of $[c(2r), +]$ with multiplicity two. The latter implies that $2r$ divides g', and g' must be even. Let $(C_{q'}, \bar{\sigma})$ the shortest unbalanced even cycle component of Γ. Clearly, q' must divide $2^t r$, but it must be of the form $2^s r$, where $s \geq 1$. In fact, let $q' = 2^s$ for some $2 \leq s \leq t$. If so, for any $s' \geq s$, neither Γ can have some cycle component $C(2^{s'} r, \bar{\sigma})$, as it would lead to common eigenvalues of multiplicity two among the cycle components, nor Λ' can have as eigenvalues of multiplicity two those of $[c(2^{s'} r), \bar{\sigma}]$, because then those of $[c(2^{s'+1} r), +]$ are in Λ' with multiplicity two and, due to $\frac{2^{s'+1} r}{2^{s'}} = 2r$ being even, we get that also the eigenvalues of $[c(2^{s'}), \bar{\sigma}]$ are eigenvalues for Λ' with multiplicity two, leading again to eigenvalues of multiplicity greater than two. Hence, q' must be of the form $2^s r$. If $s > 1$, the eigenvalue $\mu(2^{s'} r)$, with $0 \leq s' \leq s-1$ is in Λ and it must appear in Γ as well. Since $(C_{2^s r}, \bar{\sigma})$ does not contain the eigenvalues of $(C_{2^{s'} r}, \bar{\sigma})$ for $s' < s$, it implies that $\mu(2^{s'} r)$ cannot appear in some cycle component $(C_{2^{s'} r}, \bar{\sigma})$ (due to the minimality of s), so it must appear in Λ'. The latter implies that the eigenvalues of $[c(2^s r), +]$ appears with multiplicity two for Λ'. Now, for every $s \leq s' \leq t$ we have the cycle $(C_{2^{s'} r}, \bar{\sigma})$ is a component of Γ. If not, then some $\mu(2^{s'} r)$, with $s' > s$, appears in Λ' with multiplicity two, together with the eigenvalues $[c(2^{s'+1} r), +]$. Then $\mu(2^s r)$ appears in both Λ' and $(C_{2^s r}, \bar{\sigma})$, and the multiplicity of $\mu(2^s r)$ jumps to four, a contradiction. Hence, Γ is of the form (ii).

The next step is to show that both forms (i) and (ii) are not admissible for Γ, by comparing the spectral invariants b_{n-1} (cf. Lemma 5.3.10) and the polynomial computed at 4 (cf. Lemma 5.3.15). For Λ we have that $b_{n-1}(\Lambda) = gn$ and $\psi(\Lambda, 4) = -4g(n-g)$.

Assume first that Γ is of type (i). Recall that Λ' is unbalanced and g' is odd. In this case, we have that $b_{n-1}(\Gamma) = 4^{t+1} r^2$ and $\psi(\Gamma, 4) = -4^{t+2}(g'(n'-g')$. So we get the system

$$\begin{cases} gn = 4^{t+1} r^2; \\ -4g(n-g) = -4^{t+2} g'(n'-g'). \end{cases}$$

from which we get that $g^2 = 4^{t+1}(r^2 - g'(n'-g'))$ (recall that $n' > g'$). Clearly, the quantity $r^2 - g'(n'-g')$ must be positive, that is $r^2 > g'(n'-g')$ but the latter inequality has no solutions since r divides both g' and $n'-g'$. Hence Γ is not of type (i).

Assume now that Γ is of type (ii). Recall that Λ' is balanced and g' is even. In this case we have that $b_{n-1}(\Gamma) = 4^{t-s+1} g'n'$ and $\psi(\Gamma, 4) = -4^{t-s+2} g'(n'-g')$. Now we obtain the system

$$\begin{cases} gn = 4^{t-s+1} g'n'; \\ -4g(n-g) = -4^{t-s+2} g'(n'-g'). \end{cases}$$

whose solutions are $g = 2^{t-s+1}g'$ and $n = 2^{t-s+1}n'$. The latter equality implies that $n \geq 2n'$ and by Lemmas 5.3.16 and 5.3.17, we obtain that $\mu_2(\Lambda) > \mu_2(\Gamma)$. Hence, Γ is not of type (ii).

If $r = 1$, an analogous proof holds, in which $\Gamma = (L_{g',n'}, +) \bigcup_{i=s}^{t} (C_{2^i}, \bar{\sigma})$, with $s \geq 2$ being the shortest length of the unbalanced cycle component of Γ. We leave the details to the reader.

This completes the proof. ■

Theorem 5.3.19 *No two non switching isomorphic signed lollipop graphs are L-cospectral.*

PROOF. Let $\Lambda = (L_{g,n}, \sigma)$ be a signed lollipop graph. In Lemma 5.3.15 we have decomposed the L-polynomial of Λ in the combination of paths polynomials.

$$\begin{aligned}\psi(\Lambda, x) \quad &= \quad \frac{1}{x}(\psi(P_{n-g+1}) + \psi(P_{n-g}))\Big[\frac{x-3}{x}\psi(P_g) - \frac{2}{x}\psi(P_{g-1}) + 2\varsigma\Big] \\ &\quad -\frac{1}{x^2}\psi(P_g)(\psi(P_{n-g}) + \psi(P_{n-g-1})). \qquad (5.3)\end{aligned}$$

Consider Lemma 5.3.13 (iii), the formula $\psi(P_n) = (x-2)\psi(P_{n-1}) - \psi(P_{n-2})$ can be seen as a homogeneous second order recurrence equation

$$p_n = (x-2)p_{n-1} - p_{n-2},$$

with $p_0 = 0$ and $p_1 = x$ as boundary conditions. It is a matter of computation (cf. [50] for the details) to check that the solution is

$$p_n = \frac{(y^{2n}-1)(y+1)}{y^n(y-1)},$$

where y is the solution of the characteristic equation $y^2 - (x-2)y + 1 = 0$.

For any signed graph Γ, let

$$\Phi(\Gamma) = y^n\,(y-1)^2\,\psi(\Gamma, y) - (y^{2n+2} - 2y^{2n+1} - 2y + 1),$$

then, by applying the above described transformation to (5.3), we get

$$\Phi(L_{g,n}, \sigma) = 2\varsigma y^{2n-g+2} - 2\varsigma y^{2n-g+1} + y^{2n-2g+2} + y^{2g} - 2\varsigma y^{g+1} + 2\varsigma y^g. \qquad (5.4)$$

From the above polynomial, it is evident that two signed lollipop are L-cospectral if and only if both g and $\sigma(\Lambda)$ are the same, namely, the two signed lollipop graphs are also switching equivalent. This completes the proof. ■

By using the comparison technique of the above theorem, we now deal with the last case, left by Theorem 5.3.12.

Lemma 5.3.20 *The signed graphs $(L_{d,4d}, +)$ and $(L_{g',3d}, \bar{\sigma}) \cup (C_d, +)$, with d odd, are not L-cospectral.*

PROOF. We shall compare the polynomials and check whether we obtain compatible values for g'.

From Lemma 5.3.15, we can take the polynomial of the odd balanced cycle $(C_d, +)$, that is

$$\psi(C_d, +) = \frac{\psi(P_{d+1})}{x} - \frac{\psi(P_{d-1})}{x} + 2.$$

Let $\varsigma' = \varsigma(L_{g',3d})$. After some computations we get for $\Gamma = (L_{g',3d}, \bar{\sigma}) \cup (C_d, +)$ the below polynomial

$$\begin{aligned}\Phi(\Gamma) \;=\; & 2\varsigma' y^{8d-g'+2} - 2\varsigma' y^{8d-g'+1} + y^{8d-2g+2} + 4\varsigma' y^{7d-g'+2} - 4\varsigma' y^{7d-2g'+2} \\ & +2\varsigma' y^{6d-g'+2} - 2\varsigma' y^{6d-g'+1} + y^{6d-2g'+2} + y^{6d+2} - 2y^{6d+1} + y^{2d+2g} \\ & -2\varsigma' y^{2d+g'+1} + 2\varsigma' y^{2d+g'} - 2y^{2d+1} + y^{2d} + 2y^{d+2g} - 4\varsigma' y^{d+g'+1} \\ & +4\varsigma' y^{d+g'+1} + 4\varsigma' y^{d+g'} + y^{2g'} - 2\varsigma' y^{g'+1} + 2\varsigma' y^{g} - 4y^{d+1} + 2y^{d}.\end{aligned}$$

For the ease of comparison, we also write the polynomial corresponding to $\Lambda = (L_{d,4d}, +)$. Recall that d is odd and $\sigma = +$, hence $\varsigma = 1$.

$$\Phi(L_{d,4d}, +) \;=\; 2y^{7d+2} - 2y^{7d+1} + y^{6d+2} + y^{2d} - 2y^{d+1} + 2y^{d}.$$

We are going to compare the lowest degree monomials of both the above polynomials. For $\Phi(\Lambda)$ it is $2y^d$, while for $\Phi(\Gamma)$ we have three candidates, namely $y^{6d-2g'+2}$, $2\varsigma y^{g'}$ and $2y^d$. Since the polynomial must be the same, we deduce that $g' > d$ and $g' < \frac{1}{2}(5d+2)$. If we look at the monomials of degree $d+1$, we have for $\Phi(\Lambda)$ that it is $2y^{d+1}$. So $\Phi(\Gamma)$ should have the same monomial, and the only possibility is that $g' = d+1$ and $\varsigma' = 1$. But with the latter substitution the two polynomials do not coincide. Hence, Γ can not be cospectral with Λ.

