UNIVERZA NA PRIMORSKEM

FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA

(DOCTORAL THESIS)

IZBRANE GRAFOVSKO-TEORETIČNE INVARIANTE IN DEKOMPOZICIJE: OD STRUKTURE DO MEJ IN ALGORITMOV

(CERTAIN GRAPH-THEORETIC INVARIANTS AND DECOMPOSITIONS: FROM STRUCTURE TO BOUNDS AND ALGORITHMS)

KENNY ŠTORGEL

KOPER, 2023

UNIVERZA NA PRIMORSKEM

FAKULTETA ZA MATEMATIKO, NARAVOSLOVJE IN INFORMACIJSKE TEHNOLOGIJE

DOKTORSKA DISERTACIJA

(DOCTORAL THESIS)

IZBRANE GRAFOVSKO-TEORETIČNE INVARIANTE IN DEKOMPOZICIJE: OD STRUKTURE DO MEJ IN ALGORITMOV

(CERTAIN GRAPH-THEORETIC INVARIANTS AND DECOMPOSITIONS: FROM STRUCTURE TO BOUNDS AND ALGORITHMS)

KENNY ŠTORGEL

KOPER, 2023

MENTOR: IZR. PROF. DR. BORUT LUŽAR
SOMENTOR: PROF. DR. MARTIN MILANIČ

Acknowledgements

For my personal growth and all the mathematical knowledge obtained throughout my life, I have to thank my family, and friends who supported me through my difficult times and raging moments, and also being there to celebrate my achievements.

In particular, I would like to thank all the professors and teaching assistants at UP FAMNIT for their guidance on my path through the endless world of mathematics in all my years there as a student.

I am also extremely grateful to be a part of FIŠ institution where I was able to learn that passing on my knowledge to new generations of students is just as important as gaining new knowledge.

In particular, I have to thank both my supervisors Borut Lužar and Martin Milanič for all their invaluable guidance, continuous support, encouragement, and constructive criticism. Their lessons are a true inspiration and I sincerely hope that one day I will be able to inspire young minds in the same way. You both have my deepest, sincerest gratitude for always being there and mentoring me both in an academic sense as well as in life.

Contents

List of Figures **ix**

List of Tables **xii**

1 Introduction **1**

2 Preliminaries **7**
- 2.1 Graph colorings 10
- 2.2 Planar graphs 11
- 2.3 Graph problems and complexity 12

PART I **15**

3 Preliminaries to Part I **15**
- 3.1 More on planar graphs 15
- 3.2 More on colorings 16
- 3.3 Discharging method 18

4 Grötzsch Theorem and its Extensions **21**
- 4.1 Planar graphs with a small number of triangles 23
- 4.2 Our results 24
  - 4.2.1 Proof of Theorem 4.2.1 31
  - 4.2.2 Proof of Theorem 4.2.3 36
  - 4.2.3 Proof of Theorem 4.2.6 38

5 Cyclic Coloring of Plane Graphs and its Generalization **41**
- 5.1 Cyclic coloring 42
- 5.2 ℓ-facial (vertex) coloring 45
- 5.3 ℓ-facial edge coloring 48
- 5.4 Proof of Theorem 5.3.3 49

6 Facial-Parity Colorings of Plane Graphs **72**
- 6.1 Facial-parity edge coloring 74
- 6.2 Facial-parity vertex coloring 77

7 Final Remarks to Part I **79**

7.1 Remarks on proper colorings of planar graphs 79
7.2 Remarks on ℓ-facial edge coloring 81

PART II 83

8 Preliminaries to Part II 83
8.1 Ramsey's theorem 84
8.2 Tree decompositions and treewidth 85
8.3 Potential maximal cliques 87
8.4 Block-cutpoint trees 89
8.5 SPQR trees 89

9 Bounding Treewidth by a Function of the Clique Number 92
9.1 Forbidding a subgraph, topological minor, or minor 95
9.2 Forbidding an induced subgraph or an induced topological minor 97
9.3 Forbidding an induced minor 99

10 Tree Decompositions with Bounded Independence Number 106
10.1 Tree-independence number: basic properties 107
10.2 Tree-independence number: forbidding a structure 110
10.3 Tree-independence number: $K_{2,q}$-im-free graphs 112
10.4 Tree-independence number: refinements 115
10.5 Tree-independence number: reductions 118
10.5.1 Reduction to 2-connected graphs 118
10.5.2 Reduction to triconnected components 121
10.6 Tree-independence number: W_4-im-free graphs 126
10.7 Tree-independence number: K_5^--im-free graphs 131

11 Maximum Weight Independent Set and Generalizations 137
11.1 Maximum Weight Independent Set problem 139
11.2 Maximum Weight Independent Packing problem 143
11.3 The classes of $K_{2,q}$-, W_4-, and K_5^--im-free graphs 150
11.3.1 $K_{2,q}$-induced-minor-free graphs 150
11.3.2 W_4-induced-minor-free graphs 151
11.3.3 K_5^--induced-minor-free graphs 152

12 Final Remarks to Part II 153
12.1 Remarks on (tw, ω)-boundedness and open questions 153
12.2 Further algorithmic results 155
12.3 Other related work 156

13 Conclusion 160

Bibliography 162

Povzetek v slovenskem jeziku **177**

List of Figures

2.1 Two distinct plane embeddings of the same planar graph. . 12

4.1 A precoloring of three vertices in a planar graph with no triangles (example (a)) and a precoloring of two vertices in a planar graph with two triangles (example (b)) neither of which can be extended to a 3-coloring of the whole graph. . 25
4.2 Not every graph obtained from a planar graph with at most one triangle by adding a 4-vertex is 3-colorable (example (a)), nor is a graph obtained from a planar graph with two triangles by adding a 3-vertex (example (b)). In both cases the added vertex is depicted as a white vertex. 26
4.3 A precoloring of the outer 5-face which cannot be extended to a 3-coloring of the graph. 26
4.4 A planar graph K_4' with exactly one triangle and a vertex v in the center with no 3-coloring such that all the vertices in $N(v)$ receive the same color. 27
4.5 Precoloring of the neighborhood of a 4-vertex v in a K_4'-free planar graph G with one triangle cannot always be extended to a 3-coloring of G (example (a)). Similarly, precoloring of the neighborhood of a 2-vertex v in a planar graph G with two triangles cannot always be extended to a 3-coloring of G (example (b)). 28
4.6 A DHGO-composition of two complete graphs K_4. 28
4.7 The first few graphs from $\mathcal{TW}$. 29
4.8 Havel's quasi-edge uv. 30
4.9 The configuration in Lemma 4.2.13 in the case when the number of triangles increases both in G_1 and in G_2. 30
4.10 The 4-faces α and α' in the last part of Case 4. 33
4.11 The vertices in G comprising triangles in H in the last part of Case 2. 35
4.12 The coloring of f forces the colors of x, y, and z. 37
4.13 A separating 5-cycle in G containing α_1. 39

5.1 A 3-connected plane graph G with $\chi_c(G) = \Delta^*(G) + 2$. . . 44
5.2 Example of a 2-connected plane graph G with $\delta(G) = 3$ and $\chi_c(G) > \Delta^*(G) + k$. 44

5.3 A plane graph G which is a plane embedding of a K_4 with three edges sharing a common vertex subdivided exactly $\ell-1$ times and with $\chi_\ell(G) = 3\ell+1$. 46
5.4 A triangle-free plane graph G with $\chi_\ell(G) = 3\ell$. 48
5.5 A Theta graph $G = \Theta_{\ell,\ell-1,\ell-1}$ with $\chi'_\ell(G) = 3\ell+1$. 49
5.6 A reducible configuration with a 2-thread and a 2-vertex $w \in \{v_2, v_3\}$. 53
5.7 A reducible 5-face incident with a 3-vertex. 55
5.8 A reducible 7-face incident with a 2-thread with two 3-neighbors. 57
5.9 The three possible configurations of a 7-face incident with a 2-thread, a 2-vertex, and a 3-vertex. 58
5.10 A 7-face with at least two incident 2-vertices, where one of them has two 3-neighbors. 59
5.11 A reducible 9-face with an incident 2-vertex. 62
5.12 Labeling of the 10-face α. 63

6.1 A construction of Mátrai with two copies of W_4 having no cover with 3 edge-disjoint odd subgraphs. 73
6.2 The four irreducible types of Shannon triangles. 73
6.3 Two outerplane graphs requiring 10 and 9 colors, respectively, in any facial-parity edge coloring. 75
6.4 The graph $\Theta_{4,4,4}$ with 12 edges and $\chi'_{\mathrm{fp}}(G) = 12$. 75
6.5 Two outerplane graphs requiring 10 colors in any facial-parity vertex coloring. 78
6.6 The line graph of the graph $\Theta_{4,4,4}$ which has 12 vertices and $\chi_{\mathrm{fp}}(G) = 12$. 78

7.1 A wheel graph W_5 without a spoke which is not adynamically 3-colorable with u being the only mono-vertex. 80

9.1 An example of an elementary wall graph (of height 5 and width 10) on the left and a 1-subdivided wall graph on the right. 93
9.2 The graphs W_4, K_5^-, and $K_{2,3}^+$. 94
9.3 Representation of the different cases considered in the proof of Theorem 9.3.5. The induced minor contains all plain edges and is a subgraph of the graph induced by plain and dotted edges. Black squared vertices induce a C_4 and black round vertices are contracted into a single vertex (see [76]). 103

10.1 A graph containing W_4 as an induced minor obtained by contracting the dotted edge [78]. 127

List of Tables

4.1 Summary of results stating when a planar graph is 3-colorable. Checkmarks annotate for which k cycles of length k are forbidden as subgraphs. 22
4.2 Summary of results stating when a planar graph is 3-choosable. Checkmarks annotate for which k cycles of length k are forbidden as subgraphs. 23

9.1 Summary of (tw, ω)-bounded graph classes excluding a fixed graph H for six graph containment relations [76]. 95

Abstract

CERTAIN GRAPH-THEORETIC INVARIANTS AND DECOMPOSITIONS: FROM STRUCTURE TO BOUNDS AND ALGORITHMS

We study several graph invariants related to the colorings of planar graphs and also several graph invariants and decompositions related to tree decompositions of graphs in certain graph classes. Among the former are the proper coloring of planar graphs, cyclic coloring and its relaxations the ℓ-facial colorings of plane graphs, and the facial-parity colorings of plane graphs. Among the latter we study (tw, ω)-bounded graph classes, tree-independence number of graphs, and the application to the solvability of the Maximum Weight Independent Set problem.

We begin the first part of this thesis with the proper 3-coloring of planar graphs. The Grötzsch Theorem states that every triangle-free planar graph admits a proper 3-coloring. Perhaps one of the best known generalizations of the Grötzsch Theorem is the result of Grünbaum and Aksenov which states that every planar graph with at most three triangles is 3-colorable. That result is best possible due to the complete graph on 4 vertices. Thus, a lot of attention was given to study various ways in which precoloring a certain set of vertices in a planar graph without triangles with 3 colors can be extended to a 3-coloring of the whole graph. We continue on this research path and consider 3-colorings of planar graphs with at most one triangle. In particular, we show that any precoloring of any two non-adjacent vertices can be extended to a 3-coloring of the whole graph. In addition, we show that any precoloring of a face of length at most 4 can also be extended to a 3-coloring of the whole graph and that for every vertex of degree at most 3, a precoloring of its neighborhood with a single color extends to a 3-coloring of the whole graph. Furthermore, we also give examples that show tightness of our results.

Next, we study cyclic coloring of plane graphs, i.e., a coloring of the vertices of a plane graph in such a way that no face is incident with two vertices of the same color. It immediately follows that for such a coloring we require at least as many colors as is the length $\Delta^*(G)$ of the largest face and the Cyclic Coloring Conjecture states that $\lfloor \frac{3}{2}\Delta^*(G) \rfloor$ colors should suffice. An indication of the difficulty of the Cyclic Coloring Conjecture lies in the fact that it is known to be true only for a small number of cases; namely, in the cases when $\Delta^*(G) \in \{3, 4, 6\}$. Thus, an ℓ-facial vertex coloring of plane graphs was introduced as a generalization of the cyclic coloring. An ℓ-facial vertex coloring of a plane graph is a vertex coloring in which any pair of vertices at distance at most ℓ on a boundary walk of any face receive distinct colors. It is conjectured that at most $3\ell+1$ colors suffice for such a coloring, however, only the case when $\ell \leq 1$ is known to be true. An ℓ-facial edge coloring of a plane graph is a coloring of its edges such that any two edges at distance at most ℓ on a boundary walk of any face receive distinct colors. This coloring is the edge coloring variant of the ℓ-facial vertex coloring. Similarly as for the vertex version, it is conjectured that at most $3\ell+1$ colors suffice for an ℓ-facial edge coloring of any plane graph. In this case, the conjecture has been confirmed for $\ell \leq 2$. We prove that the conjecture holds also for $\ell = 3$.

The last topic of study in the first part of this thesis are facial-parity colorings of plane graphs. A facial-parity vertex coloring of a 2-connected plane graph is a facially-proper vertex coloring in which every face is incident with zero or an odd number of vertices of each color. Similarly, a facial-parity edge coloring of a connected bridgeless plane graph is a facially-proper edge coloring in which every face is incident with zero or an odd number of edges of each color. Known results state that every 2-connected plane graph admits a facial-parity vertex coloring with at most 97 colors and that every connected bridgeless plane graph admits a facial-parity edge coloring with at most 16 colors. In both cases it was conjectured that 10 colors should suffice and examples of outerplane graphs requiring 10 colors are also known. We provide an infinite family of 2-connected plane graphs that require 12 colors in any facial-parity vertex coloring and an infinite family of 2-connected plane graphs that require 12 colors in any facial-parity edge coloring.

In the second part of this thesis we focus our study on graph classes closed under taking induced subgraphs in which the absence of large cliques is both a necessary and a sufficient condition for a graph class to have bounded treewidth. This property is called (tw, ω)-boundedness. It is known that such graph classes have useful algorithmic applications related to variants of the clique and k-coloring problems. In order to study such graph classes, we consider six well-known graph containment relations: the subgraph, topological minor, minor, induced subgraph, induced topological minor, and induced minor relations. For each of the relations, we provide a complete characterization of the graphs H for which the class of graphs excluding H is (tw, ω)-bounded. The family of (tw, ω)-bounded graph classes also provides a unifying framework for various very different graph classes. Among them are graph classes of bounded treewidth, graph classes of bounded independence number, intersection graphs of connected subgraphs of graphs with bounded treewidth, and graphs in which all minimal separators are of bounded size.

We then study the tree-independence number, which is defined as follows. The independence number of a tree decomposition $\mathcal{T}$ of a graph G is the maximum independence number over all subgraphs of G induced by some bag of $\mathcal{T}$. The tree-independence number of a graph G is then defined as the minimum independence number over all tree decompositions of G. We prove several properties of the tree-independence number and, in particular, show that graph classes with bounded tree-independence number are (tw, ω)-bounded with a polynomial binding function. We then show that every (tw, ω)-bounded graph class characterized by forbidding a single graph H with respect to one of the six aforementioned graph containment relations has bounded tree-independence number. In particular, we focus on the induced minor relation and show that in the three cases when we exclude W_4, K_5^-, or $K_{2,q}$, respectively, as an induced minor, such graphs have tree-independence number at most 4 in the former two cases and at most $2(q-1)$ in the latter case. We obtain these results by using a variety of tools, including ℓ-refined tree decompositions, block-cutpoint trees, SPQR trees, and potential maximal cliques. In addition, we show how to compute tree decompositions with bounded independence number efficiently in all the identified cases with bounded tree-independence number. Moreover, we pose a conjecture that, in fact, the two properties (tw, ω)-boundedness and bounded tree-independence number are equivalent.

Finally, it is an interesting question what algorithmic implications does (tw, ω)-boundedness have with respect to various problems; in particular, our goal is to understand the extent to which this property has useful algorithmic implications for the Maximum Independent Set and related problems. We provide a partial answer to this question by identifying a sufficient condition for (tw, ω)-bounded graph classes to admit a polynomial-time algorithm for the Maximum Weight Independent Packing problem and, as a consequence, we get polynomial-time solution for the weighted variants of the Independent Set and Induced Matching problems. We show that bounded tree-independence number implies the existence of a polynomial-time algorithm for the Maximum Weight Independent Packing problem. In particular, this implies polynomial-time solvability of the Maximum Weight Independent Set problem in all (tw, ω)-bounded graph classes characterized with a single forbidden graph with respect to one of the six graph containment relations.

Math. Subj. Class. (2020): 05C15, 05C69, 05C75, 05C05, 05C10, 05C83

Keywords: facial-parity coloring, graph coloring, graph containment relation, Grötzsch Theorem, ℓ-facial coloring, Maximum Weight Independent Set problem, tree decomposition, tree-independence number.

Povzetek

IZBRANE GRAFOVSKO-TEORETIČNE INVARIANTE IN DEKOMPOZICIJE: OD STRUKTURE DO MEJ IN ALGORITMOV

V disertaciji obravnavamo različne invariante grafov povezane z barvanjem ravninskih grafov in različne invariante in dekompozicije povezane z drevesnimi dekompozicijami grafov v izbranih razredih grafov. Med prvimi so pravilna barvanja ravninskih grafov, ciklično barvanje in njegove posplošitve, kot so ℓ-lična barvanja vložitev ravninskih grafov, in lično-parna barvanja vložitev ravninskih grafov. Med drugimi pa obravnavamo (tw, ω)-omejene razrede grafov, drevesno neodvisnostno število grafov in aplikacijo le tega na razrešljivost problema najtežje neodvisne množice.

Prvi del disertacije začnemo z obravnavo 3-barvanja ravninskih grafov. Grötzschev izrek pravi, da za vsak ravninski graf brez trikotnikov obstaja pravilno 3-barvanje. Ena od najbolj znanih posplošitev Grötzschevega izreka je rezultat Grünbauma in Aksenova, ki pravi, da je vsak ravninski graf z največ tremi trikotniki 3-obarvljiv. Ta rezultat je najboljši možen, kar dokazuje polni graf na 4 vozliščih. Posledično je bilo veliko pozornosti usmerjene v obravnavo različnih načinov, da lahko predbarvanje določene množice vozlišč ravninskega grafa brez trikotnikov razširimo na 3-barvanje celotnega grafa. V tej disertaciji nadaljujemo v smeri teh raziskav, kjer se osredotočimo na 3-barvanja ravninskih grafov z največ enim trikotnikom. Bolj natančno, pokažemo, da lahko vsako predbarvanje dveh nesosednjih vozlišč vedno razširimo na 3-barvanje celotnega grafa. Dodatno pokažemo, da lahko vsako predbarvanje lica dolžine največ 4 prav tako razširimo na 3-barvanje celotnega grafa. Prav tako pokažemo, da lahko vsako predbarvanje soseščine vozlišča stopnje največ 3 z eno barvo razširimo na 3-barvanje celotnega grafa. Nenazadnje pa podamo tudi primere, ki dokazujejo tesnost naših rezultatov.

