UNIVERSITY OF PRIMORSKA
FACULTY OF MATHEMATICS, NATURAL SCIENCES AND INFORMATION TECHNOLOGIES

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ALGEBRAIC ASPECTS OF GRAPH THEORY

PhD Thesis

Koper, June 2013

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Abstract

ALGEBRAIC ASPECTS OF GRAPH THEORY

This thesis contains number of different topics in algebraic graph theory, touching and resolving some open problems that have been a center of research interest over the last decade or so. More precisely, the following open problems are considered in this thesis:

(i) Which graphs are (strongly) quasi m-Cayley graphs?

(ii) Which bicirculants are arc-transitive graphs?

(iii) Are there generalized Cayley graphs which are not Cayley graphs, but are vertex-transitive?

(iv) Are there snarks amongst Cayley graphs.

(v) Are there graphs admitting half-arc-transitive group actions with small number of alternets with respect to which they are not tightly attached?

Problem (i) is solved for circulants and $m \in \{2, 3, 4\}$. Problem (ii) is completely solved for pentavalent bicirculants. Problem (iii) is answered in the affirmative by constructing two infinite families of vertex-transitive non-Cayley generalized Cayley graphs. The graphs in the families are all bicirculants. Problem (iv) is solved for those $(2, s, t)$-Cayley graphs whose corresponding $2t$-gonal graphs are prime-valent arc-transitive bicirculants. The main step in obtaining this solution is the proof that the chromatic number of any prime-valent arc-transitive bicirculant admitting a subgroup of automorphisms acting 1-regularly, with the exception of the complete graph K_4, is at most 3. Problem (v) is solved for graphs admitting half-arc-transitive group actions with less than six alternets by showing that there exist graphs admitting half-arc-transitive group actions with four or five alternets with respect to which they are not tightly attached, whereas graphs admitting half-arc-transitive group actions with less that four alternets are all tightly attached with respect to such actions.

Math. Subj. Class (2010): 05C05, 05C10, 05C25, 05C40, 05C45, 20B15, 20F05.

Key words: circulant, bicirculant, semiregular automorphism, vertex-transitive graph, arc-transitive graph, half-arc-trasitive graph, snark, Cayley graph, quasi m-Cayley graph, generalized Cayley graph, 1-regular action, regular cover of a graph, automorphism group.

Izvleček

ALGEBRAIČNI ASPEKTI TEORIJE GRAFOV

V doktorski disertaciji so obravnavane različne teme s področja algebraične teorije grafov, in sicer pomembni odprti problemi, ki so bili predmet številnih raziskav v zadnjih dvajsetih letih:

(i) Kateri grafi so (krepko) kvazi m-Cayleyjevi grafi?

(ii) Kateri bicirkulanti so ločno tranzitivni?

(iii) Ali obstajajo posplošeni Cayleyjevi grafi, ki niso Cayleyjevi grafi, so pa točkovno tranzitivni?

(iv) Ali obstajajo snarki med Cayleyjevimi grafi?

(v) Ali obstajajo grafi, ki premorejo pol-ločno tranzitivno delovanje z majhnim številom alternetov, glede na katerega niso tesno speti?

V doktorski disertaciji je problem (i) rešen za cirkulante v primeru, ko je $m \in \{2, 3, 4\}$. Problem (ii) je rešen v celoti za petvalentne bicirkulante. Problem (iii) je rešen pritrdilno, in sicer s konstrukcijo dveh neskončnih družin točkovno tranzitivnih posplošenih Cayleyjevih grafov, ki niso Cayleyjevi grafi. V obeh družinah so grafi bicirkulanti. Problem (iv) je rešen za tiste $(2, s, t)$-Cayleyjeve grafe, katerih pripadajoči $2t$-kotni grafi so ločno tranzitivni bicirkulanti praštevilske valence. Problem (v) je rešen za grafe, ki premojo pol-ločno tranzitivno grupno delovanje z manj kot šestimi alterneti. Dokazano je, da obstajajo grafi, ki premorejo pol-ločno tranzitivno delovanje s štirimi oziroma petimi alterneti, glede na katerega niso tesno speti, medtem kot so vsi grafi, ki premorejo pol-ločno tranzitivno delovanje z manj kot štirimi alterneti, glede na to delovanje tesno speti.

Math. Subj. Class (2010): 05C05, 05C10, 05C25, 05C40, 05C45, 20B15, 20F05.

Ključne besede: cirkulant, bicirkulant, polregularen avtomorfizem, točkovno tranzitiven graf, ločno tranzitiven graf, pol-ločno tranzitiven graf, polregularen avtomorfizem, Cayleyjev graf, kvazi m-Cayleyjev graf, posplošeni Cayleyjev graf, 1-regualrno delovanje, regularen krov grafa, grupa avtomorfizmov.

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Chapter 1

Introduction

The PhD Thesis deals with graph theory from the algebraic point of view. The following families of graphs are considered: circulants, bicirculants, quasi m-Cayley graphs, generalized Cayley graphs and half-arc-transitive graphs. The structural properties of graphs in the above mentioned families are considered. Throughout the thesis graphs are finite, simple and undirected, and groups are finite, unless specified otherwise.

In 1981, Marušič [79] asked if every vertex-transitive (di)graph has a semiregular automorphism, that is, an automorphism with all cycles of equal length in its cycle decomposition. Despite the fact that this problem has been an active topic of research in the last decades, only partial results have been obtained thus far. The simplest example of a semiregular automorphism is an automorphism having just one cycle in its cycle decomposition, and a graph admitting such an automorphism is called a *circulant*. In the PhD Thesis we consider which circulants belong to a recently defined class of graphs, called *quasi m-Cayley graphs* [60], which have good symmetry properties, in the sense that they admit a group of automorphisms G that fixes a vertex of the graph and acts semiregularly on the other vertices with m orbits. In particular, in the PhD Thesis, circulants which are quasi 2-Cayley, quasi 3-Cayley and strongly quasi 4-Cayley graphs are completely classified. The classification is presented in Chapter 3 (see Theorems 3.3.2, 3.4.2 and 3.4.3).

The second simplest example of a semiregular automorphism is an automorphism with just two cycles of equal length in its cycle decomposition, and a graph admitting such an automorphism is called a *bicirculant*. Bicirculants are considered in Chapter 4. The classifications of cubic and tetravalent arc-transitive bicirculants were completed in 2007 (see [30, 86, 97]) and in 2012 (see [53, 54, 57]), respectively. In this PhD Thesis the next natural step in this direction is made by classifying pentavalent arc-transitive bicirculants (see Section 4.1). In particular, it is shown that with the exception of seven particular graphs, a connected pentavalent bicirculant is arc-transitive if and only if it isomorphic to a Cayley graph $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ on the dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$, where $r \in \mathbb{Z}_n^*$ such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod n$ (see Theorem 4.1.14).

Bicirculants also appear in Section 4.2, where generalized Cayley graphs are considered. In [89], authors asked if there exists a vertex-transitive generalized Cayley graph which is not a Cayley graph. This question is answered in affirmative. Namely, it is shown in Section 4.2.1 that the line graph of the Petersen graph is a generalized Cayley graph (see Example 4.2.2), and it is well-known that this graph is vertex-transitive but not Cayley graph. Besides the line graph of the Petersen graph, two infinite families of generalized Cayley graphs which are vertex-transitive but not Cayley graphs are constructed, and all those graphs are bicirculants (see Theorems 4.2.10 and 4.2.11). In Section 4.2.2 it is shown that every generalized Cayley graph contains a semiregular automorphism (see Theorem 4.2.5), and although not every generalized Cayley graph is vertex-transitive, this gives a particular affirmative answer to the problem regarding existence of semiregular automorphisms in vertex-transitive graphs mentioned above.

In Section 4.3 it is shown that a connected bicirculant $X \neq K_4$ of prime valency admitting a group of automorphisms containing the semiregular automorphism giving the bicirculant structure and acting regularly on the set of arcs is near-bipartite, that is, with the chromatic number at most 3 (see Theorem 4.3.2). Combining this result with the theory of Cayley maps new partial results are obtained in regards to the well-known conjecture that there are no snarks amongst Cayley graphs (see Theorem 10).

Chapter 5 investigates properties of half-arc-transitive graphs. The study of half-arc-transitive graphs started in 1966 with Tutte's result [106] that half-arc-transitive graphs have even valency. In 1970, Bouwer [12] proved that there are infinitely many half-arc-transitive graphs, and moreover, that there exist a half-arc-transitive graph with valeny $2k$ for any natural number $k \geq 2$. In 1981, Holt [43] gave an example of half-arc-transitive graph of valency 4 with 27 vertices, now known as Doyle-Holt graph (see also [21, 22]). In 1991, Alspach et al. [4] showed that this is the smallest such graph. In the late 1990s, Marušič, Waller and Nedela made remarkable progress on the study of tetravalent half-arc-transitive graphs (see [81, 83, 91]). In this PhD thesis, the properties of half-arc-transitive graphs with small number of alternets are studied. In particular, it is shown that all half-arc-transitive graphs with two and three alternets are tightly attached (see Theorems 11 and 12), whereas this does not hold for graphs admitting a half-arc-transitive action with more than three alternets (see Example 5.3.1). The structure of half-arc-transitive graphs with four and five alternets which are not tightly attached is described (see Theorems 5.3.2 and 5.3.5). It is also proven, that for a given half-arc-transitive graph with n alternets, with the use of the so-called direct product of graphs, infinitely many half-arc-transitive graphs with n alternets can be constructed (see Theorem 5.3.6).

In Chapter 2 notions concerning the thesis are introduced together with the notation and some auxiliary results that are needed in the subsequent chapters.

The results of this PhD Thesis are published in the following articles:

- I. Antončič, A. Hujdurović and K. Kutnar, A classification of pentavalent arc-transitive bicirculants, submitted.
- A. Hujdurović, Quasi m-Cayley circulants, *Ars Math. Contemp.*, **6** (2013), 147–154.
- A. Hujdurović, K. Kutnar and D. Marušič, Vertex-transitive generalized Cayley graphs which are not Cayley graphs, *Elect. J. Combin.*, submitted.

- A. Hujdurović, K. Kutnar and D. Marušič, On prime-valent symmetric bicirculants and Cayley snarks, *Proceedings of GSI2013 - Geometric Science of Information, Lecture Notes in Computer Science*, accepted.
- A. Hujdurović, K. Kutnar and D. Marušič, Half-arc-transitive group actions with small number of alternets, *J. Combin. Theory Ser. A*, submitted.

Chapter 2

Background

2.1 Groups

Through this section let $(G, \cdot)$ denote a group with the group operation $\cdot$. We will use multiplicative notation: the identity element of G will be denoted by 1 and the inverse of $g \in G$ will be denoted by g^{-1}. To simplify notation we will write gh instead of $g \cdot h$, for $g, h \in G$. For group-theoretic terms not defined here we refer the reader to [95, 99, 112]. We will use the symbol $\mathbb{Z}_r$, both for cyclic group of order r and the ring of integers modulo r. In the latter case, $\mathbb{Z}_r^*$ will denote the multiplicative group of units of $\mathbb{Z}_r$.

2.1.1 Group actions

Let Ω denote a nonempty set. A (right) group action (Ω, G) of G on Ω is a binary function:

$$\Omega \times G \to \Omega$$

defined by the rule

$$(\omega, g) \mapsto \omega^g$$

which satisfies the following two axioms:

(i) $\omega^1 = \omega$ for every $\omega \in \Omega$;

(ii) $(\omega^g)^h = \omega^{gh}$ for all $g, h \in G$ and $\omega \in \Omega$.

In a similar way we can define *(left) group action* (G, Ω).

The difference between left and right actions is in the order in which a product gh acts on an element x. For a left action h acts first and is followed by g, while for a right action g acts first and is followed by h.

Let a group G act on the set Ω. Then to each element $g \in G$ we can associate a mapping $\overline{g} : \Omega \to \Omega$, namely $\overline{g}(\omega) = \omega^g$. The mapping $\overline{g}$ is a bijection. Hence we have a mapping $\rho \colon G \to \mathrm{Sym}(\Omega)$ given by $\rho(g) = \overline{g}$. Moreover, using (i) and (ii), we see that ρ is a group homomorphism since for all $\omega \in \Omega$ and all $g, h \in G$, the image of ω under $\overline{gh}$ is the same as its image under the product $\overline{g}\ \overline{h}$. Conversely, if $f\colon G \to \mathrm{Sym}(\Omega)$ is a group homomorphism, then it gives rise to the group action of the group G on the set Ω in such a way that $f = \rho$. The group homomorphism ρ is called the *representation of the action* of G on Ω. Depending on whether this action is right or left, the corresponding representation is called *right representation* or *left representation*, and denoted by G_R and G_L, respectively. The *degree* of the right representation G_R of the group G is the cardinality of the set Ω. The *kernel* of the homomorphism ρ is the set of all those elements of the group G that act in a trivial way: $Ker\rho = \{g \in G \mid \overline{g} = id\} = \{g \in G \mid \overline{g}(\omega) = \omega,\ \forall \omega \in \Omega\} = \{g \in G \mid \omega^g = \omega,\ \forall \omega \in \Omega\}$. If the kernel is trivial then the action is said to be *faithful.*

The set

$$Orb_G(\omega) = \omega^G = \{\omega^g \mid g \in G\},$$

where $\omega \in \Omega$, is called a *G-orbit* (in short an *orbit* if the group G is clear from the context) of the element ω with respect to the action of the group G. It is not difficult to see that two orbits ω_1^G and ω_2^G are either equal (as sets) or disjoint, so the set of all orbits is a partition of Ω into mutually disjoint subsets. If the group orbit $Orb_G(\omega)$ is equal to the entire set Ω for some element ω in Ω, then G is said to be *transitive* on Ω.

A kind of dual role to orbit is played by the set of elements in G which fix a particular point ω. This is called the *stabilizer* of ω in G, and is denoted by

$$G_\omega = \{g \in G \mid \omega^g = \omega\}.$$

It is not difficult to see that G_ω is a subgroup of G and that $G_{\omega_2} = g^{-1}G_{\omega_1}g$ whenever $\omega_2 = \omega_1^g$. The orbits of G_ω on Ω are called *suborbits* of the group G (with respect to the element ω). The number of suborbits is called a *rank* of the group, and for a group of rank k we say that it is a *rank k* group. Suborbit $\{\omega\}$ is a *trivial suborbit.* If $|G_\omega| = 1$ for every element $\omega \in \Omega$ then we say that G acts *semiregularly.* If G acts on Ω transitively and $|G_\omega| = 1$ for every element $\omega \in \Omega$ we say that G acts *regularly* (G is *regular*).

The following well-know fact is known as the *Orbit - stabilizer* property:

$$|Orb_G(\omega)| = |G : G_\omega| \quad \text{for every } \omega \in \Omega. \tag{2.1}$$

Given a transitive group G acting on a set Ω, we say that a partition $\mathcal{B}$ of Ω is *G-invariant* if the elements of G permute the parts, that is, *blocks* of $\mathcal{B}$, setwise. In other words, a non-empty subset $B \subseteq \Omega$ is a block for G if for every $g \in G$ the following hold:

$$B^g = B \quad \text{or} \quad B^g \cap B = \emptyset.$$

If the trivial partitions $\{\Omega\}$ and $\{\{\omega\} \mid \omega \in \Omega\}$ are the only G-invariant partitions of Ω, then G is said to be *primitive*, and is said to be *imprimitive* otherwise. In the latter case we shall refer to the corresponding G-invariant partition as to an *imprimitivity block system* of G (see also [13]). The importance of blocks arises from the following observation. Suppose that G acts transitively on Ω and that B is a non-trivial block for G. Let $\Sigma := \{B^g \mid g \in G\}$. Then Σ is an imprimitivity block system containing B. Now G acts on Σ in an obvious way, and this new action may give useful information about G.

Given a transitive group G acting on a set Ω, and a normal subgroup $N \triangleleft G$, it is an easy exercise to check that the orbits of N form a G-invariant partition. Although there are G-invariant partitions that do not arise from orbits of a normal subgroup, the following lemma shows that if the set of orbits of a given subgroup is G-invariant, then the subgroup is indeed normal.

Lemma 2.1.1 [76] *Let G be a transitive permutation group on a finite set Ω, and let $N \leq G$ be a subgroup of G. If the set of orbits of N on Ω is G-invariant, then N is a normal subgroup of G.*

Let a group G act transitively on the set Ω. Then G acts in a natural way on the set $\Omega \times \Omega = \Omega^2$ and G-orbits on the set $\Omega \times \Omega = \Omega^2$ are called *orbitals*. The orbital $\{(\omega, \omega) \mid \omega \in \Omega\}$ is a *trivial orbital*. If $\Delta \subseteq \Omega^2$ is an orbital, then also $\Delta^* = \{(\nu, \omega) \mid (\omega, \nu) \in \Delta\}$ is an orbital, we call it the *paired orbital* of the orbital Δ. If $\Delta = \Delta^*$ then Δ is said to be *self-paired orbital*. There exists a natural correspondence between suborbits and orbitals. Let G_ω be a stabilizer of an element $\omega \in \Omega$, let Γ be the suborbit with respect to ω and let $\nu \in \Gamma$. Then the corresponding orbital is the G-orbit on Ω^2 that contains (ω, ν). It is left to the reader to checked up that this orbital is well defined. Moreover, if Δ is any orbital then the corresponding suborbit Γ is the suborbit that contains ν where ν is an arbitrary element from Ω for which we have $(\omega, \nu) \in \Delta$.

We end this section with a remarkable group-theoretic result, which says that "large" cyclic subgroups are never corefree, that is, the largest normal subgroup $\mathrm{core}_G(K)$ of a group G contained in a "sufficiently large" cyclic subgroup K if G is non-trivial.

Proposition 2.1.2 [42, Theorem B] *If H is a cyclic subgroup of a finite group G with $|H| \geq \sqrt{|G|}$, then H contains a non-trivial normal subgroup of G.*

2.2 Graphs

A *graph* or *undirected graph* X is an ordered pair $X = (V, E)$ where $V = V(X)$ is a set, whose elements are called *vertices*, and $E = E(X)$ is a set whose elements are 2-element subsets of V, called *edges*. The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge. How these dots and lines are drawn is considered irrelevant. All that matters is the information which pairs of vertices form an edge and which do not. The *order* of a graph X is the cardinality of its vertex set $|V(X)|$. Graphs are *finite* or *infinite* according to their order. If A is a subset of the Cartesian product $V \times V$ then the pair (V, A) is called *directed graph* (or *digraph*).

A vertex v is *incident* with an edge e if $v \in e$. The two vertices incident with an edge are its *endvertices*, and are said to be *adjacent*. For adjacent vertices u and v in X, we write $u \sim v$ and denote the corresponding edge by uv. If $u \in V(X)$ then $N(u)$ denotes the set of neighbors of u. A *complement* $\overline{X}$ of a graph X has the same vertex set as X, where vertices u and v are adjacent in $\overline{X}$ if and only if they are not adjacent in X.

Pairwise non-adjacent vertices or edges are called *independent*. More formally, a set of vertices or of edges is *independent* (or *stable*) if no two of its elements are adjacent. The *valency* (or *degree*) $d_X(v) = d(v)$ of a vertex v in X is the number of edges incident to the vertex v, that is $|N(v)|$. If each vertex of the graph has the same valency d, the graph is called a *regular graph of valency* d (or a d*-regular graph*). A graph X is *cubic*, *tetravalent* or *pentavalent* if it is regular of valency 3, 4 or 5, respectively.

Two graphs X and Y are equal if and only if they have the same vertex set and the same edge set. Although this is a perfectly reasonable definition, for most purposes the model of a relationship is not essentially changed if Y is obtained from X just by renaming the vertex set. This motivates the following definition: Two graphs X and Y are *isomorphic* if there is a bijection φ from $V(X)$ to $V(Y)$ such that $x \sim y$ in X if and only if $\varphi(x) \sim \varphi(y)$ in Y. We say that φ is an *isomorphism* from X to Y. If X and Y are isomorphic, then we write $X \cong Y$. It is normally appropriate to treat isomorphic graphs as if they were equal.

A *walk* in a graph X is an alternating sequence of vertices and edges, beginning and ending with a vertex, where each vertex is incident to both the edge that precedes it and the edge that follows it in the sequence. A *path* in X is a sequence of vertices $(v_1, v_2, \ldots, v_n)$ such that v_1v_2, v_2v_3, $\ldots$, $v_{n-1}v_n$ are edges of X and the vertices v_i are all distinct. A closed path $(v_1, v_2, \ldots, v_n, v_1)$ in X is called a *cycle*. The *length* of a path or a cycle is the number of edges it contains. An *even* (*odd*) cycle is a cycle of even (odd) length. A *Hamiltonian path* in X is a path which contains each vertex of X. A *Hamiltonian cycle* in X is a cycle that contains each vertex of X. A graph is *hamiltonian* if it possesses a Hamiltonian cycle. The *girth* $g = g(X)$ of X is the length of the shortest cycle contained in the graph. By an n-cycle we shall always mean a cycle with n vertices.

The *distance* $d(u, v)$ between two vertices u and v in X is the number of edges in a shortest path connecting them. With $N_i(u)$ we denote the set of vertices at distance $i > 1$ from a vertex u. A graph is *connected* if any two of its vertices are linked by a path, otherwise the graph is *disconnected*. A graph X is said to be *bipartite* if $V(X)$ can be divided into two disjoint sets U and U' $(U, U' \subseteq V(X)$, $U \cup U' = V(X))$ such that every edge has one endvertex in U and one in U'; that is, there is no edge between two vertices in the same set. It is not difficult to prove that a graph is bipartite if and only if it contains no odd cycle. A graph X is said to be *near-bipartite* if $V(X)$ can be partitioned into three disjoint sets U, U' and U'', $(U, U', U'' \subseteq V(X)$, $U \cup U' \cup U'' = V(X))$ such that there is no edge between two vertices in the same set. A graph without a cycle (that is, an *acyclic* graph) is called a *forest*. A connected forest is called a *tree*. The vertices of degree 1 in a tree are its *leaves*. Every non-trivial tree has at least two leaves.

An ordered pair (u, v) of adjacent vertices u and v in X is called an *arc*. A sequence $(u_0, u_1, u_2, \ldots, u_s)$ of distinct vertices in X is called an s*-arc* if u_i is adjacent to u_{i+1} for every $i \in \{0, 1, \ldots, s-1\}$ and $u_i \neq u_{i+2}$ for each $i \in \{0, 1, \ldots, s-2\}$. A *subgraph* of a graph X is a graph Y such that $V(Y) \subseteq V(X)$ and $E(Y) \subseteq E(X)$. If $V(Y) = Y(X)$, we say that Y is a *spanning subgraph* of X. A k-regular spanning subgraph is called a k*-factor*. If $S \subseteq V(X)$, then the subgraph $X[S]$ of X whose edges are precisely the edges of X with both endvertices in S is called the *graph induced* *by* S. More precisely, $V(X[S]) = S$ and $E(X[S]) = \{\{u,v\} \mid u,v \in S\} \cap E(X)$. Let X be a graph, and Y its subgraph. Then $X - Y$ denotes a graph with vertex set $V(X)$ and edge set $E(X - Y) = E(X) \setminus E(Y)$.

2.2.1 Action of groups on graphs

An *automorphism* α of a graph $X = (V, E)$ is an isomorphism of X into itself. Thus each automorphism α of X is a permutation of $V(X)$ which maps edges to edges and non-edges to non-edges. Denote with $\mathrm{Aut}(X)$ the set of all automorphisms of X. It is not difficult to see that this set together with the composition of the permutations forms a group, called the *automorphism group* of X. It is straightforward to see that the automorphism group of a graph is equal to the automorphism group of its complement.

Any subgroup G of $\mathrm{Aut}(X)$ acts in a natural way on the set of vertices $V(X)$, set of edges $E(X)$ and set of arcs $A(X)$ of X. A subgroup $G \leq \mathrm{Aut}(X)$ is said to be *vertex-transitive*, *edge-transitive* and *arc-transitive* provided it acts transitively on the set of vertices, edges and arcs of X, respectively. A graph X is said to be G*-vertex-transitive*, G*-edge-transitive*, and G*-arc-transitive* if G is vertex-transitive, edge-transitive and arc-transitive, respectively. If $G = \mathrm{Aut}(X)$ we simply say that X is *vertex-transitive*, *edge-transitive* or *arc-transitive*. An arc-transitive graph is also called *symmetric*.

Given a graph X, a subgroup $G \leq \mathrm{Aut}(X)$ is said to be *half-arc-transitive* if it is vertex-transitive and edge-transitive but not arc-transitive. A graph X is said to be G*-half-arc-transitive* if the subgroup $G \leq \mathrm{Aut}(X)$ is half-arc-transitive. If a graph X is $\mathrm{Aut}(X)$-half-arc-transitive, then we simply say that X is half-arc-transitive.

A subgroup $G \leq \mathrm{Aut}(X)$ is said to be s*-regular* if it acts transitively on the set of s-arcs and the stabilizer of an s-arc in G is trivial. If $G = \mathrm{Aut}(X)$ the graph X is said to be s*-regular*. A graph X is said to be (G,s)*-arc-transitive* and (G,s)*-regular* if G is transitive and regular on the set of s-arcs of X, respectively. A (G,s)-arc-transitive graph is said to be (G,s)-transitive if the graph is not $(G,s+1)$-arc-transitive.

If $G \leq \mathrm{Aut}(X)$ acts (im)primitively on $V(X)$ we say that X is G*-(im)primitive*. If $\mathcal{B}$ is imprimitivity block system for G then, clearly any two blocks $B, B' \in \mathcal{B}$ induce isomorphic vertex-transitive subgraphs of X.

2.2.2 Cayley graphs

Given a group G and a subset S of G such that $S = S^{-1}$ and $1 \notin S$, the *Cayley graph* $\mathrm{Cay}(G,S)$ on G relative to S has vertex set G and edges of the form $\{g, gs\}$ where $g \in G$ and $s \in S$. Cayley graphs on cyclic groups are called *circulants*. A Cayley graph $\mathrm{Cay}(G,S)$ is connected if and only if $\langle S \rangle = G$. If $X = \mathrm{Cay}(G,S)$ is a Cayley graph on a group G, then G acts transitively on $V(X)$ via the left multiplication action, implying that Cayley graphs are vertex-transitive. This action of G on $V(X) = G$ is regular, and is called the *left regular action*. Sabidussi [98] characterized Cayley graphs as follows. A graph is a Cayley graph on a group G if and only if its automorphism group contains a regular subgroup isomorphic to G. Not every vertex-transitive graph is a Cayley graph. The smallest such graph is the Petersen graph (see Figure 2.1).

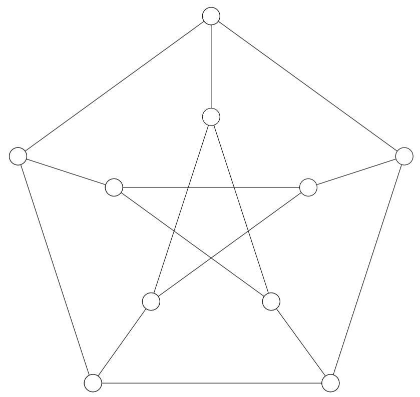

Figure 2.1: The Petersen graph, the smallest vertex-transitive graph which is not a Cayley graph.

Let $X = Cay(G, S)$ be a Cayley graph on a group G with respect to the set S. Denote by $\mathrm{Aut}(G, S)$ the set of all automorphisms of G which fix S setwise, that is,

$$\mathrm{Aut}(G, S) = \{\sigma \in \mathrm{Aut}(G) \mid \sigma(S) = S\}.$$

It is easy to check that $\mathrm{Aut}(G, S)$ is a subgroup of $\mathrm{Aut}(X)$ and that it is contained in the stabilizer of the identity element $1 \in G$. It follows from the definition of Cayley graphs that the left regular representation G_L of G induces a regular subgroup of $\mathrm{Aut}(X)$. Following Xu [120], $X = Cay(G, S)$ is called a *normal Cayley graph* if G_L is normal in $\mathrm{Aut}(X)$, that is, if $\mathrm{Aut}(G, S)$ coincides with the vertex stabilizer $1 \in G$. Moreover, if X is a normal Cayley graph, then $\mathrm{Aut}(X) = G_L \rtimes \mathrm{Aut}(G, S)$.

2.2.3 Arc-transitive circulants

As was already mentioned, circulants are Cayley graphs on cyclic groups. The classification of connected arc-transitive circulants has been obtained independently by Kovács [52] and Li [69]. Before stating this classification we need to introduce certain graph products.

Let K_n be the *complete* graph on n vertices, that is, a graph of order n in which any two distinct vertices are adjacent. The *wreath (lexicographic) product* $Y[X]$ of a graph X by a graph Y is the graph with vertex set $V(Y) \times V(X)$ such that $\{(u_1, u_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(Y)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(X)$. For a positive integer b and a graph Y, denote by bY the graph consisting of b vertex-disjoint copies of the graph Y. The graph $Y[\overline{K_b}] - bY$ is called the *deleted wreath (deleted lexicographic) product* of Y and $\overline{K_b}$, where $\overline{K_b} = bK_1$.

Proposition 2.2.1 [52, 69] *Let X be a connected arc-transitive circulant of order n. Then one of the following holds:*

(i) $X \cong K_n$;

(ii) $X = Y[\overline{K}_d]$, *where* $n = kd$, $k, d > 1$ *and* Y *is a connected arc-transitive circulant of order* k;

(iii) $X = Y[\overline{K}_d] - dY$, *where* $n = kd$, $d > 3$, $gcd(d,k) = 1$ *and* Y *is a connected arc-transitive circulant of order* k*;*

(iv) X *is a normal circulant.*

2.2.4 Semiregular automorphisms and Frucht's notation

For a graph X and a partition $\mathcal{W}$ of $V(X)$, we let $X_{\mathcal{W}}$ be the associated *quotient graph* of X relative to $\mathcal{W}$, that is, the graph with vertex set $\mathcal{W}$ and edge set induced naturally by the edge set $E(X)$. When $\mathcal{W}$ is the set of orbits of a subgroup H in $\mathrm{Aut}(X)$ the quotient graph $X_{\mathcal{W}}$ we will denoted by X_H. In particular, if $H = \langle h \rangle$ is generated by a single element h then we let $X_h = X_H$. As already mentioned the subgraph of X induced by $W \in \mathcal{W}$ will be denoted by $X[W]$. Similarly, we let $X[W, W']$ (in short $[W, W']$), $W, W' \in \mathcal{W}$, denote the bipartite subgraph of X induced by the edges having one endvertex in W and the other endvertex in W'. Moreover, for $W, W' \in \mathcal{W}$ we let $d(W)$ and $d(W, W')$ denote the valency of $X[W]$ and $X[W, W']$, respectively.

Given integers $k \geq 1$ and $n \geq 2$ we say that an automorphism of a graph is (k,n)-*semiregular* if it has k cycles of length n in its cycle decomposition.

Let X be a connected graph admitting a (k,n)-semiregular automorphism

$$\rho = (u_0^0 u_0^1 \cdots u_0^{n-1})(u_1^0 u_1^1 \cdots u_1^{n-1}) \cdots (u_{k-1}^0 u_{k-1}^1 \cdots u_{k-1}^{n-1}), \tag{2.2}$$

and let $\mathcal{W} = \{W_i \mid i \in \mathbb{Z}_k\}$ be the set of orbits $W_i = \{u_i^s \mid s \in \mathbb{Z}_n\}$ of $\langle \rho \rangle$. Using Frucht's notation [29] X may be represented in the following way. Each orbit of $\langle \rho \rangle$ is represented by a circle. Inside a circle corresponding to the orbit W_i the symbol n/T, where $T = T^{-1} \subseteq \mathbb{Z}_n \setminus \{0\}$, indicates that for each $s \in \mathbb{Z}_n$, the vertex u_i^s is adjacent to all the vertices u_i^{s+t} where $t \in T$. When $|T| \leq 2$ we use a simplified notation n/t, $n/(n/2)$ and n, respectively, when $T = \{t, -t\}$, $T = \{n/2\}$ and $T = \emptyset$. Finally, an arrow pointing from the circle representing the orbit W_i to the circle representing the orbit W_j, $j \neq i$, labeled by $y \in \mathbb{Z}_n$ means that for each $s \in \mathbb{Z}_n$, the vertex $u_i^s \in W_i$ is adjacent to the vertex u_j^{s+y}. When the label is 0, the arrow on the line may be omitted. In Figure 2.2 the Petersen graph is drawn in Frucht's notation with respect to the (2, 5)-semiregular automorphism.

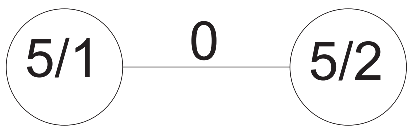

Figure 2.2: Frucht's notation of the Petersen graph.

2.2.5 Paley graphs

Let q be a prime power such that $q \equiv 1 \pmod 4$, and let $\mathbb{F}_q$ denote a finite field of order q. The *Paley graph* $P(q)$ of order q, is a graph whose vertices are elements of the finite field $\mathbb{F}_q$, in which two vertices are adjacent if and only if their difference is a non-zero square in $\mathbb{F}_q$ (see Figure 2.3 where Paley graph $P(13)$ is drown). Notice that the assumption $q \equiv 1 \pmod 4$ implies that -1 is a square in $\mathbb{F}_q$. Therefore, if $a-b$ is a square, then also $b-a$ is a square. The Paley graph of order q has valency $(q-1)/2$. The Paley graphs are *self-complementary*, that is, any Paley graph is isomorphic to its complement.

Paley graphs can also be defined as Cayley graphs. Let $\mathbb{F}_q$ be a finite field of order q and V_q^+ its additive group. Let ω be a primitive root in $\mathbb{F}_q$, and $S = \{\omega^2, \omega^4, \ldots, \omega^{q-1}\}$. Then the Paley graph $P(q)$ is isomorphic to the Cayley graph $\mathrm{Cay}(V_q^+, S)$.