This completes the proof. ■

We can finally state the main result of this section.

Theorem 5.3.21 *The signed lollipop graph $(L_{g,n}, \sigma)$ is determined by the spectrum of its Laplacian matrix.*

Proof. Let Γ be a tentative L-cospectral mate of $(L_{g,n}, \sigma) = \Lambda$. According to Theorem 5.3.3, Γ is a signed lollipop graph with possibly one or more signed cycles. If $\sigma(\Lambda) = \bar{\sigma}$, by Theorem 5.3.11 we get that Γ is connected, and it reduces to a signed lollipop graph. If $\sigma(\Lambda) = +$ by Theorems 5.3.12 and 5.3.18, we get that, excluding the special case $n = 4g$ and $\sigma = +$, the tentative cospectral mate Γ is connected, and it reduces to a signed lollipop graph. By Theorem 5.3.19, if Γ is a signed lollipop graph, then it is switching isomorphic to $(L_{g,n}, \sigma)$. The remaining special case is considered in Lemma 5.3.20 and it leads to non cospectral graphs.

This completes the proof. ■

From Theorem 5.2.6 (iii), we deduce that the (signed) subdivisions of signed lollipop graphs that A-cospectral mates cannot be subdivision graphs. Since the subdivision of lollipop graph is a lollipop graph with even order and even girth not less than 6, the following corollary holds:

Corollary 5.3.22 *Let Γ be A-cospectral with a signed lollipop graph $(L_{2g,2n}, \sigma)$, where $g \geq 3$. If Γ is a subdivision graph, then Γ is switching isomorphic to $(L_{g,2n}, \sigma)$.*

Chapter 6

Signed graphs whose second largest Laplacian eigenvalue does not exceed 3

Results of this chapter are published in [8].

6.1 Signed graphs whose second largest Laplacian eigenvalue does not exceed 3

In this section we determine the family of connected signed graphs whose second largest L-eigenvalue does not exceed 3. Observe that in view of Interlacing theorem, the latter property is hereditary (or, monotone), in fact if $\mu_2(\Gamma) \leq 3$ then $\mu_2(\Gamma') \leq 3$ for any Γ' subgraph of Γ. On the other hand, if a signed graph Γ has $\mu_2(\Gamma) > 3$ then the same holds for any Γ' containing Γ. In order to better deal with this problem, we will consider two subsections one for $n \leq 6$ and the other for $n \geq 7$.

6.1.1 Signed graphs with $\mu_2 \leq 3$ and $n \leq 6$

In this subsection we investigate the signed graphs with the property $\mu_2 \leq 3$ and $n \leq 6$. In Fig 6.1 we depict the connected signed graphs on at most 6 vertices which are forbidden for the property $\mu_2 \leq 3$, it is routine to verify that each of them has $\mu_2 > 3$.

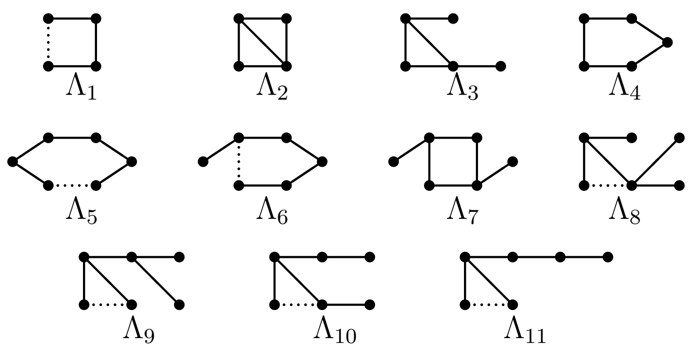

Figure 6.1: Forbidden subgraphs for the property $\mu_2 \leq 3$.

In Fig 6.2 we depict the maximal signed graphs for the property $\mu_2 \leq 3$ and order equal 6 (to be proved later in this section). The triangles in the signed graph Γ_5 can be balanced or unbalanced, in fact Γ_5 represents three non-switching equivalent signed graphs on the same underlying graph. The maximality is easy to check, in fact the addition of an edge (any sign) leads to some (switching isomorphic) forbidden subgraph given in Fig 6.1.

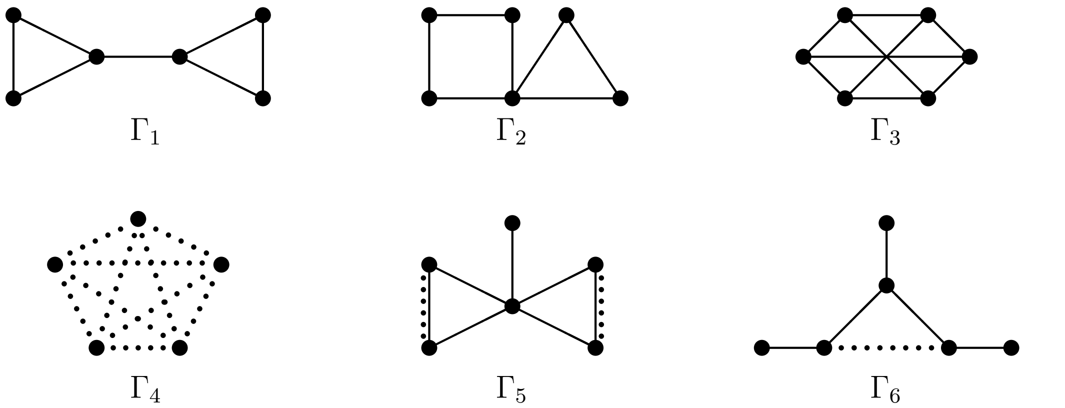

Figure 6.2: Maximal signed graphs of order 6 for the property $\mu_2 \leq 3$.

We are now ready to prove that the graphs depicted in Fig 6.1 are indeed all forbidden subgraphs for $n \leq 6$, while those in Fig 6.2 are all maximal graphs for the property $\mu_2 \leq 3$ and $n \leq 6$. To prove the latter claims, we proceed by case analysis.

Theorem 6.1.1 *Let Γ be a signed graph such that $\mu_2(\Gamma) \leq 3$ and $n \leq 6$. Then Γ is a subgraph of one of the graphs $\Gamma_1 - \Gamma_6$.*

Proof. Without loss of generality we can restrict to connected signed graphs. It is routine to check that any tree with at most 6 vertices has $\mu_2 \leq 3$, and any such a tree is a subgraph of some signed graph among $\Gamma_1 - \Gamma_6$. Therefore, in the remainder of the proof we will focus on signed graphs containing at least one cycle. We split the proof by the order and the girth. The case $n \leq 4$ is left to the reader.

Assume first that $n = 5$. Since $\Gamma_4 = (K_5, -)$, any signed graph on 5 vertices whose odd cycles are unbalanced and even cycles are balanced has a signature equivalent to the all negative signature, and hence $\mu_2 \leq 3$. Hence, we just need to consider signed graphs with at least an unbalanced even cycle or a balanced odd cycle. But $(C_4, \bar{\sigma}) = \Lambda_1$ and $(C_5, +) = \Lambda_4$ are forbidden, so the only possibility is that the signed graph contains at least one odd balanced triangle. In the latter case, (a) if the signed graph contains just a (balanced) triangle then it is a subgraph of either Γ_1 or Γ_5, any other configuration leads to Λ_3, (b) if the signed graphs contains two triangles (one being balanced), then it is a subgraph of Γ_5, otherwise one between Λ_1 and Λ_2 appears as a subgraph.

Assume next that $n = 6$. Recall that Λ_1, Λ_4, Λ_5 and Λ_6 are forbidden, so we do not have unbalanced cycles of length ≥ 4 or positive pentagons. The latter means that balanced or unbalanced triangles, balanced quadrangles or balanced hexagons are allowed.

Let the girth be 3. Consider first that an unbalanced triangle appears, then $(C_4, +)$, $(C_6, +)$ and $(C_5, -)$ cannot appear in the signed graph since they lead to forbidden configurations Λ_6 and Λ_8–Λ_{11}. Possibly, another triangle is in the signed graph, but they have to share one vertex otherwise Λ_9 arises, hence the signed graph is a subgraph of Γ_5. Finally, if exactly one unbalanced triangle appears, then in view of Λ_8–Λ_{11}, then the signed graph is a subgraph of Γ_5 or Γ_6. Consider next that a balanced triangle appears, due to the previous case, we can assume that no unbalanced triangles appear in the graph. If there is a balanced quadrangle, then they only share a vertex, otherwise Λ_2 or Λ_3 appear, hence we arrive to Γ_2. If there is a balanced hexagon, then Λ_3 again arises. To conclude this subcase, if only positive triangles are allowed, then, in order to avoid Λ_3, only subgraphs of Γ_1 or Γ_5 are allowed.

Now, let the girth be 4. If so, since Λ_7 is forbidden then the signed graph must be a subgraph of Γ_2 or Γ_3. If the girth is 5 then Λ_6 appears and we are done. Finally, if the girth is 6, then the unique possible signed graph is $(C_6, +)$ that is subgraph of Γ_3.