Nato nadaljujemo z obravnavo cikličnega barvanja vložitev ravninskih grafov, to je, barvanje vozlišč vložitve ravninskega grafa, da nobeno lice ni sosednje z dvema vozliščema enake barve. Iz definicije direktno sledi, da za tako barvanje potrebujemo vsaj toliko barv, kot je dolžina $\Delta^*(G)$ najdaljšega lica. Domneva cikličnega barvanja pa pravi, da $\lfloor \frac{3}{2}\Delta^*(G) \rfloor$ barv vedno zadostuje. O zahtevnosti domneve cikličnega barvanja priča dejstvo, da je znano le to, da domneva velja za majhno število primerov, in sicer $\Delta^*(G) \in \{3, 4, 6\}$. Posledično je bilo definirano ℓ-lično barvanje vozlišč vložitve ravninskega grafa kot posplošitev cikličnega barvanja. Vozliščno barvanje vložitve ravninskega grafa je ℓ-lično, če za vsak par vozlišč na razdalji največ ℓ na sprehodu po robu lica velja, da sta pobarvani z različno

barvo. Domneva pravi, da bi moralo zadostovati $3\ell + 1$ barv za takšno barvanje, vendar je znano le, da je domneva resnična za $\ell \leq 1$. Povezavno barvanje vložitve ravninskega grafa je ℓ-lično, če za vsak par povezav na razdalji največ ℓ na sprehodu po robu lica velja, da sta pobarvani z različno barvo. To barvanje je povezavna različica ℓ-ličnega vozliščnega barvanja. Podobno kot za vozliščno različico, obstaja domneva, da bi moralo zadostovati $3\ell + 1$ barv za ℓ-lično povezavno barvanje poljubne vložitve ravninskega grafa. O tej domnevi je znanega nekoliko več, saj je dokazano, da velja za $\ell \leq 2$. Ob tem pokažemo, da domneva velja prav tako za $\ell = 3$.

Zadnja tematika prvega dela te disertacije so lično-parna barvanja vložitev ravninskih grafov. Lično-parno barvanje vozlišč vložitve 2-povezanega ravninskega grafa je po licih pravo barvanje vozlišč, kjer je vsako lice sosednje z nič ali pa z lihim številom vozlišč posamezne barve. Lično-parno barvanje povezav vložitve povezanega ravninskega grafa brez mostov je po licih pravilno barvanje povezav, kjer je vsako lice sosednje z nič ali pa z lihim številom povezav posamezne barve. Znani rezultati pravijo, da ima vsaka vložitev 2-povezanega ravninskega grafa lično-parno barvanje vozlišč z največ 97 barvami in da ima vsaka vložitev povezanega ravninskega grafa brez mostov lično-parno barvanje povezav z največ 16 barvami. V obeh primerih je bila postavljena domneva, da 10 barv zadostuje in podani so bili primeri vložitev zunanje ravninskih grafov, ki potrebujejo natanko 10 barv. Pokažemo, da obstaja neskončna družina vložitev 2-povezanih ravninskih grafov, ki potrebujejo 12 barv v vsakem lično-parnem barvanje vozlišč in da obstaja neskončna družina vložitev 2-povezanih ravninskih grafov, ki potrebuje 12 barv v vsakem lično-parnem barvanju povezav.

V drugem delu te disertacije se usmerimo na obravnavo razredov grafov, zaprtih za inducirane podgrafe, v katerih je odsotnost velike klike tako potreben kot tudi zadosten pogoj za omejeno drevesno širino. Ta lastnost je imenovana (tw, ω)-omejenost. Znano je, da imajo taki razredi grafov uporabne algoritmične aplikacije, povezane z različnimi problemi, kot sta problem klike in problem k-barvanja. Z namenom obravnave takih razredov grafov se osredotočimo na šest različnih znanih relacij vsebovanosti v grafih, ki so: podgraf, topološki minor, minor, induciran podgraf, induciran topološki minor in induciran minor. Za vsako od omenjenih relacij v celoti karakteriziramo grafe H, za katere je razred grafov, ki ne vsebujejo grafa H glede na izbrano relacijo, (tw, ω)-omejen. Družine (tw, ω)-omejenih razredov grafov povezujejo in opišejo več različnih razredov grafov. Med njimi so razredi grafov z omejeno drevesno širino, razredi grafov z omejenim neodvisnostnim številom, razredi presečnih grafov povezanih podgrafov z omejeno drevesno širino in razredi grafov, kjer so vsi minimalni prerezi omejene velikosti.

Nato nadaljujemo z obravnavo drevesnega neodvisnostnega števila, ki je definirano na naslednji način. Neodvisnostno število drevesne dekompozicije $\mathcal{T}$ grafa G je največje neodvisnostno število med vsemi podgrafi grafa G, ki so inducirani z neko vrečo drevesne dekompozicije $\mathcal{T}$. Drevesno neodvisnostno število grafa G je nato definirano kot najmanjše neodvisnostno število med vsemi drevesnimi dekompozicijami grafa G. Med rezultati pokažemo nekaj lastnosti drevesnega neodvisnostnega števila in pokažemo, da so razredi grafov z omejenim drevesnim neodvisnostnim število (tw, ω)-omejeni s polinomsko funkcijo. Nato pokažemo, da ima vsak (tw, ω)-omejen razred grafov, karakteriziran z enim prepovedanim grafom H, glede na eno od prej omenjenih šestih relacij vsebovanosti, omejeno drevesno neodvisnostno število. Še posebej se osredotočimo na relacijo induciranega minorja in pokažemo, da v treh primerih, ko izključimo W_4, K_5^- ali $K_{2,q}$ kot induciran minor, imajo dobljeni razredi grafov drevesno neodvisnostno število največ 4 v prvih dveh primerih in največ $2(q-1)$ v zadnjem primeru. Te rezultate dobimo s pomočjo različnih orodij, med katerimi so ℓ-izpopolnjene drevesne dekompozicije, bločno-prerezno vozliščno drevesa, SPQR drevesa in največje potencialne klike. Dodatno pokažemo, kako izračunati drevesne dekompozicije z omejenim neodvisnostnim številom v polinomskem času v vseh identificiranih razredih grafov z omejenim drevesnim neodvisnostnim številom. Med drugim podamo tudi domnevo, da sta lastnosti (tw, ω)-omejenost in omejenost drevesnega neodvisnostnega števila pravzaprav ekvivalentni.

Na koncu se osredotočimo na vprašanje o tem, kakšne algoritmične implikacije ima (tw, ω)-omejenost na različne probleme. Bolj natančno se usmerimo na problem največje neodvisne množice in z njim povezane probleme. Na to vprašanje delno odgovorimo z identifikacijo zadostnega pogoja za (tw, ω)-omejene razrede grafov, da omogočajo rešitev problema najtežjega neodvisnega pakiranja v polinomskem času. Posledično nam to da rešljivost uteženih različic problema neodvisne množice in problema induciranega prirejanja v polinomskem času. Kot rezultat pokažemo, da omejeno neodvisnostno število implicira obstoj polinomskega algoritma za problem najtežjega neodvisnega pakiranja. To implicira tudi polinomsko rešljivost problema najtežje neodvisne množice v vseh (tw, ω)-omejenih razredih grafov, ki smo jih karakterizirali z enim prepovedanim grafom glede na eno od šestih relacij vsebovanosti v grafu.

Math. Subj. Class. (2020): 05C15, 05C69, 05C75, 05C05, 05C10, 05C83
Ključne besede: lično-parno barvanje, barvanje grafa, relacija vsebovanosti grafa, Grötzschev izrek, ℓ-lično barvanje, problem najtežje neodvisne množice, drevesna dekompozicija, drevesno neodvisnostno število.

Chapter 1

Introduction

$\mathcal{A}$ young and rapidly developing branch of discrete mathematics called graph theory was born all the way back in 1736 when Leonhard Euler formulated and solved the first problem on graphs known as the Königsberg Bridge Problem [103]. Essentially, graph theory is the study of relations between objects in an abstract way where relations are represented by an abstract object called a graph. It is known that graph theory has many applications in various areas such as computer science, biology, chemistry, social studies, etc., and offers many solutions to everyday problems. As in all areas of mathematics, the main focus is classification of graphs which revolves around the concept of a *graph invariant*, that is, a function defined on graphs taking the same value on isomorphic graphs. Throughout the years a large number of graph invariants were introduced. Some of them were introduced in order to deepen our understanding of previously known graph invariants, some were introduced in order to generalize known properties, some were introduced in order to help solve certain everyday problems, and some were introduced to help improve various algorithmic results in terms of time complexity. Numerous graph invariants exist, which are difficult to understand when considering all possible graphs. Thus, it is commonly the case that when studying a certain graph invariant we restrict ourselves only to certain *graph classes*. A graph class is a family of graphs closed under isomorphism (see, e.g., [45, 111, 186]).

Perhaps one of the most commonly known graph invariants, is called the *chromatic number* of a graph G, denoted by $\chi(G)$. That is, the least number of colors needed to color the vertices of G in such a way that no two adjacent vertices receive the same color (such a coloring is called a *proper coloring* of a graph). The history of this graph invariant began with the Four Color Problem. This problem, which asks whether the vertices of every planar graph can be properly colored with four colors, was introduced in 1852 by Francis Guthrie. More than a century passed before the problem was resolved in affirmative in 1976–1977 by Appel and Haken [12, 13] and Appel, Haken and Koch [14]. In general, the chromatic number of a graph gives the smallest number of sets in which we can partition the vertices of a graph in such a way that each of the sets contains only mutually non-adjacent vertices. Partitioning of the vertices of a graph can be done in many different ways by requiring various conditions to be met for each set. Thus, various notions of graph colorings can be meaningfully defined together with their corresponding graph coloring invariants. However, it is often the case that understaning such invariants is difficult in the class of all graphs. Due to that reason, research is often restricted only to some specific graph classes, e.g., the class of planar graphs, as was done in the case of the Four Color Problem.

On another note, instead of partitioning the vertex set, we may instead consider decompositions, that is, assignments of vertices of a graph to different sets, also called *bags*, where we allow each vertex to be assigned to more than one bag, in such a way that for every pair of adjacent vertices there is a bag containing both vertices. One such decomposition is a *tree decomposition* where, in fact, together with the described condition we also require a few other conditions to be fulfilled. A graph invariant which is closely related to tree decompositions is called the *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$. Roughly speaking, treewidth measures how close to a "tree" a graph is. The concept of tree decompositions, which was originally introduced by Halin in 1976, became widely known since its rediscovery by Robertson and Seymour in 1984 in their Graph Minors III paper [175], which is one of the many papers in the Graph Minors series. Many graph invariants are NP-hard to compute (see, e.g., [105]). However, it is known that tree decompositions play an important role in designing efficient algorithms, where by the word *efficient* we mean polynomial-time complexity. Many problems, that are NP-hard in general are known to be polynomial-time solvable for graphs in any class of bounded treewidth graphs. Thus, this graph invariant is a central focus of many studies.

In the remainder of this chapter we shortly present the two main directions of research we will follow throughout this thesis. One direction is to consider different graph coloring invariants in certain graph classes, that is to say, different partitions of the vertices of a graph, while the other direction is to try and better understand tree decompositions and present their further applications to problems of computing certain graph invariants that are generally NP-hard.

- Part I: The Four Color Problem motivated many mathematicians to work on graph colorings, including Brooks, Vizing, Kempe, Tait, Petersen, and Heawood. Already in 1880 Tait [188] showed that this problem is equivalent to proving that every bridgeless planar cubic graph admits a proper edge-coloring with three colors, that is, a coloring of the edges using three colors such that edges incident with a common vertex receive distinct colors. Decades later, in 1941 Brooks proved that the chromatic number of any graph is at most its maximum degree plus one, with equality only in the case of complete graphs and odd cycles. Later, in 1964 Vizing [195] showed that the least number of colors needed for a proper edge-coloring of a simple graph G (also known as the *chromatic index*) of a graph is equal to either the maximum degree or the maximum degree plus one. Graphs that achieve the former are commonly known as *class* I graphs, while those that achieve the latter are *class* II graphs. Determining both the chromatic number and the chromatic index is known to be NP-complete. In fact, it is also known that even when restriced to the class of planar graphs, deciding whether a graph can be properly colored with three colors remains NP-complete [105]. In addition, in 1981 Holyer [133] showed that deciding whether a graph is class I is NP-complete even in the case of cubic graphs. These two results show that even when restricted to certain relatively simple graph classes, computing graph coloring invariants remains difficult. As is the case with both vertex-coloring of planar graphs and edge-coloring in the class of all graphs, the tight bounds are already established. However, when trying to understand the structure of graphs that require a certain exact number of colors, or by adding additional assumptions or conditions to define new types of colorings, the problems often become harder. The goal of this part of the thesis is thus to improve the understanding of when three colors are sufficient to color the vertices of a planar graph, to find a generalization for both vertex-coloring and edge-coloring of plane graphs restricted to faces first by adding distance constraints (the ℓ-facial colorings) and second by adding parity constraints (the facial-parity colorings).

- Part II: In 1949, the term *clique*, that is, a subset of mutually adjacent vertices in a graph, was introduced by Luce and Perry. However, the study of cliques was known already in 1935 when Erdős and Szekeres [102] reformulated Ramsey theory in terms of graphs. Together with the notion of clique comes also the graph invariant called the *clique number* and denoted by $\omega(G)$, defined as the maximum size of a clique in a graph. It follows directly from the definitions that $\omega(G) \leq \chi(G)$ for every graph G. Following this observation, a question arises "When does the equality between the two invariants hold?". This motivated Berge to define one of the most well-known graph classes, the class of *perfect graphs*. Perfect graphs are defined as the graphs G for which $\omega(H) = \chi(H)$ for every induced subgraph H of G. A graph class that is closed under taking induced subgraphs is called *hereditary*. Clearly, the class of perfect graphs is hereditary by definition. As shown in 1981 by Grötschel, Lovász, and Schrijver in their seminal paper [113], various graph invariants, such as the clique number, the independence number, the chromatic number, and several others, are computable in polynomial time in this class of graphs. On the other hand, there are also many other problems that remain NP-hard in the class of perfect graphs, e.g., the maximum cut or the feedback vertex set problem, see, e.g., Karp's paper on 21 NP-complete problems [146]. Over the decades, the study of perfect graphs continued and became even more interesting with the Strong Perfect Graph Theorem proved in 2006 by Chudnovsky et al. [59]. The good algorithmic properties of perfect graphs, together with the quest for understanding conditions under which certain inequalities between graph invariants hold, prompted Gyárfás, in 1987, to introduce the notion of χ-*boundedness* [117]. A graph class is said to be χ-bounded if there exists a function f such that for every graph G in the class, $\chi(G) \leq f(\omega(G))$ for every induced subgraph H of G. While there exist graphs which have an arbitrarily large gap between the chromatic number and the clique number (see, e.g., [100]), which shows that there exist graph classes that are not χ-bounded, perfect graphs clearly satisfy this condition with the identity function. Even though many research papers deal with this concept, the concept of χ-boundedness remains, up to this day, not completely understood. Just as the chromatic number is an upper bound on the clique number, for every graph G we also have that $\chi(G) \leq \mathrm{tw}(G)+1$. Similarly as for χ-boundedness, one can then define (tw, ω)-boundedness. The class of graphs that achieve the equality, in a hereditary sense, are exactly the *chordal graphs*. Chordal graphs are, by definition, the graphs with no induced cycles of length at least four. Thus, in a similar sense that χ-boundedness generalizes perfection, (tw, ω)-boundedness generalizes chordality. The goal of the second part of this thesis is thus develop a better understanding of the notion of (tw, ω)-boundedness as well as finding new efficient algorithms to solve certain algorithmic problems.

The rest of the thesis is structured as follows. In Chapter 2, we first present several general definitions, notations, and terminology used throughout this thesis.

We begin the first part of this thesis with Chapter 3, in which we present additional definitions, notations, terminology, and results needed for Part I. In particular, in Chapter 3, we provide additional definitions and notation needed to work on plane graphs, as well as give several tools that we use in the proofs of our results. In Chapter 4, we then study the structure of planar graphs that can be colored with three colors by extending the Grötzsch Theorem by showing that any precoloring of any two non-adjacent vertices can be extended to a 3-coloring of the whole graph. In addition, we show that any precoloring of a face of length at most 4 can also be extended to a 3-coloring of the whole graph and that for every vertex of degree at most 3, a precoloring of its neighborhood with a single color extends to a 3-coloring of the whole graph. In all the cases, we also give examples that show tightness of our results. In Chapter 5, we turn our focus to colorings of the vertices and edges on the faces of plane graphs. We begin with the study of cyclic colorings, from which we then jump to their generalization the ℓ-facial vertex coloring and finally, we present a particular case of the ℓ-facial vertex coloring, namely, the ℓ-facial edge coloring. For the latter type of coloring we confirm an open conjecture which states that for any $\ell \geq 0$, we require at most $3\ell+1$ colors, for the smallest open case when $\ell = 3$. In Chapter 6, we then study such colorings of plane graphs, restricted to the faces of the graph, with added parity condition. Namely, we require that each color appears either zero or an odd number of times on a boundary walk of a face. It was conjectured that 10 colors would be sufficient both in the case of facial-parity vertex coloring as for the case of facial-parity edge coloring and examples attaining this bound are given in both cases. On the other hand, the best known resuts prove that 97 colors are enough for the facial-parity vertex coloring and 16 colors are enough for the facial-parity edge coloring. The main results in Chapter 6 are examples which prove that for any t such that $6 \leq t \leq 12$, there exist an infinite family of graphs requiring t colors for the vertex version and an infinite family of graphs requiring t colors for the edge version of facial-parity colorings. In particular, this shows that there exist graphs that require 12 colors, which slightly closes the gap. To conclude Part I, we give some concluding remarks and a short discussion in Chapter 7.

The second part of this thesis starts with Chapter 8, in which we provide various definitions that are especially related to tree decompositions of graphs. In Chapter 9, we turn our focus to the study of (tw, ω)-bounded graph classes where we consider six well-known graph containment relations: the subgraph, topological minor, minor, induced subgraph, induced topological minor, and induced minor relations. For each of the relations, we provide a complete characterization of the graphs H for which the class of graphs excluding H is (tw, ω)-bounded. We then continue to study tree decompositions with bounded independence number in Chapter 10. For this invariant, called tree-independence number, we prove several results and, in particular, show that graph classes with bounded tree-independence number are (tw, ω)-bounded with a polynomial binding function. We also show that every (tw, ω)-bounded graph class characterized by forbidding a single graph H with respect to one of the six graph containment relations studied in Chapter 9 has bounded tree-independence number. In Chapter 11, we then present some algorithmic results by providing polynomial-time algorithms for the Maximum Weight Independent Set Problem for graphs classes presented in the previous two chapters. In particular, we provide a sufficient condition for (tw, ω)-bounded graph classes to admit a polynomial-time algorithm for the Maximum Weight Independent Packing problem and, as a consequence, we get an algorithm that computes an optimal solution in polynomial time for the weighted variants of the Independent Set and Induced Matching problems. We also show that bounded tree-independence number implies the existence of a polynomial-time algorithm for the Maximum Weight Independent Packing problem and also show polynomial-time solvability of the Maximum Weight Independent Set problem in all (tw, ω)-bounded graph classes characterized with a single forbidden graph with respect to one of the six aforementioned graph containment relations. We then conclude Part II of this thesis by giving some concluding remarks and a discussion in Chapter 12.

Finally, we round finish with some short concluding remarks in Chapter 13.