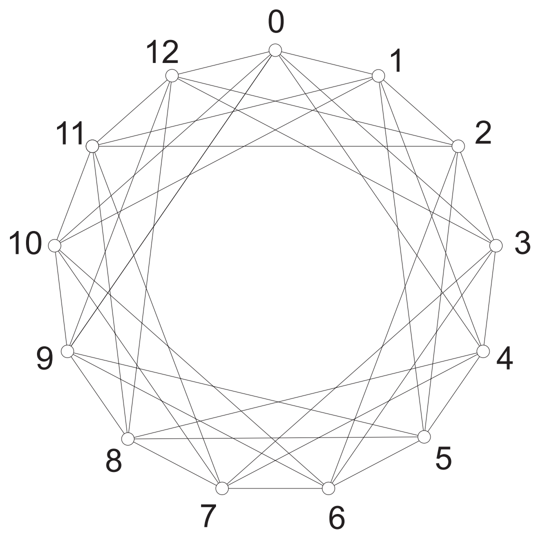

Figure 2.3: The Paley graph $P(13)$.

2.2.6 Snarks

A graph is said to be *cyclically k-edge-connected* if at least k edges must be deleted to obtain a disconnected graph, in which at least two components contain a cycle. A *snark* is a connected, cyclically 4-edge-connected cubic graph which is not 3-edge-colorable, that is, a connected, cyclically 4-edge-connected cubic graph whose edges cannot be colored by three colors in such a way that adjacent edges receive distinct colors.

Tait [103] initiated the study of snarks in 1880, when he proved that the four color theorem (see [9]) is equivalent to the statement that no snark is planar (*planar graph* is a graph which can be drawn on a plane with no edges crossing). The first known snark was the Petersen graph (see Figure 2.1), discovered in 1898. In 1946, Croatian mathematician Blanuša discovered two more snarks, both on 18 vertices, now named the Blanuša snarks [10] (see Figure 2.4). In 1975, Isaacs [50] generalized Blanuša's method to construct two infinite families of snarks . Although most known

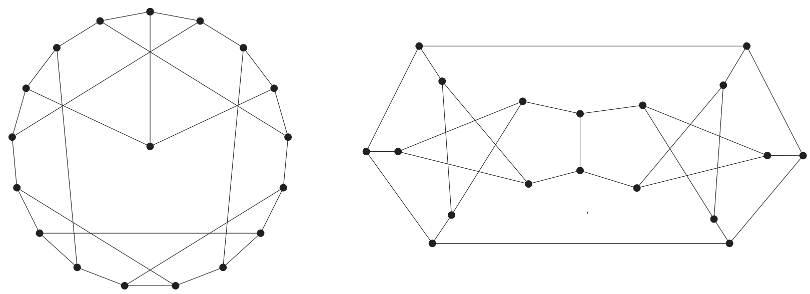

Figure 2.4: The first and the second Blanuša snark.

examples of snarks exhibit a lot of symmetry, none of them is a Cayley graph. In fact it was conjectured in [3] that no such graph exist. The proof of this conjecture would contribute significantly to various open problems regarding Cayley graphs [17, 36, 93, 118]. One of such problems is the well-known conjecture that every connected Cayley graph contains a Hamiltonian cycle which has received particular interest in the mathematical society over the last few decades (see [1, 2, 6, 26, 34, 35, 37, 51, 62, 65, 80, 92, 116, 117]). Namely, every hamiltonian cubic graph is easily seen to be 3-edge-colorable. It is also worth mentioning that the conjecture about non-existence of Cayley snarks is in fact a special case of the conjecture that all Cayley graphs on groups of even order are 1-factorizable (see [100]). A graph is 1-*factorizable* if its edge set can be partitioned into edge-disjoint 1-factors (perfect matchings).

A large number of articles, directly or indirectly related to this problem, have appeared in the literature affirming the non-existence of Cayley snarks. For example, in [94] it is proved that the smallest example of a Cayley snark, if it exists, comes either from a non-abelian simple group or from a group which has a single non-trivial proper normal subgroup. The subgroup must have index two and must be either non-abelian simple or the direct product of two isomorphic non-abelian simple groups. In Section 4.3.2 new partial results are obtained affirming the non-existence of Cayley snarks.

2.2.7 Graph covers

A *covering projection* of a graph $\widetilde{X}$ is a surjective mapping $p\colon \widetilde{X} \to X$ such that for each $\tilde{u} \in V(\widetilde{X})$ the set of arcs emanating from $\tilde{u}$ is mapped bijectively onto the set of arcs emanating from $u = p(\tilde{u})$. The graph $\widetilde{X}$ is called a *covering graph* of the *base graph* X. The set $\mathrm{fib}_u = p^{-1}(u)$ is the *fibre* of the vertex $u \in V(X)$. The subgroup K of all automorphisms of $\tilde{X}$ which fix each of the fibres setwise is called the *group of covering transformations*. The graph $\tilde{X}$ is also called a K-*cover* of X. It is a simple observation that the group of covering transformations of a connected covering graph acts semiregularly on each of the fibres. In particular, if the group of covering transformations is regular on the fibres of $\tilde{X}$, we say that $\tilde{X}$ is a *regular* K-*cover*. We say that $\alpha \in \mathrm{Aut}(X)$ *lifts* to an automorphism of $\tilde{X}$ if there exists an automorphism $\tilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$, called a *lift* of α, such that $\tilde{\alpha}p = p\alpha$. If the covering graph $\widetilde{X}$ is connected then K is the lift of the trivial subgroup of $\mathrm{Aut}(X)$. Note that a subgroup $G \leq \mathrm{Aut}(\widetilde{X})$ projects if and only if the partition of $V(\tilde{X})$ into the orbits of K is G-invariant.

A combinatorial description of a K-cover was introduced through so-called voltages by Gross and Tucker [40] as follows. Let X be a graph and K be a finite group. A *voltage assignment* on X is a mapping $\zeta \colon A(X) \to K$ with the property that $\zeta(u,v) = \zeta(v,u)^{-1}$ for any arc $(u,v) \in A(X)$ (here, and in the rest of the paper, $\zeta(u,v)$ is written instead of $\zeta((u,v))$ for the sake of brevity). The voltage assignment ζ extends to walks in X in a natural way. In particular, for any walk $D = u_1u_2\cdots u_t$ of X we let $\zeta(D)$ to denote the product $\zeta(u_1,u_2)\zeta(u_2,u_3)\cdots\zeta(u_{t-1},u_t)$, that is, the ζ-voltage of D.

The values of ζ are called *voltages*, and K is the *voltage group*. The *voltage graph* $X \times_\zeta K$ derived from a voltage assignment $\zeta \colon A(X) \to K$ has vertex set $V(X) \times K$, and edges of the form $\{(u,g),(v,\zeta(x)g)\}$, where $x = (u,v) \in A(X)$. Clearly, $X \times_\zeta K$ is a covering graph of X with respect to the projection to the first coordinate. By letting K act on $V(X \times_\zeta K)$ as $(u,g)^{g'} = (u,gg')$, $(u,g) \in V(X \times_\zeta K)$, $g' \in K$, one obtains a semiregular group of automorphisms of $X \times_\zeta K$, showing that $X \times_\zeta K$ can in fact be viewed as a K-cover of X.

Given a spanning tree T of X, the voltage assignment $\zeta \colon A(X) \to K$ is said to be *T-reduced* if the voltages on the tree arcs are trivial, that is, if they equal the identity element in K. In [39] it is shown that every regular covering graph $\tilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ζ with respect to an arbitrary fixed spanning tree T of X.

The problem of whether an automorphism α of X lifts or not is expressed in terms of voltages as follows (see Proposition 2.2.2). Given $\alpha \in \mathrm{Aut}(X)$ and the set of fundamental closed walks $\mathcal{C}$ based at a fixed vertex $v \in V(X)$, we define $\bar{\alpha} = \{(\zeta(C), \zeta(C^\alpha)) \mid C \in \mathcal{C}\} \subseteq K \times K$. Note that if K is abelian, $\bar{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X. Also, from the definition, it is clear that for a T-reduced voltage assignment ζ the derived graph $X \times_\zeta K$ is connected if and only if the voltages of the cotree arcs generate the voltage group K.

We conclude this section with four propositions dealing with lifting of automorphisms in graph covers. The first one may be deduced from [74, Theorem 4.2], the second one from [44] whereas the third one is taken from [23, Proposition 2.2], but it may also be deduced from [78, Corollaries 9.4, 9.7, 9.8].

Proposition 2.2.2 [74] *Let K be a finite group, and let $X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K. Then an automorphism α of X lifts if and only if $\bar{\alpha}$ is a function which extends to an automorphism α^* of K.*

For a connected regular cover $X \times_\zeta K$ of a graph X derived from a T-reduced voltage assignment ζ with an abelian voltage group K and an automorphism $\alpha \in \mathrm{Aut}(X)$ that lifts, $\bar{\alpha}$ will always denote the mapping from the set of voltages of the fundamental cycles on X to the voltage group K and α^* will denote the automorphism of K arising from $\bar{\alpha}$.

Two coverings $p_i\colon \widetilde{X}_i \to X$, $i \in \{1,2\}$, are said to be *isomorphic* if there exists a graph isomorphism $\phi\colon \widetilde{X}_1 \to \widetilde{X}_2$ such that $\phi p_2 = p_1$.

Proposition 2.2.3 [44] *Let K be a finite group. Two connected regular covers $X \times_\zeta K$ and $X \times_\varphi K$, where ζ and φ are T-reduced, are isomorphic if and only if there exists an automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\zeta(u,v)^\sigma = \varphi(u,v)$ for any cotree arc (u,v) of X.*

Proposition 2.2.4 [23] *Let K be a finite group, and let $X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K, and let the lifts of $\alpha \in \mathrm{Aut}(X)$ centralize K, considered as the group of covering transformations. Then for any closed walk W in X, there exists $k \in K$ such that $\zeta(W^\alpha) = k\zeta(W)k^{-1}$. In particular, if K is abelian, $\zeta(W^\alpha) = \zeta(W)$ for any closed walk W of X.*

Given a voltage assignment ζ on X and $\beta \in \mathrm{Aut}(X)$, we let ζ^β be the voltage assignment on X given by $\zeta^\beta(u,v) = \zeta(u^{\beta^{-1}}, v^{\beta^{-1}})$, $(u,v) \in A(X)$; and we let $\widetilde{\beta}$ be the permutation of $V(X) \times K$ acting as $(u,k)^{\widetilde{\beta}} = (u^\beta, k)$. Our last proposition is straightforward.

Proposition 2.2.5 *Let K be a finite group, and let $\widetilde{X} = X \times_\zeta K$ be a connected regular cover of a graph X derived from a voltage assignment ζ with the voltage group K, and let $\beta \in \mathrm{Aut}(X)$. Then the following hold.*

(i) *$\widetilde{\beta}$ is an isomorphism from $\widetilde{X}$ to $X \times_{\zeta^\beta} K$.*

(ii) *If $\widetilde{\alpha}$ is in $\mathrm{Aut}(\widetilde{X})$ which projects to α, then $\widetilde{\beta}^{-1}\widetilde{\alpha}\widetilde{\beta}$ is in $\mathrm{Aut}(X \times_{\zeta^\beta} K)$, and it projects to $\beta^{-1}\alpha\beta$.*

(iii) *If $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ centralizes the group K of covering transformations, then also $\widetilde{\beta}^{-1}\widetilde{\alpha}\widetilde{\beta}$ centralizes K.*

Chapter 3

Circulants

Results of this chapter are published in [45]. We consider circulants with additional properties described below. In Section 3.1 the concepts of quasi-semiregular actions and quasi m-Cayley graphs are introduced. Section 3.2 gives properties of strongly quasi m-Cayley graphs. In Section 3.3 all quasi 2-Cayley circulants are classified, while in Section 3.4 all quasi 3-Cayley circulants are classified and all strongly quasi 4-Cayley circulants are classified.

3.1 Quasi m-Cayley graphs

In this section we consider quasi-semiregular actions on graphs, a natural generalization of semiregular actions on graphs, which have been an active topic of research in the last decades. Following [60] we say that a group G acts *quasi-semiregularly* on a set Ω if there exists an element ∞ in Ω such that G fixes ∞, and the stabilizer G_ω of any element $\omega \in \Omega \backslash \{\infty\}$ is trivial. The element ∞ is called the *point at infinity*. A graph X is called *quasi m-Cayley* on G if the group G acts quasi-semiregularly on $V(X)$ with m orbits on $V(X)\backslash\{\infty\}$. If G is cyclic and $m = 1$ (respectively, $m = 2$, $m = 3$ and $m = 4$) then X is said to be *quasi circulant* (respectively, *quasi bicirculant*, *quasi tricirculant* and *quasi tetracirculant*). In addition, if the point at infinity ∞ is adjacent with only one orbit of G_∞ then we say that X is a *strongly quasi m-Cayley graph* on G.

Quasi m-Cayley graphs were first defined in 2011 by Kutnar, Malnič, Martinez and Marušič [60], who showed which strongly quasi $m-$Cayley graphs are strongly regular graphs.

If a graph X of order v is a quasi m-Cayley graph on a group G then $m = (v-1)/n$ where $n = |G|$. Hence, a necessary condition for a graph to be a quasi m-Cayley graph is that n divides $v - 1$. Also, if X is a regular quasi m-Cayley graph then it is of valency sn for some $s \geq 1$.

Example 3.1.1 If $n = mk+1 \geq 2$ then the complete graph K_n is a quasi m-Cayley graph on a cyclic group $\mathbb{Z}_k$.

If X is a quasi m-Cayley graph on a group G, then evidently the complement of X is also a quasi m-Cayley graph on the same group G. Recall that a *Frobenius* group is a transitive permutation group on a finite set, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. The following example shows how to obtain a quasi-semiregular group from a given Frobenius group.

Example 3.1.2 Let G be a Frobenius group acting on a finite set Ω. Let $\omega \in \Omega$ be arbitrary and let $1 < H \leq G_\omega$. Then H acts quasi-semiregularly on Ω with ω as the point at infinity.

The following example shows how to obtain quasi-semiregular actions from finite fields.

Example 3.1.3 Let $\mathbb{F}_q$ be a finite field of order q and $q-1 = n \cdot m$. Then the multiplicative group $G = \{x^n \mid x \in \mathbb{F}_q^*\}$ formed by the n-th powers of a non-zero element x of $\mathbb{F}_q$ acts by multiplication on the additive group of $\mathbb{F}_q$. The orbits of the mentioned action are $\{0\}, C_0, \ldots, C_{n-1}$, where $C_i = \theta^i G$ and θ is a primitive root of $\mathbb{F}_q$.

We end this section with the following lemma which considers vertex-transitive strongly quasi m-Cayley graphs.

Lemma 3.1.4 *Let X be a connected vertex-transitive strongly quasi $m-$Cayley graph. Then X is arc-transitive.*

Proof. Since X is vertex-transitive, it is sufficient to observe that there exists a vertex v such that the stabilizer $\mathrm{Aut}(Y)_v$ acts transitively on the neighborhood of v. It is obvious that for the point at infinity this condition is satisfied. ■

3.2 Strongly quasi m-Cayley circulants

Lemma 3.1.4 shows that whenever we have a vertex-transitive strongly quasi m-Cayley graph, the graph is also arc-transitive. Therefore, in order to determine which circulants are strongly quasi m-Cayley graphs, it suffices to consider arc-transitive circulants. Thus, we can use Proposition 2.2.1, where all connected arc-transitive circulants are classified.

In Sections 3.3 and 3.4 two lemmas showing that arc-transitive circulants described in Proposition 2.2.1(ii) and (iii) are not strongly quasi m-Cayley graphs, will be needed.

Lemma 3.2.1 *Let X be an arc-transitive circulant, described in Proposition 2.2.1 (ii). Then X is not a strongly quasi m-Cayley graph for any $m \in \mathbb{N}$.*

Proof. We have $X = Y[\overline{K}_d]$, where $n = kd$, $k, d > 1$ and Y is a connected arc-transitive circulant of order k. Suppose that X is a strongly quasi m-Cayley graph on a group G. Then $val(X) = (n-1)/m = (kd-1)/m$. On the other hand, since $X = Y[\overline{K}_d]$, we have $val(X) = val(Y) \cdot d$. These two facts combined together imply that $d(k - m \cdot val(Y)) = 1$, and so $d = 1$, a contradiction. ■

Lemma 3.2.2 *Let X be an arc-transitive circulant, described in Proposition 2.2.1 (iii). Then X is not a strongly quasi m-Cayley graph for any $m \in \mathbb{N}$.*

Proof. We have $X = Y[\overline{K_d}] - dY$, where $n = kd$, $d > 3$, $gcd(d,k) = 1$, and Y is an arc-transitive circulant of order k. Suppose that X is also a strongly quasi m-Cayley graph on a group G. By [69, Theorem 1.1] the k copies of the graph $\overline{K_d}$ form an imprimitivity block system $\mathcal{B}$ for $\mathrm{Aut}(X)$. Clearly the block $B \in \mathcal{B}$ containing the point at infinity, that is, the trivial orbit of G, is fixed by G. This implies that $|G|$ divides $d-1$. On the other hand, since the valency of X is $|G|$, we have $|G| \geq d-1$. Combining these results we obtain $|G| = d-1$. Thus, connectedness of X implies that $k = 2$. However, there are $2d-1$ vertices in X different from the point at infinity, and they cannot be divided into m orbits of size $d-1$ for any natural number m. Therefore, there are no strongly quasi m-Cayley graphs amongst the graphs from Proposition 2.2.1(iii) for any natural number $m \geq 1$. ■

The following lemma gives an upper bound on the number of divisors of the order of an arc-transitive normal circulant which is quasi m-Cayley graph.

Lemma 3.2.3 *Let X be an arc-transitive circulant, described in Proposition 2.2.1 (iv). If X is also a strongly quasi m-Cayley graph on a group G, then the order of X has at most $m+1$ divisors.*

Proof. Let $X = Cay(\mathbb{Z}_n, S)$ be a normal circulant. Let $A = \mathrm{Aut}(X)$. Since X is a normal Cayley graph, $A \cong \mathbb{Z}_n \rtimes \mathrm{Aut}(\mathbb{Z}_n, S)$. We may, without loss of generality, assume that the point at infinity corresponds to the vertex $0 \in \mathbb{Z}_n$, and so $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Therefore, $G \lesssim \mathbb{Z}_n^*$. Since G has m orbits on $\mathbb{Z}_n \backslash \{0\}$, then $\mathrm{Aut}(\mathbb{Z}_n)$ has at most m orbits on $\mathbb{Z}_n \backslash \{0\}$, and at most $m+1$ orbits on $\mathbb{Z}_n$. Elements in the same orbit of $\mathrm{Aut}(\mathbb{Z}_n)$ are clearly of the same order in $\mathbb{Z}_n$. There exist an element in $\mathbb{Z}_n$ of order d if and only if d divides n. Therefore the number of divisors of n, denoted by $\tau(n)$, is not greater than $m+1$, i.e. $\tau(n) \leq m+1$. ■

3.3 Quasi 2-Cayley graphs

In this section, connected circulants that are also quasi 2-Cayley graphs are classified (see Theorem 3.3.2). If a graph Y of order n is a quasi 2-Cayley graph on a group G, which is not a strongly quasi 2-Cayley graph, then it is isomorphic to the complete graph K_n. Namely, in such a graph, the point at infinity ∞ is adjacent to both non-trivial orbits of G, and thus it is adjacent to all the vertices different from ∞. Consequently, we can conclude that Y has valency $|V(Y)|-1$, and so Y is a complete graph. In order to classify all connected circulants that are also quasi 2-Cayley graphs it therefore suffices to characterize strongly quasi 2-Cayley graphs that are also connected circulants. We do this in Theorem 3.3.1.

Theorem 3.3.1 *Let X be a connected circulant. Then X is also a strongly quasi 2-Cayley graph if and only if X is isomorphic to the Paley graph $P(p)$, where p is a prime such that $p \equiv 1 \pmod 4$. Moreover, X is a quasi bicirculant.*

PROOF. Let X be the Paley graph $P(p)$, where p is a prime, such that $p \equiv 1 \pmod 4$. It is well known that the Paley graphs are connected arc-transitive circulants, and, as was observed in [60], they are also strongly quasi 2-Cayley graphs.

Conversely, let X be a connected circulant $Cay(\mathbb{Z}_n, S)$ of order n not isomorphic to the complete graph K_n, which is also a strongly quasi 2-Cayley graph on a group G. Then $|G| = (n-1)/2$ and X is of valency $(n-1)/2$. By lemma 3.1.4, X is an arc-transitive graph, and moreover Proposition 2.2.1, Lemma 3.2.1 and Lemma 3.2.2 combined together imply that X is a normal circulant. The theorem now follows from the three claims below.

CLAIM 1: n is an odd prime.

It is obvious that n must be odd, since 2 divides $n-1$. By Lemma 3.2.3 we have that $\tau(n) \leq 3$. Thus we have the following two possibilities for n:

- $n = p$, where p is a prime;
- $n = p^2$, where p is a prime.

Suppose that the latter case hold. Let $A = \mathrm{Aut}(X)$. Since X is a normal Cayley graph, we have $A \cong \mathbb{Z}_n \rtimes \mathrm{Aut}(\mathbb{Z}_n, S)$. We may, without loss of generality, assume that the point at infinity corresponds to the vertex $0 \in \mathbb{Z}_n$, and so $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n) \cong \mathbb{Z}_n^*$. Therefore, $\mathbb{Z}_n^*$ contains a subgroup G of order $(n-1)/2$. Since $|\mathbb{Z}_n^*| \leq n-1$ and $|G|$ divides $|\mathbb{Z}_n^*|$ we obtain that either $|\mathbb{Z}_n^*| = n-1$ or $(n-1)/2$. Since, by assumption, n is not a prime, we have $|\mathbb{Z}_n^*| = (n-1)/2$. This gives the following equation

$$\frac{p^2-1}{2} = p(p-1)$$

which has the unique solution $p = 1$, a contradiction.

CLAIM 2: $n \equiv 1 \pmod 4$.

Since $S = -S$, and no element in $\mathbb{Z}_n$ can be its own inverse, we have that the number of elements in S is even, and since $|S| = \frac{n-1}{2}$, we have $n \equiv 1 \pmod 4$.

CLAIM 3: X is isomorphic to the Paley graph $P(n)$.

By Claim 1, n is a prime. Therefore the group $\mathbb{Z}_n^*$ is cyclic, and thus since G is a subgroup of $\mathbb{Z}_n^*$, G is cyclic as well. By [52, Remark 2], we have $\mathrm{Aut}(X) = \{g \mapsto g^\sigma + h \mid \sigma \in K, h \in \mathbb{Z}_n\}$, for a suitable group $K < \mathrm{Aut}(\mathbb{Z}_n)$, and S is the orbit under K of a generating element of $\mathbb{Z}_n$, that is, $S = Orb_K(g)$ for some generating element g of $\mathbb{Z}_n$. Now we have that $\mathrm{Aut}(X)_0 = \{g \mapsto g^\sigma + h \mid \sigma \in K, h \in \mathbb{Z}_n : 0^\sigma + h = 0\} = \{g \mapsto g^\sigma \mid \sigma \in K\} \cong K$. So we see that $G \lesssim K$. Since $S = Orb_K(g) \lesssim Orb_G(g)$, and $|S| = |Orb_G(g)|$ we have $S \cong Orb_G(g)$, which implies that $S \cong G$ (taking $g = 1$). Now, since G is the index 2 subgroup of the cyclic group $\mathbb{Z}_n^*$, G is of the form $G = \langle x^2 \rangle$ where x generates $\mathbb{Z}_n^*$. Therefore G consists of all squares in $\mathbb{Z}_n^*$ and $S \cong G$, implying that X is isomorphic to the Paley graph $P(n)$ as claimed.

It is obvious that G must be cyclic, so the graph X is in fact a quasi bicirculant. ■

We are now ready to state the classification of quasi 2-Cayley circulants.

Theorem 3.3.2 *Let X be a quasi 2-Cayley graph of order n which is also a connected circulant. Then either X is isomorphic to the complete graph K_n, or $n \equiv 1 \pmod 4$ is a prime and X is isomorphic to the Paley graph $P(n)$. Moreover, X is a quasi bicirculant.*

PROOF. It follows from Theorem 3.3.1 and the paragraph preceding it. ■

In general, if X is a vertex-transitive quasi 2-Cayley graph on a group G, not isomorphic to the complete graph, then it is a strongly regular graph of a rank 3 group. Namely, the orbits of G are contained in the orbits of the stabilizer of the $\mathrm{Aut}(X)_\infty$ and since there are just two non-trivial orbits of G, then there are exactly two non-trivial orbits of $\mathrm{Aut}(X)_\infty$ which in fact must coincide with the orbits of G. Therefore $\mathrm{Aut}(X)$ must be a rank 3 group, and the graphs of the rank 3 groups are strongly regular graphs. We summarize this in the following proposition.

Proposition 3.3.3 *Vertex-transitive quasi 2-Cayley graphs are strongly regular graphs.*

3.4 Quasi 3-Cayley and 4-Cayley graphs

In this section we will deal with the question which connected circulants are also quasi 3-Cayley graphs or strongly quasi 4-Cayley graphs. We first consider the case of strongly quasi 3-Cayley graphs.

Theorem 3.4.1 *Let X be a connected circulant. Then X is also a strongly quasi 3-Cayley if and only if $X \cong Cay(\mathbb{Z}_n, S)$ where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$. Moreover, X is a quasi tricirculant.*

PROOF. Let $X = Cay(\mathbb{Z}_p, S)$ where $p \equiv 1 \pmod 3$ is a prime and S is the set of all non-zero cubes in $\mathbb{Z}_p$. Since p is a prime, it is well known that $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$ is a cyclic group of order $p-1$. Let $G = \langle a^3 \rangle$, where a is a generating element of $\mathbb{Z}_p^*$. Then G consists of all non-zero cubes in $\mathbb{Z}_p$, and $|G| = \frac{p-1}{3}$. The action of G on $\mathbb{Z}_p$ defined by $x^g = g \cdot x$ gives G as the subgroup of $\mathrm{Aut}(X)$. The group G acts quasi-semiregularly on $\mathbb{Z}_p$ with $0 \in \mathbb{Z}_p$ as the point at infinity. Namely, it is easy to check that $G_0 = G$, and that the stabilizer of any element $x \in \mathbb{Z}_p \backslash \{0\}$ is trivial. Since $|G| = \frac{p-1}{3}$, it follows that G has 3 orbits on $\mathbb{Z}_p \backslash \{0\}$, and therefore X is a quasi 3-Cayley graph. Since one of the orbits of G is the set S, the point at infinity is adjacent to only one orbit of G, so X is in fact a strongly quasi 3-Cayley graph. By the construction X is an arc-transitive circulant since $G \leq \mathrm{Aut}(X)_0$ acts transitively on the set of vertices adjacent to the vertex 0.

Conversely, let X be a connected circulant of order n, which is also a strongly quasi 3-Cayley graph on a group G. Then $|G| = \frac{n-1}{3}$. From Lemma 3.1.4 we have that X is arc-transitive, and therefore Proposition 2.2.1, Lemma 3.2.1 and Lemma 3.2.2 combined together imply that X is a normal circulant. Therefore, we can assume that $X = Cay(\mathbb{Z}_n, S)$, and that $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n)$, implying that $\frac{n-1}{3} | \varphi(n)$, where $\varphi(n)$ is the Euler totient function.

Claim 1: n is a prime number.

Let
$$n = p_1^{k_1} \cdot p_2^{k_2} \cdots p_t^{k_t},$$
be a canonic factorization of a positive integer n. From Lemma 3.2.3, we have $\tau(n) \leq 4$. Now we can calculate
$$\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_t + 1).$$
We have the following possibilities for n:

- $n = p$,
- $n = p^2$;
- $n = p^3$;
- $n = pq$,

where p and q are different primes.

If $n = p^2$, then the only solution of $\frac{n-1}{3} | \varphi(n)$ is $p = 2$ and $n = 4$. However, if $n = 4$, the graph X is of valency 1, so it is not a connected graph.

If $n = p^3$, then there is no solution of the above equation.

If n is a product of two different primes, then we have $|\mathbb{Z}_n^*| = (n-1)/3$ or $2(n-1)/3$. In the first case $\mathbb{Z}_n^* \cong G$, so $\mathbb{Z}_n^*$ acts semiregularly on $\mathbb{Z}_n \backslash \{0\}$, and it is not difficult to see that this is not the case for $n = pq$. If $|\mathbb{Z}_n^*| = 2(n-1)/3$, then we obtain the following equation
$$(p-1)(q-1) = \frac{2(pq-1)}{3}.$$
The only solutions in natural numbers of the above equation are
$$(p, q) \in \{(4, 7), (5, 5), (7, 4)\},$$
so there are no two different primes p, q satisfying the given equation.

Having in mind all these, we conclude that n is a prime.

Claim 2: X is isomorphic to the Cayley Graph $Cay(\mathbb{Z}_n, S)$, where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$.

Similarly as in the previous section, it can be shown that $G \cong S$. Since G is an index 3 subgroup of $\mathbb{Z}_n^*$, we have $G = \langle x^3 \rangle$, where x is a generating element of $\mathbb{Z}_n^*$. It follows that G consists of all cubes in $\mathbb{Z}_n^*$, so X is isomorphic to $Cay(\mathbb{Z}_n, S)$, where S is the set of all non-zero cubes in $\mathbb{Z}_n$ and $n \equiv 1 \pmod 3$ is a prime. It is obvious that the group G must be cyclic, therefore, X is in fact a quasi tricirculant. ■

We are now ready to give the classification of quasi 3-Cayley circulants.

Theorem 3.4.2 *Let X be a connected circulant. Then X is also a quasi 3-Cayley graph if and only if either $X \cong K_n$, or replacing X with its complement if necessary, $X \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all non-zero cubes in $\mathbb{Z}_n$, and n is a prime such that $n \equiv 1 \pmod 3$. Moreover, X is a quasi tricirculant.*

Proof. Let X be a connected circulant of order n, which is also a quasi 3-Cayley on a group G. The point at infinity is adjacent to all three non-trivial orbits of G, if and only if X is isomorphic to K_n. If the point at infinity is adjacent to just one non-trivial orbit of G, then X is a strongly quasi 3-Cayley graph, therefore, Theorem 3.4.1 applies. If the point at infinity is adjacent to two non-trivial orbits of G, then we consider the complement $Y = \overline{X}$ of the graph X. The graph Y is a quasi 3-Cayley graph on G, and actually it is a strongly quasi 3-Cayley graph on G. Since Y is the complement of a circulant it is also a circulant. Suppose that Y is not connected. Then, since it is vertex-transitive, it is the disjoint union of some isomorphic graphs. The point at infinity is adjacent to one orbit of G, so the connected components of Y must have at least $1 + \frac{n-1}{3}$ points. Therefore $n = k \cdot n_1$, where k is the number of connected components of Y, and n_1 is the number of points in each of the components. We have noticed that $n_1 \geq 1 + \frac{n-1}{3}$, thus $k \leq 2$. If $k = 1$ then Y is connected. Suppose that $k = 2$. Then there are two connected components of Y, say Y_1 and Y_2, each containing $n/2$ points. Suppose that $\infty \in Y_1$. Let Δ_1, Δ_2 and Δ_3 be non-trivial orbits of G, and let the point at infinity be adjacent to Δ_1. Then $\Delta_1 \subset Y_1$. Since Y_1 and Y_2 have the same size, at least one of Δ_2 and Δ_3 have points both Y_1 and Y_2. Suppose that $u, v \in \Delta_2$, such that $u \in Y_1$ and $v \in Y_2$. Since u and v are in the same orbit of G, there exist $g \in G$ which maps u to v. However, g fixes ∞, and consequently g fixes Y_1, a contradiction.

Having in mind all the written above, one can see that Y is a connected circulant, which is also a strongly quasi 3-Cayley graph. Therefore we have the desired result. ■

We end this section with a classification of strongly quasi 4-Cayley circulants.

Theorem 3.4.3 *Let X be a connected circulant. Then X is a strongly quasi 4-Cayley graph on a group G if and only if $X \cong C_9$ or $X \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all fourth powers in $\mathbb{Z}_n \backslash \{0\}$, and n is a prime such that $n \equiv 1 \pmod 4$. Moreover, X is a quasi tetracirculant.*

Proof. Let $X = C_9$. Then $X \cong Cay(\mathbb{Z}_9, \{\pm 1\})$ and the group $G = \{1, -1\} \subset \mathbb{Z}_9^*$ acts quasi semiregularly on $\mathbb{Z}_9$ with 0 as the point at infinity.

Let now $X \cong Cay(\mathbb{Z}_p, S)$, where S is the set of all fourth powers in $\mathbb{Z}_p \backslash \{0\}$, and p is a prime such that $p \equiv 1 \pmod 4$. Define $G = \langle a^4 \rangle$, where a is some generating element of $\mathbb{Z}_p^*$. Then G acts quasi-semiregularly on $\mathbb{Z}_p^*$ with 0 as the point at infinity. Since $|G| = \frac{p-1}{4}$, it follows that G has 4 orbits on $\mathbb{Z}_p \backslash \{0\}$, and therefore X is a quasi 4-Cayley graph. It is also easy to see that 0 is adjacent to only one orbit of G on $\mathbb{Z}_p \backslash \{0\}$, therefore X is a strongly quasi 4-Cayley graph. By the construction, X is a connected arc-transitive circulant.

Conversely, let X be a connected circulant of order n which is also a strongly quasi 4-Cayley graph on a group G. Then $|G| = (n-1)/4$. Using Lemma 3.1.4 we have that X is arc-transitive, and so Proposition 2.2.1, Lemma 3.2.1 and Lemma 3.2.2 combined together imply that X is a normal circulant. Therefore, we can assume that $X = Cay(\mathbb{Z}_n, S)$, and that $G \leq \mathrm{Aut}(\mathbb{Z}_n, S) \leq \mathrm{Aut}(\mathbb{Z}_n)$, implying that

$$\frac{n-1}{4} | \varphi(n). \tag{3.1}$$

Using Lemma 3.2.3 we obtain $\tau(n) \leq 5$. So we have the following five possibilities:

- $n = p$,
- $n = p^2$,
- $n = p^3$,
- $n = p^4$,
- $n = pq$,

where p and q are different primes.