This completes the proof. ■

6.1.2 Signed graphs with $\mu_2 \leq 3$ and $n \geq 7$

In this subsection we consider the connected signed graphs with the property $\mu_2 \leq 3$ and $n \geq 7$. Aouchiche et al. in [3] proved that the trees depicted in Fig 6.3 have their second largest (signless) Laplacian eigenvalue greater than 3. On the other hand, for trees the spectral theory of signed and unsigned graphs is just the same, so these trees are also forbidden for the property $\mu_2 \leq 3$ and $n \geq 7$.

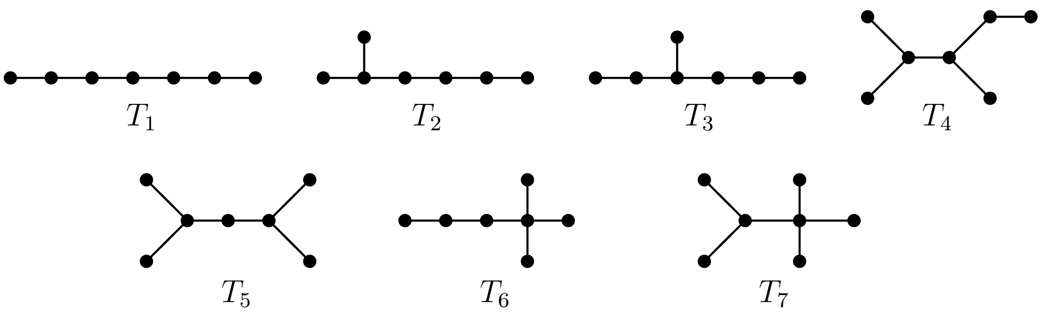

Figure 6.3: Forbidden trees for the property $\mu_2 \leq 3$.

In the same paper, it is also proved that the trees of Fig 6.3 are all the minimal forbidden graphs against the property $\mu_2 \leq 3$ and $n \geq 7$. The family of (unsigned) graphs which does not contain some tree from Fig 6.3 consists of triangles and paths of length at most 2 all sharing a common vertex. Alternatively, we can say that the diameter is less than 5, at most one vertex of degree greater than 2 is allowed and its eccentricity is at most 2. Aouchiche et al. in [3] called the above family *firefly graphs*. Clearly, the latter result can be interpreted in the context of signed graphs as valid for firefly graphs, whose triangles are all unbalanced. However, we will later show that also balanced triangles are allowed, and we arrive to the class of *signed firefly graphs*, whose triangles can be balanced or unbalanced.

For this purpose, let us define a notation for signed firefly graphs. A signed firefly graph $F(s,t,p,q)$ is the signed graph consisting of s balanced triangles and t unbalanced triangles (possibly, $s = t = 0$) all of them sharing exactly a vertex v, and at v we have p pendant vertices and q pendant paths of length 2. If $q = 0$, then the signed graph $F(s,t,p,0) = B(s,t,p)$ will be called a signed butterfly graph. If both $p = q = 0$, then $F(s,t,0,0) = \mathcal{F}(s,t)$ will be called a signed friendship graph. Finally, if $s = t = 0$, then $F(0,0,p,q) = S(p,q)$ is a stretched star, or simply a star when $q = 0$. Observe that the cycles are edge-disjoint, hence the signature is univocally determined by the number of balanced and unbalanced triangles. In Fig 6.4 we depict examples of signed firefly, butterfly and friendship graphs.

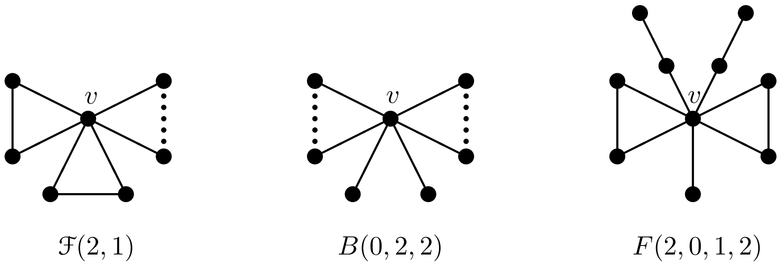

Figure 6.4: Signed firefly graphs.

In the following lemma we discuss the connected signed graphs with maximum degree $\Delta \leq 2$. The proof is omitted.

Lemma 6.1.2 *For the path P_n and the cycle (C_n,σ) we have: (a) $\mu_2(P_n) \leq 3$ if and only if $n \leq 6$, (b) $\mu_2(C_n,\sigma) \leq 3$ only for $(C_3,\pm)$, $(C_4,+)$, $(C_5,-)$ and $(C_6,+)$.*

In view of the above lemma, in the sequel we will restrict to graphs with $\Delta \geq 3$. We now give the analogous result of Aouchiche et al. in terms of signed graphs (cf. Theorem 2.5 in [3] and Theorem 3.5 in [39]).

Theorem 6.1.3 *Let Γ be a connected signed graph on $n \geq 7$ vertices with $\mu_2(\Gamma) = \mu_2 \leq 3$. Then $\Gamma = F(s,t,p,q)$, and in particular*

(1) $\mu_2 = 1$ *if and only if* $\Gamma = F(0,0,p,0)$;

(2) $\dfrac{3+\sqrt{5}}{2} - \dfrac{1}{n} < \mu_2 < \dfrac{3+\sqrt{5}}{2}$ *if and only if* $\Gamma = F(0,0,p,1)$;

(3) $\mu_2 = \dfrac{3+\sqrt{5}}{2}$ *if and only if* $\Gamma = F(0,0,p,q)$ *and* $q > 1$;

(4) $3 - \dfrac{5}{2n} < \mu_2 < 3$ *if and only if* $\Gamma = F(0,1,p,q)$;

(5) $\mu_2 = 3$ *if and only if* $\Gamma = F(s,t,p,q)$, *with* $s \neq 0$ *or* $t \geq 2$.

Proof. Recall that the Laplacian of trees is switching similar to the Laplacian of signed trees, hence the proof of items regarding trees can be taken from [3]; the same applies also when $s = 0$, since the Laplacian of $F(0,t,p,q)$ is switching similar to the signless Laplacian of the unsigned graphs. Hence, we need to check the items in the statement in the other cases. It can be done by computing the Laplacian polynomial of signed firefly graphs. For this purpose, we will make use of Theorem 5.1.9. Recall, from $L = D - A$, the Laplacian of a signed graph can be interpreted as the adjacency matrix of a signed graph with signature reversed and loops (which represent the degrees). Since the signature is reversed, then balanced triangles will give negative contribution and unbalanced triangles will give positive contributions.

To show the procedure, we first compute the characteristic polynomial of the signed friendship graph $\mathcal{F}(s,t)$. The graph corresponding to the Laplacian of $L(\mathcal{F}(s,t)) = D(\mathcal{F}(s,t)) - A(\mathcal{F}(s,t))$ will have loops of "weight" $\deg(v_i)$, positive edges become negative edges and viceversa, balanced triangles and unbalanced triangles will have weight -1 and $+1$, respectively. By keeping the latter in mind, we have that the L-polynomial of $\mathcal{F}(s,t)$ can be decomposed by the use of (5.1.9) at the vertex v (of degree $2s+2t$).

$$\psi(\mathcal{F}(s,t),x) = (x^3 - 2x^2(s+t+2) + 3x(2s+2t+1) - 4t)((x-1)(x-3))^{s+t-1}. \quad (6.1)$$

In view of the above polynomial we can say that 1 and 3 are eigenvalues of multiplicity at least $s+t-1$. Let $f_{s,t}(x) = f(x)$ be the polynomial of third degree in (6.1). It is easy to check that:

$$f(0) = -4t, \qquad f(1) = 4s, \qquad f(3) = -4t.$$

Hence, when $s,t > 0$ the polynomial $\psi(\mathcal{F}(s,t),x)$ has one eigenvalue in the real interval $(0,1)$, 1 is an eigenvalue of multiplicity $s+t-1$, there is an eigenvalue in the real interval $(1,3)$, 3 is an eigenvalue of multiplicity $s+t-1$, and there is an eigenvalue above 3. So the spectrum of $\mathcal{F}(s,t)$ is the following (the eigenvalues are listed in a non-decreasing order):

$$\mathrm{Spec}(\mathcal{F}(s,t)) = \{x_1, 1^{(s+t-1)}, x_2, 3^{(s+t-1)}, x_3\},$$

where the x_i's are the roots of the polynomial $f_{s,t}$. We still have to consider the cases in which either $s=0$ or $t=0$. It is worth mentioning that for $t=0$, the signed friendship graph $\mathcal{F}(s,0)$ is switching equivalent to the unsigned friendship graph; while for $s=0$, the the Laplacian matrix of $\mathcal{F}(0,t)$ is switching similar to the signless Laplacian of $\mathcal{F}(t,0)$.

Their L-spectra are:

$$\mathrm{Spec}(\mathcal{F}(s,0)) = \{0, 1^{(s-1)}, 3^{(s)}, 2s+1\},$$

$$\mathrm{Spec}(\mathcal{F}(0,t)) = \{1^{(t)}, \frac{1}{2}(2t+3-\sqrt{4t^2-4t+9}), 3^{(t-1)}, \frac{1}{2}(2t+3+\sqrt{4t^2-4t+9})\}.$$

Note that $\mathrm{Spec}(\mathcal{F}(s,t))$ has at most 5 distinct eigenvalues.