The first part of this thesis is based on the following papers:

- [157] H. La, B. Lužar, and K. Štorgel. Further extensions of the Grötzsch Theorem. *Discrete Math.*, 345(6):112849, 2022.
- [138] M. Horňák, B. Lužar, and K. Štorgel. 3-facial edge-coloring of

plane graphs. *Discrete Math.*, 346(5):113312, 2023.

- [198] K. Štorgel. Improved Bounds for Some Facially Constrained Colorings. *Discuss. Math. Graph Theory*, 43(1):151–158, 2023.

The second part of this thesis is based on the following papers:

- [76] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus Clique Number. I. Graph Classes with a Forbidden Structure. *SIAM J. Discrete Math.*, 35(4):2618–2646, 2021.
- [77] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus clique number. II. Tree-independence number. *arXiv preprint arXiv:2111.04543*, 2021.
- [78] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus clique number. III. Tree-independence number of graphs with a forbidden structure. *arXiv preprint arXiv:2206.15092*, 2022.

Chapter 2

Preliminaries

A (simple) *graph* G is a pair $(V(G), E(G))$ (shortly (V, E) if the considered graph is clear from the context) where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of G. An *edge* $e \in E(G)$ is an unordered pair uv, where $u, v \in V(G)$ and $u \neq v$. For an edge $e = uv$, the vertices u and v are called the *endpoints* of the edge e. If we allow $E(G)$ to be a multiset, we say that G is a *multigraph*. Repeated edges in $E(G)$ are called *parallel edges*. In addition, if we allow edges with both endpoints being the same, the so called *loops*, then G is a *pseudograph*. Unless specifically stated, all graphs considered throughout this thesis are finite and simple. The graph is *null* if it has no vertices and *edgeless* (or *empty*) if it has no edges. For two vertices $u, v \in V(G)$, we say that they are *adjacent* in G if uv is an edge in G, and *non-adjacent* otherwise. Similarly, two edges are *adjacent* if they share a common vertex, and *non-adjacent otherwise*. For a vertex v and edge e, we say that v and e are *incident* if v is an endpoint of e. The *complement* of a graph G is the graph, denoted by $\overline{G}$, with vertex set $V(G)$, in which two distinct vertices are adjacent if and only if they are non-adjacent in G. Note that the complement of the complement of G is the graph G itself. The *line graph* of a graph G, denoted by $L(G)$, is the graph obtained as follows. For every edge $e \in E(G)$ create a vertex v_e in $L(G)$ and then for every pair of distinct edges $e, f \in E(G)$ with a common vertex create an edge between their corresponding vertices in $L(G)$.

For an edge e of G, the graph obtained by deleting the edge e is the graph H with $V(H) = V(G)$ and $E(H) = E(G) \setminus \{e\}$. For a vertex v of G, the graph obtained by deleting the vertex v is the graph H with $V(H) = V(G) \setminus \{v\}$ and $E(H) = \{uw \in E(G) \mid u \neq v \text{ and } w \neq v\}$. Given a set $U \subseteq V(G)$, we denote by $G - U$ the graph obtained from G by deleting all the vertices in U. Similarly, given a set $F \subseteq E(G)$, we denote by $G - F$ the graph obtained from G by deleting all the edges in F. If $U = \{v\}$ (respectively $F = \{e\}$), we simplify the notation and write simply $G - v$ (respectively $G - e$). *Edge subdivision* of an edge $e = uv$ is the operation that deletes the edge e and adds a new vertex w and two edges uw and vw. We say that H is a *subdivision of a graph* G if H can be obtained from G by a (possibly empty) sequence of edge subdivisions.

A graph H is said to be an *induced subgraph* of G, denoted $H \subseteq_{\mathrm{is}} G$, if H can be obtained from G by a (possibly empty) sequence of vertex deletions. For a set $U \subseteq V(G)$ we denote by $G[U]$ the subgraph of G *induced by* U, i.e, the graph $G - (V(G) \setminus U)$. Note that the graph $G[U]$ is an induced subgraph of G. A graph H is a *subgraph* of G, denoted $H \subseteq_{\mathrm{s}} G$, if H can be obtained from G by a (possibly empty) sequence of vertex and edge deletions. A graph H is said to be a *topological minor* of a graph G, denoted $H \subseteq_{\mathrm{tm}} G$, if G contains a subdivision of H as a subgraph. Similarly, H is an *induced topological minor* of G, denoted $H \subseteq_{\mathrm{itm}} G$, if G contains a subdivision of H as an induced subgraph. *Edge contraction* is the operation of deleting a pair of adjacent vertices and replacing them with a new vertex whose neighborhood is the union of the neighborhoods of the two original vertices. If H can be obtained from G by a (possibly empty) sequence of vertex deletions, edge deletions, and edge contractions, then we say that G contains H as a *minor* and denote this by $H \subseteq_{\mathrm{m}} G$. Finally, we say that G contains H as an *induced minor*, denoted $H \subseteq_{\mathrm{im}} G$, if H can be obtained from G by a (possibly empty) sequence of vertex deletions and edge contractions.

For the six graph containment relations mentioned above, the following hold:

- $H \subseteq_{\mathrm{is}} G \implies H \subseteq_{\mathrm{s}} G$,
- $H \subseteq_{\mathrm{itm}} G \implies H \subseteq_{\mathrm{tm}} G$,
- $H \subseteq_{\mathrm{im}} G \implies H \subseteq_{\mathrm{m}} G$,
- $H \subseteq_{\mathrm{s}} G \implies H \subseteq_{\mathrm{tm}} G \implies H \subseteq_{m} G$, and
- $H \subseteq_{\mathrm{is}} G \implies H \subseteq_{\mathrm{itm}} G \implies H \subseteq_{\mathrm{im}} G$.

Let H_1 and H_2 be two subgraphs of a graph G. We say that the graphs H_1 and H_2 are *vertex-disjoint* if $V(H_1) \cap V(H_2) = \emptyset$. Similarly, H_1 and H_2 are *edge-disjoint* if $E(H_1) \cap E(H_2) = \emptyset$. Note that two edge-disjoint subgraphs may have common vertices. We say that a graph G is isomorphic to a graph H, denoted $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that for every pair of vertices $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If G does not contain an induced subgraph (subgraph, topological minor, induced topological minor, minor, induced minor, respectively) isomorphic to H, then we say that G is H-is-*free*, or shortly H-free, (H-s-*free*, H-tm-*free*, H-itm-*free*, H-m-*free*, H-im-*free*, respectively). The definitions naturally extend to the case when a single graph H is replaced by a family of graphs $\mathcal{F}$. For example, a graph G is said to be $\mathcal{F}$-free if G is H-free for all $H \in \mathcal{F}$. A *graph class*, usually denoted by $\mathcal{G}$, is a family of graphs closed under isomorphism. A graph class $\mathcal{G}$ is *hereditary* if for every graph $G \in \mathcal{G}$, every induced subgraph H of G is also in the class. For a graph G and a set of vertices $U \subseteq V(G)$, we say that graphs $H_1, H_2, \ldots, H_k$ *cover* U if H_i is a subgraph of G for all $i \in \{1, \ldots, k\}$, and $U \subseteq \bigcup_i V(H_i)$. Similarly, for a set of edges $F \subseteq E(G)$, we say that graphs $H_1, H_2, \ldots, H_k$ *cover* F if each H_i is a subgraph of G and $F \subseteq \bigcup_i E(H_i)$. A *vertex cover* of a graph G is a set $S \subseteq V(G)$ such that for each edge $uv \in E(G)$, $u \in S$ or $v \in S$.

Henceforth, whenever the graph is clear from the context, we omit the subscript G from notations. For a vertex $v \in V(G)$, the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ is the *(open) neighborhood* of v in G and the set $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborhood* of v in G. Similarly, for a set of vertices $U \subseteq V(G)$ both of the above definitions are naturally extended as follows. The (open) neighborhood of U is the set $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$ and the closed neighborhood of U is the set $N_G[U] = \bigcup_{v \in U} N_G[v]$. The *degree* of a vertex v, denoted by $d_G(v)$, is the cardinality of the set $N_G(v)$. A vertex v is called a *universal vertex* if $N_G[v] = V(G)$, or equivalently $d_G(v) = |V(G)| - 1$. Conversely, a vertex v is called an *isolated vertex* if $d_G(v) = 0$. If, for a vertex v of G, $d_G(v) = 1$, then v is called a *pendant vertex* or a *leaf vertex*. The *minimum degree* of a graph G, denoted by $\delta(G)$, is the smallest degree among all the vertices of G. Similarly, the *maximum degree* of G, denoted by $\Delta(G)$, is the largest degree among all the vertices of G. A graph is *odd* if all of its vertices have odd degrees and *even* if all of its vertices have even degrees. If, for a graph G, it holds that $\delta(G) = \Delta(G)$, we say that G is *regular*. As a special case, when $\delta(G) = \Delta(G) = 3$, we say that G is *cubic* and for $\delta(G) \leq \Delta(G) = 3$, we say that G is *subcubic*.

A *clique* in a graph G is a set of pairwise adjacent vertices in G (note that any set consisting of a single vertex is also a clique). A *k-clique* is a clique of size (or cardinality) k. A clique $K \subseteq V(G)$ is *maximal* if there exists no clique K' such that $K \subset K'$. A *maximum clique* is a clique of the largest cardinality. Similarly, an *independent set* in a graph G is a set of pairwise non-adjacent vertices in G. An independent set $I \subseteq V(G)$ is *maximal* if there exists no independent set I' such that $I \subset I'$. A *maximum independent set* is an independent set of the largest cardinality. For a graph G, the size of a maximum clique is denoted by $\omega(G)$ and the size of a maximum independent set is denoted by $\alpha(G)$. By definition of the complement of a graph G, we have that every clique in G is an independent set in $\overline{G}$ and every independent set in G is a clique in $\overline{G}$. It follows that $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$.

A *complete graph* on n vertices, denoted by K_n, is the graph in which every pair of vertices is adjacent, i.e., $V(K_n)$ forms a clique and $\omega(K_n) = n$. The complete graph K_3 is often referred to as a *triangle*. A *path* on n vertices, denoted by P_n, is the graph with the vertex set $V(P_n) = \{v_i \mid 1 \leq i \leq n\}$ and the edge set $E(P_n) = \{v_j v_{j+1} \mid 1 \leq j \leq n-1\}$. The vertices v_1 and v_n of P_n are called the *endpoints* of P_n and the other vertices are called *internal vertices*. A *cycle* on $n \geq 3$ vertices, denoted by C_n, is the graph with the vertex set $V(C_n) = \{v_i \mid 1 \leq i \leq n\}$ and the edge set $E(C_n) = \{v_j v_{j+1} \mid 1 \leq j \leq n\}$, where $v_{n+1} = v_1$. If n is even, we say that the cycle is *even*, otherwise we say that the cycle is *odd*. A graph is *bipartite* if there exists a *bipartition* (A, B) of its vertex set into two disjoint independent sets A and B. For non-negative integers n and m, the *complete bipartite graph*, denoted by $K_{n,m}$, is the graph with bipartition (A, B) where $|A| = n$ and $|B| = m$, and with the edge set $E(K_{n,m}) = \{uv \mid u \in A, v \in B\}$. A complete bipartite graph is *balanced* if $n = m$. A complete bipartite graph $K_{1,3}$ is called a *claw*. A *tree* is a connected graph without cycles and a *spanning tree* of a graph G is a subgraph T of G with $V(T) = V(G)$ that is a tree.

A graph G is *connected* if for every pair of vertices $u, v \in V(G)$ there exists a path between them, i.e., there exists a subgraph of G isomorphic to a path with u and v being its endpoints. A *connected component* of a graph G is a maximal subgraph of G that is connected. A set $S \subseteq V(G)$ is a *cutset* if the number of connected components in the graph $G - S$ is strictly greater than the number of connected components of G. A *k-cutset* is a cutset of cardinality k. If S contains a single vertex v, we say that v is a *cut-vertex*. In addition, if S forms a clique in G, we say that S is a *clique cutset*. Similarly, a set $F \subseteq E(G)$ is an *edge cutset* if the number of connected components of $G - F$ is strictly greater than the number of connected components of G. A *k-edge cutset* is an edge cutset of cardinality k. If F contains a single edge e, we say that e is a *cut-edge* or a *bridge*. A cutset S of either vertices or edges is *minimal* if there exists no proper subset S' of S that is a cutset. For a positive integer k, a graph G is *k-connected* if $|V(G)| \geq k + 1$ and for any set $S \subseteq V(G)$ with cardinality $|S| < k$, $G - S$ is connected. Similarly, for a positive integer k, a graph G is *k-edge-connected* if $|E(G)| \geq k + 1$ and for any set $F \subseteq E(G)$, $G - F$ is connected. A *block* in a graph G is a maximal connected subgraph of G without cut-vertices. A graph is *bridgeless* if it does not contain any bridges.

2.1 Graph colorings

A *vertex coloring* of a graph G is a mapping f from the set of vertices $V(G)$ to the set of colors C and an *edge coloring* of G is a mapping g from the set of edges $E(G)$ to the set of colors C' (usually the set of colors is represented by natural numbers). A vertex coloring f is *proper*, if for any pair of adjacent vertices u and v, $f(u) \neq f(v)$. A proper vertex coloring is usually called just a *proper coloring*. The smallest number of colors needed for a proper coloring of a graph G is called the *chromatic number* of a graph and denoted by $\chi(G)$. We say that a graph G is *k-colorable* if there exists a proper coloring of G with k colors (that is, $\chi(G) \leq k$). By definition, in a proper coloring, the vertices of the same color form an independent set. As a consequence, bipartite graphs are 2-colorable. Since all the vertices in a maximal clique are pairwise adjacent, it readily follows that $\omega(G) \leq \chi(G)$ for every graph G. For the upper bound on the chromatic number, in 1941 [47], Brooks proved the following result.

Theorem 2.1.1 (Brooks [47]). *Let G be a connected graph with maximum degree Δ. Then, $\chi(G) \leq \Delta + 1$, with the equality if and only if G is a complete graph or an odd cycle.*

An edge coloring of a graph G is *proper* if for any pair of edges e_1 and

e_2 with a common vertex, $f(e_1) \neq f(e_2)$. The smallest number of colors needed for a proper edge coloring is called the *chromatic index* of a graph and denoted by $\chi'(G)$. We say that a graph G is *k-edge-colorable* if there exists a proper edge coloring of G with at most k colors (that is, $\chi'(G) \leq k$). By definition, edges of the same color form a so-called *matching*, i.e., a set of pairwise non-adjacent edges. In 1964, Vizing proved the following result.

Theorem 2.1.2 (Vizing [195]). *Let G be a graph with maximum degree Δ. Then,*

$$\Delta \leq \chi'(G) \leq \Delta + 1\,.$$

Given a graph G together with an assignment of lists of colors to each vertex, a *list coloring* is a mapping that assigns to each vertex v a color from its list $L(v)$ of colors. For a list coloring to be proper, the same condition must hold as for the proper coloring, i.e., every pair of adjacent vertices receives distinct colors. We say that a graph is *k-choosable*, or *k-list-colorable*, if for any assignment of lists of colors of size k to each vertex, there exists a proper list coloring of G. The smallest integer k such that G is k-choosable is called the *choice number* of G. In particular, if a graph is k-choosable, it is also k-colorable as can be seen by a particular assignment of lists of colors for each vertex such that each list contains the exact same k colors. However, not every k-colorable graph is k-choosable.

Example 2.1.3. Consider a complete bipartite graph $K_{2,4}$ with the vertex set $A = \{v_1, v_2\}$ on one side of the bipartition and $B = \{u_1, u_2, u_3, u_4\}$ on the other side of the bipartition. Since $K_{2,4}$ is bipartite, it is also 2-colorable. To see that $K_{2,4}$ is not 2-choosable, assign lists of colors to the vertices in the following way: $L(v_1) = \{1, 2\}$, $L(v_2) = \{3, 4\}$, $L(u_1) = \{1, 3\}$, $L(u_2) = \{1, 4\}$, $L(u_3) = \{2, 3\}$, and $L(u_4) = \{2, 4\}$. It is easy to see that no matter the choice of colors for v_1 and v_2, there will always be a vertex in B that cannot be colored.

2.2 Planar graphs

Let G be a graph. We say that G is *planar* if there exists an embedding of G in the plane without crossing edges. Note that deleting a vertex preserves planarity, thus the class of planar graphs is hereditary. When we talk about a fixed plane embedding H of a planar graph G, we say that H is a *plane graph*. For a plane graph $G = (V(G), E(G), F(G))$, $F(G)$ denotes the set of faces of G, i.e., regions of the plane bounded by the edges of G in the plane embedding. We say that a vertex, or edge, is *incident* with a face α if it lies on the boundary of α. For a vertex, or an edge, incident with a face α we will often say that α *contains* that vertex, or edge. If there exists a plane embedding of G such that every vertex is incident with the outer face, then G is an *outerplanar* graph. When we talk about a fixed plane embedding H of an outerplanar graph G, we say that H is an *outerplane graph*. Note that a planar graph may have different plane embeddings (see Figure 2.1).

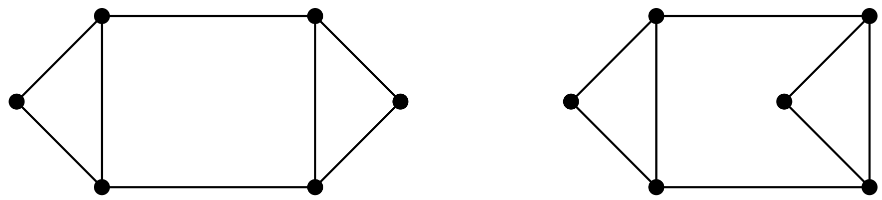


Figure 2.1: Two distinct plane embeddings of the same planar graph.

In 1930, Kuratowski [156] provided the first forbidden graph characterization of planar graphs.

Theorem 2.2.1 (Kuratowski [156]). *A graph G is planar if and only if G contains no subdivision of K_5 or $K_{3,3}$.*

Later, in 1937, Wagner [200] proved the following.

Theorem 2.2.2 (Wagner [200]). *A graph G is planar if and only if G contains neither K_5 nor $K_{3,3}$ as a minor.*

Planar graphs also have several important properties. One of them connects the number of vertices, edges, and faces of a plane graph which is known under the name Euler's formula.

Theorem 2.2.3 (Euler's formula). *Let G be a connected plane graph. Then*

$$|V(G)| - |E(G)| + |F(G)| = 2\,.$$

Another important property is known as the Four Color Theorem, proven by Appel and Haken [12, 13] and Appel, Haken, and Koch [14].

Theorem 2.2.4 (Four Color Theorem [12, 13, 14]). *Let G be a planar graph. Then,*

$$\chi(G) \leq 4\,.$$

Finally, for a non-negative integer k, a k-planar graph is a graph that can be drawn in the plane such that each edge is crossed at most k times.

2.3 Graph problems and complexity

When studying various algorithmic problems on graphs, the main question is what is the time and space complexity of the problem at hand in terms of the size of the input. To this end, the big-O, big-Ω, and big-Θ notation was introduced.

Let $f(n)$ and $g(n)$ be two functions of n.

- We say that $f(n) = O(g(n))$ if there exist constants C and N such that $f(n) \leq c \cdot g(n)$ for all $n \geq N$.