If $n = p^2$, then the only solution of (3.1) is $n = 9$. In this case, the valency of X is $(9-1)/4 = 2$, so $X \cong C_9$.

For $n = p^3$ and $n = p^4$ there is no prime satisfying (3.1).

For $n = pq$, we have $(p-1)(q-1) = \alpha \cdot (pq-1)/4$, where $\alpha \in \{1, 2, 3\}$. If $\alpha = 1$, then we have $\mathbb{Z}_n^* = G$, so $\mathbb{Z}_n^*$ must act semiregularly on $\mathbb{Z}_n \backslash \{0\}$, which is not the case. If $\alpha = 2$, then there are no two different primes satisfying $(p-1)(q-1) = (pq-1)/2$, and finally, when $\alpha = 3$, $n = 5 \cdot 13$ is the only possibility. In this case, X is a connected arc-transitive circulant on 65 vertices, which has valency 16. Since G is an index 3 subgroup of $\mathbb{Z}_{65}^* \cong \mathbb{Z}_4 \times \mathbb{Z}_{12}$, we can calculate that $G \cong \{\pm 1, \pm 8, \pm 12, \pm 14, \pm 18, \pm 21, \pm 27, \pm 31\}$, and we can see that G does not act semiregularly on $\mathbb{Z}_{65} \backslash \{0\}$. Namely, the non identity element $21 \in G$ fixes the point $13 \in \mathbb{Z}_{65} \backslash \{0\}$.

Assume now that n is a prime. Similarly as in the proof of Theorem 3.3.1, we obtain $G \cong S$, and therefore, since G is an index 4 subgroup of $\mathbb{Z}_n^*$, we have $G = \langle x^4 \rangle$, where x is some generating element of $\mathbb{Z}_n^*$. Therefore, $X \cong Cay(\mathbb{Z}_n, S)$, where S is the set of all fourth powers in $\mathbb{Z}_n \backslash \{0\}$.

It is now clear that G is a cyclic group, so X is in fact a quasi tetracirculant. ■

Chapter 4

Bicirculants

Results of this chapter are published in [7, 47, 48]. As was already pointed out, an n-bicirculant (bicirculant, in short) is a graph admitting a $(2, n)$-semiregular automorphism. A graph X is said to be G-bicirculant if the group G contains a $(2, n)$-semiregular automorphism of X. The existence of a $(2, n)$-semiregular automorphism in a bicirculant enables us to label its vertex set and edge set in the following way. Let X be a connected $n-$*bicirculant*. Then there exists a $(2, n)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$ and the vertices of X can be labeled by x_i and y_i with $i \in \mathbb{Z}_n$, such that $\rho = (x_0\, x_1\, \ldots\, x_{n-1})(y_0\, y_1\, \ldots\, y_{n-1})$. Moreover, the edge set $E(X)$ can be partitioned into three subsets

$$\begin{aligned}
\mathcal{L} &= \bigcup_{i\in\mathbb{Z}_n} \{\{x_i, x_{i+l}\} \mid l \in L\} \quad \text{(left hand side edges)},\\
\mathcal{M} &= \bigcup_{i\in\mathbb{Z}_n} \{\{x_i, y_{i+m}\} \mid m \in M\} \quad \text{(middle edges - spokes)},\\
\mathcal{R} &= \bigcup_{i\in\mathbb{Z}_n} \{\{y_i, y_{i+r}\} \mid r \in R\} \quad \text{(right hand side edges)},
\end{aligned}$$

where L, M, R are subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cup R$. We shall denote this graph by $BC_n[L, M, R]$ (this notation has been introduced in [56]). The vertices x_i, $i \in \mathbb{Z}_n$, will be referred to as *left hand side vertices* and vertices y_i, $i \in \mathbb{Z}_n$, will be referred to as *right hand side vertices.* Edge $\{x_i, y_{i+m}\}$ will be called an m-*spoke.*

A bicirculant $X = BC_n[L, M, R]$ of order $2n$ is said to be *core-free* if there exists a $(2, n)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$ such that the cyclic subgroup $\langle\rho\rangle$ has a trivial core in $\mathrm{Aut}(X)$.

The following obvious isomorphisms between bicirculants will be used in our further analysis.

Proposition 4.0.4 [56] *Let L, M and R be subsets of $\mathbb{Z}_n$ such that $L = -L$, $R = -R$, $M \neq \emptyset$ and $0 \notin L \cup R$. Then*

$$BC_n[L, M, R] \cong BC_n[\lambda L, \lambda M + \mu, \lambda R] \ (\lambda \in \mathbb{Z}_n^*,\ \mu \in \mathbb{Z}_n)$$

with the isomorphism $\phi_{\lambda,\mu}$ given by $\phi_{\lambda,\mu}(x_i) = x_{\lambda i+\mu}$ and $\phi_{\lambda,\mu}(y_i) = y_{\lambda i}$.

4.1 Pentavalent arc-transitive bicirculants

The classification of connected cubic arc-transitive bicirculants follows from results obtained in [30, 86, 97] whereas the classification of connected tetravalent arc-transitive bicirculants follows from results obtained in [56, 57, 53]. The first step in classifying pentavalent arc-transitive bicirculants was the classification of arc-transitive Tabačjn graphs obtained in [8]. A *Tabačjn graph* $T(n, a, b, r)$ is a pentavalent bicirculant isomorphic to $BC_n[\{\pm 1\}, \{0, a, b\}, \{\pm r\}]$. Because of this we will say that a bicirculant $BC_n[L, M, R]$ with $|M| = 3$ is a *generalized Tabačjn graph*. In a pentavalent bicirculant $X = BC_n[L, M, R]$ with $|M| = 5$ a mapping τ defined by the rule $\tau(x_i) = x_{-i}$ and $\tau(y_i) = y_{-i}$ is an automorphism of X, and the group $G = \langle \tau, \rho \rangle \cong D_{2n}$ acts regularly on $V(X)$, implying that X is a Cayley graph on the group G. Such graphs are called *dihedrants*.

Let us now introduce a nice tool that visualize the structure of a bicirculant. We already introduced Frucht's notation in Section 2.2.4 which is quite similar to the following. (This tool can be used for any graph admitting an (m, n)-semiregular automorphism, but we will only define it in the framework of bicirculants.) Let X be a bicirculant $BC_n[L, M, R]$ with a $(2, n)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$. Let $\mathcal{W} = \{W, W'\}$ be the set of orbits of $\langle \rho \rangle$. Then clearly the subgraph of X induced on W (as well as the subgraph induced on W') and the bipartite subgraph of X induced by the edges having one endvertex in W and the other endvertex in W' are regular. We let $d(W)$ and $d(W, W')$ denote the valency of these subgraphs. (Observe that $d(W, W') = |M|$.) We let the *quotient multigraph* X_ρ corresponding to ρ be the multigraph whose vertex set consists of $\mathcal{W}$, the two vertices $W, W' \in \mathcal{W}$ are joined by $d(W, W')$ edges, and at a vertex $W \in \mathcal{W}$ there are $d(W)/2$ loops if $d(W)$ is even and $(d(W) - 1)/2$ loops and one semiedge if $d(W)$ is odd. In case X is a connected pentavalent bicirculant there are five different multigraphs on two vertices that can occur as quotient multigraph of X. These five possibilities are shown in Figure 4.1. However, we will prove in Theorem 4.1.14 that the multigraphs shown in Figure 4.1(a) and (d) cannot occur as quotient multigraphs of arc-transitive bicirculants.

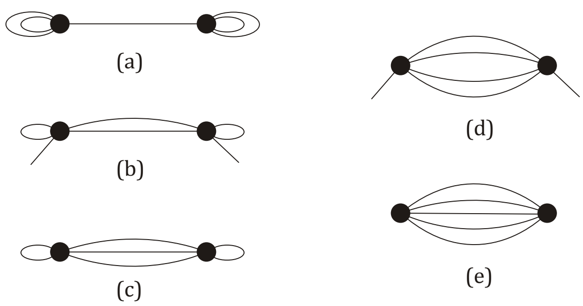

Figure 4.1: Multigraphs that can occur as quotient multigraphs of pentavalent bicirculants with respect to a $(2, n)$-semiregular automorphism ρ.

The classification of connected pentavalent arc-transitive bicirculants is carried out over Section 4.1.1, considering separately several cases depending on the number of spokes (the size of $|M|$). First it is shown that no connected pentavalent arc-transitive bicirculant $BC_n[L, M, R]$ with $|M| = 1$ or $|M| = 4$ exist (see Theorem 4.1.4). Next, in Theorem 4.1.5, it is shown that there are only two connected pentavalent arc-transitive bicirculants with $|M| = 2$ (the graphs in Theorem 4.1.14(i)). Bicirculants with $|M| = 3$ and $|M| = 5$ (that is, the generalized Tabačjn graphs and the dihedrants) are classified by considering first the core-free bicirculants. A group-theoretic result of Herzog and Kaplan [42], which says that "sufficiently large" cyclic subgroups are never core-free (see Proposition 2.1.2) combined together with a result which gives the upper bounds for the order of the automorphism group of a pentavalent arc-transitive graph (see Proposition 4.1.1), enable us to determine all core-free pentavalent arc-transitive bicirculants with $|M| \in \{3, 5\}$.

By Weiss [110, 111], for a pentavalent (G, s)-transitive graph, $s \geq 1$, the order of the vertex stabilizer G_v in G is a divisor of $2^{17} \cdot 3^2 \cdot 5$. In addition, the following result can be deduced from his work, as was recently observed by Guo and Feng [41].

Proposition 4.1.1 [41, Theorem 1.1.] *Let X be a connected pentavalent (G, s)-transitive graph for some $G \leq Aut(X)$ and $s \geq 1$. Let $v \in V(X)$. Then $s \leq 5$ and one of the following holds:*

(i) For $s = 1$, $G_v \cong \mathbb{Z}_5, D_{10}$ or D_{20};

(ii) For $s = 2$, $G_v \cong F_{20}$, $F_{20} \times \mathbb{Z}_2$, A_5 or S_5;

(iii) For $s = 3$, $G_v \cong F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $S_4 \times S_5$ or $(A_4 \times A_5) \rtimes \mathbb{Z}_2$ with $A_4 \rtimes \mathbb{Z}_2 = S_4$ and $A_5 \rtimes \mathbb{Z}_2 = S_5$;

(iv) For $s = 4$, $G_v \cong \mathrm{ASL}(2, 4)$, $\mathrm{AGL}(2, 4)$, $\mathrm{A\Sigma L}(2, 4)$ or $\mathrm{A\Gamma L}(2, 4)$;

(v) For $s = 5$, $G_v \cong \mathbb{Z}_2^6 \rtimes \Gamma\mathrm{L}(2, 4)$.

As for non-core-free pentavalent arc-transitive bicirculants, we use the fact that any such graph is a regular cyclic cover either of a core-free pentavalent arc-transitive bicirculant or of a dipole with five parallel edges. This then enables us to use graph covering techniques described in Subsection 2.2.7 to classify all pentavalent arc-transitive bicirculants.

4.1.1 Classification of pentavalent arc-transitive bicirculants

Throughout this section let $X = BC_n[L, M, R]$ be a connected pentavalent bicirculant, where $n \geq 5$. The following proposition about arc-transitivity of pentavalent bicirculants of small order, obtained using the program package MAGMA [11], will be useful throughout this section. (In the computer work the methods explained in [8] were used.)

Proposition 4.1.2 *Let $X = BC_n[L, M, R]$ be a pentavalent bicirculant. Then:*

(i) If $|M| = 4$ and $n \leq 14$ then X is not arc-transitive;

(ii) If $|M| = 2$ and $n \leq 12$ then X is arc-transitive if and only if it is either isomorphic to $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$ or to $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$;

(iii) If $|M| = 3$ and $n < 240$ then X is arc-transitive if and only if it is isomorphic to one of the following bicirculants: $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$, $BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$ and $BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. In addition, in the first two cases X is 2-transitive, and in the third case X is 1-transitive.

The following lemma, which will be used in Sections 4.1.3 and 4.1.4 is a straightforward generalization of [72, Theorem 9].

Lemma 4.1.3 *Let $X = BC_n[L, M, R]$ be a pentavalent arc-transitive bicirculant with a $(2, n)$-semiregular automorphism $\rho \in \mathrm{Aut}(X)$, and let N be the core of $\langle \rho \rangle$ in $\mathrm{Aut}(X)$. Then N is the kernel of $\mathrm{Aut}(X)$ acting on the set of orbits of N, $\mathrm{Aut}(X)/N$ acts arc-transitively on X_N, and either*

(i) X_N is a core-free pentavalent arc-transitive bicirculant with a $(2, n/|N|)$-semiregular automorphism $\bar{\rho}$ and $X_\rho \cong (X_N)_{\bar{\rho}}$, or

(ii) X_N is isomorphic to the dipole $\mathcal{D}_5$ with five parallel edges and $X_\rho \cong \mathcal{D}_5$.

By the following theorem in a pentavalent arc-transitive bicirculant depending on the size of the set M only three different cases can occur, which will be considered in separate subsubsections.

Theorem 4.1.4 *If $X = BC_n[L, M, R]$ is a connected pentavalent arc-transitive bicirculant, then $|M| \in \{2, 3, 5\}$.*

Proof. Let $X = BC_n[L, M, R]$ be a connected pentavalent arc-transitive bicirculant. By [58, Theorem 1.1], $|M| \neq 1$. We thus only need to show that $|M| \neq 4$. By way of contradiction suppose that $|M| = 4$. Then n is even and $L = R = \{\frac{n}{2}\}$. Also, we can, without loss of generality, assume that $M = \{0, a, b, c\}$, where $a, b, c \in \mathbb{Z}_n \setminus \{0\}$ are pairwise distinct.

Suppose first that $a = \frac{n}{2}$. Then the subgraph of X induced by the vertices x_0, $x_{\frac{n}{2}}$, y_0 and $y_{\frac{n}{2}}$ is isomorphic to the complete graph K_4. Consequently also the edge $\{x_0, y_b\}$ is contained in a subgraph of X isomorphic to K_4, in particular it belongs to at least two 3-cycles. Thus x_0 and y_b have at least two common neighbors. Since $R = L = \{\frac{n}{2}\}$ and $b \notin \{0, a\}$, the common neighbors of x_0 and y_b are y_c and $x_{\frac{n}{2}}$, implying that $c = b + \frac{n}{2}$. Applying the automorphism $\rho \in \mathrm{Aut}(X)$ one can now easily see that $X \cong C_n[K_2]$, which is clearly not arc-transitive. We can conclude that none of a, b and c is equal to $\frac{n}{2}$.

Now consider the edge $\{x_0, x_{\frac{n}{2}}\}$. Observe that every 4-cycle that contains this edge also contains exactly one right edge $\{y_i, y_{i+\frac{n}{2}}\}$ ($i \in \{0, a, b, c\}$), and thus it is contained on exactly four 4-cycles. Any two of these 4-cycles intersect only in $\{x_0, x_{\frac{n}{2}}\}$. The same must hold for the four 4-cycles containing the edge $\{x_0, y_0\}$, and consequently the path

$$(y_0,\ x_0,\ y_a) \text{ is contained on exactly one 4-cycle.} \tag{4.1}$$

Therefore we may assume that exactly one of the following holds: $b = 2a$ or $a+b = 0$ or $a + c = b$.

If $a + b = 0$ then three 4-cycles containing the edge $\{x_0, y_0\}$ are exactly determined: $(x_0,\ y_0,\ y_{\frac{n}{2}},\ x_{\frac{n}{2}}, x_0)$, $(x_0,\ y_0,\ x_a,\ y_a, x_0)$, and $(x_0,\ y_0,\ x_b,\ y_b,\ x_0)$. Hence, the remaining 4-cycle containing $\{x_0, y_0\}$ must be $(x_0,\ y_0,\ x_{-c},\ y_c,\ x_0)$, which implies that either $a = 2c$ or $b = 2c$.

If $a + c = b$ then again three of the four 4-cycles containing the edge $\{x_0, y_0\}$ are exactly determined. They are $(x_0,\ y_0,\ y_{\frac{n}{2}},\ x_{\frac{n}{2}},\ x_0)$, $(x_0,\ y_0,\ x_{a-b},\ y_a,\ x_0)$, and $(x_0,\ y_0,\ x_{c-b},\ y_c,\ x_0)$. Therefore the remaining 4-cycle containing $\{x_0, y_0\}$ is $(x_0,\ y_0,\ x_{-b},\ y_b,\ x_0)$ which implies that either $a = 2b$ or $c = 2b$.

This implies that we may, without loss of generality, assume that $b = 2a$. Clearly

$$(x_0,\ y_0,\ y_{\frac{n}{2}},\ x_{\frac{n}{2}},\ x_0) \text{ and } (x_0,\ y_0,\ x_{a-b},\ y_a,\ x_0)$$

are two 4-cycles containing the edge $\{x_0, y_0\}$, and the edges $\{x_0,\ y_b\}$, $\{x_0,\ y_c\}$, $\{y_0,\ x_{-b}\}$ and $\{y_0,\ x_{-c}\}$ must be contained in the remaining two 4-cycles containing the edge $\{x_0, y_0\}$. Therefore either $\{y_b,\ x_{-c}\}$ and $\{y_c,\ x_{-b}\}$ are edges in X, or $\{y_b,\ x_{-b}\}$ and $\{y_c,\ x_{-c}\}$ are edges in X. Inspecting the possibilities for these edges the following cases need to be considered. (Recall that $\frac{n}{2} \notin \{a, b, c\}$.)

Case 1. $\{y_b,\ x_{-c}\}$ is a 0-spoke.

Then $b + c = 0$, and consequently $M = \{0, a, 2a, -2a\}$. This implies that $(x_0,\ y_0,\ x_{-2a},\ y_{-2a},\ x_0)$ is the only 4-cycle containing the edge $\{x_0,\ y_{-2a}\}$ (and not containing the edge $\{x_0,\ x_{\frac{n}{2}}\}$) that exists regardless of the order of a. However, since there must exist another two 4-cycles, it follows that $4a = 0$, $5a = 0$, $6a = 0$ or $7a = 0$. Since $\langle M,\ \frac{n}{2}\rangle \cong \mathbb{Z}_n$ we can conclude that $n \leq 14$, contradicting Proposition 4.1.2.

Case 2. $\{y_b,\ x_{-c}\}$ is an a-spoke.

Then $b + c = a$, $M = \{0, a, 2a, -a\}$, and consequently $(x_0,\ y_0,\ x_a,\ y_a,\ x_0)$ and $(x_0,\ y_0,\ x_a,\ y_{2a},\ x_0)$ are two distinct 4-cycles containing the 2-arc $(x_0,\ y_0,\ x_a)$, contradicting (4.1).

Case 3. $\{y_b,\ x_{-b}\}$ is an a-spoke.

Then $2b = a$, and since $2a = b$ we have $a + b = 0$. Moreover, since $\{y_c,\ x_{-c}\}$ is an a-spoke or a b-spoke, either $a = 2c$ or $b = 2c$. In both cases, we obtain that $M = \{0,\ c,\ 2c,\ -2c\}$ and that $6c = 0$. Since $\langle M,\ \frac{n}{2}\rangle \cong \mathbb{Z}_n$ we can conclude that $n \leq 12$, contradicting Proposition 4.1.2.

Case 4. $\{y_b,\ x_{-b}\}$ is a c-spoke.

Then $2b = c$, and consequently $M = \{0, a, 2a, 4a\}$. Similarly as in Case 1, by considering the 4-cycles containing the edge $\{x_0,\ y_{4a}\}$ one can obtain that $n \leq 14$, contradicting Proposition 4.1.2. ■

4.1.2 Pentavalent bicirculants with |M| = 2

In the following theorem, pentavalent arc-transitive bicirculants with $|M| = 2$ are classified.

Theorem 4.1.5 *Let X be a connected pentavalent arc-transitive bicirculant $X = BC_n[L, M, R]$ with $|M| = 2$. Then either*

(i) $X \cong BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$, *or*

(ii) $X \cong BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$.

PROOF. Without loss of generality we can assume that $L = \{\frac{n}{2}, \pm l\}$, $M = \{0, a\}$ and $R = \{\frac{n}{2}, \pm r\}$, where $a \in \mathbb{Z}_n \setminus \{0\}$ and $l, r \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$. Observe that the edge $\{x_0, x_{\frac{n}{2}}\}$ is contained on at least four 4-cycles. Namely,

$$\begin{aligned} &(x_0, x_{\frac{n}{2}}, y_{\frac{n}{2}}, y_0, x_0),\ (x_0, x_{\frac{n}{2}}, y_{\frac{n}{2}+a}, y_a, x_0),\\ &(x_0, x_{\frac{n}{2}}, x_{\frac{n}{2}+l}, x_l, x_0) \text{ and } (x_0, x_{\frac{n}{2}}, x_{\frac{n}{2}-l}, x_{-l}, x_0) \end{aligned} \tag{4.2}$$

are all 4-cycles containing the edge $\{x_0, x_{\frac{n}{2}}\}$. Moreover, observe that every 2-arc containing the arc $(x_0, x_{\frac{n}{2}})$ lies on a 4-cycle, and by arc-transitivity of X this must then hold for every 2-arc in X. The existence of a 4-cycle containing the 2-arc (y_0, x_0, y_a) implies that $2a = 0$, $a = \pm 2r$ or $a = \pm r + \frac{n}{2}$.

If $2a = 0$ then $a = \frac{n}{2}$ and the edge $\{x_0, y_0\}$ is contained in a subgraph of X isomorphic to K_4. Consequently, by arc-transitivity, any edge in X is contained in such a subgraph. In particular, the edges $\{x_0, x_l\}$ and $\{y_0, y_r\}$ must lie on two 3-cycles in X, implying that $2l = \frac{n}{2}$ and $2r = \frac{n}{2}$. It follows that $\mathbb{Z}_n = \langle \frac{n}{2}, l, r\rangle$, where $4l = 4r = 0$, implying that $n = 4$ and $X \cong BC_4[\{\pm 1, 2\}, \{0, 2\}, \{\pm 1, 2\}]$, contradicting Proposition 4.1.2.

Therefore, either $a = \pm 2r$ or $a = \pm r + \frac{n}{2}$. Moreover, we may, without loss of generality, assume that $a = 2r$ or $a = r + \frac{n}{2}$. The existence of a 4-cycle containing the 2-arc (x_0, y_0, x_{-a}) implies that we may also assume that either $a = 2l$ or $a = l + \frac{n}{2}$. In total there are therefore four possibilities which we consider in the four cases below. In the case analysis it will be helpful to know that the existence of 4-cycles containing the 2-arc (x_0, x_l, x_{2l}) and the 2-arc (y_0, y_r, x_{2r}) implies, respectively, that

$$4l = 0 \text{ or } 3l + \frac{n}{2} = 0 \text{ or } 2l = a \text{ or } 2l = -a, \tag{4.3}$$

and

$$4r = 0 \text{ or } 3r + \frac{n}{2} = 0 \text{ or } 2r = a \text{ or } 2r = -a. \tag{4.4}$$

CASE 1. $a = 2r = 2l$.

Then either $l = r$ or $l = r + \frac{n}{2}$, and the connectedness of X implies that $\langle r, \frac{n}{2}\rangle = \mathbb{Z}_n$. If $l = r$ then the edge $\{x_0, y_a\}$ lies on at least five 4-cycles: $(x_0, y_a, y_r, y_0, x_0)$, $(x_0, y_a, x_{2l}, x_l, x_0)$, $(x_0, y_a, y_{a+\frac{n}{2}}, x_{\frac{n}{2}}, x_0)$, $(x_0, y_a, y_{3l}, x_l, x_0)$ and $(x_0, y_a, y_l, x_{-l}, x_0)$. Hence also the edge $\{x_0, x_{\frac{n}{2}}\}$ lies on at least five 4-cycles. Four of them are listed in (4.2) whereas the existence of the fifth 4-cycle combined together with (4.3) and (4.4) gives $3l = \frac{n}{2}$. This implies that $\mathbb{Z}_n = \langle r = l\rangle = \mathbb{Z}_6$, and thus, by Proposition 4.1.2, X is isomorphic to $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$. If, however, $l = r + \frac{n}{2}$ then

$$(x_0, x_{\frac{n}{2}}, y_{\frac{n}{2}+a}, y_a, x_0),\ (x_0, x_l, x_{2l}, y_a, x_0) \text{ and } (x_0, y_a, y_r, y_0, x_0)$$

are 4-cycles containing the edge $\{x_0, y_a\}$. Since there also exist a 4-cycle containing the 2-arc (x_{-l}, x_0, y_a), one of the neighbors of x_{-l}, that is one of $x_{-l+\frac{n}{2}}, x_{-2l}, y_l, y_{-l}$, is adjacent to the vertex y_a. Since $a \neq \frac{n}{2}$ it follows that $4r = 0$, $3r = 0$ or $8r = 0$, giving that $n \leq 8$, and thus, by Proposition 4.1.2, $X \cong BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$.

Case 2. $a = 2r = l + \frac{n}{2}$.

Then $l = 2r + \frac{n}{2}$ and $\langle r, \frac{n}{2} \rangle = \mathbb{Z}_n$. Applying (4.3) yields $n \in \{6, 8, 12\}$, and thus Proposition 4.1.2 applies.

Case 3. $a = r + \frac{n}{2} = 2l$.

Similarly as in Case 2, $\langle l, \frac{n}{2} \rangle = \mathbb{Z}_n$. Applying (4.4) yields $n \in \{6, 8, 12\}$, and thus Proposition 4.1.2 applies again.

Case 4. $a = r + \frac{n}{2} = l + \frac{n}{2}$.

Then $l = r$ and $\langle r, \frac{n}{2} \rangle = \mathbb{Z}_n$. Applying (4.4) yields $n \in \{4, 6, 8, 12\}$, and thus Proposition 4.1.2 applies also in this case. ■

Let us mention that the graph $BC_6[\{\pm 1, 3\}, \{0, 2\}, \{\pm 1, 3\}]$ is isomorphic to $K_{6,6} - 6K_2$, that is the complete bipartite graph minus a matching. The graph $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{\pm 3, 4\}]$ is shown in Figure 4.2.

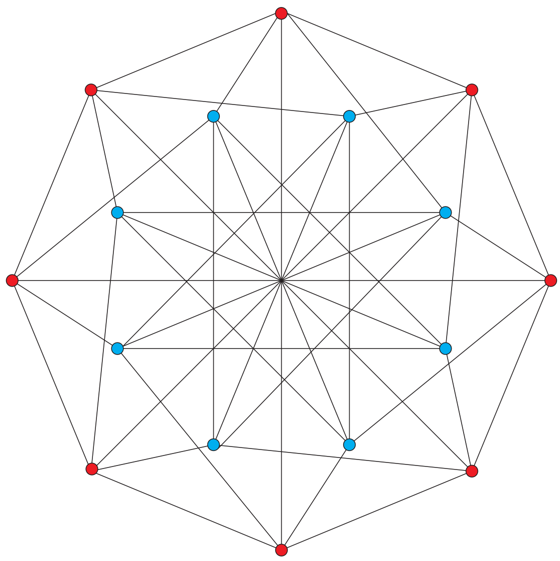

Figure 4.2: $BC_8[\{\pm 1, 4\}, \{0, 2\}, \{3, 4\}]$.

4.1.3 Generalized Tabačjn graphs

Let X be a generalized Tabačjn graph, that is, a bicirculant $BC_n[L, M, R]$ with $|M| = 3$. Then we may, without loss of generality, assume that $L = \{\pm l\}$, $M = \{0, a, b\}$ and $R = \{\pm r\}$, where $l, r \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$ and $a, b \in \mathbb{Z}_n \setminus \{0\}$. We first show that X is not 3-arc-transitive.

Theorem 4.1.6 *There exist no* 3*-arc-transitive generalized Tabačjn graph.*

PROOF. Suppose on the contrary, that for some $n \geq 3$ and $1 \leq l, r, a, b \leq n-1$ the bicirculant $X = BC_n[\{\pm l\}, \{0, a, b\}, \{\pm r\}]$ is 3-arc-transitive. Clearly, a 3-arc-transitive graph cannot be of girth 3, in particular $\mathrm{girth}(X) \geq 4$. Moreover, if $\mathrm{girth}(X) = 4$ then, by [38, Lemma 4.1.4], X is bipartite and of diameter 2, which implies that $n = 5$. However, since X is bipartite, l must be of even order in $\mathbb{Z}_5$ which is clearly impossible.

Suppose now that $\mathrm{girth}(X) = 5$. Then the 3-arc $A = (x_0, x_l, x_{2l}, x_{3l})$ lies on a 5-cycle in X. Suppose first that $5l \neq 0$. Then the 5-cycle containing A must contain one of y_i vertices, and we may assume that this 5-cycle is $(x_0, x_l, x_{2l}, x_{3l}, y_{3l}, x_0)$. That is $a = 3l$. Similarly, the 3-arc (y_0, x_0, x_l, y_l) lies on a 5-cycle of X, and so the fact that $b - a \neq \pm l$ (since $\mathrm{girth}(X) = 5$) implies that $l = \pm 2r$. Without loss of generality, we can assume that $l = 2r$. Observe that $5r \neq 0$, since otherwise we would have $5l = 0$. Therefore, the 3-arc $(y_0, y_r, y_{2r}, y_{3r})$ lies on a 5-cycle containing a left hand vertex x_i. This further implies that one of the following holds:

$$a = \pm 3r \text{ or } a - b = \pm 3r \text{ or } b = \pm 3r.$$

If $a = 3r$, then since $a = 3l = 6r$, we obtain $3r = 0$ which is in contradiction with $\mathrm{girth}(X) = 5$. If $a = -3r$ then since $l = 2r$ it follows that $(x_0, x_{2r}, y_{-r}, y_0, x_0)$ is a 4-cycle in X, a contradiction. Both $a - b = \pm 3r$ and $b = \pm 3r$ imply that $b \in \langle r \rangle$, and since also $a, l \in \langle r \rangle$, in both cases r generates $\mathbb{Z}_n$, and therefore X is isomorphic to a Tabačjn graph. But in view of [8, Lemma 3.2] this is impossible.

Suppose now that $5l = 0$. Then also $5r = 0$. Namely, since the 3-arc (x_0, y_0, y_r, x_r) lies on a 5-cycle of X, and $b - a \neq \pm r$ (since $\mathrm{girth}(X) = 5$) we have $r = \pm 2l$, and thus $5r = \pm 10l = 0$. We can, without loss of generality, assume that $l = r$ or $l = 2r$. However, if $l = r$ then $(x_0, y_0, y_l, x_l, x_0)$ is a 4-cycle in X, a contradiction. Therefore $l = 2r$. Since X is 3-arc-transitive also the 3-arc (y_0, x_0, x_l, y_{l+a}) lies on a 5-cycle, and thus $N(y_0) \cap N(y_{l+a}) \neq \emptyset$. If $y_r \in N(y_0) \cap N(y_{l+a})$ then $2r - l - a = 0$, and so $a = 0$, a contradiction. If $y_{-r} \in N(y_0) \cap N(y_{l+a})$ then $2r + l + a = 0$, and so $(x_0, x_l, x_{2l}, x_{2l+a}, x_0)$ is a 4-cycle in X, again a contradiction. It follows that $y_{\pm r} \notin N(y_0) \cap N(y_{l+a})$, and thus at least one of x_{l+a} and x_{l+a-b} belongs to $N(y_0) \cap N(y_{l+a})$, implying that at least one of following holds:

$$l + 2a = 0, \tag{4.5}$$

$$l + a + b = 0, \tag{4.6}$$

$$l + 2a - b = 0. \tag{4.7}$$

Similarly, considering the possibilities for 5-cycles containing the 3-arc (y_0, x_0, x_l, y_{l+b}) one can see that at least one of the following also holds:

$$l + 2b = 0, \tag{4.8}$$

$$l + a + b = 0, \tag{4.9}$$

$$l + 2b - a = 0. \tag{4.10}$$

If (4.5) holds then $l + 2a = 0$, $10a = -5l = 0$ and $r = 4a$, since $2(2r + 2a) = 4r + 4a = -r + 4a = 0$. Thus $\langle a, b \rangle = \mathbb{Z}_n$, and considering the equations (4.8), (4.9) and (4.10), we can conclude that either $\mathbb{Z}_n = \langle a \rangle$ and $n \leq 10$, or $\mathbb{Z}_n = \langle b \rangle$ and $n \leq 20$. In both cases Proposition 4.1.2 gives a contradiction. If $l + a + b \neq 0$ then both $l + 2a - b = 0$ and $l + 2b - a = 0$ must hold. Using the fact that $5l = 0$ we get $10a = 5b$ and $10b = 5a$, and so $15a = 15b = 0$. This implies that $l, a \in \langle b \rangle$. Since $l = 2r$ and $5l = 5r = 0$ we also have $3l = 6r = r$, implying that $n \leq 15$, and Proposition 4.1.2 applies. Therefore we can assume that $l + a + b = 0$. But then since the two 3-arcs (x_0, y_0, y_r, y_{r-a}) and (x_0, y_0, y_r, y_{r-b}) both belong to a 5-cycle one can see that $r = a + b$, and thus $l = -r$, which is impossible since $\mathrm{girth}(X) = 5$.

We can conclude that $\mathrm{girth}(X) > 5$. Moreover, since there exist a 6-cycle in X (one of such 6-cycles is $(x_0, y_0, x_{-a}, x_{l-a}, y_l, x_l, x_0)$), we have $\mathrm{girth}(X) = 6$. This implies that

$$r \notin \{\pm l, \pm 2l\} \text{ and } a, b, b - a \notin \{\pm l, \pm 2l, \pm 3l\}. \tag{4.11}$$

Note that $\mathrm{girth}(X) = 6$ implies that no two distinct 6-cycles contain a common 4-arc. Consider the 3-arc $A_0 = (y_0, x_0, x_l, y_l)$. This 3-arc lies on two 6-cycles $(y_0, x_0, x_l, y_l, x_{l-a}, x_{-a}, y_0)$ and $(y_0, x_0, x_l, y_l, x_{l-b}, x_{-b}, y_0)$.