Similarly for the signed butterfly graph $F(s,t,p,0) = B(s,t,p)$ we have

$$\psi(B(s,t,p),x) = (x-1)^{s+t+p-1}(x-3)^{s+t-1}g_{p,s,t}(x), \quad (6.2)$$

where $g_{s,t,p}(x) = g(x) = x^3 - x^2(p+2(s+t+2)) + 3x(p+2s+2t+1) - 4t)$. Let y_1, y_2, y_3 be the three roots of $g(x)$. Since

$$g(0) = -4t, \qquad g(1) = 2p+4s, \qquad g(3) = -4t,$$

we deduce that the spectrum of the signed butterfly graph is the following:

$$\mathrm{Spec}(B(s,t,p)) = \{y_1, 1^{(p+s+t-1)}, y_2, 3^{(s+t-1)}, y_3\}.$$

Note that for $t=0$, we obtain an integral signed graph

$$\mathrm{Spec}(B(s,0,p)) = \{0, 1^{(p+s+t-1)}, 3^{(s)}, 2s+p+1\}.$$

Consider next $F(s,t,p,q)$, with $(s,t) \neq (0,0)$, $t>0$ and $q \geq 1$. By applying Theorem 5.1.9 at the vertex v, we get

$$\psi(F(s,t,p,q),x) = (x-1)^{p+s+t-1}(x-3)^{s+t-1}(x^2-3x+1)^{q-1}\, h_{p,s,t,q}(x), \quad (6.3)$$

where $h_{s,t,p,q}(x) = h(x) = x^5 - x^4(p+q+2s+2t+7) + 2x^3(3p+3q+2(3s+3t+4)) - x^2(10p+11q+20s+24t+13) + 3x(p+2q+2s+6t+1) - 4t$. Let $z_1, z_2 \dots, z_5$ be the roots of $h(x)$. Since

$$h(0) = -4t, \quad h\Big(\frac{3-\sqrt{5}}{2}\Big) = q, \quad h(1) = -2p-4s, \quad h\Big(\frac{3+\sqrt{5}}{2}\Big) = q, \quad h(3) = -4t,$$

we deduce that the spectrum of the signed firefly graph is the following:

$$\mathrm{Spec}(F_{s,t,p,q}) = \{z_1, \Big(\frac{3-\sqrt{5}}{2}\Big)^{(q-1)}, z_2, 1^{(p+s+t-1)}, z_3, \Big(\frac{3+\sqrt{5}}{2}\Big)^{(q-1)}, z_4, 3^{(s+t-1)}, z_5\}.$$

Also in this case, for $t = 0$ we have a slightly different spectrum:

$$\mathrm{Spec}(F_{s,0,p,q}) = \{0, \Big(\frac{3-\sqrt{5}}{2}\Big)^{(q-1)}, u_1, 1^{(s+p-1)}, u_2, \Big(\frac{3+\sqrt{5}}{2}\Big)^{(q-1)}, 3^{(s)}, u_3\},$$

where u_1, u_2 and u_3 are the roots of $x^3 - x^2(p+q+2(s+2)) + x(3p+3q+2(3s+2)) - p - 2q - 2s - 1$.

Finally, we have to consider the firefly with no triangles $F_{0,0,p,q}$. Using the same procedure, we get the following spectrum for $q \geq 1$

$$\mathrm{Spec}(F_{0,0,p,q}) = \{0, \frac{3-\sqrt{5}}{2}^{(q-1)}, w_1, 1^{(p-1)}, w_2, \frac{3+\sqrt{5}}{2}^{(q-1)}, w_3\},$$

where the w_i's are the roots of $x^3 - x^2(p+q+4) + x(3p+3q+4) - p - 2q - 1$.

To complete the proof, we just need to show that any signed graph whose underlying graph is not a firefly graph, then it contains as a subgraph one of the forbidden trees given in Fig 6.2. Let Γ be a connected signed graph of order $n \geq 7$ with $\mu_2(\Gamma) \leq 3$ and $\Delta \geq 3$. If the diameter of Γ is at least 5, then one among T_1, T_2 and T_3 appear as a subgraph. Hence the diameter of Γ is 4 or less. If Γ has two vertices of degree at least 3, then one among T_4, T_5 and T_7 appear as a subgraph. Hence, there is only one vertex of degree at least 3 and the diameter is at most 4. Cycles of length ≥ 4 can be discarded since they lead to T_3, hence only triangles are allowed. Finally, the unique vertex of degree greater than 2 must have eccentricity 2 otherwise T_6 appears as a subgraph.

Therefore, Γ is a signed firefly graph. This completes the proof. ∎

From the polynomials computed in the previous proof we can deduce the multiplicities of the eigenvalues 3 and 1.

Lemma 6.1.4 *Let $F(s,t,p,q)$ be a signed firefly graph. Then*

a) 3 is an eigenvalue of multiplicity either s for $t = 0$, or $s+t-1$ for $t > 0$;

b) 1 is an eigenvalue of multiplicity $s+t+p-1$.

Now we have a complete characterization of signed graphs whose second largest L-eigenvalue does not exceed 3. Of course, this characterization includes and extends those given in [3, 39, 48].

Theorem 6.1.5 *Let Γ be a connected signed graph such that $\mu_2(\Gamma) \leq 3$. Then Γ is a subgraph of Γ_1–Γ_6 or Γ is a signed firefly graph. The minimal forbidden graphs against the property $\mu_2 \leq 3$ are the signed graphs Λ_1–Λ_{11} and the trees T_1–T_7.*

6.2 Spectral determination of signed firefly graphs

In the previous section we have identified the (connected) signed graphs whose second largest L-eigenvalue does not exceed 3, and by excluding the exceptions on at most 6 vertices, we find that such signed graphs are indeed signed firefly graphs. In this final section we study their spectral determination with respect to the Laplacian matrix. For this purpose, we need some additional notation and preliminary results.

Definition 6.2.1 *A signed graph $\Gamma = (G, \sigma)$ is determined by the spectrum, or the eigenvalues, of its matrix $M(\Gamma)$ if and only if any other signed graph $\Lambda = (H, \sigma')$ such that $M(\Lambda)$ has the same spectrum of $M(\Gamma)$ implies that Γ and Λ are two switching isomorphic graphs. If so, we say that* Γ is determined by the spectrum of the matrix M, *or Γ is a DMS graph.*

If Λ is not switching isomorphic to Γ, we say that the two graphs are *M-cospectral*, or, equivalently, Λ is a *M-cospectral mate* of Γ. The matrix suffix M- will be omitted if clear from the context.

The following theorem is proved in [7] and it provides some basic results for L-cospectral signed graphs. Note that the second item is the well-known fact that for connected signed graphs the least eigenvalues is 0 if and only if the graph is balanced (see [51]); a study on the least eigenvalue can be found in [5].

Theorem 6.2.2 *Let $\Gamma = (G, \sigma)$ and $\Lambda = (H, \sigma')$ be two Laplacian cospectral signed graphs, and let t_Γ^+ (t_Γ^-) be the number of balanced (resp., unbalanced) triangles in Γ. Then*

(i) Γ and Λ have the same number of vertices and edges;

(ii) Γ and Λ have the same number of balanced components;

(iii) Γ and Λ have the same Laplacian spectral moments;

(iv) Γ and Λ have the same sum of squares of degrees, $\sum_{i=1}^n d_G(v_i)^2 = \sum_{i=1}^n d_H(v_i)^2$;

(v) $6(t_\Gamma^- - t_\Gamma^+) + \sum_{i=1}^n d_G(v_i)^3 = 6(t_\Lambda^- - t_\Lambda^+) + \sum_{i=1}^n d_H(v_i)^3$.

Friendship graphs have been very studied in the literature. So far it has been proved that friendship graphs are determined by the spectrum of the Laplacian and of the signless Laplacian [41, 48]. For the adjacency matrix we have that the friendship is determined by its spectrum unless the number of triangles equals sixteen [21]. Here, we show that, for n enough large, signed friendship graphs and, in general, signed firefly graphs are determined by the spectrum of their Laplacian matrix, which extends the results given in [3, 39, 48].

Lemma 6.2.3 *Let Γ be a cospectral mate of $F(s,t,p,q)$, with maximum degree $\Delta = 2s + 2t + p + q \geq 9$. Then Γ is connected.*

Proof. Assume that Γ is disconnected, so $\Gamma = \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_k$ where Σ_i is connected and $\mu_1(\Sigma_i) \geq \mu_1(\Sigma_j)$ for $1 \leq i < j \leq k$. Since $2s + 2t + p + q \geq 9$, then $K_{1,9}$ is a subgraph and $\mu_1(F(s,t,p,q)) \geq 9$. So Σ_1 can not be a subgraph of Γ_1–Γ_6, since their maximum spectral radius is $8 = \mu_1(K_5, -)$. Hence in the remainder of the proof, $\Sigma_1 = F(s',t',p',q')$. From $\mu_2(F(s,t,p,q)) \leq 3$ we have $\mu_1(\Sigma_i) \leq 3$ for all $i \geq 2$. In view of Theorem 5.1.7, we know that $\mu_1(\Sigma_i) \leq 3$ if and only if $\Sigma_i \in \{K_1, P_2, P_3, (C_3, +)\}$, consequently Σ_i is a balanced signed graphs for any $i \geq 2$. On the other hand $F(s,t,p,q)$ and Γ share the same number of balanced components (cf. Theorem 6.2.2 (ii)), which implies that for $t \neq 0$ the signed firefly $F(s,t,p,q)$ is unbalanced. Therefore Γ has no balanced components it must be connected.