- We say that $f(n) = \Omega(g(n))$ if there exist constants c and N such that $f(n) \geq c \cdot g(n)$ for all $n \geq N$.
- We say that $f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

We say that a problem is *efficiently solvable* if there exists an algorithm solving the problem in time $O(n^k)$, where n is the size of the input and k is some constant, i.e., the problem is solvable in polynomial time. A *decision problem* is a problem that can only be answered *yes* or *no*. An example of a decision problem is the COLORING problem which, given a graph G and a positive integer k, asks whether we can properly color the vertices of G using k colors. Another example of a decision problem is the INDEPENDENT SET problem which, given a graph G and a positive integer k, asks whether G contains an independent set of size k.

Based on the time complexity required to solve the problem, we divide the problems into various complexity classes. The class P consists of all decision problems solvable in polynomial time. The class NP consists of all decision problems for which a solution can be verified in polynomial time. Clearly, it holds that $\mathsf{P} \subseteq \mathsf{NP}$, and a famous open problem whether $\mathsf{P} = \mathsf{NP}$ is still widely open. A problem $\mathcal{P}$ is NP-*hard* if every problem in NP is reducible in polynomial time to $\mathcal{P}$. A problem is NP-*complete* if it is in NP and it is NP-hard. It is known that both the aforementioned problems, the COLORING and the INDEPENDENT SET are NP-complete.

Part I

Chapter 3

Preliminaries to Part I

In this section, we define notions and present auxiliary results that we will use in Part I of this thesis.

3.1 More on planar graphs

We begin by giving further definitions and notations for plane graphs.

Let G be a plane graph. Each face α of G is bounded by a *closed walk*, i.e., an alternating sequence of vertices and edges incident with α starting and ending in the same vertex going in one direction only, that is to say, no edge incident with α is traversed twice, unless it is a bridge edge. Such a walk is called a *boundary walk* of the face α. A *facial path* is any consecutive subsequence of a boundary walk that starts and ends with a vertex. The *length (or size)* of a face α, denoted by $\ell(\alpha)$, is the number of edges on a boundary walk of α (bridges are counted twice). Note that the length of a face α in a 2-connected plane graph is equal to the number of edges (or vertices) incident with α. The largest length of a face in a plane graph G is denoted by $\Delta^*(G)$, or simply Δ^*, when the graph is clear from the context. Two vertices are at *facial-distance* k if the shortest facial path between them contains k edges. The facial distance between two edges e and f is defined as the smallest number of vertices in any facial path between a pair of vertices u, v, where u is an endpoint of e and v is an endpoint of f. In this sense, two *facially adjacent* edges, i.e., two distinct edges appearing consecutively on a boundary walk of a face α, are at facial-distance 1. We say that two vertices, or edges, are *k-facially adjacent* or *within facial-distance k* if they are at facial-distance at most k. In the case when $k = 1$, we say that the two edges are *facially adjacent.* The *facial distance between a vertex v and an edge e* is defined as the minimum distance from v to any endpoint of e.

We say that a vertex of degree k (at least k, at most k) is a *k-vertex* (a *k^+-vertex*, a *k^--vertex*) and a face of size k (at least k, at most k) is a *k-face* (a *k^+-face*, a *k^--face*). We call a k-vertex (a k^--vertex, a k^+-vertex) adjacent to a vertex v a *k-neighbor* (a *k^--neighbor*, a *k^+-neighbor*) of v.

A *k-thread* is a subgraph in G, isomorphic to the path P_k, in which all the vertices have degree 2 in G. When the endpoints of a k-thread are known, say u and v are the endpoints, we denote it as the *k-thread* (u, v). A k-thread is *incident with a face* α if all its vertices are incident with α. A k-thread (u, v) is k-facially adjacent to a vertex w if the facial distance between u and w or between v and w is at most k. Let C be a cycle in a plane graph G. We denote by $\mathrm{int}(C)$ the graph induced by the vertices lying strictly in the interior of C. Similarly, we denote by $\mathrm{ext}(C)$ the graph induced by the vertices lying strictly in the exterior of C. We say that C is a *separating cycle* if both $\mathrm{int}(C)$ and $\mathrm{ext}(C)$ contain at least one vertex.

A *dual graph* of a plane graph G is a plane (pseudo)graph H obtained by creating a vertex for each face of G, and for each edge e of G creating an edge whose endpoints correspond to the two faces incident with e in G. Since a bridge is incident with a single face on both sides, each bridge corresponds to a loop. In addition, if two faces are incident with a common 2-vertex, then H will have parallel edges. As was proven in 1997 by Balakrishnan [16], the simple plane graphs whose duals are again simple plane graphs are 3-edge-connected. Observe also that $\Delta^*(G) = \Delta(H)$ and $\Delta(G) = \Delta^*(H)$.

A *medial graph* of a connected plane graph G, denoted by $M(G)$, is a plane (pseudo)graph obtained from G by creating a vertex for each edge of G and adding an edge between two vertices of $M(G)$ every time their correspoding edges in G appear consecutively (have a common endpoint) on a boundary walk of some face. By definition, if G is a connected plane graph that contains a pendant vertex, then $M(G)$ will contain a loop. A simple argument shows that a medial graph of any plane graph is always a regular graph with maximum degree 4 and if a plane graph is cubic, then its line graph and medial graph coincide.

Let G be a plane graph. A *plane triangulation* of G is a graph H with $V(H) = V(G)$, containing G as a subgraph, and a maximal number of edges while preserving planarity. It is easy to see that in a plane triangulation every face has size 3.

3.2 More on colorings

Let σ be a partial coloring (under certain conditions) of the vertices, or edges, of a graph G with the color set C. We say that a color $c \in C$ is *σ-available* (or *available* if σ is clear from the context) for a non-colored vertex, or edge, provided that coloring that vertex, or edge, with the color c would not violate the conditions of the coloring. The set of σ-available colors for a vertex v, or an edge e, is denoted by $A_\sigma(v)$, or $A_\sigma(e)$, respectively ($A(v)$, or $A(e)$, for short when σ is clear from the context). Given a set $U \subseteq V(G)$ of non-colored vertices, the set

$$A_\sigma(U) = \bigcup_{u \in U} A_\sigma(u)$$

is called the *set of σ-available colors for V.* Similarly, given a set $F \subseteq E(G)$ of non-colored edges, the set

$$A_\sigma(F) = \bigcup_{f \in F} A_\sigma(f)$$

is called the *set of σ-available colors for E.*

Given a graph G, a *conflict graph* H of G, with respect to the desired coloring, is a graph obtained by taking $V(H)$ to be the set of elements we want to color (vertices, edges, etc.), and adding an edge between two vertices of H if and only if the corresponding elements must be colored distinctly. Let H be a conflict graph of G with respect to some coloring and let $V(H) = \{v_1, \ldots, v_n\}$. A conflict polynomial, denoted by $P(H)$, is defined as follows:

$$P(H) = \prod_{\substack{1 \le i < j \le n,\\ v_i v_j \in E(H)}} (X_i - X_j)\,.$$

Note that $P(H)$ may differ with respect to the ordering of the vertices of H.

We now state three useful theorems. First, we make use of the following generalization of Theorem 2.1.1 to list coloring.

Theorem 3.2.1 (Borodin [34]; Erdős, Rubin, Taylor [101])**.** *Let G be a connected graph and let L be a list-assignment where $|L(v)| \ge d(v)$ for each $v \in V(G)$. If*

- *$|L(v)| > d(v)$ for some vertex v, or*
- *G contains a block which is neither a complete graph nor an induced odd cycle (i.e., G is not a Gallai tree),*

then G admits a list coloring from the given lists of colors.

Another useful tool in proving various coloring results is Hall's Theorem, which guarantees distinct colors for a set of vertices.

Theorem 3.2.2 (Hall [120])**.** *A bipartite graph a bipartition (A, B) admits a matching M such that every vertex of A is an endpoint of some edge in M if and only if for every set $S \subseteq A$ the number of vertices of B with a neighbor in S is at least $|S|$.*

In other words, given a partial proper vertex coloring σ of a graph G with a color set C, if a set U consisting of n non-colored vertices is such that for every $S \subseteq U$ of size k the set of available colors $A_\sigma(S) \subseteq C$ for S has size at least k, then σ can be extended to a proper vertex coloring of the whole graph G.

The third theorem helps with determining whether one can always choose colors from the lists of available colors such that all conflicts are avoided. The following theorem is due to Alon [10] and is referred to as the *Combinatorial Nullstellensatz.*

Theorem 3.2.3 (Alon [10]). *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(X_1, \ldots, X_n)$ be a polynomial in $\mathbb{F}[X_1, \ldots, X_n]$. Suppose that the coefficient of a monomial $\prod_{i=1}^{n} X_i^{k_i}$, where each k_i is a nonnegative integer, is non-zero in $\mathbb{F}$ and the degree* $\deg(P)$ *of P equals $\sum_{i=1}^{n} k_i$. Moreover, if $S_1, \ldots, S_n$ are any subsets of $\mathbb{F}$ with $|S_i| > k_i$, then there exist $s_1 \in S_1, \ldots, s_n \in S_n$ such that $P(s_1, \ldots, s_n) \neq 0$.*

Note that Combinatorial Nullstellensatz does not provide a coloring, but rather only proves its existence. In order to better understand Theorem 3.2.3 let us consider the following example.

Example 3.2.4. Let G be a triangle with $V(G) = v_1, v_2, v_3$. Suppose that we want to properly color the vertices of G and the sets of available colors for the vertices have the following cardinality: $|A_{v_1}| = 1$, $|A_{v_2}| = 2$, and $|A_{v_3}| = 3$.

First, we can compute the conflict graph H of G with respect to proper coloring. Note that, in this case, H is isomorphic to G. Next, we compute the conflict polynomial

$$P(H) = (X_1 - X_2)(X_1 - X_3)(X_2 - X_3)\,.$$

Expanding the polynomial we get

$$P(H) = X_1^2 X_2 - X_1^2 X_3 + X_1 X_3^2 - X_1 X_2^2 + X_2^2 X_3 - X_2 X_3^2\,.$$

Note that the monomial $X_2 X_3^2$ satisfies the properties of Theorem 3.2.3 and thus, it follows that we can properly color the vertices of G avoiding all of the conflicts.

3.3 Discharging method

One of the ways of using a structure of a graph in order to prove some results is by discharding method. The most famous use of this method is in the proof of the Four Color Theorem. For the following definitions and a guide to discharging method we refer the reader to [64].

A *configuration* in a graph G is a subgraph, possibly with some additional properties on its neighborhood. A configuration is *reducible* with respect to a certain studied property P if it cannot be present in a minimal graph without property P.

Discharging method proceeds in three steps. In the first step, we assign to certain objects of a graph (e.g., vertices, edges, faces) some charge, called the *initial charge*, denoted by ch_0, in such a way that we can estimate a total count of all charges. For example, in the case of planar graphs, we may assign charges to each vertex and each face in order to use Euler's formula to compute the total charge to be negative. In the second step, we prove that certain configurations in a minimal counterexample are reducible. Using this knowledge, we then design the rules, in which the charge is relocated from one object to another by preserving the total charge. This step is called *discharging* and the rules for discharging are called *discharging rules*. In the final step, we prove, using reducible configurations, that the final charge, denoted by $\mathrm{ch_f}$, is different than the computed value from the first step (e.g., in the case of planar graphs, that often means that it no longer satisfies the Euler's formula by showing that the final charge of each vertex and face is non-negative). This shows that a minimal counterexample does not exist, thus proving the result in question.

Example 3.3.1. Let us prove that every connected planar graph G with $\delta(G) \geq 2$ and with no cut-vertices in which every face is of size at least 8 has an edge uv such that $d(u) + d(v) \leq 5$.

Suppose for a contradiction that this is not the case, i.e., $d(u) + d(v) \geq 6$ for every edge $uv \in E(G)$. Assign the initial charge as follows. For every vertex $v \in V(G)$, we set

$$ch_0(v) = 2d(v) - 6\,,$$

and for every face $\alpha \in F(G)$, we set

$$ch_0(\alpha) = \ell(\alpha) - 6\,.$$

By Euler's formula we have the following:

$$\begin{aligned}\sum_{v\in V(G)} ch_0(v) + \sum_{\alpha\in F(G)} ch_0(\alpha) &= \sum_{v\in V(G)} (2d(v) - 6) + \sum_{\alpha\in F(G)} (\ell(\alpha) - 6)\\ &= 4|E(G)| - 6|V(G)| + 2|E(G)| - 6|F(G)|\\ &= -6(|V(G)| - |E(G)| + |F(G)|)\\ &= -12\,.\end{aligned}$$

Observe that, since $d(u) + d(v) \geq 6$, every neighbor of a 2-vertex is a 4^+-vertex. Additionally, since $\delta(G) \geq 2$ and G has no cut-vertices, every vertex is incident with at least two faces. Let $n_2(\alpha)$ be the number of 2-vertices incident with α. Due to the previous observation, it follows that $n_2(\alpha) \leq \lfloor \frac{\ell(\alpha)}{2} \rfloor$ for every face $\alpha \in F(G)$. Let us now state the discharging rules.

R_1 Every 4^+-vertex sends $\frac{1}{2}$ to every adjacent 2-vertex.

R_2 Every 8^+-face sends $\frac{1}{2}$ to every incident 2-vertex.

First, we prove that the final charge of every vertex is non-negative.

- If v is a 2-vertex, then v has the initial charge of -2, and it receives $2\times\frac{1}{2}$ charge from its two adjacent 4^+-vertices by rule R_1 and $2\times\frac{1}{2}$ from its two incident 8^+-faces by rule R_2. Thus, $\mathrm{ch_f}(v) = -2 + 4 \times \frac{1}{2} = 0$.
- If v is a 3-vertex, then v has the initial charge of 0, and does not send or receive any charge.

- If v is a 4^+-vertex, then v sends at most $\frac{d(v)}{2}$ charge to adjacent vertices by rule R_1. Thus, $\mathrm{ch_f}(v) \geq 2d(v) - 6 - \frac{d(v)}{2} = \frac{3}{2}d(v) - 6 \geq 0$.

Finally, we prove that the final charge of every face is also non-negative.

- Pick any face α of G. Since every face of G is an 8^+-face, by rule R_2, α sends at most $n_2(\alpha) \times \frac{1}{2}$ to every incident 2-vertex. Now, since $n_2(\alpha) \leq \lfloor \frac{\ell(\alpha)}{2} \rfloor$, it follows that $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 - n_2(\alpha) \times \frac{1}{2} \geq \ell(\alpha) - 6 - \lfloor \frac{\ell(\alpha)}{2} \rfloor \times \frac{1}{2} \geq 0$.

Since all the charges are non-negative, we arrive at a contradiction with the initial total charge computed by Euler's formula, thus there must exist an edge uv in G with $d(u) + d(v) \leq 5$.

Chapter 4

Grötzsch Theorem and its Extensions

Coloring of planar graphs has long been an interesting problem. By the Four Color Theorem [13, 14], we know that all planar graphs can be colored with at most 4 colors. In addition, planar graphs which need at most 1 or at most 2 colors are fully characterized. The former are graphs with no edges and the latter are bipartite planar graphs. On the other hand, deciding whether a planar graph is 3-colorable is an NP-complete problem [74, 106]. As a result, the search for properties that guarantee 3-colorability of a planar graph became a widely popular research direction (see, e.g., [38] for a survey). To this end, two important results have been proven by Heawood and Grötzsch establishing the relevance of triangles. Recall that a graph is *even* if all of its vertices have even degree.

Theorem 4.0.1 (Heawood [128]). *A triangulation of a plane graph is 3-colorable if and only if it is even.*

For generalizations of Theorem 4.0.1 see [82, 98, 149]. While Heawood's result deals with planar graphs containing a "large" number of triangles, Grötzsch proved that completely forbidding triangles in a planar graph also yields a 3-colorable graph.

Theorem 4.0.2 (Grötzsch [114]). *Every triangle-free planar graph is 3-colorable.*

These results led the investigation to focus on different ways, in which triangles can appear in 3-colorable planar graphs. Obviously, having triangles close together is problematic and an easy example is already a K_4.

A conjecture by Grünbaum [115] stated that it is enough to forbid intersecting triangles in planar graphs to obtain a 3-colorable graph. However, as was shown by Havel [124], this is not the case. In the same paper Havel also proposed a weaker conjecture, in particular, he conjectured that a planar graph may contain an arbitrary number of triangles and be 3-colorable, as long as the distance between any two triangles is sufficiently large. Later, Havel [125] showed that if such a constant, call it d, exists, then $d \geq 3$. This bound was slightly improved by Aksenov and Mel'nikov [8], who showed that if such a constant exists, then $d \geq 4$. Recently, Dvořák, Kráľ, and

Thomas [94] answered Havel's conjecture in affirmative.

Theorem 4.0.3 (Dvořák et al. [94]). *There exists a constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d, then G is 3-colorable.*

This result answers the existence of such a constant, however, it is still open what is its actual value. As mentioned in the same paper by the authors, their proof gives an explicit upper bound of approximately 10^{100}, which is rather large compared to the lower bounds mentioned above.

A result of a similar flavor due to Dvořák [91] states the following.

Theorem 4.0.4 (Dvořák [91]). *If G is a planar graph such that the distance between any two cycles of length at most 4 is at least 26 then G is 3-choosable.*

On the other hand, there are 3-colorable planar graphs that may have close triangles (even incident) and have no short cycles forbidden. As was proved in [87], every planar graph obtained as a subgraph of the medial graph of a bipartite plane graph is 3-colorable (in fact, even 3-choosable). When allowing triangles to appear in an arbitrary way in planar graphs, a lot of research has been done by considering other structures, particularly cycles of lengths 4–9. Steinberg [187] conjectured that every planar graph with no cycles of length 4 and 5 is 3-colorable. This conjecture was answered in the negative by Cohen-Addad et al. [62]. On the other hand, the following problem, formulated by Lu et al. [161] (see also [145]) is far from settled.

Problem 4.0.5. What is the set I of integers $i \geq 5$, such that for any $i \in I$, every planar graph with no cycles of length 4 and i is 3-colorable?

Results regarding 3-colorability of planar graphs when forbidding cycles of four distinct lengths between 4 and 9 were obtained in several papers and summarized in [161] with the following theorem.

Theorem 4.0.6. *If G is a planar graph with no cycles of length $4, i, j, k$, where $5 \leq i < j < k \leq 9$, then G is 3-colorable.*

Although Theorem 4.0.6 settles all the cases when cycles of four distinct lengths are forbidden (up to length 9), even the following problem stated in [161] and reformulated in [145] is still not completely resolved.

Forbidden cycles (length)	3	4	5	6	7	8	9
Grötzsch [114]	✓						
Xu [207]		✓	✓		✓		
Borodin et al. [40]		✓		✓	✓		
Wang and Chen [202]		✓		✓		✓	
Kang et al. [145]		✓		✓			✓
Lu et al. [161]		✓			✓		✓

Table 4.1: Summary of results stating when a planar graph is 3-colorable. Checkmarks annotate for which k cycles of length k are forbidden as subgraphs.

Problem 4.0.7. What is the set J of pairs of integers (i, j) with $5 \leq i < j \leq 9$, such that planar graphs without cycles of length $4, i, j$ are 3-colorable?