Claim: Each 3-arc of X lies on exactly two 6-cycles.

To prove the claim suppose that there exist an additional 6-cycle C containing the 3-arc A_0. (Therefore, we are supposing that any 3-arc in X lies on at least three 6-cycles.) None of the edges $e \in E(C) \setminus \{\{x_0, y_0\}, \{x_l, y_l\}\}$ containing y_0 or y_l, can be a spoke edge (since no two different 6-cycles in X have a common 4-arc). The only possibility is thus $l = \pm 3r$. Without loss of generality, we may assume that $l = 3r$. Similarly (inspecting the 3-arc (x_0, y_0, y_r, x_r)), we obtain $r = \pm 3l$. Therefore we have $8l = 0$ or $10l = 0$. Also, since $\mathrm{girth}(X) = 6$, l and r are elements of the same order in $\mathbb{Z}_n$, and they induce cycles of the same length k ($k = 8$ or $k = 10$). Consider two such cycles $C_l = (x_0, x_l, \ldots, x_{(k-1)l}, x_0)$ and $C_r = (y_0, y_r, \ldots, y_{(k-1)r}, y_0)$. Observe that 0-spokes induce a matching between C_l and C_r. If there is an edge joining a vertex from C_l and a vertex from C_r "different from a 0-spoke", then there is another matching between C_l and C_r, and it is not difficult to see that in this case there exists a cycle in X of length less than 6. Therefore, two different spoke edges with the same endvertex cannot occur between C_l and C_r. The same holds for arbitrary two cycles of length k induced by l and r, respectively.

Suppose first that $r = 3l$. Consider the 3-arc $B_0 = (x_0, x_l, y_l, y_{4l})$. Since there is a 6-cycle $(x_0, x_l, y_l, y_{4l}, y_{7l}, x_{-l})$, then there are at least two other 6-cycles containing B_0, none of which contains edges $\{x_0, x_{-l}\}$ or $\{y_{4l}, y_{7l}\}$. If there exists a 6-cycle containing B_0 and the edge $\{x_0, y_0\}$, then if the remaining vertex at this 6-cycle is x_t then x_t is adjacent to both y_0 and y_{4l}. However, y_0 and y_{4l} lie at the same cycle of length 8 induced by $r = 3l$, and we have a contradiction with the argument from the previous paragraph. Similarly, there cannot exist a 6-cycle containing B_0 and edge $\{y_{4l}, x_{4l}\}$. Therefore, either $\{x_{4l-a}, y_a\}$ and $\{x_{4l-b}, y_b\}$ are the edges which lie on 6-cycles containing B_0, or $\{x_{4l-a}, y_b\}$ and $\{x_{4l-b}, y_a\}$ are the edges which lie on 6-cycles containing B_0. In the former case, we have $2a - 4l \in \{b, 0\}$ and $2b - 4l \in \{a, 0\}$. If $2a - 4l = b$ and $2b - 4l = a$, then $b = -a$, and we obtain a 4-cycle $(x_0, y_a, x_a, y_0, x_0)$, a contradiction. If $2a - 4l = 0$ and $2b - 4l = a$, then $4a = 0$ and $8b = 0$, therefore $n = 8$, a contradiction with Proposition 4.1.2. Similarly, if $2a - 4l = 0$ and $2b - 4l = 0$, or $2a - 4l = b$ and $2b - 4l = 0$, we obtain $n = 8$, which is impossible. If the latter holds, then $\{x_{4l-a}, y_b\}$ must be a 0-spoke, hence $a+b=4l$. Considering possible 6-cycles containing 3-arc (x_0, x_l, y_l, y_{-2l}), contradictions follows straightforward. This shows that $r=3l$ is impossible.

Suppose now that $r=-3l$. Consider the 3-arc $B=(y_0, x_0, x_l, x_{2l})$. It lies on at least three 6-cycles of X. Therefore, besides $(y_0, x_0, x_l, x_{2l}, x_{3l}, y_{3l}, y_0)$ there are at least two other 6-cycles, say D_1 and D_2, containing B. Moreover, since no two different 6-cycles in X have a common 4-arc, none of D_1 and D_2 contains x_{3l}. It is not difficult to see that y_{2l} is not contained in neither D_1 or D_2. Namely, if this was the case, say that D_1 contains y_{2l}, then the remaining vertex on D_1 cannot be right hand vertex, since the distance from y_0 and y_{2l} using only right edges is 4. On the other hand, if the remaining vertex on D_1 is left hand vertex, say x_t, then we obtain two matchings between the cycle induced by l which contains x_t and the cycle induced by r which contains y_0. Hence, we can assume that D_1 contains y_{2l+a}, and that D_2 contains y_{2l+b}. Observe that none of D_1 and D_2 contains right hand edges, since otherwise there would be a 0-spoke and an a-spoke between two cycles induced by l and r. Therefore, $D_1=(y_0, x_0, x_l, x_{2l}, y_{2l+a}, x_i, y_0)$ and $D_2=(y_0, x_0, x_l, x_{2l}, y_{2l+b}, x_j, y_0)$. Applying the same argument to the 3-arc $B'=(y_0, x_0, x_{-l}, x_{-2l})$, we see that $D_1'=(y_0, x_0, x_{-l}, x_{-2l}, y_{-2l+a}, x_{i'}, y_0)$ and $D_2'=(y_0, x_0, x_{-l}, x_{-2l}, y_{-2l+b}, x_{j'}, y_0)$ are 6-cycles in X, where $\{i', j'\}=\{i, j\}$. If $i'=i$ and $j'=j$, then we obtain that y_{-2l+a} and y_{2l+a} have a common neighbor x_i. This further implies that $b=\pm 4l$. Similarly, y_{-2l+b} and y_{2l+b} have the common neighbor x_j, which implies that $a=\pm 4l$. It follows that $\mathbb{Z}_n=\langle l\rangle$, implying that X is isomorphic to a Tabačjn graph. But this is impossible, since, by [8, Lemma 3.2], there are no 3-arc-transitive Tabačjn graphs. If, however, $i'=j$ and $j'=i$ then the vertices y_0, y_{-2l+b} and y_{2l+a} have a common neighbor x_i, while the vertices y_0, y_{-2l+a} and y_{2l+b} have a common neighbor x_j. The vertex y_0 can be adjacent to x_i via a-spoke or via b-spoke. If y_0 is adjacent to x_i via a-spoke, then we have the following: $\{x_i, y_{2l+a}\}$ and $\{x_j, y_0\}$ are b-spokes, $\{x_i, y_{-2l+b}\}$ and $\{x_j, y_{-2l+a}\}$ are 0-spokes and $\{x_j, y_{2l+b}\}$ is an a-spoke. Considering 2-arcs (y_{2l+a}, x_i, y_0) and $(y_{-2l+a}, x_j, y_{2l+b})$ gives $2l+a-b+a=0$ and $-2l+a-0+a=2l+b$. These two equations combined together imply that $6l=0$, which is impossible, since l is of order 10 in $\mathbb{Z}_n$. The same argument gives a contradiction if $\{y_0, x_i\}$ is a b-spoke. Therefore, we can conclude that each 3-arc lies on precisely two 6-cycles as claimed.

Now again consider the 3-arc $A=(x_0, x_l, x_{2l}, x_{3l})$. It is contained in exactly two 6-cycles, say C_1 and C_2. This implies that $r\neq \pm 3l$ (otherwise A would be contained on at least three different 6-cycles, namely the ones using two 0-spokes, two a-spokes and two b-spokes, respectively). We distinguish three cases depending on whether the vertices x_{-l} and x_{4l} are contained on one of C_1 and C_2 or not.

Case 1. C_1 or C_2 contains both of x_{-l} and x_{4l}.

Then we may assume that $C_1=(x_{-l}, x_0, x_l, x_{2l}, x_{3l}, x_{4l}, x_{-l})$, and therefore $6l=0$. Since no 4-arc in X belongs to two different 6-cycles the 6-cycle C_2 must contain two right vertices. In particular, we may, without loss of generality, assume that $C_2=(x_0, x_l, x_{2l}, x_{3l}, y_{3l}, y_a, x_0)$, and so $a=3l-r$. However, since $6l=0$, we obtain a third 6-cycle containing A, namely $(x_0, x_l, x_{2l}, x_{3l}, y_{-r}, y_0, x_0)$, contradicting Claim 1.

Case 2. C_1 or C_2 contains one of x_{-l} and x_{4l}.

Then we may, without loss of generality, assume that $C_1 = (x_0, x_l, x_{2l}, x_{3l}, x_{4l}, y_{4l}, x_0)$, and thus $4l = a$. It follows that $C_2 = (x_{-l}, x_0, x_l, x_{2l}, x_{3l}, y_{3l}, x_{-l})$. Recall also that this implies that there exist no 6-cycle containing A and two right vertices. In particular, the equation $3l+i\pm r-j=0$, where $i,j \in \{0, 4l, b\}$, thus has no solution. In addition to (4.11) we thus also have

$$r \notin \{\pm 3l, \pm 7l, \pm(b-l), \pm(b+3l), \pm(b-3l), \pm(b-7l)\}. \tag{4.12}$$

Consider now the 3-arc

$$B' = (x_0, x_l, y_l, y_{l+r}),$$

and let C_1' and C_2' be the two 6-cycles in X containing B'. Since none of the two 6-cycles containing $(x_l, y_l, y_{r+l}, x_{r+l})$ contains x_0, none of C_1' and C_2' contains x_{r+l}. Thus two of the vertices $x_{r+l-a} = x_{r-3l}, x_{r+l-b}$ and y_{2r+l} belong to C_1' and C_2', one to C_1' and one to C_2'. Similarly, none of C_1' and C_2' can contain y_0, and so two of the vertices x_{-l}, y_{4l} and y_b belong to C_1' and C_2', one to each.

We first show that $x_{r-3l} \notin V(C_1') \cup V(C_2')$. Suppose on the contrary that, say C_1' contains x_{r-3l}. If the remaining vertex v of C_1' is x_{-l}, then $-2l = r-3l$, contradicting (4.11). If $v = y_{4l}$, then x_{r-3l} and y_{4l} are either adjacent by a 0-spoke or by a b-spoke. In the former case $r-3l = 4l$ and in the latter case $r-3l+b = 4l$, both contradicting (4.12). Finally, if $v = y_b$, then x_{r-3l} and y_b are adjacent by a 0-spoke, and so $r-3l = b$, contradicting (4.12). Therefore $x_{r-3l} \notin V(C_1')$. Clearly, the same argument can be used to show that we also have $x_{r-3l} \notin V(C_2')$. Therefore, we may, without loss generality, assume that C_1' contains the vertex x_{r+l-b}. Now, if the remaining vertex v of C_1' is x_{-l}, then $-2l = r+l-b$, contradicting (4.12). Similarly, if $v = y_{4l}$, then x_{r+l-b} and y_{4l} are adjacent via a 0-spoke, so that $r+l-b = 4l$, again contradicting (4.12). Thus $v = y_b$, and so either $r+l-b = b$ or $r+l-b+4l = b$. In other words, we have that either

$$r = 2b - l \quad \text{or} \quad r = 2b - 5l.$$

We can now repeat the argument for the 3-arc (x_0, x_l, y_l, y_{l-r}) to find that either $-r = 2b-l$ or $-r = 2b-5l$ holds. Since $2r \neq 0$, we thus must have that $2b-l = 5l-2b$, that is $4b = 6l$. Thus either $2r = 4b-2l = 4l = a$ or $-2r = 4l = a$, both contradicting $\mathrm{girth}(X) = 6$. This completes the analysis of Case 2.

Case 3. None of C_1 and C_2 contains x_{-l} or x_{4l}.

Then, since $r \neq \pm 3l$, we can assume that $C_1 = (y_0, x_0, x_l, x_{2l}, x_{3l}, y_{a+3l}, y_0)$ and

$$r = a + 3l. \tag{4.13}$$

This of course implies that C_2 contains one of y_a and y_b, and one of y_{3l} and y_{b+3l}. If it contains y_b then, as $r \neq \pm 3l$, we have $C_2 = (y_b, x_0, x_l, x_{2l}, x_{3l}, y_{3l}, y_b)$, which forces $3l \pm r = b$. Moreover (4.13) applies that we cannot have $3l - r = b$, since in this case $b = -a$, and so $(x_0, y_{-a}, x_{-a}, y_0, x_0)$ is a 4-cycle, contradicting $\mathrm{girth}(X) = 6$. Thus $3l + r = b$, and so $b - a = 6l$. If, however, C_2 contains y_a, then it cannot contain y_{3l}. Namely, in this case $a \pm r = 3l$ would have to hold. But (4.13) combined together with $a - r = 3l$ imply $6l = 0$, while combined together with $a + r = 3l$

imply $2a = 0$, the former contradicts the assumption of this case and the latter contradicts girth$(X) = 6$ (as $(x_0, y_a, x_a, y_0, x_0)$ would be a 4-cycle). Thus $C_2 = (y_a, x_0, x_l, x_{2l}, x_{3l}, y_{b+3l}, y_a)$, and so $a \pm r = b + 3l$. If $a + r = b + 3l$ then (4.13) implies $2a = b$, contradicting girth$(X) = 6$ (as $(x_0, y_a, x_a, y_b, x_0)$ would be a 4-cycle). Hence $a - r = b + 3l$, and so $b + 6l = 0$. Therefore, either $b = a + 6l$ or $b = -6l$. However, by Proposition 4.0.4, $BC_n[\{\pm l\}, \{0, a, a+6l\}, \{\pm r\}] \cong BC_n[\{\pm l\}, \{0, a, -6l\}, \{\pm r\}]$, and thus we can assume that $C_2 = (y_a, x_0, x_l, x_{2l}, x_{3l} y_{b+3l})$ and

$$b + 6l = 0. \tag{4.14}$$

Consider now the 3-arc $F = (y_0, y_{3l+a}, y_{6l+2a}, y_{9l+3a})$. Since F lies on exactly two 6-cycles, two of the following six equations must hold

$$\pm l = 3l + 2a, \tag{4.15}$$

$$\pm l = 3l + 3a, \tag{4.16}$$

$$\pm l = 15l + 4a, \tag{4.17}$$

$$\pm l = 9l + 4a, \tag{4.18}$$

$$\pm l = 9l + 2a, \tag{4.19}$$

$$\pm l = 15l + 3a. \tag{4.20}$$

If (4.15) holds then $(x_0, y_a, y_{3l+2a}, x_{3l+2a}, x_0)$ is a 4-cycle in X, a contradiction. If (4.16) and one of (4.17), (4.18) and (4.19) hold then $a = kl$, and therefore $\mathbb{Z}_n = \langle l \rangle$. But then X is isomorphic to a Tabačjn graph, contradicting [8, Lemma 3.2]. The same contradiction is obtained if (4.20) and one of (4.17), (4.18) and (4.19) hold. Suppose now that (4.16) and (4.20) hold. Then $12l \in \{2l, -2l, 0\}$, and so $10l = 0$, $14l = 0$ or $12l = 0$. Since $3a = -3l \pm l$, and $\mathbb{Z}_n = \langle l, a \rangle$ we can conclude that $n \leq 42$, contradicting Proposition 4.1.2. The same contradiction is obtained when (4.17) and (4.18) hold. Suppose next that (4.17) and (4.19) hold. Then $2a \in \{-8l, -6l, -4l\}$. If $2a = -8l$ then since by (4.11) $r \neq \pm l$ we have $a = -4l + n/2$ and $r = -l + n/2$, and so $(x_0, y_{-4l+n/2}, y_{-5l}, x_l, x_0)$ is a 4-cycle in X, a contradiction. If $2a = -6l$ then $2r = 2(3l + a) = 0$, which is clearly impossible. If $2a = -4l$ then since by (4.11) $r \neq \pm l$ we have $a = -2l + n/2$ and $r = l + n/2$, and so $(x_0, y_{-2l+n/2}, y_{-l}, x_{-l}, x_0)$ is a 4-cycle in X, a contradiction. Therefore the only case left to consider is when (4.18) and (4.19) hold. In this case we have $2a = \pm l \pm l$. If $2a = -2l$ then $(x_0, y_a, y_{3l+2a}, x_{3l+2a}, x_0)$ is a 4-cycle in X, a contradiction. If $2a = 0$ then $(x_0, y_a, x_a, y_0, x_0)$ is a 4-cycle in X, a contradiction. If, however, $2a = 2l$ then $11l = \pm l$, and hence $n \leq 12$, contradicting Proposition 4.1.2. This completes the proof that there are no 3-arc-transitive generalized Tabačjn graphs. ■

Theorem 4.1.7 *Let X be an arc-transitive generalized Tabačjn graph. Then X is core-free if and only if $X \cong BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$.*

PROOF. Let X be an arc-transitive generalized Tabačjn graph. Let $2n$ and $m = |Aut(X)_{x_0}|$ be the order of X and the order of the vertex stabilizer of x_0 in $\mathrm{Aut}(X)$, respectively. Since X is core-free, Proposition 2.1.2 implies that $n^2 < |\mathrm{Aut}(X)| = 2nm$, and consequently $n < 2m$. Next, by Theorem 4.1.6, we have that X is s-transitive, for some $s \in \{1, 2\}$. By Proposition 4.1.1, for $s = 2$ we have $m \leq 120$, while for $s = 1$ we have $m \leq 20$. Therefore, if X is 2-transitive then $n < 240$, and if X is 1-transitive then $n < 40$. Using Proposition 4.1.2 we can conclude that X is isomorphic to $BC_3[\{\pm1\}, \{0, 1, 2\}, \{\pm1\}]$, $BC_6[\{\pm1\}, \{0, 2, 4\}, \{\pm1\}]$ or $BC_6[\{\pm1\}, \{0, 1, 5\}, \{\pm2\}]$. However, one can easily see that amongst these three graphs only the graph $BC_3[\{\pm1\}, \{0, 1, 2\}, \{\pm1\}] \cong K_6$ is core-free. ■

We are now ready to classify arc-transitive generalized Tabačjn graphs. The classification gives the graphs appearing in Theorem 4.1.14(ii), and shows that every arc-transitive generalized Tabačjn graph is in fact a Tabačjn graph (see Figure 4.3).

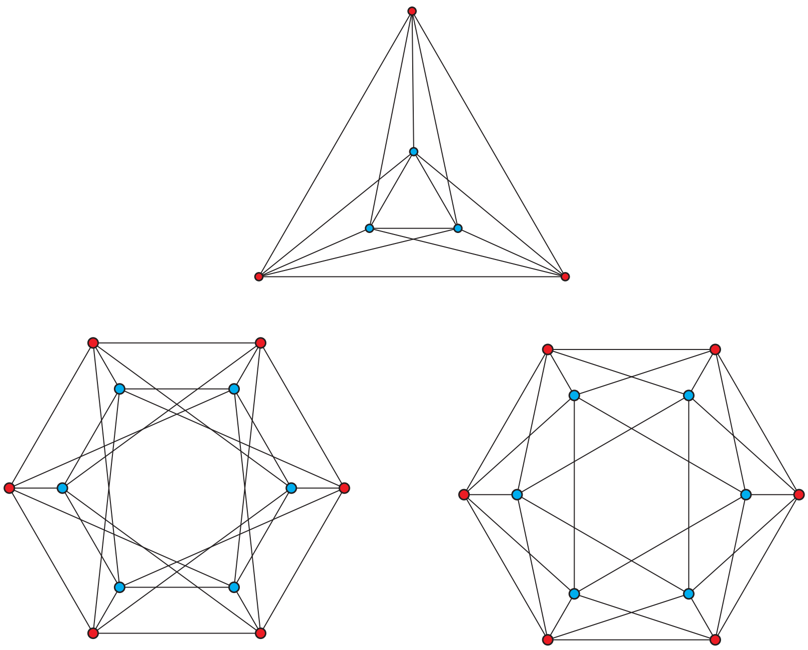

Figure 4.3: Three arc-transitive generalized Tabačjn graphs which are isomorphic to K_3, $K_{6,6} - 6K_2$ and the icosahedron, respectively.

Theorem 4.1.8 *A bicirculant $BC_n[L, M, R]$ with $|M| = 3$ is arc-transitive if and only if it is isomorphic to one of the graphs*

(i) $BC_3[\{\pm1\}, \{0, 1, 2\}, \{\pm1\}] \cong K_6$,

(ii) $BC_6[\{\pm1\}, \{0, 2, 4\}, \{\pm1\}] \cong K_{6,6} - 6K_2$ *or*

(iii) $BC_6[\{\pm1\}, \{0, 1, 5\}, \{\pm2\}]$.

Moreover, the first two graphs are 2-transitive and the third graph is 1-transitive.

Proof. Let $X = BC_n[\{\pm l\}, \{0, a, b\}, \{\pm r\}]$ be an arc-transitive generalized Tabačjn graph with a $(2, n)$-semiregular automorphism ρ giving the prescribed bicirculant structure. If X is core-free then, by Theorem 4.1.7, $X \cong BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$.

Suppose now that X is not core-free. Then there exists a non-trivial subgroup N of $\langle \rho \rangle$ which is normal in $\mathrm{Aut}(X)$. By Lemma 4.1.3, the quotient graph X_N is a connected core-free arc-transitive generalized Tabačjn graph, and hence, by Theorem 4.1.7, it is isomorphic to $X_N \cong BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$. In fact, X is isomorphic to a regular $\mathbb{Z}_m$-cover of this graph, where $|N| = m$. Note also that ρ projects to a $(2, n/m)$-semiregular automorphism of X_N giving the generalized tabačjn structure. (Below, all arithmetic operations are to be taken modulo m if at least one argument is from $\mathbb{Z}_m$ and the symbol mod m is always omitted.)

The graph $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$ is illustrated in Figure 4.4. Let us choose the following automorphisms of $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$

$$\alpha = (y_0\ y_2\ x_2\ x_0\ x_1)(y_1) \text{ and } \beta = (y_0\ y_1\ y_2)(x_0\ x_1\ x_2),$$

and let $G = \langle \alpha, \beta \rangle$. It can be checked directly, using Magma [11], that every $(2, 3)$-semiregular automorphism of $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}]$ is conjugate to β, and that every arc-transitive subgroup of its automorphism group contains the subgroup G. Because of Proposition 2.2.5 we may assume, without loss of generality, that ρ projects to β (therefore, the lifts of β centralize the group N of covering transformations) and that G lifts to a subgroup of $\mathrm{Aut}(X)$.

Any such cover X can be derived from K_6 through a suitable voltage assignment $\zeta : A(K_6) \to \mathbb{Z}_m$. To find this voltage assignment ζ fix the spanning tree T of K_6 as the one consisting of the edges

$$\{y_0, y_1\}, \{y_0, y_2\}, \{y_0, x_0\}, \{x_0, x_1\}, \{x_0, x_2\}$$

(see also Figure 4.4).

There are ten fundamental cycles in K_6, which are generated, respectively, by ten cotree arcs

$$(y_0, x_1),\ (x_0, y_1),\ (y_1, x_1),\ (y_1, x_2),\ (y_1, y_2),$$

$$(y_2, x_2), (y_2, x_0),\ (x_2, y_0),\ (x_2, x_1), \text{ and } (x_1, y_2).$$

Since X is connected, we have $\mathbb{Z}_m = \langle t_i \mid i \in \{1, \ldots, 10\} \rangle$. In addition, by Proposition 2.2.2, α extends to an automorphism α^* of $\mathbb{Z}_m$, and, by Proposition 2.2.4, β^* is the identity automorphism of $\mathbb{Z}_m$. We get from Table 4.1, where all fundamental cycles and their voltages are listed, that $\mathbb{Z}_m = \langle t_3, t_9 \rangle$. In particular, if $t_1 = t$, $t_3 = r$

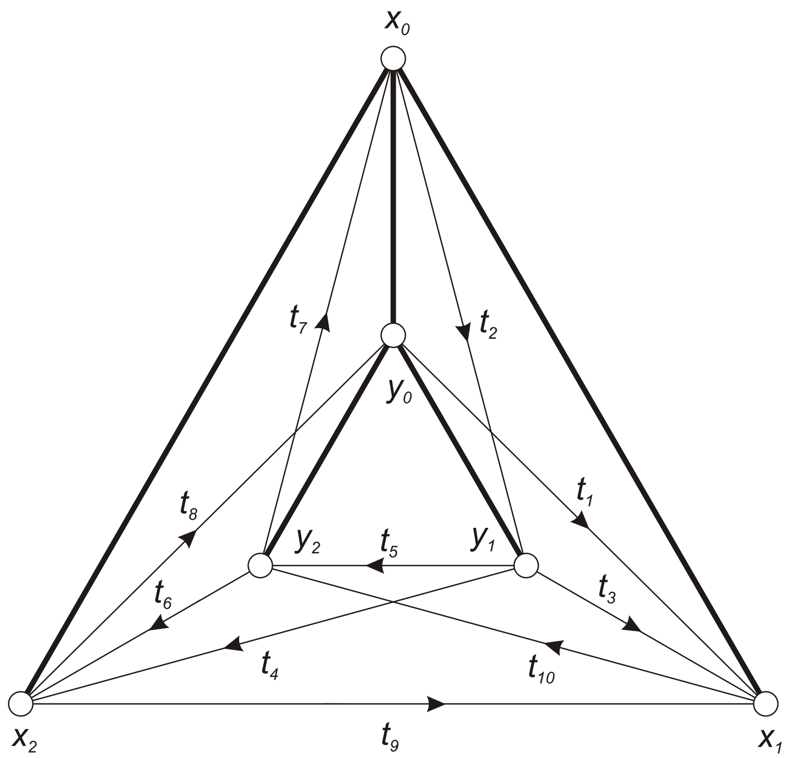

Figure 4.4: The voltage assignment ζ on $BC_3[\{\pm 1\}, \{0, 1, 2\}, \{\pm 1\}] \cong K_6$. The spanning tree consists of undirected bold edges, all carrying trivial voltage.

and $t_9 = k$ then

$$\begin{aligned}
t_1 &= t = \alpha^*(k) \in \langle k \rangle \text{ (by Row 9 of Table 4.1)},\\
t_2 &= -\alpha^*(t) \text{ (applying } \alpha \text{ to Row 11 of Table 4.1)},\\
t_4 &= t - k + r \text{ (by Row 11 of Table 4.1)},\\
t_5 &= 3r - k \text{ (combining together Rows 11, 14 and 17 of Table 4.1)},\\
t_6 &= -r \text{ (by Row 16 of Table 4.1)},\\
t_7 &= t - r \text{ (by Row 17 of Table 4.1)},\\
t_8 &= -\alpha^*(t) + r \text{ (by Row 18 of Table 4.1)},\\
t_{10} &= 2r - k - \alpha^*(t) = 2r - 2t\\
&\quad \text{(by Row 20 and applying } \alpha \text{ to Row 18 of Table 4.1)},\\
t_2 &= -\alpha^*(t) = k - 2t,\\
2t_3 = 2r &= 2k - 2t \in \langle k \rangle \text{ (applying } \alpha \text{ to Row 17 of Table 4.1)}.
\end{aligned}$$

Further, applying α to Row 5 of Table 4.1 we get $2r = -5t+3k$, and so it follows that $3t = k$ and $2r = 4t$. The voltages of the fundamental cycles using these equalities are given in Table 4.2.

		C	$\zeta(C)$	C^α	$\zeta(C^\alpha)$
1	C_1	(y_0, x_1, x_0, y_0)	t_1	(y_2, y_0, x_1, y_2)	$t_1 + t_{10}$
2	C_2	(y_0, x_0, y_1, y_0)	t_2	(y_2, x_1, y_1, y_2)	$-t_{10} - t_3 + t_5$
3	C_3	$(y_0, y_1, x_1, x_0, y_0)$	t_3	$(y_2, y_1, y_0, x_1, y_2)$	$-t_5 + t_1 + t_{10}$
4	C_4	$(y_0, y_1, x_2, x_0, y_0)$	t_4	$(y_2, y_1, x_0, x_1, y_2)$	$-t_5 - t_2 + t_{10}$
5	C_5	(y_0, y_1, y_2, y_0)	t_5	(y_2, y_1, x_2, y_2)	$-t_5 + t_4 - t_6$
6	C_6	$(y_0, y_2, x_2, x_0, y_0)$	t_6	$(y_2, x_2, x_0, x_1, y_2)$	$t_6 + t_{10}$
7	C_7	(y_0, y_2, x_0, y_0)	t_7	(y_2, x_2, x_1, y_2)	$t_6 + t_9 + t_{10}$
8	C_8	(y_0, x_0, x_2, y_0)	t_8	(y_2, x_1, x_0, y_2)	$-t_{10} - t_7$
9	C_9	(x_0, x_2, x_1, x_0)	t_9	(x_1, x_0, y_0, x_1)	t_1
10	C_{10}	$(y_0, x_0, x_1, y_2, y_0)$	t_{10}	$(y_2, x_1, y_0, x_2, y_2)$	$-t_{10} - t_1 - t_8 - t_6$
		C	$\zeta(C)$	C^β	$\zeta(C^\beta)$
11	C_1	(y_0, x_1, x_0, y_0)	t_1	(y_1, x_2, x_1, y_1)	$t_4 + t_9 - t_3$
12	C_2	(y_0, x_0, y_1, y_0)	t_2	(y_1, x_1, y_2, y_1)	$t_3 + t_{10} - t_5$
13	C_3	$(y_0, y_1, x_1, x_0, y_0)$	t_3	$(y_1, y_2, x_2, x_1, y_1)$	$t_5 + t_6 + t_9 - t_3$
14	C_4	$(y_0, y_1, x_2, x_0, y_0)$	t_4	$(y_1, y_2, x_0, x_1, y_1)$	$t_5 + t_7 - t_3$
15	C_5	(y_0, y_1, y_2, y_0)	t_5	(y_1, y_2, y_0, y_1)	t_5
16	C_6	$(y_0, y_2, x_2, x_0, y_0)$	t_6	$(y_1, y_0, x_0, x_1, y_1)$	$-t_3$
17	C_7	(y_0, y_2, x_0, y_0)	t_7	(y_1, y_0, x_1, y_1)	$t_1 - t_3$
18	C_8	(y_0, x_0, x_2, y_0)	t_8	(y_1, x_1, x_0, y_1)	$t_3 + t_2$
19	C_9	(x_0, x_2, x_1, x_0)	t_9	(x_1, x_0, x_2, x_1)	t_9
20	C_{10}	$(y_0, x_0, x_1, y_2, y_0)$	t_{10}	$(y_1, x_1, x_2, y_0, y_1)$	$t_3 - t_9 + t_8$

Table 4.1: Fundamental cycles and their images with corresponding voltages in K_6.

		x	$\alpha^*(x)$
1	t_1	t	$2r - t$
2	t_2	t	$-t$
3	t_3	r	$-r + 2t$
4	t_4	$r - 2t$	$-r$
5	t_5	$3r - 3t$	$t - r$
6	t_6	$-r$	$r - 2t$
7	t_7	$t - r$	$r + t$
8	t_8	$r - 3t$	$t - r$
9	t_9	$3t$	t
10	t_{10}	$2t$	0

Table 4.2: Voltages of the fundamental cycles given by t, k, r and their images under α^*.

From Row 10 of Table 4.2 we have that $2t = 0$, and therefore we have $k = 3t = t$. Combining together Rows 1 and 2 of Table 4.2 we obtain $2r = 0$. Hence, $\mathbb{Z}_m = \langle k, r \rangle$, where $2k = 0$ and $2r = 0$, and therefore $m = 2$ and $k, r \in \{0, 1\}$. One can now easily see that for $(k, r) = (1, 0)$ we have $X \cong BC_6[\{\pm 1\}, \{0, 2, 4\}, \{\pm 1\}]$, while for $(k, r) \in \{(0, 1), (1, 1)\}$ we have $X \cong BC_6[\{\pm 1\}, \{0, 1, 5\}, \{\pm 2\}]$. ∎

4.1.4 Bipartite dihedrants

As observed in Section 4.1 a bicirculant $BC_n[\emptyset, M, \emptyset]$ is a bipartite Cayley graph on the dihedral group D_{2n}, that is, a bipartite dihedrant. To obtain the classification of pentavalent arc-transitive bipartite dihedrants the following lemma (which considers prime-valent bipartite dihedrants) will be useful.

Lemma 4.1.9 *Let $X = Cay(D_{2n}, S)$ be a p-valent arc-transitive bipartite dihedrant, where p is a prime and $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$. If $H = \langle a\rangle$ is a normal subgroup of* $\mathrm{Aut}(X)$*, then $X \cong Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, \ldots, ba^{r^{p-2}+\ldots+r+1}\})$, where $r \in \mathbb{Z}_n^*$ is such that $r^{p-1} + \ldots + r + 1 \equiv 0 \pmod n$.*

Proof. Let $X = Cay(D_{2n}, S)$, where $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$ and $|S| = p$, where p is a prime, and let $H = \langle a\rangle$ be a normal subgroup of $\mathrm{Aut}(X)$. Consider the quotient graph $Y = X_H$. Observe that Y is a multigraph with 2 vertices and p parallel edges (the so-called *dipole* graph $\mathcal{D}_p$), and X is a regular $\mathbb{Z}_n$-cover of Y.

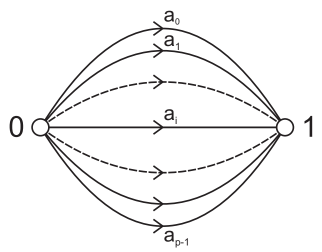

Figure 4.5: The dipole graph $\mathcal{D}_p$ with p parallel edges.