Hence, it remains to consider $F(s,0,p,q)$. The latter is balanced, so $\Gamma = \Sigma_1 \cup \Sigma_2$, with $\Sigma_1 = F(s',t',p',q')$ unbalanced ($t' > 0$) and $\Sigma_2 \in \{K_1, P_2, P_3, (C_3, +)\}$.

Case 1. $s > 0$.

To discard the remaining cases, we consider the multiplicity of 1's and 3's in the spectrum of $F(s, 0, p, q)$. In view of Lemma 6.1.4, we get that the multiplicity of 1 is $p+s-1$ and the multiplicity of 3 is s (note, $t = 0$). We consider several subcases based on Σ_2.

Assume that $\Sigma_2 = (C_3, +)$, then $\Gamma = F(s', t', p', q') \cup (C_3, +)$ for some $t' > 0$. In view of Theorem 6.2.2 (i), both signed graphs share the same order and the size. So we deduce that $s = s' + t'$ (cf. the cyclomatic number), but the latter implies that Γ has 3 as an eigenvalue of multiplicity $s + 1$, namely $s - 1$ from Σ_1 and 2 from Σ_2. So $(C_3, +)$ is discarded.

Similarly to the previous case, by letting $\Sigma_2 = P_3$, we obtain that for Γ the multiplicity of 3 is $s + 1$, and P_3 is discarded as well.

Assume that $\Sigma_2 = P_2$, now Γ contains 2 as an eigenvalue. However $\psi(F(s, 0, p, q), 2) = -2(2s + p - 1)$, which implies that 2 does not belong to the spectrum of $F(s, 0, p, q)$.

Finally, let $\Sigma_2 = K_1$, so $\Gamma = F(s', t', p', q')$. By Theorem 6.2.2 (i), and by equating the multiplicity of 1, we obtain that $s' + t' = s + 1$, $p' = p - 1$ and $q' = q - 1$. Now, we have the degree sequence of Γ, that is $\{2s + p + q, 2^{(2s+q+1)}, 1^{(p+q-2)}, 0\}$. By Theorem 6.2.2 (iv), the sum of squares of degrees of vertices from $F(s, 0, p, q)$ and Γ differs by 2, and we deduce that the two signed graphs cannot be cospectral.

Case 2. $s = 0$.

In this case $F(0, 0, p, q)$ is a tree. In view of Theorem 6.1.3 (1) and (2), $\mu_2(F(0, 0, p, q)) \leq \frac{3+\sqrt{5}}{2} < 3$, so Σ_2 can be neither P_3 nor $(C_3, +)$, since their spectral radius is 3. Hence Σ_2 is a tree, and consequently $\Sigma_1 = F(s', t', p', q')$ is an unbalanced unicyclic graph. The only possibility is that $\Sigma_1 = F(0, 1, p', q')$, but, in view of Theorem 6.1.3 (4), we get $\mu_2(\Gamma) > 3 - \frac{5}{2n} > \frac{3+\sqrt{5}}{2} \geq \mu_2(F(0, 0, p, q))$, and again they can not be cospectral.

Hence Γ is a connected signed graph. This completes the proof. ∎

In the next theorem we discuss the spectral determination of signed friendship graphs.

Theorem 6.2.4 *Let $\mathcal{F}(s, t)$ be a signed friendship graph of order $n \geq 9$. Then $\mathcal{F}(s, t)$ is determined by the spectrum of its Laplacian matrix.*

Proof. Let Γ be a tentative cospectral mate of $\mathcal{F}(s, t)$. In view of Lemma 6.2.3, Γ is a connected signed graph. Furthermore, $\mu_1(\mathcal{F}(s, t)) \geq 9$ and $\mu_2(\mathcal{F}(s, t)) \leq 3$, hence, by Theorem 6.1.5, Γ is a signed firefly graph $F(s', t', p', q')$. From Theorem 6.2.2 (i) Γ has the same cyclomatic number of $\mathcal{F}(s, t)$, that is $p' = q' = 0$ and $\Gamma = \mathcal{F}(s', t')$ with $s' + t' = s + t$.

Assume first that $t = 0$ and $s \geq 4$. If so, $\mathcal{F}(s, 0)$ is a balanced graph with just 4 different integral eigenvalues and $\mu_2 = 3$ with multiplicity s. If $t' > 0$ then $\mu_2(\Gamma) = 3$ with multiplicity $s + t - 1$, and they cannot be cospectral. Hence, $t' = 0$ and $s' = s$, and they are switching equivalent.

Assume finally, that $t \neq 0$ and $s + t \geq 4$. If $\mathcal{F}(s, t)$ and $\Gamma = \mathcal{F}(s', t')$ are cospectral, then the polynomials $f_{s',t'}$ and $f_{s,t}(x)$ must be the same. By equating the least coefficient we get $t' = t$. Hence, also $s' = s$, which implies Γ being switching equivalent to $\mathcal{F}(s, t)$.

This completes the proof. ∎

Theorem 6.2.5 *Let $B(s, t, p) = F(s, t, p, 0)$ be a signed butterfly graph with $s+t > 0$, $p > 0$ of order $n \geq 9$. Then $B(s, t, p)$ is determined by the spectrum of its Laplacian matrix.*

Proof. Let Γ be a tentative cospectral mate of $B(s, t, p)$. In view of Lemma 6.2.3, Γ is a connected signed graph. Furthermore, $\mu_1(B(s, t, p)) \geq 9$ and $\mu_2(B(s, t, p)) \geq 3 - \frac{5}{2n}$, hence, by Theorem 6.1.5, Γ is a signed firefly graph $F(s', t', p', q')$ with $s' + t' > 0$.

From Theorem 6.2.2 (i) and Lemma 6.1.4, we get the following equalities:

$$\begin{cases} 2s' + 2t' + p' + 2q' = 2s + 2t + p \\ s' + t' = s + t \\ s' + t' + p' - 1 = s + t + p - 1 \end{cases}$$

whose solution is $s' + t' = s + t$, $p' = p$ and $q' = 0$. Hence, $\Gamma = B(s', t', p)$. By comparing their cubic polynomials $g_{s,t,p}$ and $g_{s',t',p}$ we see that $t' = t$ (cf. the free term), and then $s' = s$. So Γ is switching equivalent to $B(s, t, p)$. ■

Theorem 6.2.6 *Let $F(s, t, p, q)$ be the signed firefly graph with $s + t > 0$, $q > 0$ and $s + t + p + q \geq 9$. Then $F(s, t, p, q)$ is determined by the spectrum of its Laplacian matrix.*

Proof. Let Γ be a tentative cospectral mate of $F(s, t, p, q)$. By Lemma 6.2.3 and by the restrictions on the order, we get $\Gamma = F(s', t', p', q')$. We again consider the system coming from equating their orders, sizes, and multiplicities of the eigenvalues 1 and $\frac{3+\sqrt{5}}{2}$:

$$\begin{cases} 2s' + 2t' + p' + 2q' = 2s + 2t + p + 2q \\ s' + t' = s + t \\ s' + t' + p' - 1 = s + t + p - 1 \\ q' - 1 = q - 1 \end{cases}$$

Hence, $s' + t' = s + t$ $p' = p$ and $q' = q$. To get that $t' = t$ and $s' = s$, it is enough to compare the free term of the polynomials $h_{s,t,p,q}$ in Theorem 6.1.3. And we conclude that Γ is switching equivalent to $F(s, t, p, q)$. ■

It remains to show that signed fireflies $F(s, t, p, q)$ with $s = t = 0$ are also determined by their Laplacian spectra. However, in view of Lemma 6.2.3, a cospectral mate of $F(0, 0, p, q)$ must be connected, which implies that we are reducing to both the L-theory or Q-theory of unsigned graphs. So, the spectral determination of $F(0, 0, p, q)$ can be taken from [3], and we will not prove it again. Finally, by collecting the the above results, we can state the main result of this section.

Theorem 6.2.7 *Let $F(s, t, p, q)$ be a signed firefly graph with $2s + 2t + p + q \geq 9$. Then $F(s, t, p, q)$ is determined by the spectrum of its Laplacian matrix.*

The above theorem can be extended to the remaining cases. Unfortunately, we were not able to compare the spectra of signed graphs with $\mu_2 \leq 3$ and order at most 6. It is worth mentioning that some pair of cospectral non switching isomorphic graphs (in fact, with the all negative signature) has been already found (cf. H_9 and H_{10} in [48]), so it would not be surprising that some further pair arises from the more general settings of signed graphs.

Chapter 7

Conclusions

In the thesis a special case of a long standing conjecture saying that the necessary conditions are also sufficient ones for Hamiltonian decomposition of complete uniform hypergraphs has been solved. Recently D. Kühn and D. Osthus in [38] solved this problem completely. The next step in the research of Hamiltonian decompositions should be checking other families of hypergraphs.

Classifying the full automorphism groups of a family of double generalized Petersen graphs increased our knowledge about non-Cayley-vertex transitive graphs. There are at least two ways how these results might be extended. We can consider a triple, quadro, penta, etc. generalized Petersen graphs define as follows.