We summarize known results of this type in Tables 4.1 and 4.2. In the case of 3-choosability even more is left open. However, Voigt showed that there exist non 3-choosable triangle-free planar graphs [196] and also non 3-choosable planar graphs without cycles of length 4 and 5 [197]. In the case of 3-choosability, a theorem similar to Theorem 4.0.6 was stated in [204].

Theorem 4.0.8. *If G is a planar graph with no cycles of length $4, i, j, 9$, where $5 \leq i < j \leq 8$, then G is 3-choosable.*

For a summary of known results of this type on 3-choosability see Table 4.2.

Forbidden cycles (length)	3	4	5	6	7	8	9
Thomassen [191]	✓	✓					
Lam et al. [158]	✓		✓	✓			
Dvořák et al. [96]	✓			✓	✓		
Dvořák et al. [95]	✓				✓	✓	
Zhang et al. [212]; Zhu et al. [215]	✓					✓	✓
Zhang and Wu [214]		✓	✓	✓			✓
Zhang and Wu [213]		✓	✓		✓		✓
Wang et al. [205]		✓	✓			✓	✓
Wang et al. [204]		✓		✓	✓		✓
Shen and Wang [183]		✓		✓		✓	✓
Wang et al. [206]		✓			✓	✓	✓

Table 4.2: Summary of results stating when a planar graph is 3-choosable. Checkmarks annotate for which k cycles of length k are forbidden as subgraphs.

4.1 Planar graphs with a small number of triangles

Only four years after the appearance of the Grötzsch theorem, Grünbaum [115] observed that even if a planar graph contains up to three triangles then it is 3-colorable. His original proof had a mistake and was later corrected by Aksenov [4].

Theorem 4.1.1 (Aksenov [4]). *Every planar graph with at most three triangles is 3-colorable.*

Borodin [33] and Borodin et al. [41] later gave new shorter proofs of Theorem 4.1.1. In the latter case, the authors relied on the following result obtained by Kostochka and Yancey [150] which gives the minimum number of edges in a k-critical graph for any $k \geq 4$. A graph G is *k-critical* if $\chi(G) = k$ and for any induced subgraph H of G, $\chi(H) < k$.

Theorem 4.1.2 (Kostochka and Yancey [150]). *If $k \geq 4$ and G is a* *k-critical graph then*

$$E(G) \geq \left\lceil \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)} \right\rceil .$$

When $k = 4$, we immediately get the following corollary to Theorem 4.1.2.

Corollary 4.1.3. *If G is a 4-critical graph on n vertices then*

$$|E(G)| \geq \frac{5n-2}{3} .$$

Using the result of Kostochka and Yancey, together with the shorter proof of Theorem 4.1.1, Borodin et al. [41] presented several extensions of the Grötzsch theorem (Theorem 4.0.2) which we discuss below.

In [141], the authors proved that by adding a vertex of degree 3 to a triangle-free planar graph G, the graph retains 3-colorability. Extending this result, Borodin et al. [41] proved the following.

Theorem 4.1.4 (Borodin et al. [41]; Jensen and Thomassen [141])**.** *Let G be a triangle-free planar graph and let H be a graph such that $G = H - v$ for some vertex v of degree 4 of H. Then H is 3-colorable.*

In addition, they also provided a short proof of a result due to Aksenov, Borodin, and Glebov [6], which extends the result of Aksenov [5] and Jensen and Thomassen [141] stating that adding one edge to a triangle-free planar graph preserves 3-colorability.

Theorem 4.1.5 (Aksenov et al. [6]; Borodin et al. [41])**.** *Let G be a triangle-free planar graph. Then each coloring of any two non-adjacent vertices can be extended to a 3-coloring of G.*

Another result from [41] is the following.

Theorem 4.1.6 (Borodin et al. [41])**.** *Let G be a triangle-free planar graph and let f be a face of G of length at most 5. Then each 3-coloring of f can be extended to a 3-coloring of G.*

Conversely, one can consider precolorings of a face f of length at least k, where $k \geq 6$, however, as it turns out, not every precoloring of the vertices of f can be extended to a 3-coloring of G. Complete characterizations of the cases when $k \in \{6, 7, 8, 9\}$ were given in [109], [7], [90], and [53], respectively.

In the rest of this section, we present our results from [157].

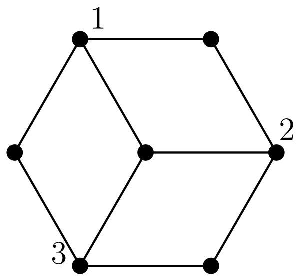
4.2 Our results

Building upon the work by Borodin et al. [41] presented above, we study planar graphs with at most one triangle. We postpone the proofs of Theorems 4.2.1, 4.2.3, and 4.2.6 and present them in Sections 4.2.1, 4.2.2, and 4.2.3, respectively.

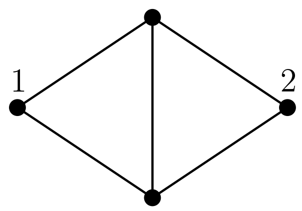
First, let us extend Theorem 4.1.5.

Theorem 4.2.1 (Theorem 1.6 in [157]). *Let G be a planar graph with at most one triangle. Then each coloring of any two non-adjacent vertices can be extended to a 3-coloring of G.*

Additionally, we show that the result is tight both in terms of the number of precolored vertices as well as in terms of the number of triangles. Both cases are shown in Figure 4.1, given together with the precoloring of three or two vertices which cannot be extended to a 3-coloring of the whole graph.



(a)



(b)

Figure 4.1: A precoloring of three vertices in a planar graph with no triangles (example (a)) and a precoloring of two vertices in a planar graph with two triangles (example (b)) neither of which can be extended to a 3-coloring of the whole graph.

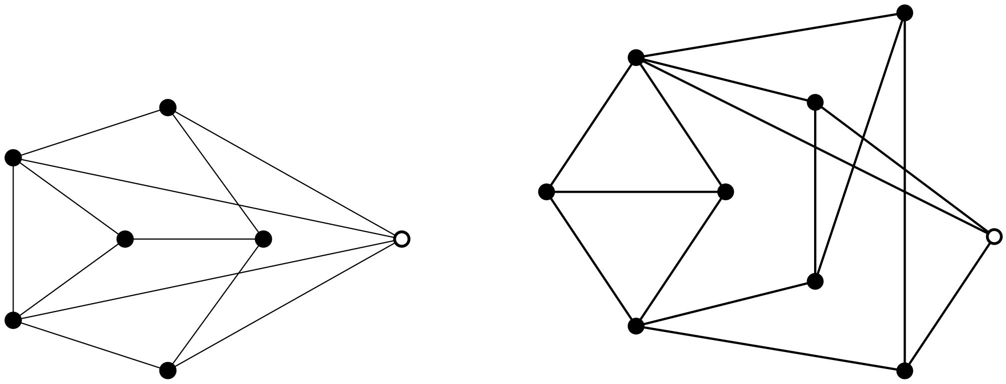
Using Theorem 4.2.1, we obtain a result similar to Theorem 4.1.4.

Theorem 4.2.2 (Theorem 1.7 in [157]). *Let G be a planar graph with at most one triangle and let H be a graph such that $G = H - v$ for some vertex v of degree at most 3 in H, which is adjacent with at most two vertices of the triangle in G if it exists. Then H is 3-colorable.*

Proof. Let G be a planar graph with at most one triangle and let H be any graph such that $G = H - v$ for some vertex v satisfying the assumptions of the theorem. Let $N_H(v) = \{v_1, v_2, v_3\}$. As v is adjacent with at most two vertices of the triangle in G (if it exists), we may assume, without loss of generality, that v_1 and v_2 are not adjacent. By Theorem 4.2.1, we can color v_1 and v_2 with the same color and extend it to a 3-coloring of G in which the three vertices in $N_H(v)$ are colored with at most two distinct colors. It follows that there is an available color with which we can color v. $\square$

Even in this case, we have tightness both in terms of the number of precolored vertices and in terms of the number of triangles (see Figure 4.2). Note that the condition that v is not adjacent to all the vertices of a triangle is necessary as that would imply the existence of K_4 in H.

In the case of precoloring extensions for small faces, we prove an analogue of Theorem 4.1.6 for faces of length at most 4 (recall that in the case of triangle-free planar graphs, faces of length 5 can also be considered).



(a) (b)

Figure 4.2: Not every graph obtained from a planar graph with at most one triangle by adding a 4-vertex is 3-colorable (example (a)), nor is a graph obtained from a planar graph with two triangles by adding a 3-vertex (example (b)). In both cases the added vertex is depicted as a white vertex.

Theorem 4.2.3 (Theorem 1.8 in [157]). *Let G be a planar graph with at most one triangle and let f be a face of G of length at most 4. Then each 3-coloring of f can be extended to a 3-coloring of G.*

In terms of the length of the face, this result is tight. As shown in Figure 4.3, a precoloring of a 5-face in a planar graph with one triangle cannot always be extended to a 3-coloring of the whole graph.

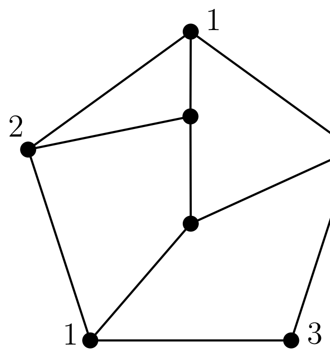


Figure 4.3: A precoloring of the outer 5-face which cannot be extended to a 3-coloring of the graph.

In the case of extending a precoloring of faces, Dvořák and Lidický [90] characterized all the situations when a precoloring of an 8-cycle can be extended to a 3-coloring of the whole graph. As was remarked in [53], the result of Dvořák and Lidický implies the following.

Theorem 4.2.4 (Dvořák and Lidický [90]). *Let G be a triangle-free planar graph and let v be a vertex of degree at most 4 in G. Then there exists a 3-coloring of G where all neighbors of v are colored with the same color.*

As a corollary of Theorem 4.1.4 one can obtain a result similar to Theorem 4.2.4, but for any three neighbors of a vertex of an arbitrary degree.

Corollary 4.2.5 (Corollary 1.10 in [157]). *Let G be a triangle-free planar graph and let v_1, v_2, and v_3 be distinct vertices with a common neighbor v. Then there exists a 3-coloring of G where v_1, v_2, and v_3 are colored with the same color.*

Proof. Let H be a graph obtained from G by adding a 4-vertex u adjacent to v, v_1, v_2, and v_3. Since G is a triangle-free planar graph, applying Theorem 4.1.4 we get that H is 3-colorable. Moreover, the vertices v_1, v_2, and v_3 must be colored with the same color in any 3-coloring of H as each forms a triangle together with u and v. $\square$

Let us denote by K_4' the graph obtained from K_4 by choosing any vertex of K_4, denote it by v, and subdividing once each edge incident with v (see Figure 4.4). By abuse of notation, let us call a graph G to be K_4'-free if it does not contain K_4' as a subgraph in such a way that the degree of v in G is the same as the degree of v in K_4'. It is easy to see that the vertices in the neighborhood of v cannot be colored with the same color.

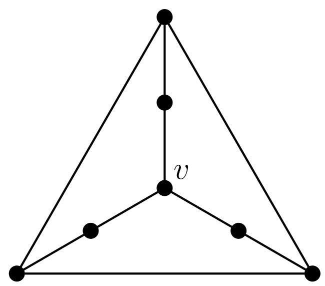


Figure 4.4: A planar graph K_4' with exactly one triangle and a vertex v in the center with no 3-coloring such that all the vertices in $N(v)$ receive the same color.

In the case of planar graphs with one triangle, we will prove the following slightly weaker result than the one from Theorem 4.2.4.

Theorem 4.2.6 (Theorem 1.11 in [157]). *Let G be a K_4'-free planar graph with at most one triangle. Then, for every vertex of degree at most 3 with an independent neighborhood, a precoloring of its neighbors with the same color can be extended to a 3-coloring of G.*

Note that Theorem 4.2.6 is tight both in terms of the degree of a vertex and in terms of the number of triangles (see Figure 4.5).

Before continuing on to the proofs, we need the following definitions and results.

Definition 4.2.7. Let G and H be two graphs. A *DHGO-composition* $DHGO(G, H)$ (sometimes also called Ore composition) of G and H is the graph obtained through the following three steps in order:

1. delete an edge $e = uv$ from G,
2. split a vertex w of H into two non-isolated vertices w_1 and w_2 (i.e., replace w with two non-adjacent vertices w_1 and w_2 and add edges

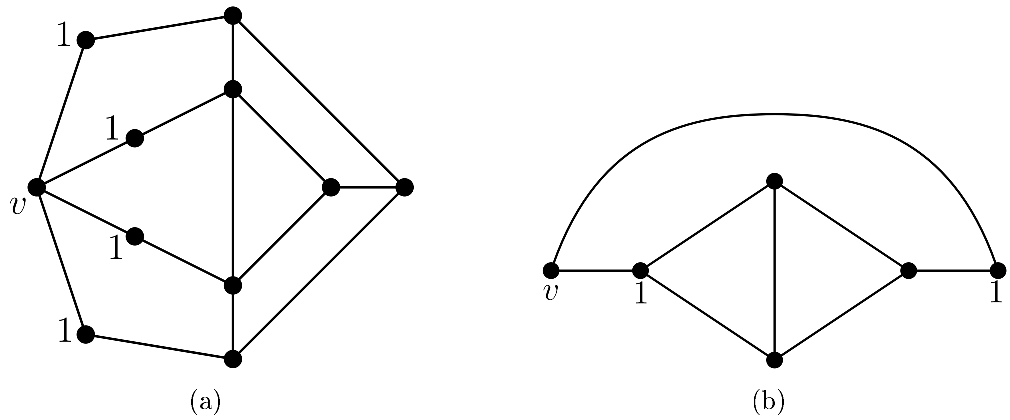


Figure 4.5: Precoloring of the neighborhood of a 4-vertex v in a K_4'-free planar graph G with one triangle cannot always be extended to a 3-coloring of G (example (a)). Similarly, precoloring of the neighborhood of a 2-vertex v in a planar graph G with two triangles cannot always be extended to a 3-coloring of G (example (b)).

incident to either w_1 or w_2 in such a way that the following conditions are satisfied; $N(w_1) \cap N(w_2) = \emptyset$, $N(w_1) \cup N(w_2) = N(w)$, and both w_1 and w_2 have degree at least 1), and

3. identify u with w_1 and v with w_2.

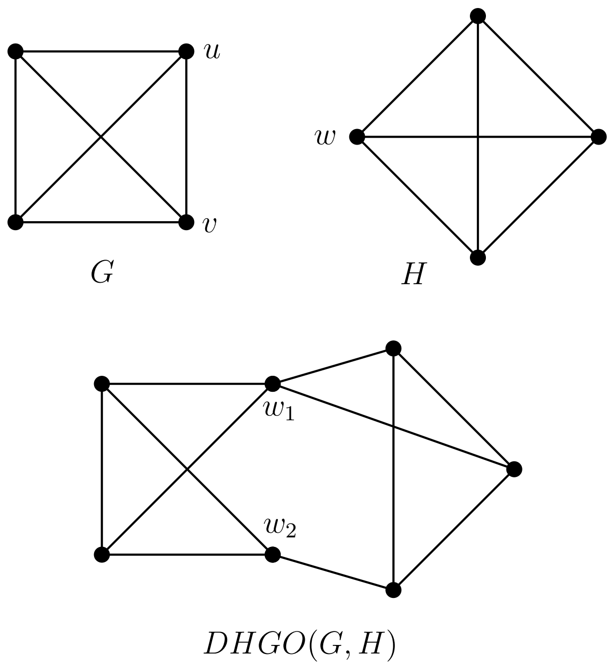


Figure 4.6: A DHGO-composition of two complete graphs K_4.

See Figure 4.6 for an example of a DHGO-composition. Note that the DHGO-compositions are not unique and they were used already in 1964 by Dirac [86], although the roots of such compositions began a decade earlier (see [84]).

Definition 4.2.8. A graph is k-Ore if it is obtained from a sequence of

DHGO-compositions of complete graphs K_k.

Using the notion of 4-Ore graphs, Kostochka and Yancey [151] proved a stronger version of Theorem 4.1.2.

Theorem 4.2.9 (Kostochka and Yancey [151]). *If G is a 4-critical graph, then*

$$|E(G)| \geq \frac{5|V(G)| - 2}{3}.$$

Moreover, the equality is achieved if and only if G is a 4-Ore graph.

Let us denote by a $\mathrm{Pl}_{4,4f}$*-graph* a planar graph with exactly four triangles and no 4-faces. In [39], the authors proved the following theorem which describes a relation between 4-Ore graphs and $\mathrm{Pl}_{4,4f}$-graphs.

Theorem 4.2.10 (Borodin et al. [39]). *Every 4-Ore graph has at least four triangles. Moreover, a 4-Ore graph has exactly four triangles if and only if it is a $\mathrm{Pl}_{4,4f}$-graph.*

We call an edge of a graph a *diamond edge* if it belongs to exactly two triangles. We can now define a special case of a DHGO-composition (see [39]).

Definition 4.2.11. Let $e = xy$ be a diamond edge of G and let $H = K_4$. A *diamond expansion* of G and H is then a DHGO-composition $DHGO(G, H)$ over the edge e.

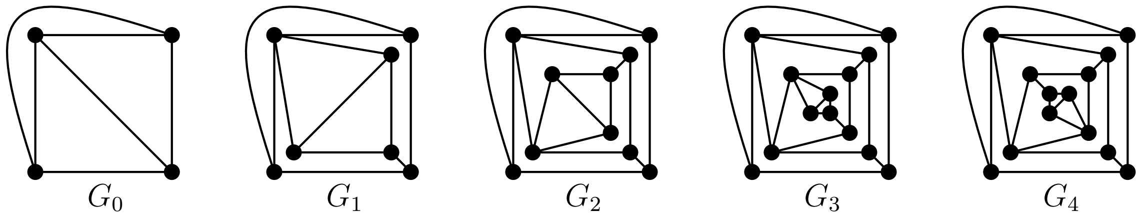


Figure 4.7: The first few graphs from $\mathcal{TW}$.

In 2004, Thomas and Walls [190] constructed an infinite family $\mathcal{TW}$ (see Figure 4.7) of $\mathrm{Pl}_{4,4f}$-graphs defined as follows. The family $\mathcal{TW}$ consists of all graphs obtained from K_4 as a sequence of diamond expansions. By definition of k-Ore graphs, as diamond expansion is a special case of a DHGO-composition, it follows directly that every graph in $\mathcal{TW}$ is a 4-Ore graph. Furthermore, by construction, every graph in $\mathcal{TW}$ has exactly four triangles. Thus, by Theorem 4.2.10, every graph in $\mathcal{TW}$ is also a $\mathrm{Pl}_{4,4f}$-graph.

Let $\mathcal{TW}_1$ be a family of graphs obtained from $\mathcal{TW}$ by replacing a single diamond edge with a Havel's quasi-edge, i.e., a graph such that in each 3-coloring of it, the two vertices u and v must receive distinct colors (see Figure 4.8) and let $\mathcal{TW}_2$ be a family of graphs obtained from $\mathcal{TW}_1$ in the same way. Note that $\mathcal{TW}_2$ can also be obtained from $\mathcal{TW}$ by replacing two vertex-disjoint diamond edges.