Let $V(Y) = \{0, 1\}$ and let $a_0, a_1, \ldots, a_{p-1}$ be the p parallel arcs (directed from 0 to 1) in Y (see Figure 4.5). Let $\alpha = (a_0\, a_1\, \ldots\, a_{p-1})$ be an automorphism of Y cyclically permuting these arcs. Since X is arc-transitive, the automorphism α must lift. Choose a_0 to be the spanning tree of Y, and let $t_1, t_2, \ldots, t_{p-1}$ be the voltages of the arcs $a_1, a_2, \ldots, a_{p-1}$, respectively. The fundamental cycle containing the arc a_i is a 2-cycle consisting of a_i and a_0^- (that is, the opposite arc of a_0). The image of this cycle under α is the 2-cycle $[a_{i+1}, a_1^-]$ if $i \in \{1, \ldots, p-2\}$, and $[a_0, a_1^-]$ if $i = p-1$. Therefore, α^*, which is an automorphism of $\mathbb{Z}_n^*$, maps the voltages $t_i \in \mathbb{Z}_n$ by the following rule

$$\alpha^*(t_i) = t_{i+1} - t_1, \quad (i \in \{1, \ldots, p-2\}) \tag{4.21}$$

$$\alpha^*(t_{p-1}) = -t_1. \tag{4.22}$$

Since $t_2 = t_1 + \alpha^*(t_1)$, we have $t_2 \in \langle t_1\rangle$. Similarly, one can see that $t_i \in \langle t_1\rangle$, and hence $\mathbb{Z}_n = \langle t_1\rangle$. By Proposition 2.2.3, we can therefore assume that $t_1 = 1$. Let $r \in \mathbb{Z}_n^*$ be such that $\alpha^*(x) = r \cdot x$. Using (4.21) we obtain $t_i = r^{i-1} + \ldots + r + 1$, and combining this with (4.22) we obtain $r^{p-1} + \ldots + r + 1 \equiv 0 \pmod n$. Using the obtained voltages in Y, it is not difficult to see that $X \cong$

$Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, \ldots, ba^{r^{p-2}+\ldots+r+1}\})$. ■

Remark 4.1.10 For arbitrary prime number p there are infinitely many p-valent graphs described in the previous lemma. Namely, if n is a prime such that $n \equiv 1 \pmod p$, then the order of the group $\mathbb{Z}_n^*$ is divisible by p, and hence there exist an element $r \in \mathbb{Z}_n^*$ of order p in $\mathbb{Z}_n^*$. Therefore, $r^p - 1$ is divisible by n, and since $r - 1 < n$ we have $r^{p-1} + \ldots + r + 1 \equiv 0 \pmod n$. Since there are infinitely many primes n such that $n \equiv 1 \pmod p$ (for a fixed p), there are infinitely many p-valent arc-transitive bipartite dihedrants described in the previous lemma.

The following result about pentavalent 2-arc-transitive dihedrants can be extracted from [82, Theorem 1].

Proposition 4.1.11 *Let X be a connected pentavalent 2-arc-transitive dihedrant on $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$. Then one of the following is true:*

(i) X is isomorphic to the complete bipartite graph $K_{5,5}$, or the complete bipartite graph minus a matching $K_{6,6} - 6K_2$, or the incidence graph of projective space $B(PG(2,4))$;

(ii) $\mathrm{core}_{\mathrm{Aut}(X)}(\langle a\rangle) \neq 1$ *and X is a cyclic regular cover of some graph mentioned in (i).*

In Figure 4.3 the graph $K_{6,6} - 6K_2$ is shown with respect to the semiregular automorphism giving rise to the generalized tabačjn structure of this graph. In Figure 4.6, this graph is represented, with respect to the semiregular automorphism giving rise to dihedrant structure.

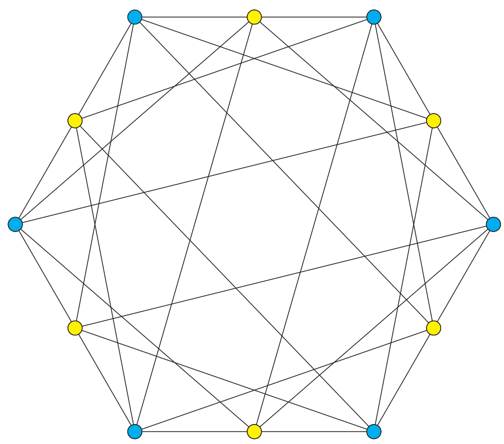

Figure 4.6: Graph $K_{6,6} - 6K_2$ drawn as dihedrant.

The graph $B(PG(2,4))$ is the points-lines incidence graph of the projective space $PG(2,4)$. In particular, consider a 3-dimensional vector space V over a finite field of order 4. Then vertices of $B(PG(2,4)$ are all 1-dimensional and all 2-dimensional subspaces of V, and two vertices are connected, when one is contained in another. This graph is of order 42.

Lemma 4.1.12 *Let X be a connected pentavalent arc-transitive bipartite dihedrant. Then X is core-free if and only if X is isomorphic to the complete bipartite graph minus a matching $K_{6,6}-6K_2$, or to the incidence graph of projective space $B(PG(2,4))$.*

PROOF. With the use of program package Magma [11], one can check that $K_{6,6}-6K_2$ and $B(PG(2,4))$ are core-free bipartite dihedrants. Let X be a core-free connected pentavalent arc-transitive bipartite dihedrant $X = BC_n[\emptyset, M, \emptyset]$ ($|M| = 5$). If X is 2-arc-transitive then, by Proposition 4.1.11, either X is isomorphic to the complete bipartite graph minus a matching $K_{6,6} - 6K_2$ or to the incidence graph of projective space $B(PG(2,4))$. Suppose now that X is not 2-arc-transitive, and let $m = |\mathrm{Aut}(X)_{x_0}|$ be the order of the stabilizer of the vertex $x_0 \in V(X)$ in $\mathrm{Aut}(X)$. Then, by Proposition 4.1.1, $m \leq 20$. Since X is core-free, Proposition 2.1.2 implies that $n < 2m$, and thus $n < 40$. However, with the use of the program package MAGMA [11] one can see that no such graph exist. ∎

Theorem 4.1.13 *Let X be a connected pentavalent arc-transitive bipartite dihedrant $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle$. Then X is isomorphic to one of the following graphs:*

(i) $K_{6,6} - 6K_2$,

(ii) $BC_{12}[\emptyset, \{0,1,2,4,9\}, \emptyset]$,

(iii) $BC_{24}[\emptyset, \{0,1,3,11,20\}, \emptyset]$,

(iv) $B(PG(2,4))$,

(v) $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ *where* $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$, *where* $r \in \mathbb{Z}_n^*$ *such that* $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod n$.

PROOF. Let X be a connected pentavalent arc-transitive bipartite dihedrant on $D_{2n} = \langle a, b \mid a^n = b^2 = (ab)^2 = 1\rangle$. Then a is a $(2,n)$-semiregular automorphism of X. If X is core-free then, by Lemma 4.1.12, either $X \cong K_{6,6} - 6K_2$ or $X \cong B(PG(2,4))$. Suppose now that X is not core-free. Then there exist a subgroup N of $\langle a\rangle$ which is normal in $\mathrm{Aut}(X)$. If $N = \langle a\rangle$ then, by Lemma 4.1.9, $X \cong Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$, where $r \in \mathbb{Z}_n^*$ is such that $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod n$.

If N is a non-trivial subgroup of $\langle \rho\rangle$ then, by Lemma 4.1.3, the quotient graph X_N is a connected pentavalent core-free arc-transitive bipartite dihedrant, and hence, by Lemma 4.1.12, either $X_N \cong K_{6,6} - 6K_2$ or $X_N \cong B(PG(2,4))$. Suppose first that the latter holds. With the use of MAGMA [11] one can see that $B(PG(2,4))$ is 4-transitive, and that there are no s-arc-transitive subgroups of $\mathrm{Aut}(B(PG(2,4)))$ for $s \leq 3$. This implies that X is 4-transitive as well. Since there exist a 6-cycle in X, by [38, Lemma 4.1.3], girth$(X) = 6$, and consequently, by [38, Lemma 4.1.4], X is bipartite with diameter 3. All these combined together imply that X has 42 vertices, and so $X \cong B(PG(2,4))$, a contradiction. We may therefore assume that $X_N \cong K_{6,6} - 6K_2$.

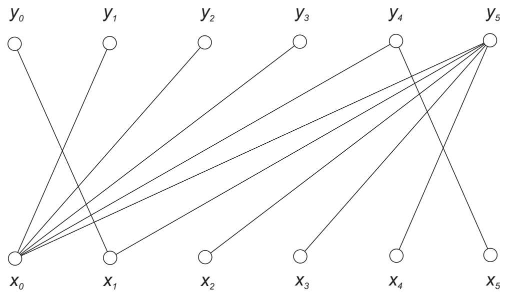

Figure 4.7: The spanning tree in $K_{6,6} - 6K_2$.

Let the vertices of X_N be labeled with x_i and y_i, where $i \in \{0,1,2,3,4,5\}$. Then we can define the edge set of X_N as $E(X_N) = \{\{x_i, y_j\} \mid i,j \in \{0,1,2,3,4,5\}, i \neq j\}$. Let us choose the following automorphisms of X_N

$$\begin{aligned} \alpha &= (x_0)(x_1\ x_2\ x_4\ x_5\ x_3)(y_0)(y_1\ y_2\ y_4\ y_5\ y_3) \text{ and} \\ \beta &= (x_0\ x_1\ x_2\ x_3\ x_4\ x_5)(y_0\ y_1\ y_2\ y_3\ y_4\ y_5), \end{aligned}$$

and let $G = \langle \alpha, \beta \rangle$. It can be checked directly, using Magma, that every $(2,6)$-semiregular automorphism of $X_N \cong K_{6,6} - 6K_2$ giving the "dihedrant" structure is conjugate to β, and that every arc-transitive subgroup of $\mathrm{Aut}(X_N)$ which contains a $(2,6)$-semiregular automorphism giving the "dihedrant" structure contains a subgroup conjugate to G. Fix the spanning tree T of X_N as the one consisting of the edges (see Figure 4.7)

$$\begin{aligned} &\{x_0,y_1\},\{x_0,y_2\},\{x_0,y_3\},\{x_0,y_4\},\{x_0,y_5\},\{x_1,y_0\}, \\ &\qquad \{x_1,y_5\},\{x_2,y_5\},\{x_3,y_5\},\{x_4,y_5\},\{x_5,y_4\}. \end{aligned}$$

The fundamental cycles in X_N are generated, respectively, by the following cotree arcs: $(x_1,y_2),(x_1,y_3),(x_1,y_4),(x_2,y_0),(x_2,y_1),(x_2,y_3),(x_2,y_4),(x_3,y_0),(x_3,y_1),$ $(x_3,y_2),(x_3,y_4),(x_4,y_0),(x_4,y_1),(x_4,y_2),(x_4,y_3),(x_5,y_0),(x_5,y_1),(x_5,y_2),(x_5,y_3)$.

Since X is connected, we have $\mathbb{Z}_m = \langle t_i \mid i \in \{1,\ldots,19\}\rangle$. In addition, α extends to an automorphism α^* of $\mathbb{Z}_m$, and by Proposition 2.2.4, β^* is the identity automorphism of $\mathbb{Z}_m$. We get from Table 4.3 (using Rows 1-19) that $\mathbb{Z}_m = \langle t_1, t_2, t_3, t_5\rangle$, in

		C	$\zeta(C)$	C^β	$\zeta(C^\beta)$
1	C_1	$(x_1, y_2, x_0, y_5, x_1)$	t_1	$(x_2, y_3, x_1, y_0, x_2)$	$t_6 - t_2 - t_4$
2	C_2	$(x_1, y_3, x_0, y_5, x_1)$	t_2	$(x_2, y_4, x_1, y_0, x_2)$	$t_7 - t_3 - t_4$
3	C_3	$(x_1, y_4, x_0, y_5, x_1)$	t_3	$(x_2, y_5, x_1, y_0, x_2)$	$-t_4$
4	C_4	$(x_2, y_0, x_1, y_5, x_2)$	t_4	$(x_3, y_1, x_2, y_0, x_3)$	$t_9 - t_5 + t_4 - t_8$
5	C_5	$(x_2, y_1, x_0, y_5, x_2)$	t_5	$(x_3, y_2, x_1, y_0, x_3)$	$t_{10} - t_1 - t_8$
6	C_6	$(x_2, y_3, x_0, y_5, x_2)$	t_6	$(x_3, y_4, x_1, y_0, x_3)$	$t_{11} - t_3 - t_8$
7	C_7	$(x_2, y_4, x_0, y_5, x_2)$	t_7	$(x_3, y_5, x_1, y_0, x_3)$	$-t_8$
8	C_8	$(x_3, y_0, x_1, y_5, x_3)$	t_8	$(x_4, y_1, x_2, y_0, x_4)$	$t_{13} - t_5 + t_4 - t_{12}$
9	C_9	$(x_3, y_1, x_0, y_5, x_3)$	t_9	$(x_4, y_2, x_1, y_0, x_4)$	$t_{14} - t_1 - t_{12}$
10	C_{10}	$(x_3, y_2, x_0, y_5, x_3)$	t_{10}	$(x_4, y_3, x_1, y_0, x_4)$	$t_{15} - t_2 - t_{12}$
11	C_{11}	$(x_3, y_4, x_0, y_5, x_3)$	t_{11}	$(x_4, y_5, x_1, y_0, x_4)$	$-t_{12}$
12	C_{12}	$(x_4, y_0, x_1, y_5, x_4)$	t_{12}	$(x_5, y_1, x_2, y_0, x_5)$	$t_{17} - t_5 + t_4 - t_{16}$
13	C_{13}	$(x_4, y_1, x_0, y_5, x_4)$	t_{13}	$(x_5, y_2, x_1, y_0, x_5)$	$t_{18} - t_1 - t_{16}$
14	C_{14}	$(x_4, y_2, x_0, y_5, x_4)$	t_{14}	$(x_5, y_3, x_1, y_0, x_5)$	$t_{19} - t_2 - t_{16}$
15	C_{15}	$(x_4, y_3, x_0, y_5, x_4)$	t_{15}	$(x_5, y_4, x_1, y_0, x_5)$	$-t_3 - t_{16}$
16	C_{16}	$(x_5, y_0, x_1, y_5, x_0, y_4, x_5)$	t_{16}	$(x_0, y_1, x_2, y_0, x_1, y_5, x_0)$	$-t_5 + t_4$
17	C_{17}	$(x_5, y_1, x_0, y_4, x_5)$	t_{17}	$(x_0, y_2, x_1, y_5, x_0)$	$-t_1$
18	C_{18}	$(x_5, y_2, x_0, y_4, x_5)$	t_{18}	$(x_0, y_3, x_1, y_5, x_0)$	$-t_2$
19	C_{19}	$(x_5, y_3, x_0, y_4, x_5)$	t_{19}	$(x_0, y_4, x_1, y_5, x_0)$	$-t_3$
		C	$\zeta(C)$	C^α	$\zeta(C^\alpha)$
20	C_1	$(x_1, y_2, x_0, y_5, x_1)$	t_1	$(x_2, y_4, x_0, y_3, x_2)$	$t_7 - t_6$
21	C_2	$(x_1, y_3, x_0, y_5, x_1)$	t_2	$(x_2, y_1, x_0, y_3, x_2)$	$t_5 - t_6$
22	C_3	$(x_1, y_4, x_0, y_5, x_1)$	t_3	$(x_2, y_5, x_0, y_3, x_2)$	$-t_6$
23	C_4	$(x_2, y_0, x_1, y_5, x_2)$	t_4	$(x_4, y_0, x_2, y_3, x_4)$	$t_{12} - t_4 + t_6 - t_{15}$
24	C_5	$(x_2, y_1, x_0, y_5, x_2)$	t_5	$(x_4, y_2, x_0, y_3, x_4)$	$t_{14} - t_{15}$
25	C_6	$(x_2, y_3, x_0, y_5, x_2)$	t_6	$(x_4, y_1, x_0, y_3, x_4)$	$t_{13} - t_{15}$
26	C_7	$(x_2, y_4, x_0, y_5, x_2)$	t_7	$(x_4, y_5, x_0, y_3, x_4)$	$-t_{15}$
27	C_8	$(x_3, y_0, x_1, y_5, x_3)$	t_8	$(x_1, y_0, x_2, y_3, x_1)$	$-t_4 + t_6 - t_2$
28	C_9	$(x_3, y_1, x_0, y_5, x_3)$	t_9	$(x_1, y_2, x_0, y_3, x_1)$	$t_1 - t_2$
29	C_{10}	$(x_3, y_2, x_0, y_5, x_3)$	t_{10}	$(x_1, y_4, x_0, y_3, x_1)$	$t_3 - t_2$
30	C_{11}	$(x_3, y_4, x_0, y_5, x_3)$	t_{11}	$(x_1, y_5, x_0, y_3, x_1)$	$-t_2$
31	C_{12}	$(x_4, y_0, x_1, y_5, x_4)$	t_{12}	$(x_5, y_0, x_2, y_3, x_5)$	$t_{16} - t_4 + t_6 - t_{19}$
32	C_{13}	$(x_4, y_1, x_0, y_5, x_4)$	t_{13}	$(x_5, y_2, x_0, y_3, x_5)$	$t_{18} - t_{19}$
33	C_{14}	$(x_4, y_2, x_0, y_5, x_4)$	t_{14}	$(x_5, y_4, x_0, y_3, x_5)$	$-t_{19}$
34	C_{15}	$(x_4, y_3, x_0, y_5, x_4)$	t_{15}	$(x_5, y_1, x_0, y_3, x_5)$	$t_{17} - t_{19}$
35	C_{16}	$(x_5, y_0, x_1, y_5, x_0, y_4, x_5)$	t_{16}	$(x_3, y_0, x_2, y_3, x_0, y_5, x_3)$	$t_8 - t_4 + t_6$
36	C_{17}	$(x_5, y_1, x_0, y_4, x_5)$	t_{17}	$(x_3, y_2, x_0, y_5, x_3)$	t_{10}
37	C_{18}	$(x_5, y_2, x_0, y_4, x_5)$	t_{18}	$(x_3, y_4, x_0, y_5, x_3)$	t_{11}
38	C_{19}	$(x_5, y_3, x_0, y_4, x_5)$	t_{19}	$(x_3, y_1, x_0, y_5, x_3)$	t_9

Table 4.3: Fundamental cycles and their images with corresponding voltages in $K_{6,6} - 6K_2$.

particular, we have:

$$\begin{aligned}
t_4 &= -t_3 \text{ (by Row 3 of Table 4.3)}\\
t_{19} &= -t_3 \text{ (by Row 19 of Table 4.3)}\\
t_{18} &= -t_2 \text{ (by Row 18 of Table 4.3)}\\
t_{17} &= -t_1 \text{ (by Row 17 of Table 4.3)}\\
t_{16} &= -t_5 - t_3 \text{ (combining Rows 3 and 16 of Table 4.3)}\\
t_{15} &= t_5 \text{ (by Row 16 of Table 4.3)}\\
t_{14} &= t_5 - t_2 \text{ (combining Rows 16 and 19 of Table 4.3)}\\
t_{13} &= -t_1 - t_2 + t_3 + t_5 \text{ (combining Rows 16 and 18 of Table 4.3)}\\
t_{12} &= -t_1 \text{ (combining Rows 3, 16 and 17 of Table 4.3)}\\
t_{11} &= t_1 \text{ (by Row 11 of Table 4.3)}\\
t_{10} &= t_1 - t_2 + t_5 \text{ (combining Rows 12 and 15 of Table 4.3)}\\
t_9 &= -t_2 + t_5 \text{ (combining Rows 12 and 14 of Table 4.3)}\\
t_8 &= -t_2 \text{ (combining Rows 3, 12 and 13 of Table 4.3)}\\
t_7 &= t_2 \text{ (by Row 8 of Table 4.3)}\\
t_6 &= t_1 + t_2 - t_3 \text{ (combining Rows 8 and 11 of Table 4.3).}
\end{aligned}$$

Using the results obtained above we further have:

$$t_2 = -\alpha^*(t_1) \text{ (by Row 30 of Table 4.3)} \tag{4.23}$$

$$t_3 = \alpha^*(t_1) + t_1 \text{ (by Row 20 of Table 4.3)} \tag{4.24}$$

$$t_5 = -\alpha^*(t_2) = (\alpha^*)^2(t_1) \text{ (by Row 26 of Table 4.3).} \tag{4.25}$$

Therefore, $\mathbb{Z}_m = \langle t_1 \rangle$, and, by Proposition 2.2.3, we may assume that $t_1 = 1$ and that $\alpha^*(x) = kx$ for some $k \in \mathbb{Z}_m^*$. We now have $t_2 = -k$, $t_3 = k+1$ and $t_5 = k^2$. Row 22 of Table 4.3 implies that $k^2 = k$, and using Row 27 of Table 4.3 we can conclude that $k^2 = 1$, implying that $k = 1$. Further, Row 20 of Table 4.3 now implies that $4 = 0$, and thus either $m = 2$ or $m = 4$. If $m = 2$ then $X \cong BC_{12}[\emptyset, \{0,1,2,4,9\}, \emptyset]$ and if $m = 4$ then $X \cong BC_{24}[\emptyset, \{0,1,3,11,20\}, \emptyset]$. ■

We are now ready to state the classification of connected pentavalent arc-transitive bicirculants.

Theorem 4.1.14 *A connected pentavalent bicirculant $BC_n[L, M, R]$ is arc-transitive if and only if it is isomorphic to one of the following graphs:*

(i) $BC_6[\{\pm1, 3\}, \{0, 2\}, \{\pm1, 3\}]$, $BC_8[\{\pm1, 4\}, \{0, 2\}, \{\pm3, 4\}]$;

(ii) $BC_3[\{\pm1\}, \{0, 1, 2\}, \{\pm1\}]$, $BC_6[\{\pm1\}, \{0, 2, 4\} \{\pm1\}]$, $BC_6[\{\pm1\}, \{0, 1, 5\}, \{\pm2\}]$;

(iii) $BC_6[\,\emptyset, \{0, 1, 2, 3, 4\}, \emptyset\,]$, $BC_{12}[\,\emptyset, \{0, 1, 2, 4, 9\}, \emptyset\,]$, $BC_{21}[\,\emptyset, \{0, 1, 4, 14, 16\}, \emptyset\,]$, $BC_{24}[\,\emptyset, \{0, 1, 3, 11, 20\}, \emptyset\,]$, *or* $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$ *where* $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$, *and* $r \in \mathbb{Z}_n^*$ *such that* $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod n$.

Proof. Suppose that $X = BC_n[L, M, R]$ is a connected pentavalent arc-transitive bicirculant, where $n \geq 5$. Then, by Theorem 4.1.4, $|M| \in \{2, 3, 5\}$, and therefore Theorems 4.1.5, 4.1.8 and 4.1.13 combined together give that X is isomorphic to one of the graphs given in the statement of the theorem, all of which are clearly arc-transitive. ■

Remark 4.1.15 By Theorem 4.1.13, there exist a $\mathbb{Z}_4$-cover of $K_{6,6} - 6K_2$ which is a 2-arc-transitive bipartite dihedrant. This graph was missing in the classification of 2-arc-transitive dihedrants obtained in [24]. In the communication with the authors of [24], a typing error in the statement of their main result was discovered, and the correct statement is given bellow.

Theorem 4.1.16 [24] *Let X be a dihedrant. Then X is 2-arc-transitive if and only if it is one of the graphs listed bellow:*

(i) cycles C_{2n}, $n \geq 3$;

(ii) complete graphs K_{2n}, $n \geq 3$;

(iii) complete bipartite graphs $K_{n,n}$, $n \geq 3$;

(iv) complete bipartite graphs minus a matching $K_{n,n} - nK_2$, $n \geq 3$;

(v) incidence and nonincidence graphs $B(H_{11})$ and $B'(H_{11})$ of the Hadamard design on 11 points;

(vi) incidence and nonincidence graphs $B(PG(d, q))$ and $B'(PG(d, q))$, with $d \geq 2$ and q a prime power, of projective spaces;

(vii) infinite family of regular $\mathbb{Z}_d$-covers K^{2d}_{q+1} of $K_{q+1,q+1} - (q+1)K_2$, where $q \geq 3$ is an odd prime power and d is a divisor of $q-1$, obtained by identifying the vertex set of the base graph with two copies of the projective line $PG(1, q)$, where the missing matching consists of all pairs of the form $[a, a']$, $a \in PG(1, q)$, and the edge $[a, b']$ carries trivial voltage if a or b is infinity, and carries voltage $h \in \mathbb{Z}_d$, the residue class of $h \in \mathbb{Z}$, if and only if $a - b = \theta^h$, where θ generates the multiplicative group $\mathbb{F}_q^*$ of the Galois field $\mathbb{F}_q$.

4.2 Generalized Cayley graphs

Results of this section are published in [48]. In this section we study generalized Cayley graphs, first introduced in [89]. Let G be a finite group, S a subset of G and α an automorphism of G such that the following conditions are satisfied:

(i) $\alpha^2 = 1$,

(ii) if $g \in G$, then $\alpha(g^{-1})g \notin S$,

(iii) if $f, g \in G$ then $\alpha(f^{-1})g \in S$ implies $\alpha(g^{-1})f \in S$.

Then the *generalized Cayley graph* $X = GC(G, S, \alpha)$ on G with respect to the ordered pair (S, α) is a graph with vertex set G, with two vertices $f, g \in V(X)$ being adjacent in X if and only if $\alpha(f^{-1})g \in S$. In other words, a vertex $f \in G$ is adjacent to all the vertices of the form $\alpha(f)s$, where $s \in S$. Note that (ii) implies that X has no loops, and (iii) implies that X is undirected. Also, in view of (i), the condition (iii) is equivalent to $\alpha(S^{-1}) = S$. Namely, by letting $f = 1$ in (iii), we obtain $\alpha(S^{-1}) = S$, and conversely, if $\alpha(S^{-1}) = S$, then $\alpha(f^{-1})g \in S$ implies that $\alpha(g^{-1}\alpha(f)) = \alpha(g^{-1})f \in S$. For $\alpha = 1$ a generalized Cayley graph $GC(G, S, \alpha)$ is a Cayley graph. Therefore every Cayley graph is also a generalized Cayley graph, but the converse is not true (see [89, Proposition 3.2]).

In [89] properties of generalized Cayley graphs relative to double coverings of graphs are considered, and the following problem, suggesting possible ways of constructing non-Cayley vertex-transitive graphs, is posed.

Problem 4.2.1 *Are there generalized Cayley graphs which are not Cayley graphs, but are vertex-transitive?*

In the following section this problem is solved by giving an example of vertex-transitive generalized Cayley graph which is not a Cayley graph.

4.2.1 The line graph of the Petersen graph

The line graph of the Petersen graph is a vertex-transitive graph but not a Cayley graph. In the example below it is shown that it is isomorphic to a generalized Cayley graph on the cyclic group $\mathbb{Z}_{15}$, which gives an affirmative answer to Problem 5.

Example 4.2.2 Let X be the generalized Cayley graph $GC(\mathbb{Z}_{15}, S, \alpha)$ on the cyclic group $\mathbb{Z}_{15}$ with respect to the subset $S = \{1, 2, 4, 8\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_{15})$ acting according to the rule $\alpha(x) = -4x$ (see Figure 4.8). Observe that the automorphism $\gamma \in \mathrm{Aut}(\mathbb{Z}_{15})$ induced by the element $3 \in \mathbb{Z}_{15}$ (fixed by α) acts semiregularly on the vertex set of X with three orbits of length 5 (see also Lemma 4.2.4).

Let us denote the vertices of X (elements of $\mathbb{Z}_{15}$) in the following way:

$$x_i = 3i + 1, \quad y_i = 3i, \quad z_i = 3i + 2; \quad i \in \{0, 1, 2, 3, 4\}.$$

Then the orbits of the $(3, 5)$-semiregular automorphism γ are $\{x_i \mid i \in \mathbb{Z}_5\}$, $\{y_i \mid i \in \mathbb{Z}_5\}$ and $\{z_i \mid i \in \mathbb{Z}_5\}$, and moreover for each $i \in \{0, 1, 2, 3, 4\}$ we have that

$$\begin{array}{lll} x_i & \text{is adjacent to} & x_{i+1},\ x_{i-1},\ y_i \text{ and } y_{i+4}, \\ y_i & \text{is adjacent to} & x_i,\ x_{i-4},\ z_i \text{ and } z_{i+2}, \text{ and} \\ z_i & \text{is adjacent to} & y_i,\ y_{i-2},\ z_{i+2} \text{ and } z_{i-2}. \end{array}$$

Let $V = \{u_i \mid i \in \mathbb{Z}_5\} \cup \{v_i \mid i \in \mathbb{Z}_5\}$ and $E = \{\{u_i, u_{i+1}\}, \{u_i, v_i\}, \{v_i, v_{i+2}\} \mid i \in \mathbb{Z}_5\}$ be the vertex set and the edge set of the Petersen graph $GP(5, 2)$, respectively. Then the rotation $\rho = (u_0\ u_1\ u_2\ u_3\ u_4)(v_0\ v_1\ v_2\ v_3\ v_4)$ acts semiregularly on the edges of $GP(5, 2)$ with three orbits. Labeling the edges of $GP(5, 2)$ in the following way (see Figure 4.9)

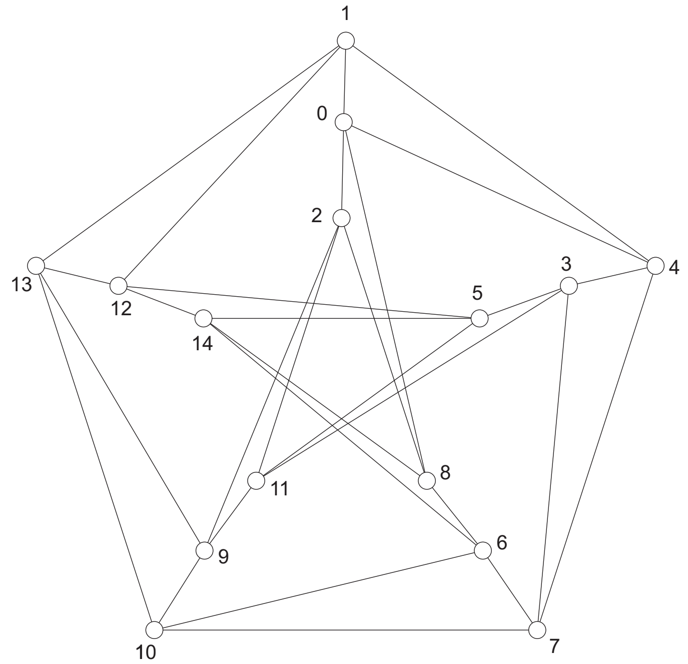

Figure 4.8: The generalized Cayley graph $GC(\mathbb{Z}_{15}, S, \alpha)$, where $S = \{1, 2, 4, 8\}$ and $\alpha(x) = -4x$.

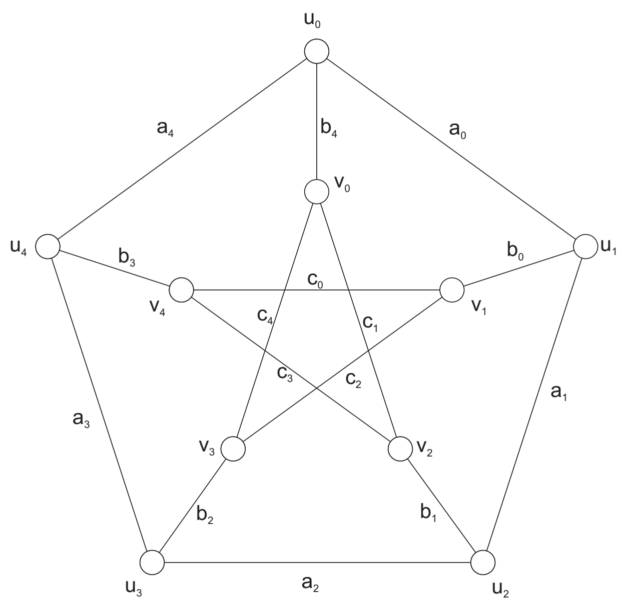

Figure 4.9: Labeling of the Petersen graph.

$$\begin{aligned} a_i &= \{u_i, u_{i+1}\} \quad (\text{for } i \in \mathbb{Z}_5) \\ b_{i+4} &= \{u_i, v_i\} \quad (\text{for } i \in \mathbb{Z}_5) \\ c_i &= \{v_{i-1}, v_{i+1}\} \quad (\text{for } i \in \mathbb{Z}_5) \end{aligned}$$

and considering the structure of the line graph $L(GP(5,2))$ with respect to the $(3,5)$-semiregular automorphism ρ of $L(GP(5,2))$ we see that the adjacencies in $L(GP(5,2))$ are as follows

$$\begin{array}{lll} a_i & \text{is adjacent to} & a_{i+1},\ a_{i-1},\ b_i \text{ and } b_{i+4}, \\ b_i & \text{is adjacent to} & a_i,\ a_{i-4},\ c_i \text{ and } c_{i+2}, \text{ and} \\ c_i & \text{is adjacent to} & b_i,\ b_{i-2},\ c_{i+2} \text{ and } c_{i-2}. \end{array}$$

Now one can easily see that the mapping $f\colon V(X) \to V(L(GP(5,2)))$ defined by the rule

$$\begin{aligned} x_i &\mapsto a_i, \\ y_i &\mapsto b_i, \\ z_i &\mapsto c_i, \end{aligned}$$

for each $i \in \{0,1,2,3,4\}$, is an isomorphism of X into $L(GP(5,2))$.

We wrap up this section by giving two additional vertex-transitive generalized Cayley graphs which are not Cayley graphs. The computations were done by the program package Magma [11].

Example 4.2.3 Using the program package Magma [11], one can see that the generalized Cayley graph $GC(\mathbb{Z}_{45}, S, \alpha)$ with respect to $S = \{1,2,4,8,16,32,19,38,31,17,34,23\}$ and $\alpha \in \mathrm{Aut}(\mathbb{Z}_{45})$ defined by the rule $\alpha(x) = -19x$, and the generalized Cayley graph $GC(\mathbb{Z}_{156}, S, \alpha)$ with respect to $S = \{6,13,48,65,108,150\}$ and $\alpha \in \mathrm{Aut}(\mathbb{Z}_{156})$ defined by the rule $\alpha(x) = 79x$ are both vertex-transitive non-Cayley graphs.