Definition 7.0.8 *Given an integer* $n \geq 3$ *and* $t_i \in \mathbb{Z}_n \setminus \{0\}$*,* $2 \leq 2t_i < n$*, where* $1 \leq i \leq s$*, the multi generalized Petersen graph* $\mathrm{MP}(n, t_1, \ldots, t_s)$ *is defined to have the vertex set* $\{x_i^j, \mid i \in \mathbb{Z}_n \;\; j \in \mathbb{Z}_{2s+2}\}$ *and the edge set the union* $\Omega \cup \Sigma \cup I$*, where*

$\Omega = \{\{x_i^0, x_{i+1}^0\}, \{x_i^{2s+1}, x_{i+1}^{2s+1}\} \mid i \in \mathbb{Z}_n\}$ *(the outer edges),*

$\Sigma = \{\{x_i^{2j}, x_i^{2j+1}\}, \mid i \in \mathbb{Z}_n, \;\; j \in \mathbb{Z}_{s+1}\}$ *(the spokes), and*

$I = \{\{x_i^{2k+1}, x_{i\pm t}^{2k+2}\}, \mid i \in \mathbb{Z}_n, \;\; k \in \mathbb{Z}_s\}$ *(the inner edges).*

Based on computer-assisted checks we are posting the following three conjectures which stimulate further research on the topic.

Conjecture 7.0.9 *Let* $\Gamma = \mathrm{MP}(n, t_1, \ldots, t_s)$*. If* n *and* s *are even and* $t_k \in \{1, \frac{n}{2} - 1\}$ *for any* $k \in \{1, \ldots, s\}$ *then* Γ *is vertex-transitive.*

Conjecture 7.0.10 *Let* $\Gamma = \mathrm{MP}(n, t_1, \ldots, t_s)$*. If* n *is even,* s *is odd, and* $t_k^2 = \pm 1 \pmod{n}$ *and* $t_k = \pm t_{k'} \pmod{\frac{n}{2}}$ *for odd* $k, k' \in\in \{1, \ldots, s\}$ *and* $t_k \in \{1, \frac{n}{2} - 1\}$ *for even* $k \in\in \{1, \ldots, s\}$ *then* Γ *is vertex-transitive.*

The main idea in proving these conjectures is to find an automorphism of the graph in question fixing spokes (as a set) and interchanging cycles on the set $\Omega \cup I$. First, one can see that $\alpha : x_i^j \mapsto x_{i+1}^j$ is an automorphism of the graph $\mathrm{MP}(n, t_1, \ldots, t_s)$. Second, observe that $\mathrm{MP}(n, t_1, \ldots, t_k, \ldots, t_s) \cong \mathrm{MP}(n, t_1, \ldots, \frac{n}{2} - t_k, \ldots, t_s)$. Third, one should prove that a mapping δ defined by

$$\begin{aligned}\delta : \;\; & x_{2i}^0 \mapsto x_{2it}^1, \; x_{2i+1}^0 \mapsto x_{(2i+1)t}^2, \; x_{2i}^2 \mapsto x_{2it}^3, \; x_{2i+1}^2 \mapsto x_{(2i+1)t}^0, \\ & x_{2i}^1 \mapsto x_{2it}^0, \; x_{2i+1}^1 \mapsto x_{(2i+1)t}^3, \; x_{2i}^3 \mapsto x_{2it}^2, \; x_{2i+1}^3 \mapsto x_{(2i+1)t}^4, \\ \ldots . \end{aligned}$$

is an automorphism interchanging cycles. And finally, it needs to be proven that the group $\langle \alpha, \delta \rangle$ is acting transitively on $\mathrm{MP}(n, t_1, \ldots, t_s)$.

Conjecture 7.0.11 *The sufficient conditions for* $\mathrm{MP}(n, t_1, \ldots, t_s)$ *being vertex-transitive are the same as necessary ones.*

One the other hand, we can treat the double generalized Petersen graphs as a special case of the so-called *split rose window graphs* (see Figure 7.1). First steps in this research direction is already in consideration in collaboration with B. Frelih, K. Kutnar, T. Pisanski.

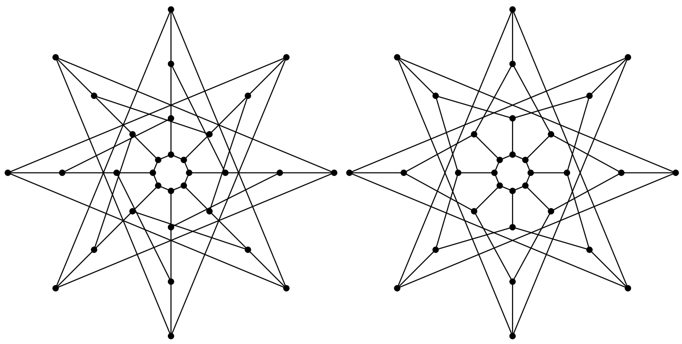

Figure 7.1: The split rose window graph $\mathrm{SW}_1(8, 1, 2, 3)$.

Definition 7.0.12 *Given an integer* $n \geq 3$ *and* $a, k, b \in \mathbb{Z}_n \setminus \{0\}$*, the split rose window graph of type 1* $\mathrm{SW}_1(n, a, k, b)$ *is defined to have vertex set* $\{x_i, u_i, v_i, y_i |\ i \in \mathbb{Z}_n\}$ *and edge set the union* $\Omega \cup \Sigma \cup I$*, where*

$$\Omega = \{\{x_i, x_{i+a}\}, |\ i \in \mathbb{Z}_n\} \cup \{\{y_i, y_{i+b}\}, |\ i \in \mathbb{Z}_n\} \text{ (the outer edges),}$$

$$\Sigma = \{\{x_i, u_i\}, |\ i \in \mathbb{Z}_n\} \cup \{\{y_i, v_i\}, |\ i \in \mathbb{Z}_n\} \text{ (the spokes), and}$$

$$I = \{\{u_i, v_i\}, |\ i \in \mathbb{Z}_n\} \cup \{\{u_i, v_{i+k}\}, |\ i \in \mathbb{Z}_n\} \text{ (the inner edges).}$$

The theorems that are proven in the signed spectral part of the thesis are generalization of an analogues theorems for standard graphs. Further attempts to generalized result to signed graph theory and creating new theorems are being already made.

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Povzetek v slovenskem jeziku

DOLOČENI RAZREDI (HIPER)GRAFOV IN NJIHOVE ALGEBRIČNE LASTNOSTI

Disertacija povezuje različna področja teorije grafov, s posebnim poudarkom na algebraični teoriji grafov, predstavi rešitve določenih odprtih problemov, kot so dekompozicija polnih hipergrafov na hamiltonske cikle, in razširi rezultate teorije grafov na teorijo predznačnih grafov. V disertaciji se še posebej posvetimo sledečim odprtim problemom:

(i) Kateri hipergrafi premorejo dekompozicijo na hamiltonske cikle?

(ii) Kako najti celotno grupo avtomorfizmov dvojno posplošenega Petersenovega grafa?

(iii) Kaj lahko karakteriziramo predznačne lizika grafe glede na njihove Laplaceove lastne vrednosti?

(iv) Je mogoče karakterizirati vse predznačne grafe z majhno drugo Laplaceovo lastno vrednostjo?

Teoretična izhodišča

Hipergraf $H = (V, E)$ je urejeni par dveh množic, množice točk $V = V(H) = \mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ in množice hiperpovezav $E = E(H) = \{e_0, e_1, \ldots, e_m\}$, kjer je $e_i \subseteq V$. Če velja $|e_i| = k$ za vse $e_i \in E$, rečemo, da je H k-*uniformen* hipergraf. *Popoln* k-*uniformen hipergraf* na n točkah vsebuje kot povezave vse k-podmnožice množice $\{0, 1, \ldots, n-1\}$. Takšen hipergraf označujem s K_n^k. Obstajajo različna pojmovanja Hamiltonskih ciklov v hipergrafih; vsa so veljavne posplošitve standardnega pojma v običajnih grafih. V tem delu smo se osredotočili na Bergevo pojmovanje [10]. *Hamiltonski cikel* v k-uniformnem hipergrafu H je zaporedje $(x_0, e_0, x_1, e_1, \ldots, x_{n-1}, e_{n-1}, x_0)$, kjer so $x_0, x_1, \ldots, x_{n-1}$ točke hipergrafa H in $e_0, e_1, \ldots, e_{n-1}$ takšne hiperpovezave hipergrafa H, da velja $x_i, x_{i+1} \in e_i$, $0 \le i \le n-1$, kjer operiramo z indeksi točk modulo n in kjer je $e_i \neq e_j$ za $i \neq j$. Rečemo, da premore hipergraf $H = (V, E)$ *dekompozicijo* na Hamiltonske cikle, če obstaja takšna družina Hamiltonskih ciklov $\mathcal{C} = \{C_1, C_2, \ldots, C_h\}$, da je $E(C_i) \cap E(C_j) = \emptyset$ za $i \neq j$ in $\bigcup E(C_i) = E(H)$.