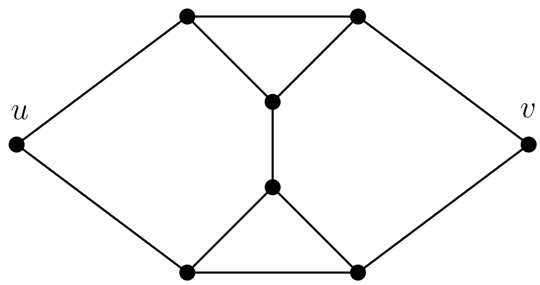


Figure 4.8: Havel's quasi-edge uv.

In [39], the authors additionally gave a complete characterization of 4-critical plane graphs having exactly four triangles as the union of families $\mathcal{TW}$, $\mathcal{TW}_1$, and $\mathcal{TW}_2$.

Theorem 4.2.12 (Borodin et al. [39]). *The class of* $\mathrm{Pl}_{4,4f}$*-graphs is equal to the union* $\mathcal{TW} \cup \mathcal{TW}_1 \cup \mathcal{TW}_2$.

This gives an infinite family of examples of graphs which confirm that Theorem 4.1.1 is tight in terms of the number of triangles.

The following lemma, together with Corollary 4.1.3, is a crucial tool both in the proofs from [41] and the proofs of Theorems 4.2.1, 4.2.3, and 4.2.6, where we use minimality of counterexamples; see, e.g., [41] for its proof.

Lemma 4.2.13 (Borodin et al. [41], Lemma 10). *Let G be a plane graph and $F = v_1v_2v_3v_4$ be a 4-face in G such that v_1v_3, $v_2v_4 \notin E(G)$. Let G_i be obtained from G by identifying v_i and v_{i+2} where $i \in \{1, 2\}$. If the number of triangles increases in both G_1 and G_2, then there exists a triangle $v_iv_{i+1}z$ for some $z \in V(G)$ and $i \in \{1, 2, 3, 4\}$. Moreover, G contains vertices x and y not in F such that $v_{i+1}zxv_{i+3}$ and v_izyv_{i+2} are paths in G (indices are modulo 4).*

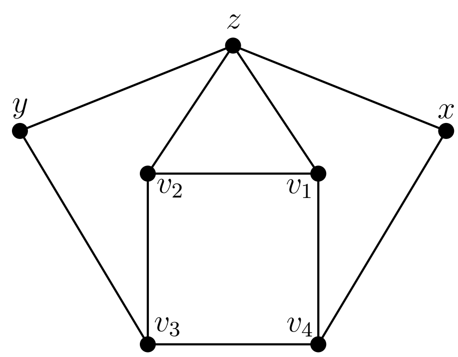


Figure 4.9: The configuration in Lemma 4.2.13 in the case when the number of triangles increases both in G_1 and in G_2.

In the case of planar graphs with one triangle, we use the following simplified statement of Lemma 4.2.13.

Corollary 4.2.14 (Corollary 2.2 in [157]). *Let G be a plane graph with at most one triangle and let α be any 4-face of G. Then, at least one of the* *following holds:*

(*a*) *α is adjacent to a triangle, or*

(*b*) *for at least one pair of opposite vertices of α, we can identify them without creating any new triangles.*

We are now ready to prove Theorems 4.2.1, 4.2.3, and 4.2.6.

4.2.1 Proof of Theorem 4.2.1

We prove Theorem 4.2.1 in two steps. First, we consider the case when the two precolored vertices receive distinct colors, which is equivalent to the statement of Theorem 4.2.15.

Theorem 4.2.15 (Theorem 3.1 in [157]). *Let G be a planar graph with at most one triangle and let H be a graph such that $G = H - e$ for some edge e of H. Then H is 3-colorable.*

Proof. We prove the theorem by contradiction. Suppose that H is a counterexample minimizing the number of vertices plus the number of edges and let G be a plane graph with at most one triangle such that $G = H - e$ for some edge e of H. Note that since G is planar and contains at most one triangle, it is 3-colorable by Theorem 4.1.1. By Theorem 4.1.5, we may assume that G contains exactly one triangle T. Moreover, by the minimality, H is 4-critical.

We consider five cases regarding 4-faces in G.

Case 1: *G has at most two 4-faces.* Let $f_{4,G}$ denote the number of 4-faces in G. By the Handshaking Lemma, we have

$$2m_G = \sum_{\alpha \in F(G)} \ell(\alpha) \geq 3 + 4 \cdot f_{4,G} + 5 \cdot (f_G - (1 + f_{4,G})) = 5f_G - 2 - f_{4,G}$$

(in the calculation, we assume that T is a face, otherwise the lower bound on the number of edges would be even higher). Then, $5f_G \leq 2m_G + 4$ and by applying the Euler's Formula and observing that $n_H = n_G$ and $m_H = m_G + 1$, we infer that

$$10 = 5n_G - 5m_G + 5f_G \leq 5n_G - 3m_G + 4 = 5n_H - 3(m_H - 1) + 4\,.$$

Thus,

$$m_H \leq \frac{5n_H - 3}{3}\,,$$

a contradiction to Theorem 4.1.2.

Case 2: *G has a 4-face $\alpha = v_1v_2v_3v_4$ such that at most one vertex of α is incident with T and at most one vertex of e is incident with α.* Let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. By the assumption and Corollary 4.2.14, we may assume, without loss of generality, that G_1 contains T as the unique triangle. Note that the graph H_1 obtained from H by identifying v_1 and v_3 contains e and is thus 3-colorable by the minimality. Thus, we can extend the coloring of H_1 to the coloring of H in which v_1 and v_3 receive the same color, a contradiction.

Case 3: *G has a 4-face $\alpha = v_1v_2v_3v_4$ such that at most one vertex of α is incident with T and both vertices of e are incident with α.* We may assume, without loss of generality, that $e = v_1v_3$. Let G_2 be the graph obtained from G by identifying v_2 and v_4. Note that if the number of triangles does not increase in G_2, then we can continue as in Case 2.

Therefore, by Lemma 4.2.13, there exist vertices $x, z \in V(G)$ such that $xv_4, xz, zv_2 \in E(G)$. Consequently, no 4-face of G, other than α, can contain both vertices v_1 and v_3 due to planarity.

Due to Cases 1 and 2, and the fact that α contains both vertices of e, there exists a 4-face $\alpha' = v_1'v_2'v_3'v_4'$ such that α' contains two vertices of T, say v_1' and v_2' (note that the two vertices incident with T are not opposite in α', otherwise there would be another triangle in G), with z' being the third vertex of T. Let G_i' be the graph obtained from G by identifying v_i' and v_{i+2}', where $i \in \{1, 2\}$. Again, if the number of triangles does not increase in G_1' or G_2', then we can color H with 3-colors.

It follows that there exist vertices $x', y' \in V(G)$ such that $x'z'$, $x'v_4'$, $y'z'$, and $y'v_3' \in E(G)$. Suppose that at least one of $C_1 = z'v_2'v_3'y'$ or $C_2 = z'v_1'v_4'x'$ is a 4-face, say C_1. By our observation above, C_1 does not contain both vertices of e. Let G' be the graph obtained from G by identifying v_2' and y'. Note that the number of triangles in G' does not increase. Let H' be the graph obtained from G' by adding the edge e. By the minimality, we can color H' with 3 colors and extend the coloring to a coloring of H, in which y' and v_2' receive the same color, a contradiction.

Thus, we may assume that both C_1 and C_2 are separating 4-cycles. Note that if the vertices of α (and thus also the endvertices of e) belong to the vertex set $V_1 = V(\text{ext}(C_1)) \cup V(C_1)$, then $H[V_1]$ contains both T and e. Therefore, we can color $H[V_1]$ by the minimality and extend the coloring of C_1 to a coloring of $H[V(\text{int}(C_1)) \cup V(C_1)]$ by Theorem 4.1.6. We use an analogous argument for C_2 in the case when the vertices of α belong to the graph induced by the vertex set $V(\text{int}(C_1)) \cup V(C_1)$, which implies that the vertices of α belong to the vertex set $V(\text{ext}(C_2)) \cup V(C_2)$. Thus, H is 3-colorable, a contradiction.

Case 4: *G has a 4-face $\alpha = v_1v_2v_3v_4$ such that exactly two of its vertices, say v_1 and v_2, are incident with T, and at most one vertex of the edge e is incident with α.* Let z be the third vertex of T. Using similar arguments as in the previous cases, we infer that there exist vertices $x, y \in V(G)$ such that xz, xv_4, yz, and $yv_3 \in E(G)$.

Suppose that $C_1 = zv_2v_3y$ is a 4-face. If $e \neq v_2y$, then consider the graph G' obtained from G by identifying v_2 and y. Note that the number of triangles in G' does not increase. Let H' be the graph obtained from G'

by adding the edge e. By the minimality, we can color H' with 3-colors and extend the coloring to a coloring of H, in which y and v_2 receive the same color, a contradiction. Therefore, $e = v_2y$. But then, either $C_2 = zv_1v_4x$ is a 4-face, in which case we can apply the same procedure on v_1 and x as we did on v_2 and y, or C_2 is a separating 4-cycle. However, since both $V(T)$ and $V(e)$ belong to the vertex set $V_1 = V(\text{ext}(C_2)) \cup V(C_2)$, we can complete the coloring in a similar manner as in the last paragraph of Case 3, a contradiction.

Thus, by symmetry, both C_1 and C_2 are separating 4-cycles. Moreover, each of C_1 and C_2 contains exactly one vertex of e in its interior. Furthermore, T is a 3-face, otherwise we can color $H[V(\text{ext}(T)) \cup V(T)]$ by the minimality, and then extend the coloring to the interior of T by Theorem 4.1.6. Additionally, due to Case 1, there exists a 4-face $\alpha' = v_1'v_2'v_3'v_4'$ in G, distinct from α. If identifying either v_1' and v_3', or v_2' and v_4' results in a graph with one triangle, namely T, then by the minimality, it is 3-colorable and the coloring can be extended to H. Therefore, by the fact that G has only one triangle and Lemma 4.2.13, two vertices of α' are incident with T, say $v_1' = v_1$ and $v_2' = z$ (meaning that at least one of v_3' and v_4' is in $V(\text{int}(C_2))$, see Figure 4.10) and there are vertices x' and y' in G such that $x'v_4'$, $x'v_2$, $y'v_3'$, and $y'v_2 \in E(G)$. This is not possible due to the planarity of G, a contradiction.

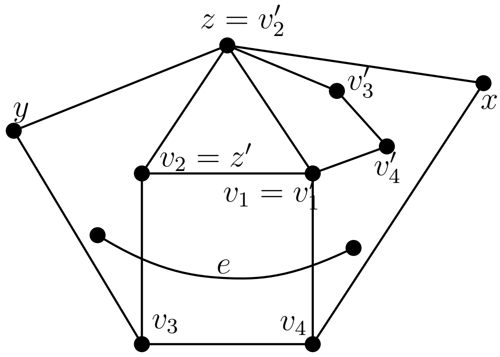


Figure 4.10: The 4-faces α and α' in the last part of Case 4.

Case 5: *G has at least three 4-faces and each of them is incident with two vertices of T and both vertices of e.* Let $\alpha = v_1v_2v_3v_4$ be such a face and let $T = v_1v_2z$. Without loss of generality, we may assume that $e = v_1v_3$. Let G_2 be the graph obtained from G by identifying v_2 and v_4. If the number of triangles does not increase in G_2, then we are done. Thus, by Lemma 4.2.13, there exists a vertex $x \in V(G)$ such that xz, $xv_4 \in E(G)$. Note that by the assumptions, $C = zv_1v_4x$ is not a 4-face, since it is incident to exactly one vertex of e. Therefore, C is a separating 4-cycle. But then, the vertices of both T and e are contained in the vertex set $V_1 = V(\text{ext}(C)) \cup V(C)$. Let $V_2 = V(\text{int}(C)) \cup V(C)$. By the minimality, we can color $G[V_1]$ and extend the coloring of C to the coloring of $G[V_2]$ by Theorem 4.1.6, a contradiction.

Since no 4-face can be incident with all three vertices of T, the proof is completed. □

In the second step of proving Theorem 4.2.1, we show that any two non-adjacent vertices in a planar graph with one triangle can be colored with the same color.

Theorem 4.2.16 (Theorem 3.2 in [157]). *Let G be a planar graph with at most one triangle. Then each coloring of any two non-adjacent vertices with the same color can be extended to a 3-coloring of G.*

Proof. We prove the theorem by contradiction. Suppose that a counterexample G is a plane graph with the minimum number of vertices. By Theorem 4.1.5, we may also assume that G contains exactly one triangle T. Let u and v be two non-adjacent vertices of G.

Let H be the graph obtained from G by identifying the vertices u and v. Clearly, $n_G = n_H + 1$ and $m_G = m_H$. By the minimality, H is 4-critical. To reach a contradiction, we only need to prove that H is 3-colorable, which implies that there exists a 3-coloring of G in which u and v receive the same color.

We consider three cases regarding 4-faces in G.

Case 1: *G has no 4-faces.* By the Handshaking Lemma, we have

$$2m_G = \sum_{\alpha \in F(G)} \ell(\alpha) \geq 3 + 5 \cdot (f_G - 1) = 5f_G - 2\,.$$

Then, $5f_G \leq 2m_G + 2$ and by applying the Euler's Formula, we infer that

$$10 = 5n_G - 5m_G + 5f_G \leq 5n_G - 3m_G + 2 = 5n_H + 5 - 3m_H + 2\,.$$

Thus,

$$m_H \leq \frac{5n_H - 3}{3}\,,$$

a contradiction to Theorem 4.1.2.

Case 2: *G has exactly one 4-face.* Similarly as in Case 1, we can compute that $5f_G \leq 2m_G + 3$ and by applying Euler's Formula, we infer that

$$m_H \leq \frac{5n_H - 2}{3}\,.$$

In the case when $m_H < \frac{5n_H - 2}{3}$, we obtain a contradiction to Theorem 4.1.2, and therefore, H has exactly $\frac{5n_H-2}{3}$ edges.

Let $\alpha = v_1v_2v_3v_4$ be the 4-face in G and let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$.

Suppose first that the number of triangles does not increase in G_1 or G_2, say G_1. In the case $\{u, v\} \neq \{v_1, v_3\}$, we identify v_1 and v_3 in H to obtain the graph H_1. By the minimality, we can color H_1 with 3 colors and extend the coloring to a coloring of H, and therefore also to G, a contradiction. Hence, we may assume that $\{u, v\} = \{v_1, v_3\}$. In this case, H is a planar graph with exactly one triangle. Thus, by Theorem 4.1.1, there exists a 3-coloring of H, and therefore also of G, a contradiction.

We may thus assume that the number of triangles increases in both G_1 and G_2. By Lemma 4.2.13, without loss of generality, we may assume that there exist vertices $x, y, z \in V(G)$ such that zv_1, zv_2, xz, xv_4, yz, and $yv_3 \in E(G)$, where zv_1v_2 is T. Since G contains exactly one 4-face, it follows that both $C_1 = zv_1v_4x$ and $C_2 = zv_2v_3y$ are separating 4-cycles.

Note that if both u and v belong to the subgraph of G induced by the vertex set $V_1 = V(\text{ext}(C_1)) \cup V(C_1)$, then we can color $G[V_1]$ by the minimality and use Theorem 4.1.6 to extend the coloring of C_1 to the coloring of the interior of C_1. By symmetry, we may thus assume, without loss of generality, that $u \in V(\text{int}(C_1))$ and $v \in V(\text{int}(C_2))$.

Since $m_H = \frac{5n-2}{3}$, by Theorems 4.2.9 and 4.2.10, we infer that H must have at least 5 triangles. Therefore, since G has exactly one triangle, it follows that by identifying u and v, we create at least four new triangles. We will prove that this cannot happen.

First, observe that no new triangle can contain vertices x or y, since that would imply the existence of another triangle, distinct from T, in G. Next, observe that u is adjacent with at most one of the vertices v_1 and v_4, and v is adjacent with at most one of the vertices v_2 and v_3. Thus, at most one new triangle can be formed using the edges v_1v_2 or v_3v_4, and so there must exist at least three triangles in H which contain the vertex z and either u or v, say u, is adjacent to z. Therefore, there exist at least three vertices $w_1, w_2, w_3 \in V(G)$ such that $w_1, w_2, w_3 \in V(\text{int}(C_2))$. Moreover, each of them is adjacent to z and v (see Figure 4.11). Consider now the

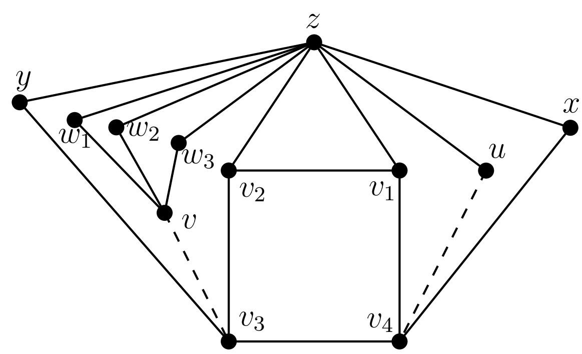


Figure 4.11: The vertices in G comprising triangles in H in the last part of Case 2.

4-cycle $C = zw_1vw_2$. Since G contains exactly one 4-face, it follows that C is a separating 4-cycle. Furthermore, the exterior of C together with the vertices of C contains both u and v, as well as T. Thus, by the minimality, we can color $G[V(\text{ext}(C)) \cup V(C)]$ and extend the 3-coloring of the vertices of C to a 3-coloring of H by Theorem 4.1.6.

Case 3: *G has at least two 4-faces.* Let $\alpha = v_1v_2v_3v_4$ be a 4-face and let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. Since G contains exactly one triangle, by Corollary 4.2.14, either, without loss of generality, v_1v_2 is an edge of T or we can identify v_1 and v_3 or v_2 and v_4 without creating any new triangles. Suppose first that v_1v_2 is not an edge of T; say that G_1 has at most one triangle. Then, in the case $\{u, v\} \neq \{v_1, v_3\}$, we identify v_1 and v_3 in H to obtain the graph H_1. By the minimality, we can color H_1 with 3 colors and extend the coloring to a coloring of H, and therefore also to G, a contradiction. Hence, we may assume that $\{u, v\} = \{v_1, v_3\}$. In this case, H is a planar graph with exactly one triangle. Thus, by Theorem 4.1.1, there exists a 3-coloring of H, and therefore also of G, a contradiction.

Thus, we may assume that $T = v_1v_2z$, with z being distinct from v_3 and v_4, and that in both G_1 and G_2 the number of triangles is at least 2. Therefore, by Lemma 4.2.13, there exist vertices $x, y \in V(G)$ such that xz, xv_4, yz, and $yv_3 \in E(G)$.

Suppose that $C_1 = zv_1v_4x$ is a 4-face. Then, due to planarity of G, in the graph G' obtained by identifying v_1 and z, no new triangle is created. Thus, by the minimality, we can color G' and infer 3-colorability of G in a similar manner as above, a contradiction.