4.2.2 Semiregular automorphisms in generalized Cayley graphs

Throughout this section let $X = GC(G,S,\alpha)$ be the generalized Cayley graph on a group G with respect to the ordered pair (S,α), where $S \subset G$ and $\alpha \in \mathrm{Aut}(G)$ satisfy the assumptions from the definition of generalized Cayley graphs. In the lemma below we show that group elements which are fixed by α act as automorphisms of X. This will then enable us to prove that X admits a semiregular automorphism (see Theorem 4.2.5).

Lemma 4.2.4 *Let $X = GC(G,S,\alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S,α), and let $N = \{g \in G \mid \alpha(g) = g\}$. Then $N_L \leq \mathrm{Aut}(X)$ acts semiregularly on $V(X)$, where N_L denotes the standard left multiplication action of N on G.*

Proof. Let $g \in N$. Then gx is adjacent to gy if and only if $\alpha((gx)^{-1})gy \in S$. Since $\alpha(g) = g$ we have $\alpha((gx)^{-1}g)y = \alpha(x^{-1})y$, and hence $\alpha((gx)^{-1})gy \in S$ if and only if $\alpha(x^{-1})y \in S$, that is, for $g \in N$ the vertex gx is adjacent to gy if and only if x is adjacent to y, implying that $g_L \in \mathrm{Aut}(X)$. ∎

It is known that each finite transitive permutation group contains a fixed-point-free element of prime power order (see [27, Theorem 1]), but not necessarily a fixed-point-free element of prime order (and, hence, a semiregular element); see for instance [15, 27]. As already mentioned, in 1981 Marušič asked if every vertex-transitive digraph admits a semiregular automorphism (see [79, Problem 2.4]). The existence of such automorphisms plays an important role in the proofs of many results concerning some important open problems in algebraic graph theory such as for example the hamiltonicity problem for connected vertex-transitive graphs (see [1, 65, 73]). In 1997 Klin extended this question by asking whether every transitive 2-closed permutation group contains a semiregular element (see [14]). This problem has spurred quite a bit of interest in the mathematical community (see for instance [15, 20, 27, 31, 33, 67]) producing the conjecture, which is usually referred to as the *polycirculant conjecture*, that every finite transitive 2-closed permutation group has a semiregular element. Although not every generalized Cayley graph is vertex-transitive, in some sense, the next theorem gives a new partial affirmative answer to the above conjecture.

Theorem 4.2.5 *Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α). Then X admits a semiregular automorphism which lies in $G_L \leq \mathrm{Aut}(X)$.*

Proof. If there exist $g \in G$ $(g \neq 1)$ such that $\alpha(g) = g$, then, by Lemma 4.2.4, g_L acts semiregularly on X. We may therefore assume that

$$\text{no non-trivial element of } G \text{ is fixed by } \alpha. \tag{4.26}$$

Let $f\colon G \to G$ be a mapping defined by

$$f(x) = \alpha(x^{-1})x.$$

To show that f is a bijection suppose on the contrary that there exist $x, y \in G$ such that $f(x) = f(y)$. Then $\alpha(x^{-1})x = \alpha(y^{-1})y$, and consequently yx^{-1} is fixed by α, that is, $\alpha(yx^{-1}) = yx^{-1}$. The assumption (4.26) then implies that $yx^{-1} = 1$, and so $x = y$, a contradiction. This shows that f is a bijection, and therefore $f(G) = G$, implying that for every element $y \in G$ there exists an element $x \in G$ such that $y = \alpha(x^{-1})x$. But, by (*ii*) from the definition of generalized Cayley graphs, no element in S can be of such a form, implying that S must be an empty set, a contradiction. ∎

As already mentioned, not every generalized Cayley graph is necessarily a Cayley graph. As a direct consequence of Theorem 4.2.5, however, non-Cayley generalized Cayley graphs of prime order do not exist.

Corollary 4.2.6 *Every generalized Cayley graph with prime number of vertices is a Cayley graph.*

We end this section with an interesting property of generalized Cayley graphs that can be extracted from the proof of Theorem 4.2.5.

Corollary 4.2.7 *Let $X = GC(G, S, \alpha)$ be a generalized Cayley graph on a group G with respect to the ordered pair (S, α). Then*

$$|S| \leq |G| - \frac{|G|}{|Fix(\alpha)|}$$

where $Fix(\alpha) = \{g \in G \mid \alpha(g) = g\}$.

4.2.3 Generalized Cayley bicirculants

By Theorem 4.2.5 every generalized Cayley graph admits a semiregular automorphism. It is of particular interest to study generalized Cayley graphs admitting a semiregular automorphism with just two cycles in its cyclic decomposition, that is, the generalized Cayley bicirculants. We note that such generalized Cayley graphs can be constructed from cyclic groups of order 0 (mod 4). In particular, for the cyclic group $G = \mathbb{Z}_{4k}$ the mapping $\alpha\colon G \to G$ defined by the rule

$$\alpha(x) = (2k + 1)x$$

is an involution in $\mathrm{Aut}(G)$ fixing the element $2 \in G$. Since $2 \in G$ is of order $2k$, by Lemma 4.2.4, it gives rise to a $(2, 2k)$-semiregular automorphism in a generalized Cayley graph $GC(\mathbb{Z}_{4k}, S, \alpha)$, where S is an arbitrary subset of G satisfying the assumptions from the definition of generalized Cayley graphs. In particular, one can easily see that the following proposition holds.

Proposition 4.2.8 *For a natural number $n = 4k$ let $X = GC(\mathbb{Z}_n, S, \alpha)$ be a generalized Cayley graph on $\mathbb{Z}_n$ where $\alpha(x) = (2k + 1)x$. Then X is isomorphic to a bicirculant $BC_{2k}[L, M, R]$, where $L = \{s/2 \mid s \in S,\ s\text{-even}\}$, $M = \{(s - 1)/2 \mid s \in S,\ s\text{-odd}\}$, and $R = k + L$.*

Remark 4.2.9 *Observe that every bicirculant $BC_{2k}[L, M, R]$, where $R = L + k$, of order twice an even number is isomorphic to a generalized Cayley graph $GC(\mathbb{Z}_{4k}, S, \alpha)$ on $\mathbb{Z}_{4k}$ with respect to $S = \{2l \mid l \in L\} \cup \{2m + 1 \mid m \in M\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_{4k})$ defined by the rule $\alpha(x) = (2k + 1)x$.*

The next theorem gives an infinite family of vertex-transitive generalized Cayley graphs, which are not Cayley graphs. It is left to the reader to check that α and S given in the statement of this theorem indeed satisfy the conditions from the definition of generalized Cayley graphs.

Theorem 4.2.10 *For a natural number $k \geq 1$ let $n = 2((2k + 1)^2 + 1)$ and let X be the generalized Cayley graph $GC(\mathbb{Z}_n, S, \alpha)$ on the cyclic group $\mathbb{Z}_n$ with respect to $S = \{\pm 2, \pm 4k^2, 2k^2 + 2k + 1\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ defined by the rule $\alpha(x) = ((2k + 1)^2 + 2) \cdot x$. Then X is a non-Cayley vertex-transitive graph.*

Proof. By Proposition 4.2.8,

$$X \cong BC_{((2k+1)^2+1)}[\{\pm 1, \pm 2k^2\}, \{0\}, \{\pm(2k+1), \pm(2k^2+2k)\}].$$

Let $m = n/2$, and let $\rho = (u_0\ u_1 \dots u_{m-1})(v_0\ v_1 \dots v_{m-1})$ be the $(2,m)$-semiregular automorphism of X giving the above described bicirculant structure. Let $U = \{u_i \mid i \in \mathbb{Z}_m\}$ and $V = \{v_i \mid i \in \mathbb{Z}_m\}$ be the two orbits of $\langle\rho\rangle$. We claim that $\{U, V\}$ is an $\mathrm{Aut}(X)$-invariant partition. Let $\gamma \in \mathrm{Aut}(X)$ of X. If for each $u_i \in U$, $\gamma(u_i)$ belongs to U then $\gamma(U) = U$ and $\gamma(V) = V$. Suppose now that for some vertex in U its image under γ belongs to V. Without loss of generality, $\gamma(u_0) = v_i \in V$. Then $\gamma(N(u_0)) = N(v_i)$. Let u_t be one of the four neighbors of u_0 belonging to U, that is, $t \in \{\pm 1, \pm 2k^2\}$. If $\gamma(u_t) \notin V$, then $\gamma(u_t) = u_i$. But then $\gamma(\{u_0, u_t\}) = \{v_i, u_i\}$ does not lie on a 4-cycle in X, which in view of the fact that for any $t \in \{\pm 1, \pm 2k^2\}$ the edge $\{u_0, u_t\}$ lies on a 4-cycle in X is impossible. Therefore, $\gamma(N(u_0) \cap U) = N(v_i) \cap V$. Using the connectedness of the subgraphs of X induced on U and V, respectively, we conclude that $\gamma(U) = V$ and $\gamma(V) = U$. Hence $\{U, V\}$ is an imprimitivity block system for $\mathrm{Aut}(X)$, as claimed. Moreover, by Lemma 2.1.1, the subgroup $N = \langle\rho\rangle$ is normal in $\mathrm{Aut}(X)$.

Let $\tau\colon G \to G$ be defined by the rule

$$\tau(u_j) = v_{(2k+1)j} \text{ and } \tau(v_j) = u_{(2k+1)j}.$$

Then one can easily see that τ is an automorphism of X of order 4. Let $G = \langle\rho, \tau\rangle$ be a subgroup of $\mathrm{Aut}(X)$ generated by ρ and τ. Observe that G acts transitively on $V(X)$, and that $|G| = 4m$.

Let now $\varphi \in \mathrm{Aut}(X)$ be an automorphism of X which fixes u_0. Since U is a block of imprimitivity for $\mathrm{Aut}(X)$ and v_0 is the only neighbor of u_0 outside U we must have $\varphi(v_0) = v_0$. Since $\langle\rho\rangle$ is normal in $\mathrm{Aut}(X)$ there exists a natural number r relatively prime to m such that $\varphi\rho\varphi^{-1} = \rho^r$. Consequently for each $j \in \{1, 2, \dots, m\}$ we have

$$\varphi\rho^j = \rho^{rj}\varphi.$$

This further implies that $\varphi(u_j) = \varphi(\rho^j(u_0)) = \rho^{rj}(u_0) = u_{rj}$, and similarly one can obtain that $\varphi(v_j) = v_{rj}$. Then since $\varphi(u_1) = u_r$ and r is relatively prime to m, it follows that $r \in \{1, -1\}$. If $r = 1$, then φ is the identity, and if $r = -1$, then φ is an involution. This shows that there are only two automorphisms fixing u_0. Since X is vertex-transitive, by the orbit-stabilizer lemma (2.1), we can conclude that $|\mathrm{Aut}(X)| = 2 \cdot |V(X)| = 4m$. This implies that $\mathrm{Aut}(X) = G = \langle\rho, \tau\rangle$. Furthermore, since τ is not acting semiregularly on $V(X)$, there exists no regular subgroup of $\mathrm{Aut}(X)$ of order $2m$, and hence X is not a Cayley graph. ∎

In the next theorem a second infinite family of vertex-transitive generalized Cayley graphs, which are not Cayley graphs, is given. The family consists of bicirculants of valency 6. As before it is left to the reader to check that α and S given in the statement of the theorem indeed satisfy the conditions from the definition of generalized Cayley graphs.

Theorem 4.2.11 *For a natural number k such that $k \not\equiv 2 \pmod 5$, $t = 2k+1$ and $n = 20t$, the generalized Cayley graph $GC(\mathbb{Z}_n, S, \alpha)$ on the cyclic group $\mathbb{Z}_n$ with respect to $S = \{\pm 2t, \pm 4t, 5, 10t-5\}$ and the automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ defined by the rule $\alpha(x) = (10t+1)x$, is a non-Cayley vertex-transitive graph.*

PROOF. By Theorem 4.2.8 we have

$$X \cong BC_{10t}[\{\pm t, \pm 2t\}, \{0, 5t-5\}, \{\pm 3t, \pm 4t\}].$$

Let $\rho = (u_0\, u_1 \ldots u_{10t-1})(v_0\, v_1 \ldots v_{10t-1})$ be a semiregular automorphism of X which generates the described bicirculant structure. Let $U = \{u_0, u_1, \ldots, u_{10t-1}\}$ and $V = \{v_0, v_1, \ldots, v_{10t-1}\}$ be the orbits of $\langle\rho\rangle$. Observe that the left hand side and the right hand side edges all lie on some triangle whereas no spoke lies on a triangle. Namely, if some spoke were on a triangle, then we would have $5t-5 \in \{\pm t, \pm 2t, \pm 3t, \pm 4t\}$. But it is easy to see that none of the arithmetic conditions arising from these conditions is possible since by assumption $t \not\equiv 0 \pmod 5$. Therefore, no automorphism of X can map a left hand side edge or a right hand side edge into a spoke.

Let γ be an automorphism of X. If $\gamma(U) = U$ then also $\gamma(V) = V$. Suppose that $\gamma(U) \neq U$. Without loss of generality, we may assume that $\gamma(u_0) = v_j$. The neighbors of u_0, contained in U, must also be mapped into V, since otherwise, γ would map a left hand side edge into a spoke, which is impossible. Similar reasoning shows that all the neighbors of u_0 contained in V must be mapped into U. Repeating this argument we eventually achieve that $\gamma(U) = V$ and $\gamma(V) = U$. This shows that $\{U, V\}$ is an imprimitivity block system for $\mathrm{Aut}(X)$, and thus, by Lemma 4.2.4, we conclude that $N = \langle\rho\rangle$ is a normal subgroup of $\mathrm{Aut}(X)$.

To see that X is vertex-transitive, it suffices to observe that the mapping $\tau \in \mathrm{Aut}(X)$ defined by the rule

$$\tau(u_i) = v_{ri} \text{ and } \tau(v_i) = u_{ri},$$

where r is relatively prime to $10t$ and such that $r \equiv \pm 3 \pmod{10}$, is always an automorphism of X. Namely, $r = \lambda t + 1$ is relatively prime to t for any λ, and it is possible to choose λ in such a way that $\lambda t + 1 \equiv \pm 3 \pmod{10}$. This automorphism τ interchanges the two orbits of $N = \langle\rho\rangle$, and hence X is vertex-transitive.

To complete the proof we need to show that X is not a Cayley graph. In view of the above it suffices to show that there is no semiregular automorphism interchanging U and V (since U and V are blocks of imprimitivity for $\mathrm{Aut}(X)$). More precisely it suffices to show that there exists no semiregular automorphism mapping u_0 into v_0. Suppose on the contrary that there exists a semiregular automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\varphi(u_0) = v_0$. Then since $N = \langle\rho\rangle$ is normal in $\mathrm{Aut}(X)$, there exists a natural number r relatively prime to $10t$, such that $\varphi\rho\varphi^{-1} = \rho^r$. This implies that for each $i \in \mathbb{Z}_n$ we have $\varphi\rho^i = \rho^{ri}\varphi$, and so $\varphi(u_i) = v_{ri}$. Since $\varphi(u_t) = v_{rt} \in \{v_{3t}, v_{-3t}, v_{4t}, v_{4t}\}$, we have that $r \equiv \pm 3 \pmod{10}$. Moreover, since φ must map the right hand side neighbors of u_0 into the left hand side neighbors of v_0, we have that $\varphi(\{v_0, v_{5t-5}\}) = \{u_0, u_{5t+5}\}$.

Suppose first that $\varphi(v_0) = u_0$. Then in its cycle decomposition φ has one cycle of size 2, and since, by assumption, φ is semiregular, it follows that φ is an involution. But since $r \equiv \pm 3 \pmod{10}$, it follows that $\varphi^2(u_t) = u_{r^2 t} = u_{-t} \neq u_t$, a contradiction.

Suppose now that $\varphi(v_0) = u_{5t+5}$. Then $\varphi(v_i) = u_{ri+5t+5}$, for every $i \in \{0, 1, \ldots, 10t-1\}$. Since φ is semiregular, all of the cycles in its cycle decomposition have the same length $2s$, where s is a divisor of t. Moreover, observe that since $t = 2k+1$ is odd, it follows that s is also odd. Now it is not difficult to verify that

$$\varphi^{2j}(u_i) = u_{(5t+5)(1+r^2+\ldots+r^{2j-2})+r^{2j}i},$$

for $i \in \{0, 1, \ldots, 10t-1\}$ and any natural number j. In particular, we have $\varphi^{2j}(u_0) = u_{(5t+5)(1+r^2+\ldots+r^{2j-2})}$, and since all cycles in the cycle decomposition of φ are of size $2s$ we have

$$(5t+5)(1+r^2+\ldots+r^{2s-2}) \equiv 0 \pmod{10t}.$$

On the other hand, $\varphi^{2s}(u_1) = u_{(5t+5)(1+r^2+\ldots+r^{2s-2})+r^{2s}} = u_{r^{2s}}$, which implies that $r^{2s} \equiv 1 \pmod{10t}$. But this is clearly impossible, since $r^2 \equiv -1 \pmod{10}$ and s is odd.

This shows that there is no semiregular automorphism of X which maps u_0 into v_0, implying that X is not a Cayley graph as claimed. ■

4.3 On prime-valent arc-transitive bicirculants and Cayley snarks

Results of this section are published in [47]. The motivation for this research comes from the well-known conjecture that there are no snarks amongst Cayley graphs. While examples of snarks were initially scarce - the Petersen graph being the first known example - infinite families of snarks are now known to exist. A *Cayley snark* is a cubic Cayley graph which is a snark. Although most known examples of snarks exhibit a lot of symmetry, none of them is a Cayley graph. It was conjectured in [3] that no such graphs exist, but so far only partial results have been obtained (see [46, 94, 97]). The proof of this conjecture would contribute significantly to various open problems regarding Cayley graphs. One such problem is the well-known conjecture that every connected Cayley graph contains a Hamiltonian cycle. Clearly, every hamiltonian cubic graph is 3-edge-colorable. Using results from Section 4.3.1 combined together with certain results on independent sets of vertices in arc-transitive graphs in the larger context of Cayley maps (see [46]), the non-existence of snarks amongst infinitely many cubic Cayley graphs is proved (see Theorem 10).

4.3.1 Prime valent bicirculants

The following relationship between invariant partitions and orbits of $(2,n)$-semiregular elements in transitive groups will be needed in the proof of Theorem 4.3.2. Let G be a transitive permutation group acting imprimitively on the set Ω of cardinality $|\Omega| = 2n$, with a G-invariant partition $\mathcal{B}$ and let $\rho \in G$ be a $(2,n)$-semiregular element of G. Fix a block $B \in \mathcal{B}$ and let $\mathcal{O}$ be an orbit of $H = \langle \rho \rangle$ which has non-empty intersection with B. Then $B \cap \mathcal{O}$ coincides with an orbit of $\langle \rho^i \rangle$, for some $i \in \mathbb{Z}_n$. Consequently, for the other orbit $\mathcal{O}'$ the intersection $B \cap \mathcal{O}'$ is either empty or an orbit of $\langle \rho^i \rangle$. In the proof of Theorem 4.3.2 the following lemma about primitive 1-regular actions will be needed.

Lemma 4.3.1 *Let X be a graph admitting a 1-regular action of a group G with a cyclic vertex stabilizer. If G acts primitively on $V(X)$ then X is a Cayley graph on an elementary abelian p-group.*

Proof. Since G is primitive on $V(X)$ the vertex stabilizer $C = G_v$, $v \in V(X)$, is a maximal subgroup of G, and consequently, since C is abelian, the normalizer of any subgroup $F \leq C$ in G is equal to C, that is $N_G(F) = C$. By [112, Theorem 3.6], $N_G(C)$ is transitive on vertices left fixed by C. Since $N_G(C) = C$ it follows that C fixes only one vertex, that is, G is a Frobenius group. Its Frobenius kernel K is a characteristic subgroup, which is also regular and nilpotent, and thus solvable. Moreover, since G is primitive one can easily see that K is characteristically simple, and thus an elementary abelian p-group (see also [18, Exercise 3.4.6]). ■

We are now ready to prove the main theorem of this section.

Theorem 4.3.2 *Let $X \neq K_4$ be a prime-valent arc-transitive G-bicirculant with $G \leq \mathrm{Aut}(X)$ acting 1-regularly on X. Then X is near-bipartite.*

PROOF. Let $2n$ and q be the order and the valency of $X = BC_n[L, M, R]$, respectively. Further, let $G \leq Aut(X)$ be a 1-regular subgroup containing a $(2, n)$-semiregular authomorphism ρ. Clearly the vertex stabilizer $C = G_v \cong \mathbb{Z}_q$ is cyclic. For $q = 2$, X is a cycle, and thus near-bipartite. For $q = 3$, [93, Theorem 17], [96, Thèoréme 5] and [46, Proposition 8] combined together imply that X is near-bipartite. We may therefore assume that $q \geq 5$.

Let $H = \langle \rho \rangle$. It suffices to consider the case when H is core-free in G. Namely, let $N = \mathrm{core}_G(H)$. If $N = H$ then there are no edges inside the two orbits of H, and therefore X is a bipartite graph. If N is a proper subgroup of H, then we can consider the quotient graph X_N, with respect to the orbits of N on $V(X)$ which is a core-free G'-bicirculant of order $2n/|N|$ with G' acting 1-regularly with cyclic vertex stabilizer. If X_N is near-bipartite then also X is near-bipartite, since we can color all the vertices in the orbit of N corresponding to $v' \in V(X_N)$ with the same color as v'. Since N is a normal subgroup of G there are no edges inside the orbits of N, and thus such a coloring is a proper coloring.

Therefore let H be core-free in G, and let $\mathcal{O}_1$ and $\mathcal{O}_2$ be the two orbits of H. Since G is 1-regular we have $|G| = 2nq$, and thus, by Lemma 2.1.2, $q > n/2$. If G is primitive, then one can see, using the argument from the proof of Lemma 4.3.1, that G is a Frobenius group with an elementary abelian Frobenius kernel containing H. This implies that $n = 2$. But then X is of order 4, and so of valency 2 or 3, contradicting the assumption that $q \geq 5$.

Suppose now that G is imprimitive on $V(X)$ and let $\mathcal{B}$ be a maximal G-invariant partition on $V(X)$. Then the group G/K is primitive on $X_{\mathcal{B}}$, where K is the kernel of the action of G on $\mathcal{B}$. By arc-transitivity of G on X, there are no edges inside the blocks of $\mathcal{B}$. Let $B, B' \in \mathcal{B}$ be two adjacent blocks. The fact that X is of prime valency implies that for any vertex $v \in B$ either v has at most one neighbor in B' or all of its neighbors are in B'. If the latter occurs, the connectedness of X implies that $\mathcal{B} = \{B, B'\}$, and thus X is bipartite. We may therefore assume that the former holds. Depending on whether blocks in $\mathcal{B}$ intersect with both of the orbits of H or with just one of them two different cases need to be considered.

CASE 1. For every orbit $\mathcal{O}$ of H and every block $B \in \mathcal{B}$ we have $|B \cap \mathcal{O}| = d$.

Then each block $B \in \mathcal{B}$ is of size $|B| = 2d$ and the graph $X_{\mathcal{B}}$ is a circulant of order $n/d = 2n/2d$. Since every vertex in B has its neighbors in different blocks of $\mathcal{B}$ the graph $X_{\mathcal{B}}$ is at least of valency $q > n/2$. This implies that $d = 1$, and moreover that $X[B, B'] \cong 2K_2$. Using Lemma 4.3.1, we get that $X_{\mathcal{B}}$ (admitting a primitive group action) is a q-valent circulant of prime order p, and thus, by Proposition 2.2.1, X is a normal circulant $X_{\mathcal{B}} = \mathrm{Cay}(\mathbb{Z}_p, S)$. Since X is a normal circulant the stabilizer of a vertex in its full automorphism group is the group $\mathrm{Aut}(\mathbb{Z}_p, S)$, where $|S| = q > p/2$. By arc-transitivity, $\mathrm{Aut}(\mathbb{Z}_p, S)$ acts transitively on S, and therefore all elements in S have the same order in $\mathbb{Z}_p$. Since $\mathrm{Aut}(\mathbb{Z}_p, S) \leq \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p^*$ and $\mathrm{Aut}(\mathbb{Z}_p, S)$ has more than $p/2$ elements, in follows that $\mathrm{Aut}(\mathbb{Z}_p, S) = \mathrm{Aut}(\mathbb{Z}_p)$. But then $S = \mathbb{Z}_p \setminus \{0\}$, and hence $X_{\mathcal{B}}$ is a complete graph on p vertices, a contradiction (since complete graphs are not normal circulants).

CASE 2. For every block $B \in \mathcal{B}$ there exists an orbit $\mathcal{O}$ of H such that $B \subseteq \mathcal{O}$.

If blocks in $\mathcal{B}$ coincide with the orbits of H, then the quotient graph $X_{\mathcal{B}}$ is isomorphic to K_2, implying that the original graph X is bipartite. Otherwise, since $q > n/2$, the quotient graph $X_{\mathcal{B}}$ is a q-valent G/K-bicirculant of order $2n/|B|$, $B \in \mathcal{B}$, with G/K acting primitively on $V(X_{\mathcal{B}})$. But then, by Lemma 4.3.1, $X_{\mathcal{B}} \cong K_4$, and so $q \leq 3$, a contradiction. ∎

Remark 4.3.3 *There exist infinite many prime-valent G-bicirculants which are $(G,1)$-regular, see, for example, [28, 59, 68].*

4.3.2 Cayley snarks

As cubic graphs are of even order the existence of a Hamiltonian cycle in a cubic Cayley graph implies that the graph is not a snark for we can color the edges of the Hamiltonian cycle with two colors and the remaining edges with the third color. In particular, the conjecture regarding non-existence of Cayley snarks is essentially a weaker form of the folklore conjecture that every connected Cayley graph with order greater than 2 possesses a Hamiltonian cycle, which is in fact a Cayley variant of Lovász's conjecture [73] that every connected vertex-transitive graph possesses a Hamiltonian path.

For a cubic Cayley graph $\mathrm{Cay}(G, S)$ the generating set S is of two forms: either it consists of three involutions or it consists of an involution, a non-involution and its inverse. In the first case, $\mathrm{Cay}(G, S)$ is clearly 3-edge-colorable, and we may therefore restrict ourselves to the study of cubic Cayley graphs with respect to generating sets with a single involution. More precisely, we may consider Cayley graphs X arising from groups having a $(2, s, t)$-generation, that is, from groups $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$. Here "$\ldots$" denotes the extra relations needed in the presentation of the group. Such graphs are called $(2, s, t)$-*Cayley graphs*. If s is even, then a $(2, s, t)$-Cayley graph is not a snark, since in this case the set of edges $\{\{g, gx\} \mid g \in G\}$ obviously forms an even 2-factor in X, that is a 2-factor consisting of cycles of even length, and thus the edges on this 2-factor can be colored with two colors and the remaining edges with the third color.

In [46] methods to construct even 2-factors in cubic Cayley graphs are given. These methods are a generalization of the methods in [37], and later also used in [34, 35, 36] where the hamiltonicity problem for $(2, s, 3)$-Cayley graphs was considered. Let $s \geq 3$ and $t \geq 3$ be positive integers and let $X = \mathrm{Cay}(G, \{a, x, x^{-1}\})$ be a $(2, s, t)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^t = 1, \ldots \rangle$. The graph X is cubic and has a canonical Cayley map $\mathcal{M}(X)$ given by an embedding in the closed orientable surface of genus $g = 1 + |G| \cdot (1/4 - 1/2s - 1/2t)$ with $|G|/s$ disjoint s-gons and $|G|/t$ $2t$-gons as the corresponding faces. This map is given using the same rotation of the x, a, x^{-1} edges at every vertex and results in one s-gon and two $2t$-gons adjacent to each vertex. A $2t$-*gonal graph* $O(X)$ is associated with this $(2, s, t)$-Cayley graph X in the following way. Its vertex set consists of all $2t$-gons in $\mathcal{M}(X)$ arising from the relation $(ax)^t = 1$, where two such $2t$-gons are adjacent if they share an edge in X. The graph $O(X)$ may also be seen as the *orbital graph* of the left action of G on the set $\mathcal{C}$ of left cosets of the subgroup $C = \langle ax \rangle$, arising from the suborbit $\{aC, caC, c^2aC, c^3aC, \ldots, c^{t-1}aC\}$ of length t, where $c = ax$. In particular, the following results can be extracted from [46, Proposition 7] and [46, Theorem 1], respectively.

Proposition 4.3.4 [46] *Let X be a $(G,1)$-regular graph of valency q, $q \geq 3$ a prime. Then X can be constructed via a Cayley graph on the group G with respect to its $(2,s,q)$-generation. In particular, X is isomorphic to a $2q$-gonal graph of a $(2,s,q)$-Cayley graph on the group G.*

Proposition 4.3.5 [46] *Let $X = Cay(G, \{a, x, x^{-1}\})$ be a $(2,s,q)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^q = 1, \ldots \rangle$, $s \geq 3$, $q \geq 3$ a prime, and let $O(X)$ be the corresponding $2q$-gonal graph of X. If $O(X)$ is near-bipartite, then X is not a snark.*

Theorem 4.3.6 *Let $X = Cay(G, \{a, x, x^{-1}\})$ be a $(2,s,q)$-Cayley graph on a group $G = \langle a, x \mid a^2 = x^s = (ax)^q = 1, \ldots \rangle$, $s \geq 3$, $q \geq 3$ a prime, and let $O(X)$ be the corresponding $2q$-gonal graph of X. If $O(X)$ is a G-bicirculant, then X is not a snark.*

Proof. By Proposition 4.3.4, $O(X)$ is a $(G,1)$-regular graph of valency q. Since $O(X)$ is a G-bicirculant, Theorem 4.3.2 implies that $O(X)$ is either isomorphic to the complete graph K_4 or it is near-bipartite. In the first case, $|G| = 12$ and so, by [65, Theorem 1.2], X is not a snark. In the second case X is not a snark by Proposition 4.3.5. ∎

Chapter 5

Half-arc-transitive graphs

Results of this chapter are published in [49]. Half-arc-transitive graphs have been defined in Section 2.2.1. In [106] Tutte observed that the valency of a half-arc-transitive graph is even. In [12] Bouwer gave a construction of a half-arc-transitive graph of valency $2k$ for any integer $k \geq 2$. (Note that the smallest half-arc-transitive graph is the Doyle-Holt graph [4, 21, 43], a quartic graph of order 27.)

Half-arc-transitive graphs, quartic half-arc-transitive graphs in particular, and graphs admitting half-arc-transitive group actions in general have recently become an active topic of research. Various constructions of such graphs together with their structural properties are known, and a classification of certain restricted families of half-arc-transitive graphs has also been obtained, see [5, 25, 63, 70, 71, 75, 81, 84, 90, 101, 102, 104, 105, 107, 108, 109, 113, 119]. There are several approaches used in that respect, ranging from more algebraic in nature, such as the investigation of (im)primitivity (of half-arc-transitive group actions on graphs), to those which are more geometric and/or combinatorial in nature, such as for example the reachability relation approach, explained in Section 5.1.

5.1 Alternets

Let X be a graph of valency $2k$, $k \geq 2$, admitting a half-arc-transitive action of a subgroup $G \leq \mathrm{Aut}(X)$. Further, let $D_G(X)$ be one of the two oppositely oriented digraphs associated to X with respect to the action of G. (Choose the orientation on an arbitrary edge, the action of G then defines the orientations of the remaining edges. These two digraphs correspond to two paired orbital digraphs associated with G.) We shall say that two directed edges are "related" if they have the same initial vertex, or the same terminal vertex. This gives rise to an equivalence relation, called the *reachability relation* (see [16, 77] where this concept is considered in a larger context of infinite graphs). The subgraphs consisting of equivalence classes of directed edges of the reachability relation are called *G-alternating cycles* when X is of valency 4 (see [81]), and *G-alternets* in the general case [113]. Clearly, the alternets are blocks of imprimitivity for the action of $\mathrm{Aut}(D_G(X))$ on the edges of $D_G(X)$.

Given an arc (u, v) of $D_G(X)$, the vertices u and v are called its *tail* and *head*, respectively. Let $\mathcal{A} = \{A_i \mid i \in \{1, 2, \ldots, k\}\}$ be the corresponding set of alternets in X. All alternets have the same even length, half of which is called the *G-radius* of X. For each i, define the *head set* H_i to be the set of all vertices at the heads of arcs in $A_i \in \mathcal{A}$ and the *tail set* T_i to be the set of all vertices at the tails of arcs in $A_i \in \mathcal{A}$. The head sets as well as the tail sets partition $V(X)$. With a slight generalization of [81], where these concepts were introduced for graphs of valency 4 with at least three alternating cycles, we call the intersection $A = A_{i,j} = H_i \cap T_j$, when non-empty, a *$G$-attachment set*. Observe that all attachment sets have the same cardinality, called the *G-attachment number* of X. If $A_{i,j}$ is a singleton then X is said to be *G-loosely attached*. On the other hand, if $A_{i,j} = H_i = T_j$ then X is said to be *G-tightly attached*. (In all of the above concepts the symbol G is omitted when the group G is clear from the context.)

The case of only one alternet, that is, the case where the reachability relation is universal, has been given considerable attention, leaving many open questions. In particular, half-arc-transitive graphs with a primitive automorphism group are necessarily of this kind. Such graphs were studied in [70] with a construction of an infinite family of graphs of valency 14. Further, it was shown there that no primitive half-arc-transitive graphs of valency less than or equal to 8 exist, leaving the valencies 10 and 12 open. More recently, an infinite family of half-arc-transitive graphs of valency 12, with one alternet and an imprimitive group of automorphisms was constructed in [63]. No examples of half-arc-transitive graphs with one alternet and valency less than 12 are known to the author. The existence of half-arc-transitive graphs with a prescribed number of alternets $n \geq 3$ follows, for example, from [81] where the classification of all quartic tightly attached half-arc-transitive graphs of odd radius is given (see also [101] for the classification of such graphs of even radius). Examples of half-arc-transitive graphs with two alternets are given in Example 5.2.4. Note that such graphs are necessarily tightly attached, as is proved in Theorem 11 in the more general contexts of graphs admitting a half-arc-transitive group actions. The same holds for the case of three alternets (see Theorem 12). As for four and five alternets, graphs admitting a half-arc-transitive group action with respect to which they are not tightly attached, do exist and are covers of the rose window graph $R_6(5, 4)$ and the graph X_5 defined in Example 5.3.1, respectively (see Theorems 5.3.2 and 5.3.5). Finally, we show that the number of alternets in the direct product $X \times Y$ of an arc-transitive graph X and a half-arc-transitive graph Y is the same as the number of alternets in Y if X is non-bipartite, and twice the number of alternets in Y if X is bipartite (see Theorem 5.3.6). This allows constructions of infinitely many half-arc-transitive graphs with a prescribed number $n \geq 2$ of alternets (see Remark 5.3.7) provided one such graph exist.