Leta 1884 je Walecki dokazal, da za liha cela števila n popoln graf K_n^2 premore Hamiltonsko dekompozicijo, torej particijo povezav grafa na Hamiltonske cikle, medtem ko za soda števila n premore popoln graf K_n^2 dekompozicijo na popolno prirejanje in Hamiltonske cikle (glej [15]). Za $k = 3$ je Bermond [11] prikazal Hamiltonsko dekompozicijo popolnega 3-uniformnega hipergrafa K_n^3 za $n \equiv 2 \pmod 3$ in $n \equiv 4 \pmod 6$. Verrall [46] je kasneje dopolnil rešitev za primer $n \equiv 1 \pmod 6$ in dokazal, da v primeru $n \equiv 0 \pmod 3$ hipergraf $K_n^3 - I$, kjer je I popolno prirejanje, premore Hamiltonsko dekompozicijo. Kühn in Osthus [38] sta dokazala obstoj Hamiltonske dekompozicije Bergovega tipa za poljubna n in k. Podoben problem na popolnih grafih sta obravnavala Buratti in Del Fra [16]. V disertaciji smo predstavili potrebne in zadostne pogoje za obstoj *ciklične* Hamiltonske dekompozicije hipergrafa K_n^k za poljubna k in n, torej dekompozicije $\mathcal{C} = \{C_1, C_2, ..., C_h\}$, kjer je vsak cikel $C_i \in \mathcal{C}$, $i \in \{1, 2, \ldots, h\}$ Hamiltonski in obstaja permutacija σ točk grafa K_n^k, ki premore natanko eno takšno ciklično dekompozicijo, da se množica točk vsakega cikla $C_i \in \mathcal{C}$ ujema z neko orbito $\langle\sigma\rangle$, ki deluje na množici točk grafa K_n^k.

Posplošeni Petersenovi grafi $\mathrm{GP}(n, k)$, ki jih je prvi predstavil Coxeter v [18], so naravna posplošitev dobro znanega Petersenovega grafa. Naj bosta $n \geq 3$ in $k \in \mathbb{Z}_n \setminus \{0\}$, $2 \leq 2k < n$. Potem je posplošeni Petersenov graf $\mathrm{GP}(n, k)$ definiran z množico točk $\{u_i, v_i | \ i \in \mathbb{Z}_n\}$ in množico povezav $\Omega \cup \Sigma \cup I$, kjer je $\Omega = \{\{u_i, u_{i+1}\}, | \ i \in \mathbb{Z}_n\}$ (zunanje povezave), $\Sigma = \{\{u_i, v_i\}, | \ i \in \mathbb{Z}_n\}$ (špice) in $I = \{\{v_i, v_{i+k}\}, | \ i \in \mathbb{Z}_n\}$ (notranje povezave). Naravna posplošitev posplošenih Petersenovih grafov so dvojni posplošeni Petersenovi grafi $\mathrm{DP}(n, t)$, ki so bili prvič predstavljeni v [57] kot primeri točkovno tranzitivnih ne-Cayleyjevih grafov. Naj bosta $n \geq 3$ in $t \in \mathbb{Z}_n \setminus \{0\}$, $2 \leq 2t < n$. Potem je dvojni posplošeni Petersenov graf $\mathrm{DP}(n, t)$ definiran z množico točk $\{x_i, y_i, u_i, v_i | \ i \in \mathbb{Z}_n\}$ in množico povezav $\Omega \cup \Sigma \cup I$, kjer je $\Omega = \{\{x_i, x_{i+1}\}, \{y_i, y_{i+1}\} \mid i \in \mathbb{Z}_n\}$ (zunanje povezave), $\Sigma = \{\{x_i, u_i\}, \{y_i, v_i\} \mid i \in \mathbb{Z}_n\}$ (špice) in $I = \{\{u_i, v_{i+t}\}, \{v_i, u_{i+t}\} \mid i \in \mathbb{Z}_n\}$ (notranje povezave).

Ta disertacija predstavi podatke o povezavi med strukturnimi lastnostmi dvojnih posplošenih Petersenovih grafov in strukturnimi lastnostmi posplošenih Petersenovih grafov [4, 17]. Ogledali smo si tudi Hamiltonskost, barvanje točk ter barvanje povezav dvojnih posplošenih Petersenovih grafov. Dokazali smo, da vsak graf $\mathrm{DP}(2n, t)$ premore Hamiltonski cikel, medtem ko je za grafe $\mathrm{DP}(2n+1, t)$ obstoj Hamiltonskih ciklov dokazan samo za primer, ko je t generator grupe $\mathbb{Z}_{2n+1}$. Vsak graf $\mathrm{DP}(2n, t)$ je dvodelen, torej za barvanje točk zadostujeta dve barvi, medtem ko so za graf $\mathrm{DP}(2n+1, t)$ potrebne tri barve. Za konec poglavja pokažemo —še še, da med dvojnimi posplošenimi Petersenovimi grafi ni snarkov, to je povezan kubični graf brez mostov, katerega kromati čno število povezav je enako 4.

Naj bo graf $\Gamma = (G, \sigma)$ predznačen graf, kjer je G vpeti enostavni graf in preslikava $\sigma : E(G) \to \{+, -\}$ predznačna preslikava na povezavah grafa G. V literaturi so enostavni grafi proučevani preko lastnih vrednosti številnih matrik, povezanih z grafi. Ena najbolj proučevanih matrik je sosednostna matrika $A(G) = (a_{ij})$, kjer je $a_{ij} = 1$, ko sta točki i in j sosedni, in $a_{ij} = 0$ sicer, poleg Laplaceove oziroma Kirchoffove matrike $L(G) = D(G) - A(G)$, kjer je matrika $D(G) = \mathrm{diag}(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ diagonalna matrika s stopnjami točk. V zadnjih letih je še ena matrika pritegnila dosti pozornosti raziskovalcev; tako imenovana Laplaceove matrika brez predznakov, definirana kot $Q(G) = A(G) + D(G)$. Matrike lahko priredimo tudi predznačnim grafom. Sosednostno matriko $A(\Gamma) = (a_{ij}^\sigma)$, kjer je $a_{ij}^\sigma = \sigma(ij)a_{ij}$, imenujemo (predznačna) sosednostna matrika in matrika $L(\Gamma) = D(G) - A(\Gamma)$ je pripadajoča Laplaceova matrika. Tako sosednostna kot Laplaceova matrika sta realni, simetrični matriki, torej so njune lastne vrednosti realne. V disertaciji smo si ogledali problem spektralne karakterizacije, ki ga razširimo na sosednostno in Laplaceovo matriko predznačnih grafov. Preučimo spektralno determiniranost predznačnih lizika grafov in dokažemo, da je vsak predznačni lizika graf determiniran s spektrom svoje Laplaceove matrike. Drugi problem spektralne teorije grafov, kateremu se posvetimo v disertaciji, je Laplaceova teorija predznačnih grafov. Osredotočimo se na predznačne grafe, katerih druga največja Laplaceova lastna vrednost je relativno majhna. V literaturi se najde podobne raziskave. Včasih je motivacija za raziskavo aplikativna [35] in številni raziskovalci so proučevali strukturo (ne-predznačnih) grafov z majhno drugo največjo lastno vrednostjo neke predpisane matrike grafa [25, 30, 40, 44]. Nedavno [9] so avtorji posvetili več luči na opazko, da je spektralna teorija predznačnih grafov naravna posplošitev spektralne teorije enostavnih grafov, še posebej, če upoštevamo Laplaceovo teorijo predznačnih grafov. Laplaceova teorija predznačnih grafov posplošuje tako spektralno teorijo Laplaceovih, kot ne-predznačnih Laplaceovih matrik grafov in lahko ponudi kakšno razlago za pojave z nepredvidljivim vzorcem vedenja (na primer, v [9] za ekstremne grafe glede na magnitudo koeficientov Laplaceovega polinoma). Po drugi strani so bili grafi z majhno drugo največjo Laplaceovo in ne-predznačno Laplaceoavo lastno vrednostjo, ki ne presega 3, proučevani tako za Laplaceove kot ne-predznačne Laplaceove grafe [3, 39, 48]. Torej se je naravno vprašati, kako bi se lahko naredilo revizijo teh rezultatov v širšem kontekstu Laplaceovih matrik predznačnih grafov. Omembe vredno je tudi, da med drugim srečamo (predznačne) grafe prijateljstva, kjer so trikotniki lahko uravnoteženi ali neuravnoteženi. Obravnavamo tudi problem spektralne determiniranosti pri predznačnih grafih (glej tudi [7]). Ta problem je običajno težji za predznačen kot ne-predznačen primer. Laplaceov spekter predznačnih grafov pravzaprav sploh ne loči med povezanimi in nepovezanimi predznačnimi grafi. V disertaciji karakteriziramo in identificiramo vse predznačne grafe, katerih druga največja Laplaceova lastna vrednost ni večja od 3, in proučimo problem spektralne determiniranosti za primer predznačnih kresnica grafov. Natančneje, dokažemo, da so skoraj vsi predznačni kresnica grafi določeni s spektrom Laplaceove matrike.

Ciklična Hamiltonska dekompozicija popolnih k-uniformnih hipergrafov

Posvetimo se določanju zadostnih in potrebnih pogojev za obstoj ciklične Hamiltonske dekompozicije hipergrafa K_n^k za poljubni celi števili k in n. V disertaciji dokažemo sledeča izreka.