Therefore, by symmetry, we may assume that both C_1 and $C_2 = zv_2v_3y$ are separating 4-cycles. Note that if both u and v belong to the vertex set $V_1 = V(\text{ext}(C_1)) \cup V(C_1)$ (resp., $V_2 = V(\text{ext}(C_2)) \cup V(C_2)$), then, by the minimality, we can color the graph H_1 (resp., H_2) obtained from $G[V_1]$ (resp., $G[V_2]$) by identifying u and v and extend the coloring to a coloring of H by Theorem 4.1.6, hence also obtaining a 3-colorability of G.

Thus, we may assume, without loss of generality, that $u \in \text{int}(C_1)$ and $v \in \text{int}(C_2)$. Now, consider a 4-face $\alpha' = v_1'v_2'v_3'v_4'$. If α' satisfies the property (b) of Corollary 4.2.14, then we proceed as above to obtain a contradiction. Therefore, α' is incident with T and, by planarity of G, the vertices of α' are all contained in $V(\text{int}(T)) \cup V(T)$. But then, both u and v belong to the exterior of T and we can color, by the minimality, the graph obtained from $G[V(\text{ext}(T)) \cup V(T)]$ by identifying u and v. Finally, we extend the obtained coloring to a coloring of H by Theorem 4.1.6. Hence, from the coloring of H, we again obtain 3-colorability of G, a contradiction. This completes the proof. □

Theorems 4.2.15 and 4.2.16 combined settle Theorem 4.2.1.

4.2.2 Proof of Theorem 4.2.3

Next, we prove Theorem 4.2.3.

Proof of Theorem 4.2.3. Let G be a planar graph with at most one triangle and let f be a precolored face of length at most 4.

Suppose first that f is of length 3. Since there is only one coloring of f (up to a permutation of colors), the result follows from Theorem 4.1.1.

Thus, we may assume that $f = v_1v_2v_3v_4$ is a 4-face. Suppose that the precoloring of f uses all three colors. Then, two non-adjacent vertices of f, say v_1 and v_3, receive distinct colors and the other two vertices are colored with the third. Note that the same coloring of f (up to a permutation of colors) can be obtained by adding an edge between v_1 and v_3. The obtained graph is 3-colorable by Theorem 4.2.15.

Therefore, we may assume that the vertices of f are precolored with two colors. We proceed by contradiction. Let G be a plane graph with at most one triangle such that a precoloring of some 4-face f with two colors cannot be extended to a 3-coloring of G. Moreover, let G be the smallest such graph in terms of the vertices. Clearly, G has exactly one triangle T, otherwise the precoloring can be extended by Theorem 4.1.6.

Let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. If the number of triangles does not increase in G_1 or G_2, say G_1, then there is a 3-coloring of G_1, guaranteed by Theorem 4.2.1, which induces a 3-coloring of G such that the vertices of f are colored with two colors.

Thus, by Lemma 4.2.13, without loss of generality, we may assume that there exist vertices $x, y, z \in V(G)$ such that zv_1, zv_2, xz, xv_4, yz, and $yv_3 \in E(G)$, where $T = zv_1v_2$. Observe that coloring of f forces also the colors on x, y, and z (see Figure 4.12).

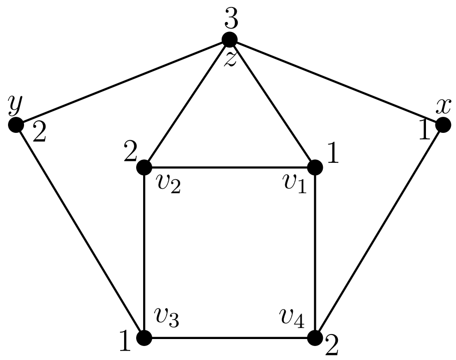


Figure 4.12: The coloring of f forces the colors of x, y, and z.

Let $C_1 = zv_1v_4x$ and $C_2 = zv_2v_3y$. Suppose that at least one of C_1 or C_2, say C_1, is a separating 4-cycle. Then, by the minimality, the coloring of f extends to a 3-coloring of $G[V(\text{ext}(C_1)) \cup V(C_1)]$. Since the obtained coloring of C_1 extends to a 3-coloring of $G[V(\text{int}(C_1)) \cup V(C_1)]$ by Theorem 4.1.6, we obtain a 3-coloring of G, a contradiction.

Thus, we may assume that both C_1 and C_2 are 4-faces in G. In a similar manner as above, we infer that T must be a 3-face. But then, the precoloring of the 5-cycle $C_3 = v_3v_4xzy$ given in Figure 4.12 extends to a 3-coloring of $G[V(\text{ext}(C_3) \cup V(C_3))]$ (which might as well be an empty graph) by Theorem 4.1.6 and we color the two vertices in the interior of C_3 as in Figure 4.12, hence obtaining a 3-coloring of G, a contradiction. This completes the proof. □

4.2.3 Proof of Theorem 4.2.6

We conclude this section with a proof of Theorem 4.2.6.

Proof of Theorem 4.2.6. We prove the theorem by contradiction. Let G be a minimal counterexample to the theorem, i.e., G is a K_4'-free planar graph with at most one triangle and the minimum number of vertices such that there is a vertex u of degree at most 3 with an independent neighborhood, such that precoloring the vertices in $N(u)$ with a same color does not extend to a 3-coloring of G.

First, observe that by Theorem 4.1.4, G has exactly one triangle T, and by Theorem 4.2.1, u is a 3-vertex. Let $N[u] = \{u, u_1, u_2, u_3\}$ and let H be the graph obtained by identifying $N[u]$ into a vertex w. Let α_1, α_2, and α_3 be the three faces incident to u in G that contain respectively $\{u_1, u_2\}$, $\{u_2, u_3\}$, and $\{u_1, u_3\}$. Furthermore, let α_1', α_2', and α_3' be the faces incident to w in H corresponding to α_1, α_2, and α_3.

Clearly, every 3-coloring of H induces a 3-coloring of G with u_1, u_2, and u_3 colored with a same color, while u can be colored with either of the remaining two colors. Additionally, since G is a planar graph, H is also a planar graph and by the minimality of G, H is 4-critical. Observe also that $n_G = n_H + 3$, $m_G = m_H + 3$, and $f_G = f_H$.

Now, we prove two structural properties of H.

Claim 4.2.17. *H has no separating triangles.*

Proof of Claim 4.2.17. Suppose the contrary and let C be a separating triangle in H. First, suppose that C is the triangle of G. Without loss of generality, we may assume that $w \in V(\mathrm{int}(C))$. By the minimality, there is a 3-coloring of $H[V(\mathrm{int}(C)) \cup V(C)]$, and by Theorem 4.1.1, we can extend it to a 3-coloring of H, since $H[V(\mathrm{ext}(C)) \cup V(C)]$ has exactly one triangle, a contradiction.

Therefore, we may assume that $C \neq T$. In that case, C has been created from a 5-cycle C_G after we identified $N[u]$ into w and thus $w \in V(C)$. Since $C \neq T$, we may assume, without loss of generality, that $H[V(\mathrm{int}(C)) \cup V(C)]$ contains α_1' but not α_2' or α_3' (see Figure 4.13).

By the minimality, there is a 3-coloring ϕ of $H[V(\mathrm{ext}(C)) \cup V(C)]$. Now, we show that we can extend ϕ to the interior of C. Let $H_1 = H[V(\mathrm{int}(C)) \cup V(C)]$. We proceed by induction on the number of separating triangles in H_1. First, recall that all separating triangles in H_1 are incident to w; more precisely, they were obtained from 5-cycles in G containing $\{u_1, x, u_2\}$.

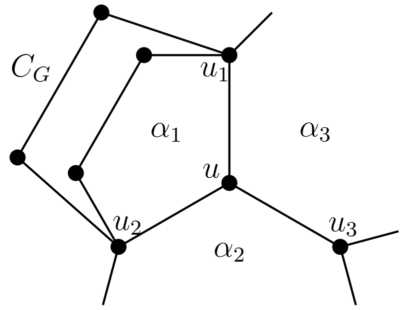


Figure 4.13: A separating 5-cycle in G containing α_1.

Suppose that $H_1 = H[V(\text{int}(C)) \cup V(C)]$ has no separating triangle. Then it has at most three triangles: C as its outer face, possibly α_1', and possibly T. Therefore, H_1 is a planar graph with at most three triangles and thus 3-colorable by Theorem 4.1.1.

So, we may assume that H_1 has at least one separating triangle; we select a separating triangle C' such that all separating triangles in H_1 are contained in $H_1' = H_1[V(\text{int}(C')) \cup V(C')]$. Then, by induction, there is a 3-coloring ϕ' of H_1'. Finally, using the colorings ϕ and ϕ', we can complete the coloring of H by coloring $H[V(H_1) \setminus V(\text{int}(C'))]$ using Theorem 4.1.1 and an eventual permutation of colors in ϕ', a contradiction. ■

Claim 4.2.18. If H has a separating 4-cycle, then both its interior and exterior must contain w or a triangle.

Proof of Claim 4.2.18. Suppose the contrary and let C be a separating 4-cycle of H such that $H[V(\text{int}(C)) \cup V(C)]$ is a triangle-free planar graph that does not contain w. By the minimality, there is a 3-coloring ϕ of $H[V(\text{ext}(C)) \cup V(C)]$. By Theorem 4.1.6, we can extend ϕ to the whole graph H, a contradiction. ■

Now, we are ready to finish the proof by considering three cases regarding 4-faces of G.

Case 1: *G has no 4-faces.* By the Handshaking Lemma, we have $2m_G \geq 5f_G - 2$ and so $2m_H + 6 \geq 5f_H - 2$. Then, $5f_H \leq 2m_H + 8$ and by applying the Euler's Formula on G, we infer that

$$m_H \leq \frac{5n_H - 2}{3}.$$

Since H is 4-critical, by Theorem 4.2.9, we have that $m_H = \frac{5n_H-2}{3}$ and that H is a 4-Ore graph. Moreover, since H does not have separating triangles by Claim 4.2.17, there are at most four triangles in H (T and the faces α_1', α_2', and α_3'). Thus, by Theorem 4.1.1, H has exactly four triangles and by Theorem 4.2.10, H is a $\text{Pl}_{4,4f}$-graph. Recall that three of the triangles are incident to the same vertex w. The only $\text{Pl}_{4,4f}$-graph for which this is true is K_4 [39, Theorem 4]. However, to obtain K_4, all three neighbors of u in G must be of degree 2, meaning that G must be K_4', a contradiction.

Case 2: *G has a 4-face that is incident to u.* As a result, after identifying u and its neighbors, H has at most three triangles by Claim 4.2.17. Therefore, H is 3-colorable by Theorem 4.1.1.

Case 3: *G has a 4-face $\alpha = v_1v_2v_3v_4$ and α is not incident to u.* The edges v_1v_3 and v_2v_4 are not present in G, otherwise G would have at least two triangles. Moreover, if u is adjacent to two (opposite) vertices of α, say $v_1 = u_1$ and $v_3 = u_3$, then, by Case 2, neither $C_1 = uv_1v_2v_3$ nor $C_2 = uv_1v_4v_3$ is a 4-face. Moreover, without loss of generality, we may assume that $u_2 \in V(\text{ext}(C_1))$. However, by the minimality, there is a 3-coloring of $G[V(\text{ext}(C_1)) \cup V(C_1)]$, and it can easily be extended to the whole G by Theorem 4.2.3. Therefore, at most one of the vertices of α is adjacent to u.

Let G_i be the graph obtained from G by identifying v_i and v_{i+2}, where $i \in \{1, 2\}$. Suppose that the only triangle in G_1 is T. Then, by the minimality, the graph H_1 obtained by identifying the vertices v_1 and v_3 in H is 3-colorable. Clearly, its coloring can be extended to H and thus also to G, a contradiction.

Therefore, by symmetry, we may assume that in G_1 and G_2 the number of triangles increases. It follows by Lemma 4.2.13 that there are vertices x, y, and z such that $v_1z, v_2z, xz, xv_4, yz, yv_3 \in E(G)$, where $T = v_1v_2z$. If one of $C_1 = zv_1v_4x$ and $C_2 = zv_2v_3y$ is a 4-face, it has the same properties as α and two of its vertices are incident with T. But that is not possible due to planarity.

Thus, C_1 and C_2 are separating 4-cycles of G. Since, at most one of them can contain $u \neq z$ (by definition, u is not incident with a triangle), the other one remains a separating 4-cycle of H, which does not contain T nor w, a contradiction to Claim 4.2.18. This completes the proof. □

Chapter 5

Cyclic Coloring of Plane Graphs and its Generalization

In the previous chapter we considered the "usual" proper colorings of the vertices of planar graphs. In this and the next chapter we will consider various colorings of planar graphs defined by some other restrictions. In particular, we will consider colorings of plane graphs in which various constraints are given with respect to the faces. Note that we require the word plane instead of planar as a single planar graph may have different embeddings in the plane. We say that a vertex (edge) coloring of a plane graph G is *facially-proper* if for every face α of G there are no two distinct vertices (edges) appearing consecutively on a boundary walk of α that are colored with the same color. Note that every proper vertex coloring of a plane graph is also facially-proper and vice versa. On the other hand, every proper edge coloring is also facially-proper, but the converse does not always hold. It is easy to see that in the case of subcubic plane graphs every facially-proper edge coloring is also a proper edge coloring. This is not necessarily true in the case of plane graphs with maximum degree at least 4.

Facially-proper edge coloring was first considered for the family of bridgeless cubic plane graphs. In 1880, Tait [188] noticed that the vertices of every plane graph are 4-colorable if and only if the edges of every bridgeless cubic planar graph are 3-colorable. In general every connected plane graph admits a facially-proper edge coloring with at most four colors [140] and the graph which achieves this bound is a wheel with five spokes W_5.

Throughout the years, a number of facially constrained colorings were defined with various additional constraints. Many such colorings are presented in recent survey papers by Czap and Jendroľ [71] from 2017, and Czap, Horňák, and Jendroľ [68] from 2021. In the rest of this chapter we will consider the cyclic coloring, the ℓ-facial coloring (both vertex and edge versions), the odd edge coloring, and the facial-parity coloring (also both vertex and edge versions).

5.1 Cyclic coloring

A *cyclic coloring* (also known as a *rainbow vertex coloring*, see [71]) is a proper vertex coloring of a plane graph such that no two vertices incident with the same face are colored with the same color. The minimum number of colors needed for a cyclic coloring of a plane graph is called the *cyclic chromatic number* and denoted by $\chi_c(G)$. This notion was first introduced in 1969 by Ore and Plummer [169]. They considered cyclic coloring of plane pseudographs and later noticed that it is enough to consider only connected plane graphs. As a matter of fact, they showed that it is enough to consider only 2-connected plane graphs. By the definition, it easily follows that $\chi_c(G) \geq \Delta^*(G)$ (recall that $\Delta^*(G)$ denotes the length of the longest face of G). On the other hand, Ore and Plummer [169] proved the following upper bound for the cyclic chromatic number of any plane graph.

Theorem 5.1.1 (Ore and Plummer [169]). *Let d_1 and d_2 be the two largest face sizes in a plane graph G. Then,*

$$\chi_c(G) \leq d_1 + d_2 \leq 2\Delta^*(G).$$

Later, in 1984, Borodin [35] implicitly conjectured the following.

Conjecture 5.1.2 (Borodin [35]). Let G be a plane graph with $\Delta^*(G) \geq 3$. Then,

$$\chi_c(G) \leq \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor.$$

Conjecture 5.1.2, which is known under the name Cyclic Coloring Conjecture, has become a focus for many research studies. In 1987, Plummer and Toft [172] presented the first family of plane graphs attaining the upper bound given by the Cyclic Coloring Conjecture which are exactly graphs G that are the line graphs of the Theta graph $\Theta_{k,k,k}$ for which $\Delta^*(G) = 2(k+1)$ and $\chi_c(G) = 3(k+1)$. Five years later, Borodin [36] slightly improved the general upper bound.

Theorem 5.1.3 (Borodin [36]). *Let G be a plane graph with $\Delta^*(G) \geq 3$. Then,*

$$\chi_c(G) \leq \begin{cases} 2\Delta^*(G) - 3 & \text{for } \Delta^*(G) \geq 8, \\ 12 & \text{for } \Delta^*(G) \leq 7, \\ 11 & \text{for } \Delta^*(G) \leq 6, \\ 9 & \text{for } \Delta^*(G) \leq 5. \end{cases}$$

It was several years later when Borodin, Sanders, and Zhao [42] managed to improve the upper bound by showing that $\lfloor \frac{9}{5}\Delta^*(G) \rfloor$ colors suffice for a cyclic coloring of any plane graph and two years later Sanders and Zhao [179] improved the upper bound even further.

Theorem 5.1.4 (Sanders and Zhao [179]). *Let G be a plane graph $\Delta^*(G) \geq 3$. Then,*

$$\chi_c(G) \leq \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil .$$

Currently, Theorem 5.1.4 gives the best known upper bound depending only on Δ^* and the Cyclic Coloring Conjecture (Conjecture 5.1.2) is known to be true only for the values $\Delta^*(G) \in \{3, 4, 6\}$. The case $\Delta^*(G) = 3$ follows directly from the Four Color Theorem, while the case $\Delta^*(G) = 4$ follows from the fact that we can properly color the vertices of every 1-planar graph with at most 6 colors (see [35, 37]). The last case $\Delta^*(G) = 6$ was solved by Hebdige and Kráľ [129] in 2016. As a remark, for the case $\Delta^*(G) = 7$ both Theorems 5.1.3 and 5.1.4 state that 12 colors are enough. This result was further improved to 11 colors as a consequence of another result (see Corollary 5.2.4 in the next section). Although the Cyclic Coloring Conjecture is still widely open, Amini, Esperet, and van den Heuvel [11] showed that it is asymptotically true.

Theorem 5.1.5 (Amini et al. [11]). *For every $\epsilon > 0$, there exists Δ_ϵ such that every plane graph of maximum face degree $\Delta^* \geq \Delta_\epsilon$ admits a cyclic coloring with at most $(\frac{3}{2} + \epsilon)\Delta^*(G)$ colors.*

As the Cyclic Coloring Conjecture proved to be a difficult problem, many researchers focused on the cyclic coloring of 3-connected plane graphs, which appeared to be simpler. In 1987, Plummer and Toft [172] obtained the first results in this direction and proved that $\Delta^*(G) + 9$ colors always suffice. Furthermore, they also obtained several slightly better results in some special cases with respect to $\Delta^*(G)$ summarized in the following theorem.