5.2 Two and three alternets

In this section it is proved that every half-arc-transitive graph with two or three alternets is tightly attached.

Theorem 5.2.1 *Let X be a G-half-arc-transitive graph with two alternets. Then the group G contains a normal subgroup H of index 2 having two orbits on vertices* *as well as on edges of X.*

PROOF. Let A_1 and A_2 be the two alternets of X. Then $\{A_1, A_2\}$ is a G-invariant partition. Define $H = \{g \in G \mid A_1^g = A_1\}$ as to be the stabilizer of A_1 in G. Then $[G : H] = 2$, since for any $g_1, g_2 \in G \setminus H$ we have $g_1H = g_2H$. Namely, if $g_1, g_2 \notin H$, then $A_2^{g_2} = A_1$ and $A_2^{g_1} = A_1$. Hence $A_1^{g_2^{-1}g_1} = A_2^{g_1} = A_1$ and so $g_2^{-1}g_1 \in H$. Therefore $g_1H = g_2H$.

It is easy to see that H has two orbits on $V(X)$, corresponding to the sets of all heads and tails of arcs in A_1. Further, H has two orbits on $E(X)$, one orbit containing all incoming edges into heads of arcs in A_1 and the second orbit contains all outgoing edges from tails of arcs in A_1. ∎

From [85, Lemma 2.6] the following proposition about bipartite half-arc-transitive graphs can be extracted.

Proposition 5.2.2 *Let X be a bipartite G-half-arc-transitive graph. Then the corresponding reachability relation is not universal.*

The following theorem gives an infinite family of graphs admitting a half-arc-transitive group action with two alternets. However, all these graphs are arc-transitive. Namely, by [81, Proposition 2.2], a bipartite graph having an automorphism ρ such that $\langle\rho\rangle$ has two orbits on the vertex set of the graph of length more than or equal to 2 cannot be half-arc-transitive. Also, for $p = 5$ the corresponding graph coincide with the Cayley graph of the cyclic group $\mathbb{Z}_{10}$ relative to $\{\pm1, \pm3\}$ given in [81, Proposition 2.4(ii)].

Theorem 5.2.3 *The graph $K_{p,p} - pK_2$, p an odd prime, admits a half-arc-transitive group action with two alternets.*

PROOF. Observe that $X = K_{p,p} - pK_2 \cong Cay(\mathbb{Z}_{2p}, \{\pm1, \pm3, \ldots, \pm(p-2)\})$, and that the mapping $\rho : \mathbb{Z}_{2p} \to \mathbb{Z}_{2p}$ defined by the rule $\rho(x) = x + 2$, $x \in \mathbb{Z}_{2p}$ is an automorphism of X of order p.

Let a be a generator of $\mathbb{Z}_{2p}^*$, and let $\alpha : \mathbb{Z}_{2p} \to \mathbb{Z}_{2p}$ be the mapping defined by the rule $\alpha(x) = 1 + ax$. Then we have

$$\alpha^k(x) = \sum_{j=0}^{k-1} a^j + a^k x.$$

Since $\langle a\rangle = \mathbb{Z}_{2p}^*$ it follows that $a^{p-1} = 1$, implying that $\alpha^{p-1}(x) = x$, and so α is of order $p - 1$. A straightforward calculation shows that $\langle\rho\rangle$ is normal in $G = \langle\rho, \alpha\rangle$. Namely,

$$\alpha^{-1}\rho\alpha(x) = \alpha^{-1}\rho(1 + ax) = \alpha^{p-2}(3 + ax)$$

$$= \sum_{j=0}^{p-3} a^j + a^{p-2}(3 + ax) = x + 2a^{p-2} = \rho^{a^{p-2}}(x).$$

Furthermore, note that $\langle\rho\rangle \cap \langle\alpha\rangle = \{1\}$, and hence $G = \langle\rho\rangle \rtimes \langle\alpha\rangle$.

Consider the action of G on the vertex set $V(X)$. The orbits of ρ in this action are $\{1,3,5,\ldots,2p-1\}$ and $\{0,2,4,\ldots,2p-2\}$, and the orbit of α containing 0 is

$$\{0,1,1+a,1+a+a^2,\ldots,1+a+\ldots+a^{p-3}\}.$$

Therefore G acts transitively on $V(X)$. Now consider the action of G on the edge set $E(X)$. The orbits of $\langle\rho\rangle$ on $E(X)$ have length p. We claim that the orbits of $\langle\alpha\rangle$ on $E(X)$ are of length $p-1$. Suppose on the contrary that these orbits are not of length $p-1$. Then there exists an edge $\{x,y\}\in E(X)$ which is fixed by α. Without loss of generality we can assume that $x\in\mathbb{Z}_{2p}$ is odd, and $y\in\mathbb{Z}_{2p}$ is even. Both a and x are odd, so it is impossible that $\alpha(x)=x$, since $1+ax$ is even. Therefore, we must have $\alpha(x)=y$, and $\alpha(y)=x$. Combining the given equalities $1+ax=y$ and $1+ay=x$, we obtain $(a-1)(x-y)=0$. Further, since $a-1$ is not divisible by p, we have $x-y=0$ or $x-y=p$, contradicting the assumption that $\{x,y\}$ is an edge in X. This shows that α fixes no edge in $E(X)$, and consequently the orbits of $\langle\alpha\rangle$ on $E(X)$ are of length $p-1$, as claimed.

Since the orbits of $\langle\alpha\rangle$ on $E(X)$ are of length $p-1$, and the orbits of $\langle\rho\rangle$ on $E(X)$ are of length p, it follows that the lengths of orbits of G on $E(X)$ are divisible by both p and $p-1$, and hence by their product $p(p-1)=|E(X)|$, consequently G acts transitively and regularly on $E(X)$. Further, since the order of G is less than the number of arcs in X, the group G cannot be transitive on the set of arcs $A(X)$. Therefore, X is G-half-arc-transitive.

Alternets are blocks of imprimitivity for G, and so the number of heads contained in an alternet is a divisor of $2p=|V(X)|$. In view of the fact that X is of valency $p-1$ these alternets cannot have just two heads. Moreover, since X is bipartite, the number of heads cannot be $2p$, by Proposition 5.2.2. Therefore the number of heads is precisely p, and we conclude that there are just two alternets in X. ■

The next example gives three half-arc-transitive graphs with two alternets.

Example 5.2.4 With the use of Magma [11] one can see that for the groups

$$\begin{aligned}
H_1 &= \langle a,b,c,d,r \mid a^2=b^2=c^2=d^2=r^7=1, a^r=a, b^r=c, c^r=d, d^r=bd\rangle\\
&\cong \mathbb{Z}_2^4 \rtimes \mathbb{Z}_7,\\
H_2 &= \langle a,b,c,d,e,f \mid a^2=b^2=d, c^2=d^2=e^3=f^3=1, b^a=bd, e^a=ef^2, e^b=f,\\
&\quad e^d=e^2, f^a=e^2f^2, f^b=e^2, f^d=f^2\rangle \cong \mathbb{Z}_3^2\rtimes(Q_8\times\mathbb{Z}_2),\ \text{and}\\
H_3 &= \langle a,b,c,d,e \mid a^2=c^3=e^3=1, b^3=c, d^3=e^2, d^a=d^2e, d^b=de, e^a=e^2\rangle\\
&\cong (\mathbb{Z}_9\rtimes\mathbb{Z}_9)\rtimes\mathbb{Z}_2.
\end{aligned}$$

of orders 112, 144 and 162, respectively, the Cayley graphs $\mathrm{Cay}(H_1,S_1\cup S_1^{-1})$, $\mathrm{Cay}(H_2,S_2\cup S_2^{-1})$ and $\mathrm{Cay}(H_3,S_3\cup S_3^{-1})$, where $S_1=\{arb,ar^3cd,ar^5c\}$, $S_2=\{adf^{-1},bce,abdf\}$ and $S_3=\{abc,acde^2,ab^2e^2\}$, are 6-valent half-arc-transitive graphs with two alternets. Observe also that $\mathrm{Cay}(H_1,S_1\cup S_1^{-1})$, the smallest of the three, is a regular $\mathbb{Z}_2^4$-cover of the complete graph K_7.

It is obvious that every G-half-arc-transitive graph with two alternets is G-tightly attached. In the next theorem it is shown that the same is true for G-half-arc-transitive graphs with three alternets.

Theorem 5.2.5 *Let X be a G-half-arc-transitive graph with three alternets. Then X is G-tightly attached.*

Proof. Let X be a G-half-arc-transitive graph with three alternets, say A_1, A_2, and A_3. Let T_i and H_i be the tails and heads of the arcs in $A_i \in \{A_1, A_2, A_3\}$, respectively. Clearly, $T_i \cap H_i = \emptyset$ for every $i \in \{1, 2, 3\}$, and both sets $\{T_1, T_2, T_3\}$ and $\{H_1, H_2, H_3\}$ form a partition of $V(X)$.

Suppose that X is not G-tightly attached, and let $v \in T_1$. Since $T_1 \cap H_2 \neq \emptyset$, we may assume that $v \in H_2$. Now consider the set of out-neighbors N of v. The stabilizer G_v must act transitively on N, and since $v \in T_1 \cap H_2$, it follows that G_v fixes T_1 and H_2 setwise. It follows that G_v fixes the alternets A_1 and A_2. Therefore, either $N \subseteq T_2$ or $N \subseteq T_3$. Since v was an arbitrary vertex in T_1 this holds for the set of out-neighbors of every vertex in T_1.

Now, let $u \in T_1$ be such that the set N_1 of all out-neighbors of u is contained in T_2. Then the set N_2 of all in-neighbors of the vertices in N_1 must have their remaining out-neighbors also in T_2. Namely, if this was not the case then there would exist a vertex in T_1 with some of its out-neighbors in T_2 and some in T_3 which is not possible. Repeating the same argument, one can see that all of the vertices in T_1 have their out neighbors in T_2. Consequently we have $H_1 = T_2$, contradicting the assumption that X is not G-tightly attached. ■

In [5, Theorem 2.5] authors gave an infinite family of half-arc-transitive graphs with three alternets (and infinitely many different valencies). The family is given in the example below. The concept of a metacirculant needs to be introduced first. A graph X is an (m, n)-*metacirculant* if it has an (m, n)-semiregular automorphism ρ together with another automorphism σ such that $\sigma^{-1}\rho\sigma = \rho^r$, for some $r \in \mathbb{Z}_n^*$, which cyclically permutes the orbits of $\langle\rho\rangle$. The vertex set of an (m, n)-metacirculant X can be represented as $V(X) = \{x_i^j \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ where $x_i^j = (x_0^0)^{\sigma^i\rho^j}$ for all $i \in \mathbb{Z}_m, j \in \mathbb{Z}_n$. Let $\mu = \lfloor m/2 \rfloor$ and let $S_i = \{s \in \mathbb{Z}_n \mid x_0^0 \sim x_i^s\}$, $0 \leq i \leq \mu$. Then every metacirculant is completely determined by the tuple $(m, n, r; S_0, S_1, \ldots, S_\mu)$ which is called the *symbol* of X.

Example 5.2.6 [5] Let p be a prime and $p \equiv 1 \pmod 3$. Let $d > 1$ be a divisor of $(p-1)/3$ and let $S = \langle s \rangle$ be the subgroup of $\mathbb{Z}_p^*$ of order d. Let $r \in \mathbb{Z}_p^* \setminus S$ be a 3-element with $r^3 \in S$. We use $M(d; 3, p)$ to denote the metacirculant graph with symbol $(3, p, r; \emptyset, S)$. If $(d, p) \neq (2, 7)$ or $(3, 19)$ then the graph $M(d; 3, p)$ is a half-arc-transitive graph of order $3p$ and valency $2d$ which has three alternets. This graph is independent of the choice of r. The automorphism group $A = \mathrm{Aut}(M(d; 3, p))$ is isomorphic to a semidirect product of $\mathbb{Z}_p$ and $\mathbb{Z}_{3d}$, and A acts regularly on the edge set of $M(d; 3, p)$.

5.3 Four or more alternets

In the example below an infinite family of G-half-arc-transitive graphs which are not G-tightly attached is constructed.

Example 5.3.1 For a natural number $n \geq 4$ let X_n be the graph with vertex set

$$V(X_n) = \{(i,j) | \; i \in \mathbb{Z}_n, j \in \mathbb{Z}_n \setminus \{i\}\}$$

and edge set

$$E(X_n) = \{\{(i,j),(k,i)\} | (i,j) \in V(X_n), k \in \mathbb{Z}_n, k \neq i, k \neq j\}.$$

Observe that the rule $(i,j)^\sigma = (i^\sigma, j^\sigma)$, where $(i,j) \in V(X_n)$ and $\sigma \in S_n$, defines an action of the symmetric group S_n on the graph X_n. In fact S_n is a subgroup of $\mathrm{Aut}(X_n)$ acting transitively on $V(X_n)$. Namely, for every pair of vertices $(i_1,j_1),(i_2,j_2) \in V(X_n)$ there exists a permutation in S_n mapping i_1 to i_2 and j_1 to j_2, and for any pair of edges $e_1 = \{(i_1,j_1),(k_1,i_1)\}$ and $e_2 = \{(i_2,j_2),(k_2,i_2)\}$ in $E(X_n)$ there exists a permutation in S_n mapping i_1, j_1, k_1 to i_2, j_2, k_2, respectively.

The stabilizer of a vertex $v = (i,j) \in V(X_n)$ in S_n is isomorphic to S_{n-2}, and it is not difficult to see that it has two orbits $\{(k,i) \mid k \neq i,j\}$ and $\{(j,k) \mid k \neq i,j\}$ on the set of neighbors of v, implying that S_n acts half-arc-transitively on X_n. Moreover there are exactly n alternets A_i, $i \in \mathbb{Z}_n$, with respect to this action. The corresponding tail sets are $T_i = \{(i,j) \mid j \in \mathbb{Z}_n, j \neq i\}$ and the corresponding head sets are $H_i = \{(k,i) \mid k \in \mathbb{Z}_n, k \neq i\}$. The intersection number is 1, so these graphs are S_n-loosely attached.

By Example 5.3.1 there exists a G-half-arc-transitive graph with four alternets which is not G-tightly attached. A precise structure of such graphs is given in the following theorem. To state the theorem we need to define the so-called rose window graphs, introduced first in [114] (see also [55, 57]). For given natural numbers $n \geq 3$ and $1 \leq a, r \leq n-1$, the *rose window graph* $R_n(a,r)$ has vertex set $\{x_i \mid i \in \mathbb{Z}_n\} \cup \{y_i \mid i \in \mathbb{Z}_n\}$ and edge set $\{\{x_i, x_{i+1}\} \mid i \in \mathbb{Z}_n\} \cup \{\{y_i, y_{i+r}\} \mid i \in \mathbb{Z}_n\} \cup \{\{x_i, y_i\} \mid i \in \mathbb{Z}_n\} \cup \{\{x_{i+a}, y_i\} \mid i \in \mathbb{Z}_n\}$. The automorphism group of the rose window graph $R_6(5,4)$, the graph appearing in the theorem below, is isomorphic to $S_4 \times \mathbb{Z}_2$. There are two half-arc-transitive subgroups of $\mathrm{Aut}(R_6(5,4))$, one is isomorphic to S_4 giving rise to four alternets, radius 3, and attachment number 1, and the other one is isomorphic to $A_4 \times \mathbb{Z}_2$ giving rise to six alternets, radius 2 and attachment number 1. It is easy to see that the rose window graph $R_6(5,4)$ is isomorphic to the graph X_4 defined in Example 5.3.1.

Theorem 5.3.2 *Let X be a G-half-arc-transitive graph with four alternets which is not G-tightly attached. Then there exists a partition of $V(X)$ relative to which the quotient graph of X is isomorphic to $X_4 \cong R_6(5,4)$.*

Proof. Let A_1, A_2, A_3 and A_4 be the alternets of X under the action of the group G. Then the family of tail sets $\{T_1, T_2, T_3, T_4\}$ forms a G-invariant partition of $V(X)$. Since X is not G-tightly attached, there are two cases that need to be considered, depending of whether the head set H_1 of the alternet A_1 has non-empty intersection with two (say T_2 and T_3) or all three of the tail sets T_2, T_3, and T_4. (The same holds for the set of tail set T_1 and its intersections with the head sets H_2, H_3 and H_4, respectively.)

Case 1. $H_1 \cap T_4 = \emptyset$, and $H_1 \cap T_2$ and $H_1 \cap T_3$ are non-empty sets.

It follows that every vertex in T_1 has out-neighbors in both T_2 and T_3. Therefore there exists an element in G_v, $v \in T_1$, interchanging T_2 and T_3, and fixing T_4, setwise. In its action this automorphism fixes A_1 and A_4, and interchanges A_2 and A_3, and consequently it interchanges H_2 and H_3. Therefore, the vertex v being fixed cannot be contained in H_2 or H_3, and so $v \in H_4$. Since v was arbitrary it follows that $T_1 = T_4$, contradicting the assumption that X is not G-tightly attached.

Case 2. $H_1 \cap T_2$, $H_1 \cap T_3$, and $H_1 \cap T_4$ are non-empty sets.

We distinguish two subcases: either each vertex in T_1 has out-neighbors in T_2, T_3 and T_4, or in just two of them. Suppose that the first possibility occurs, and take $v \in T_1$. Without loss of generality we may assume that $v \in H_2$. Then every automorphism fixing v also fixes H_2, and consequently fixes T_2. But then the stabilizer G_v cannot be transitive on the set of out-neighbors of v, because its out-neighbors from T_2 cannot be moved to out-neighbors in T_3 and T_4, a contradiction. We may therefore assume that the second possibility above occurs: every vertex in T_1 has out-neighbors in just two of the tail sets T_2, T_3 and T_4. This gives us a natural partition of T_1 into three parts T_1^{23}, T_1^{24} and T_1^{34}, where T_1^{ij} denotes the set of vertices in T_1 having out-neighbors in T_i and T_j. Clearly, the stabilizer G_u of $u \in T_1^{ij}$ acts transitively on the set of its out-neighbors, and therefore, it interchanges T_i and T_j. It follows that $T_1^{ij} \cap H_i = \emptyset$ and $T_1^{ij} \cap H_j = \emptyset$ implying that $T_1^{23} \subseteq H_4$, $T_1^{24} \subseteq H_3$ and $T_1^{34} \subseteq H_2$. In an analogous way we get a decomposition of T_2, T_3 and T_4 into sets

$$T_2^{1,3},\ T_2^{1,4},\ T_2^{3,4};\ T_3^{1,2},\ T_3^{1,4},\ T_3^{2,4};\ T_4^{1,2},\ T_4^{1,3},\ T_4^{2,3},$$

and their relative containment in the sets H_1, H_2, H_3 and H_4. This gives us the decomposition $\{T_k^{i,j}\}$ of $V(X)$ relative to which the corresponding quotient graph of X is a 4-valent graph on 12 vertices which turns out to be isomorphic to the rose window graph $R_6(5,4)$. ■

By Theorem 5.3.2, a non-tightly attached half-arc-transitive graph with four alternets is a (multi)cover of the rose window graph $R_6(5,4)$. Additional information is given by the proposition below which shows that these graphs are not regular cyclic covers of $R_6(5,4)$.

Proposition 5.3.3 *Every connected regular cyclic cover of the rose window graph* $R_6(5,4)$*, along which the half-arc-transitive subgroup of its automorphism group giving rise to four alternets lifts, is arc-transitive.*

Proof. Let $X = R_6(5,4)$, $A = \mathrm{Aut}(X)$, and let G be a half-arc-transitive subgroup of A with four alternets. Then X is not G-tightly attached (see the paragraph preceding Theorem 5.3.2). Let the vertices of X be labeled as shown in Figure 5.1. With the use of Magma [11] one can obtain that $G = \langle \alpha, \beta, \gamma, \delta \rangle$ and $A = \langle G, \tau \rangle$

where

$$\begin{aligned}
\alpha &= (x_0x_2x_4)(x_1x_3x_5)(y_0y_2y_4)(y_1y_3y_5);\\
\beta &= (x_0x_3)(x_1y_3)(x_2y_1)(x_4y_0)(x_5y_4)(y_2y_5);\\
\gamma &= (x_0y_2)(x_1y_0)(x_2x_5)(x_3y_5)(x_4y_3)(y_1y_4);\\
\delta &= (x_1x_5)(x_2x_4)(y_0y_1)(y_2y_5)(y_3y_4);\\
\tau &= (x_0x_3)(x_1x_4)(x_2x_5)(y_0y_3)(y_1y_4)(y_3y_5).
\end{aligned}$$

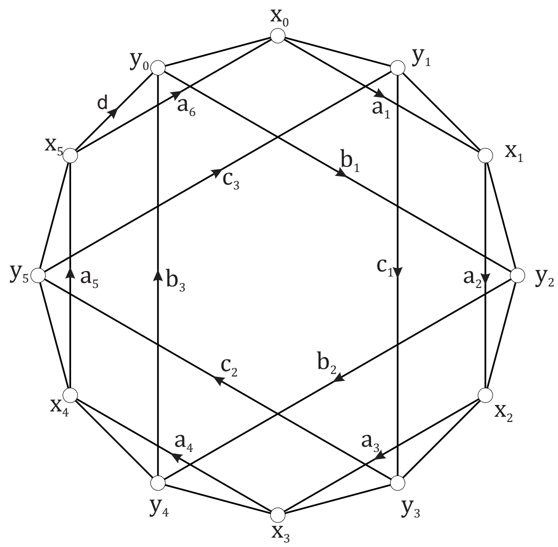

Figure 5.1: The voltage assignment ζ on $R_6(5,4)$. The spanning tree consists of undirected edges, all carrying trivial voltage.

Suppose that $\tilde{X}$ is a $\mathbb{Z}_n$-regular cover of X which is a half-arc-transitive graph with four alternets. Then the group G lifts along this covering projection, but the automorphism τ does not. Any such cover can be derived from X through a suitable T-reduced voltage assignment $\zeta : A(X) \to \mathbb{Z}_n$, where T is the spanning tree of X consisting of the edges

$$x_0y_1,\ y_1x_1,\ x_1y_2,\ y_2x_2,\ x_2y_3,\ y_3x_3,\ x_3y_4,\ y_4x_4,\ x_4y_5,\ y_5x_5.$$

There are 13 fundamental cycles in X, which are generated, respectively, by 13 cotree arcs (x_0,x_1), (x_1,x_2), (x_2,x_3), (x_3,x_4), (x_4,x_5), (x_5,x_0), (x_5,y_0), (y_0,y_2), (y_2,y_4), (y_4,y_0), (y_1,y_3), (y_3,y_5), and (y_5,y_1) (see Table 5.1 where voltages of all these cycles together with the voltages of their images under the automorphisms α, β, γ, δ and τ are listed).

The group G lifts if and only if α,β,γ and δ lift. By [78, Theorem 5.3], an automorphism ϕ lifts if and only if its action $\bar{\phi}$ on the fundamental cycles gives rise to an automorphism ϕ^* of the voltage group. From Table 5.1 one can see that $\{a_2,a_5,d,b_3,c_1\}$ generates the covering group $\mathbb{Z}_n$. Further, observe that $\tau^*(x) = -\delta^*(x)$ for every $x \in \{a_2,a_5,d,b_3,c_1\}$. Since these elements generate the covering group $\mathbb{Z}_n$ we conclude that $\tau^*(x) = -\delta^*(x)$ for every $x \in \mathbb{Z}_n$. Therefore, if δ lifts and τ does not lift, then at least one of the following holds: $a_4 \neq a_6 - d, a_1 \neq a_3, c_2 \neq c_3 - d, b_1 \neq b_2$. But, by Table 5.1, we have $a_1 = \alpha^*(a_5) = \alpha^*(\gamma^*(a_3)) = \gamma^*(\alpha^*(a_3)) = \gamma^*(a_5) = a_3$, $a_4 = -\delta^*(a_3) = -\delta^*(a_1) = a_6 - d$. Similarly $c_2 = c_3 - d$ and $b_1 = b_2$, showing that τ lifts. Therefore, the whole automorphism group A of X lifts, and consequently $\tilde{X}$ is arc-transitive. ■

By Proposition 5.3.3 there are no connected half-arc-transitive graphs with four alternets that are cyclic regular covers of $R_6(5,4)$. However, with the use of Magma [11] one can easily see that the graph X with vertex set

$$V(X) = \{(x_i, j, k), (y_i, j, k) \mid i \in \mathbb{Z}_6, j, k \in \mathbb{Z}_2\}$$

and edge set

$$\begin{aligned} E(X) = \; & \{\{(x_i, j, k), (x_{i+1}, j, k)\}, \{(y_i, j, k), (y_{i+4}, j, k)\}, \{(x_i, j, k), (y_i, j, k)\}, \\ & \{(x_{i+5}, j, k), (y_i, j, k)\}, \{(x_0, j, k), (x_1, j+1, k)\}, \\ & \{(x_0, j, k), (x_5, j+1, k)\}, \{(y_1, j, k), (y_5, j+1, k)\}, \\ & \{(y_0, j, k), (y_2, j+1, k)\}, \{(x_2, j, k), (y_2, j+1, k)\}, \\ & \{(x_3, j, k), (y_3, j+1, k)\}, \{(x_3, j, k), (y_4, j+1, k)\}, \\ & \{(x_4, j, k), (y_5, j+1, k)\}, \{(x_1, j, k), (x_2, j, k+1)\}, \\ & \{(x_2, j, k), (x_3, j, k+1)\}, \{(y_1, j, k), (y_3, j, k+1)\}, \\ & \{(y_2, j, k), (y_4, j, k+1)\}, \{(x_0, j, k), (y_1, j, k+1)\}, \\ & \{(x_4, j, k), (y_4, j, k+1)\}, \{(x_5, j, k), (y_0, j, k+1)\}, \\ & \{(x_5, j, k), (y_5, j, k+1)\}, \{(x_3, j, k), (x_4, j+1, k+1)\}, \\ & \{(x_4, j, k), (x_5, j+1, k+1)\}, \{(y_0, j, k), (y_4, j+1, k+1)\}, \\ & \{(y_3, j, k), (y_5, j+1, k+1)\}, \{(x_0, j, k), (y_0, j+1, k+1)\}, \\ & \{(x_1, j, k), (y_1, j+1, k+1)\}, \{(x_1, j, k), (y_2, j+1, k+1)\}, \\ & \{(x_2, j, k), (y_3, j+1, k+1)\} \mid i \in \mathbb{Z}_6, j, k \in \mathbb{Z}_2\} \end{aligned}$$

is a half-arc-transitive graph with four alternets. Besides, its radius is 12 and its attachment number is 4, and therefore is not tightly-attached. It is a regular $\mathbb{Z}_2^2$-cover of the multigraph obtained from $R_6(5,4)$ by replacing each edge with two parallel edges.

We propose the following question.

Question 5.3.4 *Does there exists a regular cover of $R_6(5,4)$ which is (non-tightly attached) half-arc-transitive with four alternets?*

The graph X_5, constructed in Example 5.3.1, admits a non-tightly attached half-arc-transitive action with five alternets. Its automorphism group $\mathrm{Aut}(X_5)$ is isomorphic to $S_5 \times \mathbb{Z}_2$. There are two half-arc-transitive subgroups of $\mathrm{Aut}(X_5)$, one is isomorphic to S_5 with vertex stabilizer isomorphic to S_3, and the other is isomorphic to A_5 with vertex stabilizer isomorphic to $\mathbb{Z}_3$. In addition, both of these half-arc-transitive actions give rise to five alternets, radius 4 and attachment number

Voltage	α	β	γ	δ	τ
a_1	a_3	a_3	$-b_1-b_3-b_2$	$-a_6+d$	a_4
a_2	a_4	$-c_1-c_3-c_2$	$-d+a_6$	$-a_5$	a_5
a_3	a_5	a_1	a_5	$-a_4$	a_6-d
a_4	a_6-d	$-d+a_6$	$-c_2-c_1-c_3$	$-a_3$	a_1
a_5	a_1	$-b_3-b_2-b_1$	a_3	$-a_2$	a_2
a_6	$d+a_1$	$-a_3+c_2+c_3-a_1$ $-a_6+d+b_1+b_2$	$b_2+b_3-a_6$ $-a_5+c_3+c_1-x_3$	$-d-a_1$	$d+a_3$
d	d	$-a_4-a_3+c_2+c_3$ $-a_1-a_6+d+b_1+b_2$	$-a_2+b_2+b_3-a_6$ $-a_5+c_3+c_1-x_3$	$-d$	d
b_1	b_2	$-c_2+a_3+a_4$	$-a_1-b_3-b_2$	$-c_3+d$	c_2
b_2	b_3-d	$a_6+a_1-c_3$	$-c_3+a_5+a_6$	$-c_2$	c_3-d
b_3	b_1+d	$-a_5-a_4-a_3$ $+c_2+c_3-a_1-a_6$	$b_2+b_3-a_6-a_5+c_3$	$-c_1-d$	c_1+d
c_1	c_2	$-a_2-c_3-c_2$	$a_5+a_6-b_3$	$-b_3+d$	b_3-d
c_2	c_3-d	$-b_1-d+a_6+a_1$	$a_4-c_1-c_3$	$-b_2$	b_1
c_3	c_1+d	$c_2+c_3-a_1$ $-a_6+d+b_1$	$b_3-a_6-a_5+c_3+c_1$	$-b_1-d$	b_2+d

Table 5.1: Voltages of the fundamental cycles and voltages of their images in $R_6(5,4)$.

1. In the next theorem we prove that every graph admitting a non-tightly attached half-arc-transitive group action with five alternets has a decomposition of its vertex set with X_5 as the corresponding quotient.

Theorem 5.3.5 *Let X be a G-half-arc-transitive graph with five alternets which is not G-tightly attached. Then there exists a partition of $V(X)$ relative to which the quotient graph of X is isomorphic to X_5.*

Proof. Let A_1, A_2, A_3, A_4 and A_5 be the alternets of X under the action of the group G. Then the family of tail sets $\{T_1, T_2, T_3, T_4, T_5\}$ forms a G-invariant partition of $V(X)$. Since X is not G-tightly attached, the head set H_1 of the alternet A_1 has non-trivial intersection with two (say T_2 and T_3), three (say T_2, T_3 and T_4) or all four of the tail sets T_2, T_3, T_4 and T_5. (The same holds for the set of tail set T_1 and its intersections with the head sets H_2, H_3, H_4 and H_5, respectively.)

Case 1. $H_1 \cap T_4 = H_1 \cap T_5 = \emptyset$, and $H_1 \cap T_2$ and $H_1 \cap T_3$ are non-empty sets.

It follows that every vertex in T_1 has out-neighbors in both T_2 and T_3. Therefore there exists an automorphism in G_v, $v \in T_1$, interchanging T_2 and T_3, and consequently also H_2 and H_3. We conclude that $v \in H_4$ or $v \in H_5$. Since X is not G-tightly attached it follows that T_1 has non-trivial intersection with both H_4 and H_5. Furthermore G_v fixes setwise the two alternets A_4 and A_5, as well as the corresponding head sets and tail sets. We summarize: the out-neighbors of T_1 are in T_2 and T_3 and the in-neighbors of T_1 are in T_4 and T_5. Using this argument for the other tail sets we conclude that for every tail set in X its out-neighbors and its in-neighbors belong to different pairs of the remaining four tail sets. Now, the out-neighbors of T_2 lie in two of the tail sets T_3, T_4 and T_5, and the out-neighbors of T_3 lie in two of the tail sets T_2, T_4 and T_5. Suppose first that the out-neighbors of T_2 lie in T_3 and T_4. From the argument above we know that there exists automorphism fixing the tail sets T_1, T_4 and T_5 while interchanging the tail sets T_2 and T_3. This automorphism interchanges the sets of out-neighbors of T_2 and T_3, forcing the out-neighbors of T_3 to be in T_2 and T_4. This implies, in view of the fact that the in-neighbors of T_1 are in T_4 and T_5, that the out-neighbors of T_5 belong to three different tail sets, a contradiction. The same argument applies when T_2 has its out-neighbors in T_3 and T_5, we simply switch the roles of T_4 and T_5. We may therefore assume that the last of the three possibilities for the location of the out-neighbors of T_2 occurs: they are in T_4 and T_5. Then the in-neighbors of T_2 are in T_1 and T_3, and T_3 has out-neighbors in T_2. But because of the symmetry of the roles of T_2 and T_3, one can see that T_3 must have its out-neighbors also in T_4 and T_5, a contradiction.

Case 2. $H_1 \cap T_5 = \emptyset$, and $H_1 \cap T_2$, $H_1 \cap T_3$ and $H_1 \cap T_4$ are non-empty sets.

We have two possibilities depending on whether each vertex in T_1 has out-neighbors in each of T_2, T_3 and T_4 or in just two of them. If the first possibility occurs then T_1 has no out-neighbors in T_5, and furthermore for every $v \in T_1$ the stabilizer G_v acts transitively on $\{T_2, T_3, T_4\}$, and thus also on $\{H_2, H_3, H_4\}$ and fixes setwise the tail sets T_1 and T_5 and the head sets H_1 and H_5. It follows that $T_1 = H_5$, contradicting the fact that X is not tightly attached.