Theorem 7.0.13 *Naj bosta k in n takšni pozitivni celi števili, da je k tuje številu n in da je $\lambda k > n$, kjer je λ najmanjši ne-trivialni delitelj števila n. Potem k-uniformni hipergraf K_n^k premore ciklično Hamiltonsko dekompozicijo.*

Theorem 7.0.14 *Naj bosta k in n takšni pozitivni celi števili, da k-uniformni hipergraf K_n^k premore ciklično Hamiltonsko dekompozicijo. Potem je k tuje številu n in $\lambda k > n$, kjer je λ najmanjši ne-trivialni delitelj števila n.*

Avtomorfizmi in strukturne lastnosti dvojno posplošenih Petersenovih grafov

Rešujemo problem karakterizacije celotnih grup avtomorfizmov dvojno posplošenih Petersenovhi grafov $\mathrm{DP}(n,t)$. Preverjamo, kateri so hamiltonski, iščemo vrednosti kromatičnega števila $\chi(\mathrm{DP}(n,t))$, barvni indeks $\chi'(\mathrm{DP}(n,t))$ in odkrivamo, če med grafi $\mathrm{DP}(n,t)$ obstajajo snarki. V disertaciji so dokazani sledeči izreki (definicije permutacij $alpha, \beta, \gamma, \lambda, \delta, \eta\rangle$ se nahajajo na straneh 12, 13).

Theorem 7.0.15 *Grupa avtomorfizmov $A(n,t)$ dvojno posplošenega Petersenovega grafa $\mathrm{DP}(n,t)$ je določena na sledeči način:*

(i) *Če sta $n \equiv 0 \pmod 2$, $4t = n$ in $(n,t) \neq (4,1)$, potem je $A(n,t) = \langle \alpha, \beta, \gamma, \eta \rangle$.*

(ii) $A(4,1) = \langle \alpha, \beta, \gamma, \delta, \eta \rangle$.

(iii) *Če sta $n \equiv 0 \pmod 2$, $t^2 \equiv \pm 1 \pmod n$ in $(n,t) \neq (10,3)$, potem je $A(n,t) = \langle \alpha, \beta, \gamma, \delta \rangle$.*

(iv) $A(10,3) = \langle \alpha, \delta, \lambda \rangle$, *kjer je*

$$\lambda = (x_1,u_0)(x_2,v_3)(x_3,y_3)(x_4,y_4)(x_5,y_5)(x_6,v_5)(x_7,u_8)(u_1,v_7)$$
$$(u_2,u_6)(u_3,y_2)(u_4,v_4)(u_5,y_6)(u_7,v_1)(v_0,y_1)(v_2,v_6)(v_8,y_7).$$

(v) *Če sta* $n \equiv 2 \pmod 4$, $t^2 \equiv k \pm 1 \pmod n$, *kjer je* $n = 2k$ *in* $(n,t) \neq (10,2)$, *potem je* $A(n,t) = \langle \alpha, \beta, \gamma, \psi \rangle$.

(vi) $A(10,2) = \langle \alpha, \psi, \mu \rangle$.

(vii) *Če sta* $n \equiv 0 \pmod 4$ *in* $t^2 \equiv k \pm 1 \pmod n$, *kjer je* $n = 2k$, *potem je* $A(n,t) = \langle \alpha, \beta, \gamma, \phi \rangle$.

(viii) $A(5,2) \cong S_5$.

(ix) *V vseh primerih, različnih od zgoraj naštetih osmih, velja, da je* $A(n,t) = \langle \alpha, \beta, \gamma \rangle$.

Theorem 7.0.16 *Vsak dvojno posplošeni Petersenov graf* $\mathrm{DP}(2n,t)$ *premore hamiltonski cikel. Vsak dvojno posplošeni Petersenov graf* $\mathrm{DP}(2n+1,t)$, *kjer velja* $\mathbb{Z}_{2n+1} = \langle t \rangle$, *premore hamiltonski cikel.*

Theorem 7.0.17 *Kromatično število dvojno posplošenega Petersenovega grafa* $\mathrm{DP}(n,t)$ *je enako* 2, *če je* n *sodo število, oziroma enako* 3, *če je* n *liho število. Barvni indeks dvojno posplošenega Petersenovega grafa* $\mathrm{DP}(n,t)$ *je enak* 3.

Spektralna karakterizacija predznačnih lizika grafov

Ugotavljamo, katere lastnosti lahko razberemo iz koeficientov Laplaceovega polinoma predznačnega grafa. Predstavimo določene dokaze, uporabne za problem spektralne karakterizacije. Proučujemo tudi Laplaceovo spektralno karakterizacije predznačnih lizika grafov. V disertacije dokažemo sledeče izreke, pri čemer so definicije pojmov podane v poglavjih 2.3 in 5.3.

Theorem 7.0.18 *Naj bo graf* Γ *Laplaceov kospekterski par grafa* $(L_{g,n}, \bar{\sigma})$. *Potem je graf* Γ *povezan.*

Theorem 7.0.19 *Naj bo graf* Γ *Laplaceov kospekterski par grafa* $\Lambda = (L_{g,n}, +)$, *kjer je* $d = gcd(g,n)$ *sodo število. Potem je graf* Γ *povezan.*

Theorem 7.0.20 *Noben par ne-izmenljivih izomorfnih lizika grafov ni Laplaceov kospekterski.*

Theorem 7.0.21 *Predznačni lizika graf* $(L_{g,n}, \sigma)$ *je determiniran s spektrom svoje Laplaceove matrike.*

Predznačni grafi, katerih druga največja Laplaceova lastna vrednost ne presega 3

Karakteriziramo in identificiramo vse predznačne grafe, katerih druga največja Laplaceova lastna vrednost ne presega 3. Proučujemo problem spektralne determinacije za predznačne kresnica grafe. Pokažemo, da so skoraj vsi predznačni grafi prijateljstva določeni s spektrom Laplaceove matrike. Vseh definicij tu ne bomo ponavljali, nahajajo pa se v angleškem jeziku v poglavju 2.3 in na straneh $45 - 47$.

Theorem 7.0.22 *Naj bo graf* Γ *povezan predznačen graf na* $n \geq 7$ *točkah in naj bo* $\mu_2(\Gamma) = \mu_2 \leq 3$. *Potem je* $\Gamma = F(s,t,p,q)$ *in velja sledeče:*

(1) $\mu_2 = 1$, *če in samo če* $\Gamma = F(0,0,p,0)$;

(2) $\frac{3+\sqrt{5}}{2} - \frac{1}{n} < \mu_2 < \frac{3+\sqrt{5}}{2}$, *če in samo če* $\Gamma = F(0,0,p,1)$*;*

(3) $\mu_2 = \frac{3+\sqrt{5}}{2}$, *če in samo če* $\Gamma = F(0,0,p,q)$ *in* $q > 1$*;*

(4) $3 - \frac{5}{2n} < \mu_2 < 3$, *če in samo če* $\Gamma = F(0,1,p,q)$*;*

(5) $\mu_2 = 3$, *če in samo če* $\Gamma = F(s,t,p,q)$, *kjer* $s \neq 0$ *ali* $t \geq 2$.

Theorem 7.0.23 *Naj bo graf* $\mathcal{F}(s,t)$ *predznačen graf prijateljstva reda* $n \geq 9$. *Potem je graf* $\mathcal{F}(s,t)$ *določen s spektrom svoje Laplaceove matrike.*

Theorem 7.0.24 *Naj bo graf* $F(s,t,p,q)$ *predznačen kresnica graf, kjer je* $s+t > 0$, $q > 0$ *in* $s+t+p+q \geq 9$. *Potem je graf* $F(s,t,p,q)$ *določen s spektrom svoje Laplaceove matrike.*

Metodologija

Osnovna orodja, uporabljena v raziskovanju, segajo od kombinatoričnih in algebraičnih metod v teoriji grafov do povsem abstraktnih premislekov v sklopu abstraktne in permutacijske teorije grafov. Skozi celoten potek raziskovanja smo za testiranje rezultatov uporabljali računalniški program Magma [13].

Pri problemu ciklične Hamiltonske dekompozicije popolnih k-uniformnih hipergrafov smo uporabili metode iz [11], [15], [46] v povezavi z grupnimi delovanji, kar nam je omogočilo rešitev problema ciklične dekompozicije.

Za dvojno posplošene Petersenove grafe smo uporabili posplošeno verzijo Fruchtove metode [28] za karakterizacijo celotne grupe avtomorfizmov, medtem ko so kombinatorične konstrukcije dale odgovor glede obstoja hamiltonskih ciklov, kromatičnega števila in kromatičnega indeksa.

Pred kratkim so se avtorji [9] posvetili izražanju koeficientov L-polinoma predznačnih grafov in podali izraz, ki prikaže L-polinom preko A-polinoma določenih drugih predznačnih grafov. Na te rezultate smo gledali s stališča, da določajo nekatere spektralne invariante, in nadalje proučevali njihov vpliv na kombinatorično strukturo predznačnih grafov. Proučevali smo problem Laplaceove spektralne determinacije za skupino predznačnih grafov, poznanih kot predznačni lizika grafi, in dokazali, da je vsak predznačen lizika graf določen z lastnimi vrednostmi svoje Laplaceove matrike. Pozornost smo posvetili tudi predznačnim grafom, katerih druga največja Laplaceova lastna vrednost je dokaj majhna. Podobne raziskave so bile obravnavane v literaturi, pri čemer je bila včasih motivacija aplikativna [35], in mnogi raziskovalci so raziskovali strukturo (ne-predznačnih) grafov z majhno drugo največjo lastno vrednostjo katere od pripadajočih matrik grafa [25, 30, 40, 44].

Declaration

I declare that this thesis does not contain any materials previously published or written by another person except where due reference is made in the text.

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