Now suppose that every vertex from T_1 has its out-neighbors in just two of the tail sets T_2, T_3 or T_4. This gives rise to a natural partition of T_1 into three parts T_1^{23}, T_1^{24} and T_1^{34}, where T_1^{ij} denotes the set of vertices in T_1 having out-neighbors in T_i and T_j. It is clear that $T_1^{23} \subseteq H_4 \cup H_5$, $T_1^{24} \subseteq H_3 \cup H_5$ and $T_1^{34} \subseteq H_2 \cup H_5$. Since the intersection of T_1 with precisely three of the head sets H_2, H_3, H_4 and H_5 is non-empty, with all of these intersections having the same cardinality it follows that either precisely one or none of T_1^{23}, T_1^{24}, T_1^{34} is entirely contained in H_5. Suppose first that one of T_1^{23}, T_1^{24}, T_1^{34} is entirely contained in H_5, say $T_1^{23} \subseteq H_5$. It follows that $T_1^{24} \subseteq H_3$ and $T_1^{34} \subseteq H_2$. Take a vertex $v \in T_1^{24}$, and consider its out-neighbors $u \in T_2$ and $w \in T_4$. The stabilizer G_v acts transitively on the out-neighbors of v, and therefore there exists $\alpha \in G_v$ which fixes v and maps w to u, and consequently maps H_4 to H_2. Also the in-neighbors of w, which lie in H_2 and H_3, have to be mapped by α to the in-neighbors of u, which lie in H_3 and H_5. But then α maps $\{H_2, H_3\}$ into $\{H_3, H_5\}$, a contradiction. We may therefore assume that none of T_1^{23}, T_1^{24}, T_1^{34} is entirely contained in H_5. But since T_1 has empty intersection with precisely one of H_2, H_3, H_4 and H_5, this implies that $T_1^{23} \subseteq H_4$, $T_1^{24} \subseteq H_3$ and $T_1^{34} \subseteq H_2$. This means that the vertex set of the alternet A_1 has a non-empty intersection with the vertex set of A_2, A_3 and A_4, but empty intersection with the vertex set of A_5. For symmetry reasons this means that for each i there exists a unique $j \neq i$ such that $A_i \cap A_j = \emptyset$. But this is clearly impossible since the number of alternets is five, an odd number.

Case 3. $H_1 \cap T_2$, $H_1 \cap T_3$, $H_1 \cap T_4$ and $H_1 \cap T_5$ are non-empty sets.

We have three possibilities depending on whether every vertex from T_1 has its out-neighbors in all four, or just three or two of the tail sets T_2, T_3, T_4 and T_5. Suppose that the first possibility occurs and let $v \in T_1$ have its out-neighbors in each of T_2,

T_3, T_4 and T_5. It follows that G_v is transitive on the set $\{T_2, T_3, T_4, T_5\}$. On the other hand since $v \in H_i$ for some $i \in \{2, 3, 4, 5\}$, say $v \in H_2$, it follows that G_v fixes H_2 and thus also T_2 setwise, a contradiction.

Assume now that each vertex from T_1 has its out-neighbors in three of the sets T_2, T_3, T_4 and T_5. We can partition T_1 into four sets T_1^{234}, T_1^{235}, T_1^{245} and T_1^{345} where T_1^{ijk} stands for all vertices in T_1 having its out-neighbors in T_i, T_j and T_k. Clearly $T_1^{234} \subseteq H_5$, $T_1^{235} \subseteq H_4$, $T_1^{245} \subseteq H_3$, and $T_1^{345} \subseteq H_2$. In fact $T_1^{234} = T_1 \cap H_5$, $T_1^{235} = T_1 \cap H_4$, $T_1^{245} = T_1 \cap H_3$, and $T_1^{345} = T_1 \cap H_2$. Similar equalities hold for the other intersections $T_i \cap H_j$, $i \neq j$. The set $\{H_i \cap T_j \mid i \neq j\}$ consisting of intersections of two G-invariant partitions of $V(X)$ is itself a G-invariant partition of $V(X)$. Replacing $T_i \cap H_j$ with a pair (i, j), we have an edge from (i, j) to (j, k) whenever $k \neq i$. Therefore, the quotient of X with respect to the partition $\{T_i \cap H_j \mid i \neq j\}$ is isomorphic to the graph X_5 defined in Example 5.3.1.

Finally, we assume that each vertex from T_1 has its out-neighbors in just two of T_2, T_3, T_4 and T_5. Consider the partition of T_1 arising from the intersections with head sets H_2, H_3, H_4 and H_5. Either every vertex in $T_1 \cap H_2$ has its out-neighbors in just two of the tail sets T_3, T_4 and T_5, or in all three of them. The same must hold (simultaneously) for each of the intersections $T_1 \cap H_3$, $T_1 \cap H_4$ and $T_1 \cap H_5$. Suppose that the first possibility occurs, and let $v \in T_1 \cap H_2$ have its out-neighbors in T_3 and T_4. Then there exists $\alpha \in G_v$ interchanging T_3 and T_4, but fixing T_2 and T_5 setwise. It may be seen that the out-neighbors of vertices in $T_1 \cap H_5$ must lie in T_4 and one of T_2 and T_3. It follows that G_v fixes each of T_2, T_3, T_4 and T_5 setwise. In particular, G_v is not transitive on the set of out-neighbors of v, contradicting edge-transitivity of X. Finally, we are left with the case where every vertex in $T_1 \cap H_2$ has its out neighbors in each of T_3, T_4 and T_5. Then the same must hold for every intersection $T_i \cap H_j$, $i \neq j$. Taking the quotient graph of X with respect to the partition $\{T_i \cap H_j \mid i, j \in \{1, 2, 3, 4, 5\},\ i \neq j\}$, we obtain a graph isomorphic to X_5, completing the proof of Theorem 5.3.5. ∎

When aiming for constructions of graphs admitting half-arc-transitive group actions with a prescribed number of alternets, direct products of graphs prove useful. The *direct product* $X \times Y$ of graphs X and Y has vertex set $V(X) \times V(Y)$ where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $\{x_1, x_2\} \in E(X)$ and $\{y_1, y_2\} \in E(Y)$. Let G and H be groups, and X and Y be a G-arc-transitive graph and a H-half-arc-transitive graph, respectively. Then it is not difficult to see that $X \times Y$ is $(G \times H)$-half-arc-transitive graph (see, for example, [113]). Furthermore, the direct product $X \times Y$ is connected provided X and Y are connected and at least one of them is not bipartite.

Theorem 5.3.6 *Let G and H be groups, and let X be a connected G-arc-transitive graph, and let Y be a connected H-half-arc-transitive graph with $n \geq 2$ alternets. Then*

(i) if X is not bipartite, then $X \times Y$ is a connected $(G \times H)$-half-arc-transitive graph with n alternets; and

(ii) if X is bipartite and Y is not bipartite, then $X \times Y$ is a connected $(G \times H)$-

half-arc-transitive graph with $2n$ *alternets.*

PROOF. By the comments preceding this theorem, we know that $X \times Y$ is a connected $G \times H$-half-arc-transitive graph. Assume first that X is not bipartite. Choosing any alternet A in Y it is not difficult to see that $X \times A$ is an alternet in $X \times Y$, and so there are n alternets in $X \times Y$. On the other hand, if X is bipartite then for a given alternet A in Y, the subgraph $X \times A$ has two connected components, giving rise to two alternets in $X \times Y$. Consequently the number of alternets in $X \times Y$ is $2n$ in this case. ∎

Remark 5.3.7 Since there are infinitely many non-bipartite arc-transitive graphs, Theorem 5.3.6 implies the existence of infinitely many half-arc-transitive graphs with $n \geq 2$ alternets provided one such graph exist.

Chapter 6

Conclusions

A number of research problems from algebraic graph theory are solved, in particular: a complete classification of quasi 2-, quasi 3- and strongly quasi 4-Cayley circulants, the classification of pentavalent arc-transitive bicirculants, existence of non-Cayley vertex-transitive generalized Cayley graphs, existence of semiregular automorphism in generalized Cayley graphs, characterization of non-tightly attached half-arc-transitive graphs with small number of alternets.

These results represent a significant contribution to a possible solution of some long standing open problems, such as existence of Cayley snarks and existence of semiregular autormophisms in vertex-transitive graphs.

The basic tools used in the research range from combinatorial and algebraic methods in graph theory to purely abstract considerations in group theory. In addition, computer-implemented algebraic tools, such as MAGMA [11], is also used. Specifically, an essential part of the strategy in classifying quasi 2-, quasi 3- and strongly quasi 4-circulants is the classification of arc-transitive circulants, obtained independently by Kovacs [52] and Li [69]. Next, the classification of pentavalent arc-transitive bicirculants uses graph covering techniques as a very important tool.

To wrap up, the results of this PhD Thesis represent a significant contribution to a number of long standing open problems in graph theory. In addition, the thesis discuses directions which may lead to solving the problems in question for other infinite families of graphs.

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Povzetek v slovenskem jeziku

ALGEBRAIČNI ASPEKTI TEORIJE GRAFOV

V doktorski disertaciji so obravnavane različne teme s področja algebraične teorije grafov, in sicer pomembni odprti problemi, ki so bili predmet številnih raziskav v zadnjih dvajsetih letih:

(i) Kateri grafi so (krepko) kvazi m-Cayleyjevi grafi?

(ii) Kateri bicirkulanti so ločno tranzitivni?

(iii) Ali obstajajo posplošeni Cayleyjevi grafi, ki niso Cayleyjevi grafi, so pa točkovno tranzitivni?

(iv) Ali obstajajo snarki med Cayleyjevimi grafi?

(v) Ali obstajajo grafi, ki premorejo pol-ločno tranzitivno delovanje z majhnim številom alternetov, glede na katerega niso tesno speti?

V doktorski disertaciji je problem (i) rešen za cirkulante v primeru, ko je $m \in \{2, 3, 4\}$. Problem (ii) je rešen v celoti za petvalentne bicirkulante. Problem (iii) je rešen pritrdilno, in sicer s konstrukcijo dveh neskončnih družin točkovno tranzitivnih posplošenih Cayleyjevih grafov, ki niso Cayleyjevi grafi. V obeh družinah so grafi bicirkulanti. Problem (iv) je rešen za tiste $(2, s, t)$-Cayleyjeve grafe, katerih pripadajoči $2t$-kotni grafi so ločno tranzitivni bicirkulanti praštevilske valence. Problem (v) je rešen za grafe, ki premojo pol-ločno tranzitivno grupno delovanje z manj kot šestimi alterneti. Dokazano je, da obstajajo grafi, ki premorejo pol-ločno tranzitivno delovanje s štirimi oziroma petimi alterneti, glede na katerega niso tesno speti, medtem ko so vsi grafi, ki premorejo pol-ločno tranzitivno delovanje z manj kot štirimi alterneti, glede na to delovanje tesno speti.

Osnovni pojmi in definicije

Naj bo V končna neprazna množica in E poljubna družina dvoelementnih podmnožic množice V. Paru $X = (V, E)$ pravimo *graf* na *množici točk* (oziroma *množico vozlišč*) $V = V(X)$ z *množico povezav* $E = E(X)$. Grafe navadno predstavimo tako, da narišemo točko za vsako točko grafa in dve točki povežemo, če tvorita povezavo grafa. Ni pomembno, na kakšen način so narisane točke in povezave. Pomembno je le, kateri pari točk so povezani. Točki, ki nista povezani, sta *neodvisni* točki. Množici točk, v kateri nobeni dve točki nista povezani, pravimo *neodvisna* množica. *Valenca* (ali *stopnja*) $d_X(v) = d(v)$ točke v v grafu X je število povezav, ki imajo točko v za krajišče. Če imajo vse točke grafa enako valenco d, rečemo, da je graf *regularen valence d*, ali *d-regularen.* Graf X je *kubičen*, *štirivalenten* oziroma *petvalenten*, če je regularen valence 3, 4 oziroma 5. Podgraf $X[U]$ grafa X *induciran* na podmnožici $U \subseteq V(X)$ je graf z množico točk U, v katerem sta dve točki povezani, če in samo če sta povezani v grafu X. *Komplement* grafa X je graf z množico točk $V(X)$, v katerem sta dve različni točki povezani, če in samo če nista povezani v grafu X. Graf X je *skoraj-dvodelen*, če obstaja taka neodvisna podmnožica $U \subseteq V(X)$, da je graf induciran na $V(X) \setminus U$ dvodelen graf.

Grafa X in Y sta si enaki natanko takrat, ko imata enaki množici točk in enaki množici povezav. Grafa X in Y sta si *izomorfna*, če obstaja bijektivna preslikava $\varphi\colon V(X) \to V(Y)$, ki povezave grafa X preslika v povezave grafa Y. Če sta grafa X in Y izomorfna, pišemo $X \cong Y$.

Avtomorfizem grafa X je permutacija množice točk, ki ohranja sosednost točk in povezav. Množica vseh avtomorfizmov grafa X z običajno operacijo komponiranja preslikav tvori *grupo avtomorfizmov* $\mathrm{Aut}(X)$ grafa X. Poljubna podgrupa G grupe avtomorfizmov grafa X naravno deluje na množici točk $V(X)$, množici povezav $E(X)$ in na množici lokov $A(X)$ (t.j. množici urejenih parov, ki tvorijo povezave) grafa X. Pravimo, da grupa G na grafu X deluje *točkovno tranzitivno*, *povezavno tranzitivno* oziroma *ločno tranzitivno*, če je delovanje grupe G na pripadajoči množici točk, povezav oziroma lokov grafa X tranzitivno. Za graf X, ki premore tako delovanje, pa pravimo, da je *G-točkovno tranzitiven*, *G-povezavno tranzitiven* oziroma *G-ločno tranzitiven* (ali *G-simetričen*). Če grupa G na $A(X)$ deluje regularno, pravimo, da G na grafu X deluje 1*-regularno.*

Za podgrupo grupe avtomorfizmov grafa X rečemo, da deluje *pol-ločno tranzitivno*, če deluje točkovno in povezavno tranzitivno, ne deluje pa ločno tranzitivno. Graf, katerega grupa avtomorfizmov deluje pol-ločno tranizitvno, imenujemo *pol-ločno tranzitiven graf.*

Naj bo G končna grupa z identiteto 1 in S generatorska množica grupe G, za katero velja: $S = S^{-1}$ in $1 \notin S$. Potem je *Cayleyjev graf* $\mathrm{Cay}(G, S)$ grupe G glede na množico S graf z množico točk G in množico povezav $\{\{g, gs\} \mid g \in G, s \in S\}$. Cayleyjev graf ciklične grupe imenujemo *cirkulant.* Vsak Cayleyjev graf je točkovno tranzitiven, obstajajo pa grafi, ki so točkovno tranzitivni in niso Cayleyjevi grafi. Najmanjši tak graf je Petersenov graf (glej sliko 2.1).

Naj bosta $k \geq 1$ in $n \geq 2$ naravni števili. Avtomorfizem grafa je (k, n)-*polregularen*, če ima v cikličnem dekompoziciji natanko k ciklov, vse enake dolžine n. Leta 1981 je Marušič [79] postavil vprašanje, ali vsak točkovno tranzitiven (di)graf premore polregularen avtomorfizem. Kljub temu, da je bil ta problem že predmet številnih raziskav, je še vedno nerešen. Najenostavnejši primer polregularnih avtomorfizmov je avtomorfizem, ki ima v ciklični dekompoziciji natanko en cikel. Grafi, ki premorejo tak polregularen avtomorfizem, so ravno cirkulanti.

Sabidussi [98] je Cayleyjeve grafe karakteriziral na naslednji način: Graf X je Cayleyjev graf grupe G natanko tedaj, ko njegova grupa avtomorfizmov vsebuje regularno podgrupo, ki je izomorfna grupi G. Posplošitev Cayleyjevih grafov so *kvazi*

$m-$*Cayleyjevi* grafi. Namesto obstoja regularne podgrupe grupe avtomorfizmov, se tu zahteva obstoj *kvazi polregularne* podgrupe grupe avtomorfizmov. Grupa G deluje *kvazi polregularno* na množici Ω, če v Ω obstaja tak element ∞, da G fiksira ∞ in je stabilizator G_x poljubnega elementa $x \in \Omega\backslash\{\infty\}$ trivialen. Element ∞ imenujemo *točka v neskončnosti*. Graf X je *kvazi* m*-Cayleyjev na* G, če grupa G deluje kvazi-polregularno na $V(X)$ z m orbitami na $V(X)\backslash\{\infty\}$. Če je G ciklična in je $m = 1$ (oziroma $m = 2$, $m = 3$ in $m = 4$), graf X imenujemo *kvazi cirkulant* (oziroma *kvazi bicirkulant*, *kvazi tricirkulant* in *kvazi tetracirkulant*). Če je točka v neskončnosti ∞ povezana samo z eno orbito stabilizatorja G_∞, rečemo, da je X *krepko kvazi* m*-Cayleyjev graf* na G. Kvazi $m-$Cayleyjevi grafi so bili prvič definirani leta 2011 [60].

Kvazi m-Cayleyjevi grafi

V doktorski disertaciji so dokazani trije izreki, ki klasificirajo cirkulante, ki so hkrati kvazi 2-Cayleyjevi, kvazi 3-Cayleyjevi oziroma krepko kvazi 4-Cayleyjevi grafi. *Paleyjev graf* $P(n)$, ki nastopa v prvem izreku, je graf, katerega točke so elementi končnega obsega reda n, dve točki pa sta povezani, če se razlikujeta v kvadratnem ostanku.

Izrek 1 *Naj bo* X *kvazi 2-Cayleyjev graf reda* n*, ki je hkrati povezan cirkulant. Potem je* X *izomorfen polnemu grafu* K_n *ali pa je* $n \equiv 1 \pmod 4$ *praštevilo in je* X *izomorfen Paleyjevem grafu* $P(n)$*. Še več,* X *je kvazi bicirkulant.*

Izrek 2 *Naj bo* X *povezan cirkulant. Potem je* X *tudi kvazi 3-Cayleyjev graf natako takrat, ko je* $X \cong K_n$ *ali* $X \cong Cay(\mathbb{Z}_n, S)$*, kjer zamenjamo* X *z njegovim komplementom, če je potrebno, in je* S *množica vseh neničelnih kubov v* $\mathbb{Z}_n$*,* n *pa tako praštevilo, da je* $n \equiv 1 \pmod 3$*. Še več,* X *je kvazi tricirkulant.*

Izrek 3 *Naj bo* X *povezan cirkulant. Potem je* X *krepko kvazi 4-Cayleyjev graf grupe* G *natanko takrat, ko je* $X \cong C_9$ *ali* $X \cong Cay(\mathbb{Z}_n, S)$*, kjer je* S *množica vseh četrtih potenc v* $\mathbb{Z}_n\backslash\{0\}$ *in je* n *tako praštevilo, da je* $n \equiv 1 \pmod 4$*. Še več,* X *je kvazi tetracirkulant.*

Petvalentni bicirkulanti

Bicirkulant X je graf, ki premore $(2, n)$-polregularen avtomorfizem. Če $(2, n)$-polregularen avtomorfizem pripada podgrupi $G \leq \mathrm{Aut}(X)$, pravimo, da je X G-*bicirkulant*. Obstoj $(2, n)$-polregularnega avtomorfizma nam omogoča, da označimo točke in povezave grafa na naslednji način. Naj bo X povezan bicirkulant in naj bo $\rho \in \mathrm{Aut}(X)$ $(2, n)$-polregularni avtomorfizem. Potem lahko označimo točke grafa X z x_i in y_i, kjer je $i \in \mathbb{Z}_n$, tako da je

$$\rho = (x_0\, x_1\, \ldots\, x_{n-1})(y_0\, y_1\, \ldots\, y_{n-1}).$$

Množico povezav pa lahko razdelimo na tri dele, *levi*, *srednji* in *desni* del:

$$\begin{aligned}\mathcal{L} &= \bigcup_{i\in\mathbb{Z}_n} \{\{x_i, x_{i+l}\} \mid l \in L\} \quad \text{(leve povezave)},\\ \mathcal{M} &= \bigcup_{i\in\mathbb{Z}_n} \{\{x_i, y_{i+m}\} \mid m \in M\} \quad \text{(srednje povezave)},\\ \mathcal{R} &= \bigcup_{i\in\mathbb{Z}_n} \{\{y_i, y_{i+r}\} \mid r \in R\} \quad \text{(desne povezave)},\end{aligned}$$

kjer so L, M, R take podmnožice množice $\mathbb{Z}_n$, da je $L = -L$, $R = -R$, $M \neq \emptyset$ in $0 \notin L \cup R$. Tak graf bomo označevali z $BC_n[L, M, R]$ (ta notacija bicirkulantov je bila prvič uporabljena v [56]). Točke x_i, $i \in \mathbb{Z}_n$, imenujemo *leve točke*, točke y_i, $i \in \mathbb{Z}_n$, pa *desne točke*.

Klasifikacija kubičnih ločno tranzitivnih bicirkulantov je bila dokončana leta 2007 (glej [30, 86, 97]). Klasifikacija štirivalentnih ločno tranzitivnih bicirkulantov pa je bila dokončana leta 2012 (glej [53, 54, 57]). V doktorski disertaciji je narejen naslednji korak, klasificirani so vsi petvalentni ločno tranzitivni bicirkulanti.

Izrek 4 *Povezan petvalenten bicirkulant $BC_n[L, M, R]$ je ločno tranzitiven natanko tedaj, ko je izomorfen enem izmed naslednjih grafov:*

(i) $BC_6[\{\pm1, 3\}, \{0, 2\}, \{\pm1, 3\}]$, $BC_8[\{\pm1, 4\}, \{0, 2\}, \{\pm3, 4\}]$;

(ii) $BC_3[\{\pm1\}, \{0, 1, 2\}, \{\pm1\}]$, $BC_6[\{\pm1\}, \{0, 2, 4\} \{\pm1\}]$, $BC_6[\{\pm1\}, \{0, 1, 5\}, \{\pm2\}]$;

(iii) $BC_6[\,\emptyset, \{0, 1, 2, 3, 4\}, \emptyset\,]$, $BC_{12}[\,\emptyset, \{0, 1, 2, 4, 9\}, \emptyset\,]$, $BC_{21}[\,\emptyset, \{0, 1, 4, 14, 16\}, \emptyset\,]$, $BC_{24}[\,\emptyset, \{0, 1, 3, 11, 20\}, \emptyset\,]$, *ali* $Cay(D_{2n}, \{b, ba, ba^{r+1}, ba^{r^2+r+1}, ba^{r^3+r^2+r+1}\})$, *kjer je* $D_{2n} = \langle a, b \mid a^n = b^2 = baba = 1\rangle$ *in* $r \in \mathbb{Z}_n^*$ *tak element, da je* $r^4 + r^3 + r^2 + r + 1 \equiv 0 \pmod{n}$.

Posplošeni Cayleyjevi grafi

Naj bo G končna grupa, S podmnožica grupe G in α tak avtomorfizem grupe G, da velja:

(i) $\alpha^2 = 1$,

(ii) če je $g \in G$, potem $\alpha(g^{-1})g \notin S$,

(iii) če sta $f, g \in G$ taka elementa, da je $\alpha(f^{-1})g \in S$, potem je $\alpha(g^{-1})f \in S$.

Potem je *posplošeni Cayleyjev graf* $X = GC(G, S, \alpha)$ na grupi G glede na urejen par (S, α) graf, katerega točke so elementi grupe G, dve točki $f, g \in V(X)$ pa sta povezani v grafu X natanko takrat, ko je $\alpha(f^{-1})g \in S$. Z drugimi besedami, točka $f \in G$ je povezana z vsemi točkami oblike $\alpha(f)s$, kjer je $s \in S$. Za $\alpha = 1$ je posplošeni Cayleyjev graf $GC(G, S, \alpha)$ Cayleyjev graf $Cay(G, S)$. Torej, vsak Cayleyjev graf je posplošeni Cayleyjev graf, obrat pa ne drži. Obstajajo posplošeni Cayleyjevi grafi, ki niso Cayleyjevi. Posplošeni Cayleyjevi grafi so bili prvič definirani v [89], kjer so obravnavane lastnosti teh grafov v povezavi z t.i. dvojnimi krovi grafov in je zastavljeno naslednje vprašanje.

Problem 5 *Ali obstajajo posplošeni Cayleyjevi grafi, ki so točkovno tranzitivni, niso pa Cayleyjevi grafi?*

V doktorski disertaciji na to vprašanje odgovorimo pritrdilno, in sicer s konstrukcijo dveh neskončnih družini posplošenih Cayleyjevih grafov, ki so točkovno tranzitivni, niso pa Cayleyjevi grafi. Grafi v obeh družinah so bicirkulanti, v prvi družini petvalentni in v drugi šestvalentni.

Izrek 6 *Za naravno število $k \geq 1$ naj bo $n = 2((2k+1)^2 + 1)$. Potem je posplošeni Cayleyjev graf $GC(\mathbb{Z}_n, S, \alpha)$ na ciklični grupi $\mathbb{Z}_n$, glede na $S = \{\pm 2, \pm 4k^2, 2k^2 + 2k + 1\}$ in avtomorfizem $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ definiran z $\alpha(x) = ((2k+1)^2 + 2) \cdot x$, točkovno tranzitiven graf, ki ni Cayleyjev graf.*

Izrek 7 *Za naravno število k, $k \not\equiv 2 \pmod 5$, naj bo $t = 2k + 1$ in $n = 20t$. Potem je posplošeni Cayleyjev graf $GC(\mathbb{Z}_n, S, \alpha)$ na ciklični grupi $\mathbb{Z}_n$, glede na $S = \{\pm 2t, \pm 4t, 5, 10t - 5\}$ in avtomorfizem $\alpha \in \mathrm{Aut}(\mathbb{Z}_n)$ definiran z $\alpha(x) = (10t + 1) \cdot x$, točkovno tranzitiven graf, ki ni Cayleyjev graf.*

V doktorski disertaciji dokažemo tudi, da vsak posplošeni Cayleyjev graf premore polregularen avotmorfizem in s tem naredimo pomemben korak pri reševanju že zgoraj omenjenga problema o obstoju polregularnih avtomorfizmov v točkovno tranzitivnih grafih.

Izrek 8 *Naj bo $X = GC(G, S, \alpha)$ posplošeni Cayleyjev graf na grupi G glede na urejen par (S, α). Potem X premore polregularen avtomorfizem, ki je vsebovan v grupi $G_L \leq \mathrm{Aut}(X)$.*

Snarki

Snark je povezan, ciklično 4-povezavno-povezan kubični graf, katerega povezave ni mogoče pobarvati s tremi barvami na tak način, da so sosednje povezave različnih barv.

Prvi je obravnaval snarke Tait leta 1880. Dokazal je, da je izrek štirih barv [9] ekvivalenten izreku, da ne obstajajo ravninski snarki [103]. Petersenov graf, ki je bil odkrit leta 1898, je bil prvi poznan snark. Leta 1946 je hrvaški matematik Blanuša odkril dva nova snarka, oba z 18 točkami, ki ju danes imenujemo Blanuševa snarka [10]. Leta 1975 je Isaacs posplošil Blanuševo metodo in z njo konstruiral dve neskončni družini snarkov [50]. Čeprav znani snarki nosijo določeno stopnjo simetrije, nobeden od njih ni Cayleyjev graf. V [3] je postavljena domneva, da tak graf sploh ne obstaja. Dokaz te domneve bi v veliki meri prispeval k številnim odprtim problemom o Cayleyjevih grafih [17, 93, 118]. Eden izmed teh problemov je dobro znana domneva, da vsak Cayleyjev graf vsebuje hamiltonski cikel, ki je v zadnjih letih predmet številnih raziskav (glej [1, 2, 6, 26, 34, 35, 37, 51, 62, 65, 80, 92, 116, 117]).

V doktorski disertaciji dokažemo, da snarki ne obstajajo v posebni družini Cayleyjevih grafov. Ključni korak pri dokazu nosi naslednji izrek, ki pravi, da je z izjemo polnega grafa K_4 vsak ločno tranzitiven bicirkulant praštevilske valence, ki premore 1-regularno delovanje grupe, ki vsebuje polregularen avtomorfizem, skoraj-dvodelen.

Izrek 9 *Naj bo $X \neq K_4$ tak ločno tranzitiven G-bicirkulant praštevilske valence, da $G \leq \mathrm{Aut}(X)$ deluje 1-regularno na X. Potem je X skoraj-dvodelen.*

S pomočjo zgornjega izreka in teorije Cayleyjevih zemljevidov v doktorski disertaciji dokažemo naslednji izrek.

Izrek 10 *Naj bo $X = Cay(G, \{a, x, x^{-1}\})$ $(2, s, q)$-Cayleyjev graf na grupi $G = \langle a, x \mid a^2 = x^s = (ax)^q = 1, \dots \rangle$, kjer je $s \geq 3$ in je $q \geq 3$ praštevilo. Če je pripadajoči $2q$-kotni graf grafa X G-bicirkulant, graf X ni snark.*

Pol-ločno tranzitivni grafi z majhnim številom alternetov

Obravnavo pol-ločno tranzitivnih grafov je začel Tutte leta 1966 [106]. Leta 1970 je Bouwer [12] dokazal obstoj neskončnega števila pol-ločno tranzitivnih grafov. Leta 1981 pa je Holt [43] skonstruiral primer pol-ločno tranzitivnega grafa na 27 točkah, ki je valence 4 (glej [21, 22]). Za ta graf je leta 1991 Alspach s soavtorji [4] dokazal, da je najmanjši pol-ločno tranzitiven graf. Zelo pomembno vlogo pri raziskovanju 4-valentnih pol-ločno tranzitivnih grafov so imeli Marušič, Waller in Nedela (glej [81, 83, 91]).

Grafu X valence $2k$, $k \geq 2$, ki premore pol-ločno tranzitivno delovanje podgrupe $G \leq \mathrm{Aut}(X)$, na naraven način priredimo usmerjen graf. (Izberemo orientacijo poljubne povezave in potem delovanje grupe G definira orientacijo na preostalih povezavah). Za dve usmerjeni povezavi rečemo, da sta "v relaciji", če imata isto začetno točko, ali isto končno točko. S pomočjo te relacije se definira ekvivalenčna relacija, imenovana *relacija dosegljivosti* (glej [16] in [77]). Podgrafi, ki sestojijo iz ekvivalenčnih razredov usmerjenih povezav relacije dosegljivosti, se imenujejo *alternirajoči cikli* v primeru, ko je X štirivalenten (glej [81, 87]) in *alterneti* v splošnem (glej [113]). Za pol-ločno tranzitiven graf rečemo, da je *tesno spet*, če so vsa vozlišča, ki so repi v enem alternetu, hkrati glave v nekem drugem fiksiranem alternetu. (Za lok (u, v) rečemo, da je u *rep* in v *glava*.)

V doktorski disertaciji obravnavamo lastnosti pol-ločno tranzitivnih grafov oziroma grafov, ki premorejo pol-ločno tranzitivno delovanje z majhnim številom alternetov.

Izrek 11 *Naj bo X G-pol-ločno tranzitiven graf z dvema alternetoma. Potem grupa G vsebuje podgrupo edinko H indeksa 2, ki ima dve orbiti na točkah in na povezavah grafa.*

Grafi $K_{p,p} - pK_2$, kjer je p liho praštevilo, so primeri grafov, ki premorejo pol-ločno tranzitivno delovanje z dvema alternetoma. Očitno so vsi pol-ločno tranzitivni grafi z dvema alternetoma tesno speti. V doktorski disertaciji dokažemo, da enako velja za pol-ločno tranzitivne grafe s tremi alterneti.

Izrek 12 *Naj bo X G-pol-ločno tranzitiven graf s tremi alterneti. Potem je graf X G-tesno spet.*

V doktorski disertaciji dokažemo tudi, da grafi, ki premorejo pol-ločno tranzitivno delovanje z več kot tremi alterneti, niso nujno tesno speti. Primeri so podani v spodnjem primeru. Izreka, ki sledita primeru, pa karakterizirana grafe, ki premorejo pol-ločno tranzitivno delovanje s štirimi oziroma petimi alterneti, ki niso tesno speti.

Primer 13 Za naravno število $n \geq 4$ naj bo X_n graf, ki ima množico točk

$$V(X_n) = \{(i, j) \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_n \setminus \{i\}\}$$

in množico povezav

$$E(X_n) = \{\{(i, j), (k, i)\} \mid (i, j) \in V(X_n), k \in \mathbb{Z}_n, k \neq i, k \neq j\}.$$

Potem simetrična grupa S_n deluje na X_n pol-ločno tranzitivno z n alterneti, ki niso tesno speti.

Izrek 14 *Naj bo X G-pol-ločno tranzitiven graf s štirimi alterneti, ki ni G-tesno spet. Potem obstaja kvocientni graf grafa X, ki je izomorfen grafu X_4.*

Izrek 15 *Naj bo X G-pol-ločno tranzitiven graf s petimi alterneti, ki ni G-tesno spet. Potem obstaja kvocient grafa X, ki je izomorfen grafu X_5.*

Metodologija

Osnovni orodji pri raziskovanju sta algebraična teorija grafov in teorija permutacijskih grup (glej [112]). Za računanje so uporabljene metode računalniške algebre ter računalniški program MAGMA [11]. Pomembno vlogo v obravnavanju Cayleyjevih snarkov imajo metode, ki so uporabljene v [34, 35, 36, 37], enako kot rezultati v [96] glede maksimalnih neodvisnih množic. Bistveni del v obravnavi kvazi m-Cayleyjevih grafov je klasifikacija ločno-tranzitivnih cirkulantov [52, 69]. Pri klasifikaciji petvalentnih ločno tranzitivnih bicirkulantov je uporabljena klasifikacija ločno tranzitivnih Tabačjn grafov [8]. Poleg tega pomembno vlogo igra rezultat, ki pravi, da v končnih grupah "dovolj velike" ciklične podgrupe vedno vsebujejo ne-trivialno podgrupo edinko grupe [42]. Pri obravnavi drugih problemov je, poleg metod iz teorije grup in krovnih tehnik [39, 40, 78], zelo pomembna kombinatorična analiza [73] obravnavanih struktur.

Kazalo

Stvarno kazalo

Declaration

I declare that this thesis does not contain any materials previously published or written by another person except where due reference is made in the text.

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