Theorem 5.1.6 (Plummer and Toft [172]). *Let G be a 3-connected plane graph. Then,*

$$\chi_c(G) \leq \begin{cases} \Delta^*(G) + 9 & \textit{for all } \Delta^*(G), \\ \hline \Delta^*(G) + 8 & \textit{for } \Delta^*(G) \leq 10, \\ \Delta^*(G) + 7 & \textit{for } \Delta^*(G) \leq 9, \\ \Delta^*(G) + 6 & \textit{for } \Delta^*(G) \leq 8, \\ \hline \Delta^*(G) + 8 & \textit{for } \Delta^*(G) \geq 14, \\ \Delta^*(G) + 7 & \textit{for } \Delta^*(G) \geq 15, \\ \Delta^*(G) + 6 & \textit{for } \Delta^*(G) \geq 18, \\ \Delta^*(G) + 5 & \textit{for } \Delta^*(G) \geq 24, \\ \Delta^*(G) + 4 & \textit{for } \Delta^*(G) \geq 42. \end{cases}$$

In addition to the results shown in Theorem 5.1.6, they constructed an infinite family of 3-connected plane graphs for which $\chi_c(G) = \Delta^*(G) + 2$

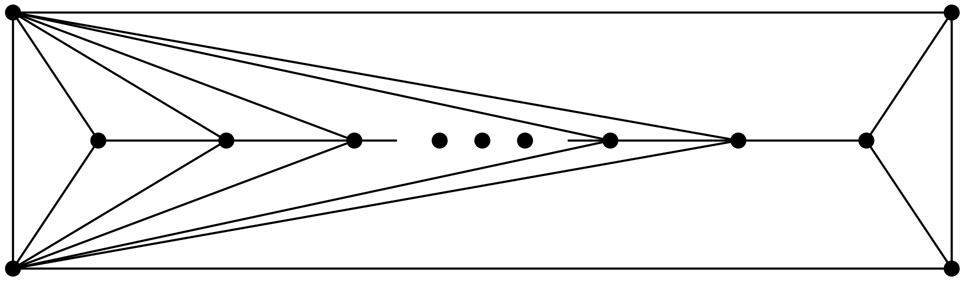


Figure 5.1: A 3-connected plane graph G with $\chi_c(G) = \Delta^*(G) + 2$.

(see Figure 5.1), as well as provided examples of 2-connected plane graphs with minimum degree 3 showing that for any given positive integer k there exist plane graphs with $\chi_c(G) > \Delta^*(G) + k$ (see Figure 5.2). Finally, they also posed the following conjecture.

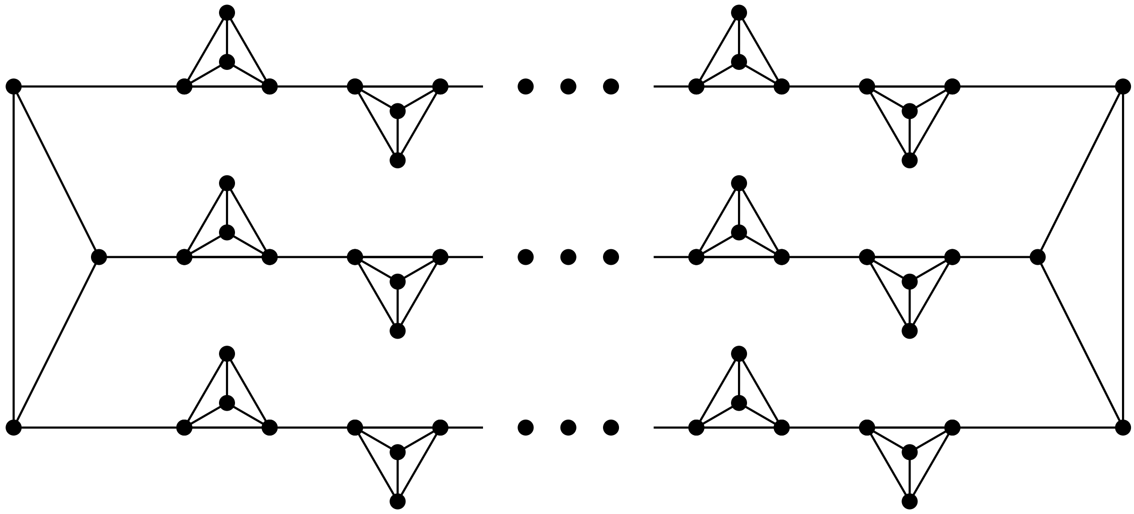


Figure 5.2: Example of a 2-connected plane graph G with $\delta(G) = 3$ and $\chi_c(G) > \Delta^*(G) + k$.

Conjecture 5.1.7 (Plummer and Toft [172]). Let G be a 3-connected plane graph. Then,

$$\chi_c(G) \le \Delta^*(G) + 2\,.$$

The first positive answer to Conjecture 5.1.7 was provided by Horňák and Jendroľ [136, 137] who proved that for all $\Delta^*(G) \ge 24$, Conjecture 5.1.7 holds. Later, in 2010, Horňák and Zlámalová [139] proved that Conjecture 5.1.7 holds for all $\Delta^*(G) \ge 18$.

Theorem 5.1.8 (Horňák and Zlámalová [139]). *Let G be a 3-connected plane graph with $\Delta^*(G) \ge 18$. Then,*

$$\chi_c(G) \le \Delta^*(G) + 2\,.$$

Finally, in 2016, Dvořák et al. [92] proved Conjecture 5.1.7 also for the cases when $\Delta^*(G) \in \{16, 17\}$.

Theorem 5.1.9 (Dvořák et al. [92]). *Let G be a 3-connected plane graph with $\Delta^*(G) \in \{16, 17\}$. Then*

$$\chi_c(G) \leq \Delta^*(G) + 2\,.$$

Observe that in the case when $\Delta^*(G) \in \{3, 4\}$, Conjecture 5.1.7 coincides with the Cyclic Coloring Conjecture which is known to be true. It follows that the rest of the cases left open are the cases when $\Delta^*(G) \in \{5, 6, \ldots, 15\}$. For these, the best result known thus far was obtained by Enomoto and Horňák [99] in 2009, who proved that the cyclic chromatic number of any 3-connected plane graph is at most $\Delta^*(G) + 5$.

Theorem 5.1.10 (Enomoto and Horňák [99]). *Let G be a 3-connected plane graph. Then*

$$\chi_c(G) \leq \Delta^*(G) + 5\,.$$

Another partial result towards solving the Conjecture 5.1.7 was obtained by
Kriesell [155], who proved that $\Delta^*(G) + 2$ colors suffice for the case of locally connected 3-connected plane graphs. A graph is *locally connected* if, for every vertex v of G, the graph $G[N(v)]$ induced on $N(v)$ is connected.

Theorem 5.1.11 (Kriesell [155]). *Let G be a locally connected 3-connected plane graph. Then,*

$$\chi_c(G) \leq \Delta^*(G) + 2\,.$$

In the same paper Kriesell posed the following conjecture in line with Conjecture 5.1.7.

Conjecture 5.1.12 (Kriesell [155]). Let G be a locally connected 3-connected plane graph. Then,

$$\chi_c(G) \leq \Delta^*(G) + 1\,.$$

5.2 ℓ-facial (vertex) coloring

An *ℓ-facial coloring* is a vertex coloring of a plane graph such that any two vertices incident with the same face at distance at most ℓ on that face receive distinct colors. The minimum number of colors needed for an ℓ-facial coloring of a plane graph G is called the *ℓ-facial chromatic number* and is denoted by $\chi_\ell(G)$.

The ℓ-facial coloring was first introduced in 2005 by Kráľ, Madaras, and Škrekovski [154], where they posed the following conjecture (of a similar flavor as the Cyclic Coloring Conjecture).

Conjecture 5.2.1 (Kráľ et al. [154]). Let G be a plane graph and $\ell \geq 1$. Then

$$\chi_\ell(G) \leq 3\ell + 1\,.$$

Note also that if true, then the bound is tight as can be seen by the plane embeddings of K_4, in which three edges with a common vertex are subdivided $\ell - 1$ times, see Figure 5.3.

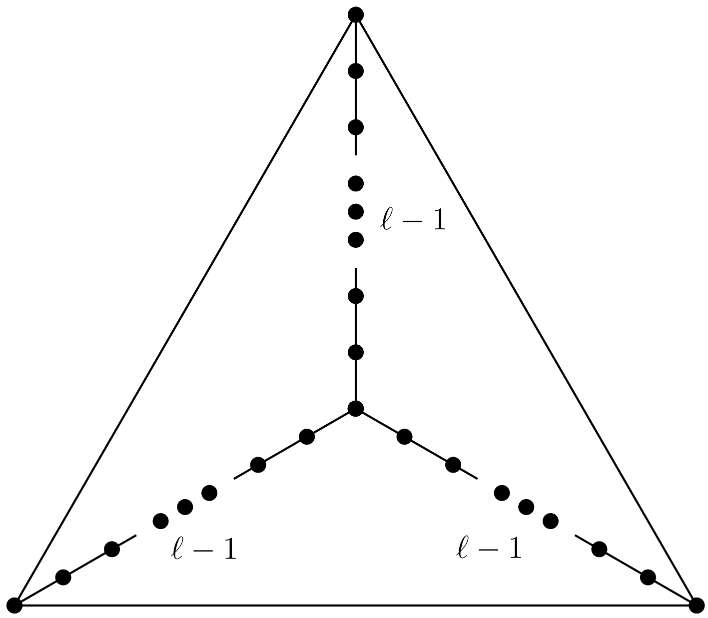


Figure 5.3: A plane graph G which is a plane embedding of a K_4 with three edges sharing a common vertex subdivided exactly $\ell - 1$ times and with $\chi_\ell(G) = 3\ell + 1$.

Conjecture 5.2.1 is known under two names, the $(3\ell{+}1)$-Conjecture and the Facial Coloring Conjecture. By the definition, given a plane graph G with $\Delta^*(G) \leq 2\ell + 1$, then every cyclic coloring of G is also an ℓ-facial coloring of G and vice versa. Thus, when $\Delta^*(G) = 2\ell + 1$, the Facial Coloring Conjecture implies the Cyclic Coloring Conjecture (Conjecture 5.1.7). In addition, unlike for the cyclic coloring we no longer have dependency on the length of the longest face, but rather only on the choice of ℓ. This implies that the ℓ-facial coloring can be seen as a generalization of the cyclic coloring to the class of all plane graphs.

First, observe that the ℓ-facial coloring is well defined even in the case when $\ell = 0$ which would also satisfy the conjectured bound of $3\ell{+}1$ (when $\ell = 0$ we can color all the vertices with a single color). Furthermore, the ℓ-facial coloring generalizes the proper coloring. As such, for $\ell = 1$, the correctness of the Facial Coloring Conjecture is implied by the Four Color Theorem. Thus, one only needs to consider the cases when $\ell \geq 2$. The first results in this direction were obtained by the same authors who introduced the ℓ-facial coloring over two papers (see Král' et al. [153, 154]).

Theorem 5.2.2 (Kráľ et al. [153, 154]). *Let G be a plane graph. Then,*

$$\chi_\ell(G) \leq \begin{cases} \frac{18\ell}{5} + 2 & \textit{for } \ell \geq 5, \\ 15 & \textit{for } \ell = 4, \\ 12 & \textit{for } \ell = 3, \\ 8 & \textit{for } \ell = 2. \end{cases}$$

Later, in 2008, Havet, Sereni, and Škrekovski [127] reduced the upper bound in the case when $\ell = 3$ to 11 (to this day this result is the best known for $\ell = 3$).

Theorem 5.2.3 (Havet et al. [127]). *Let G be any plane graph and let $\ell = 3$. Then,*

$$\chi_\ell(G) \leq 11\,.$$

Note that for $\ell = 3$, taking $\Delta^*(G)$ to be at most 7, Theorem 5.2.3 has the following corollary for the cyclic coloring.

Corollary 5.2.4 (Havet et al. [127]). *Let G be a plane graph with $\Delta^*(G) \leq 7$. Then,*

$$\chi_{\mathrm{c}}(G) \leq 11\,.$$

Havet et al. [126] further improved results from Theorem 5.2.2 for the cases $\ell = 45$, $\ell = 47$, and $\ell \geq 49$ by proving the following result.

Theorem 5.2.5 (Havet et al. [126]). *Let G be a plane graph and let $\ell \geq 1$. Then,*

$$\chi_\ell(G) \leq \left\lfloor \frac{7\ell}{2} \right\rfloor + 6\,.$$

To this day the Facial Coloring Conjecture remains widely open. In the case of triangle-free planar graphs, however, Dvořák, Škrekovski, and Tancer [97] posed the following conjecture.

Conjecture 5.2.6 (Dvořák et al. [97]). Let G be a triangle-free plane graph and let $\ell \geq 1$. Then,

$$\chi_\ell(G) \leq 3\ell\,.$$

Note that Conjecture 5.2.6 is in line with the Grötzsch Theorem, which implies the case when $\ell = 1$ just as the Four Color Theorem implies the case $\ell = 1$ for the Facial Coloring Conjecture. In addition, if true, the bound in Conjecture 5.2.6 is tight as can be seen by Figure 5.4.

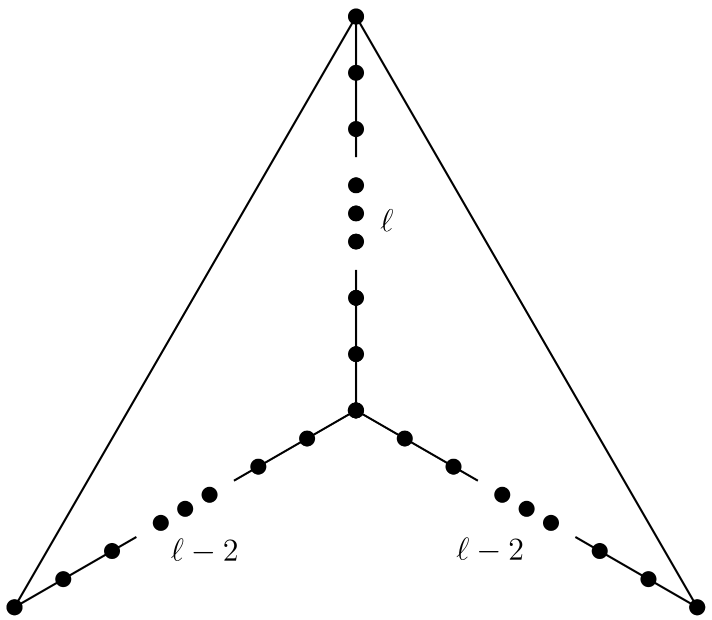


Figure 5.4: A triangle-free plane graph G with $\chi_\ell(G) = 3\ell$.

5.3 ℓ-facial edge coloring

An *ℓ-facial edge coloring* of a plane graph G is a not necessarily proper edge-coloring of G such that any two edges incident with the same face at distance at most ℓ on that face receive distinct colors. The minimum number of colors needed for an ℓ-facial edge coloring of G is called the *ℓ-facial chromatic index* and is denoted by $\chi'_\ell(G)$.

The ℓ-facial edge coloring was first introduced in 2015 by Lužar, Mockovčiaková, Soták, Škrekovski, and Šugerek [162] as the edge coloring counterpart to the ℓ-facial vertex coloring. The authors proposed the following conjecture named the Facial Edge-Coloring Conjecture.

Conjecture 5.3.1 (Lužar et al. [162]). Every plane graph admits an ℓ-facial edge-coloring with at most $3\ell + 1$ colors for every $\ell \geq 1$.

Observe that the conjectured upper bound for the ℓ-facial chromatic index is the same as for the ℓ-facial chromatic number. Moreover, if true, then the bound is tight as can be seen by the plane embedding of the Theta graph $\Theta_{\ell,\ell-1,\ell-1}$ (see Figure 5.5).

Note also that an ℓ-facial edge-coloring of a plane graph G corresponds to an ℓ-facial coloring of the medial graph $M(G)$ of G. Since $M(G)$ is a plane pseudograph the case $\ell = 1$ is implied by the Four Color Theorem (note that loops and parallel edges do not affect the vertex coloring).

In the same introductory paper [162], the authors confirmed the Facial-Edge Coloring Conjecture (Conjecture 5.3.1) for the case $\ell = 2$.

Theorem 5.3.2 (Lužar et al. [162]). *Every plane graph admits a 2-facial edge-coloring with at most 7 colors.*

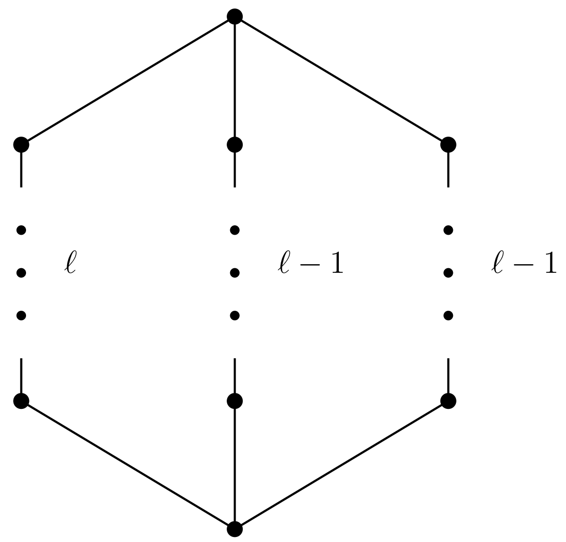


Figure 5.5: A Theta graph $G = \Theta_{\ell,\ell-1,\ell-1}$ with $\chi'_\ell(G) = 3\ell + 1$.

In the next section, we prove that the Facial-Edge Coloring Conjecture holds also for the case $\ell = 3$.

Theorem 5.3.3 (Theorem 1 in [138]). *Every plane graph admits a* 3*-facial edge-coloring with at most* 10 *colors.*

5.4 Proof of Theorem 5.3.3

To prove Theorem 5.3.3, we in fact prove a slightly stronger result, namely, we prove the theorem for plane pseudographs. Towards this end, we first prove several structural properties of a minimal counterexample with respect to the number of vertices that does not admit a 3-facial edge coloring with at most 10 colors. In the end we use the discharging method to show that such a graph does not exist. For the sake of completeness, in what follows, we give the necessary properties needed for the discharging proof and their proofs as they appear in [138].

Structure of a minimal counterexample

In what follows, instead of always saying a 3-facial edge coloring, we will instead shorten this phrase to 3-FEC. We first show that G does not contain cut-vertices.

Lemma 5.4.1. *G is* 2*-connected.*

Proof. Suppose the contrary and let v be a cut vertex of G. There exists a component H of $G - v$ such that the vertex v is in the subgraph G_1 of G induced by the vertex set $V(H) \cup \{v\}$ incident with the unbounded face. Let G_2 be the subgraph of G induced by the vertex set $V(G) \setminus V(H)$. By the minimality of G, there exist a 3-FEC σ_1 of G_1 and a 3-FEC σ_2 of G_2 with the same set C of at most 10 colors.

Consider the set E_1^1 of edges of the unbounded face of G_1 that are incident with v (note that $1 \leq |E_1^1| \leq 2$) and the set E_1^j of edges of G_1 that are in G_1 at facial-distance $j-1$ from the closest edge of E_1^1, for $j = 2, 3$. Furthermore, consider the set E_2^j of edges of G_2 that are in G at facial-distance j from the closest edge of E_1^1, for $j = 1, 2, 3$. For the set C_i^j of colors of edges in E_i^j we have $|C_i^j| \leq 2$, and we may assume without loss of generality that $|C_1'| \leq |C_2'|$ for $C_i' = C_i^1 \cup C_i^2 \cup C_i^3$, for $i = 1, 2$. If $|C_1'| + |C_2'| \leq 10$, then (again without loss of generality) $C_1' \cap C_2' = \emptyset$ and so the common extension of σ_1 and σ_2 is a 3-FEC of G with the set of colors C, a contradiction.

So, $5 \leq |C_1'| \leq 6$, $|C_2'| = 6$, $|C_2^1| = |C_2^2| = |C_2^3| = 2$ and $E_2^3 = \{e_1, e_2\}$. Let $C(e)/C[c]$ for $e \in E(G_2)/c \in C$ be the color class of σ_2 containing the edges of G_2 colored with $\sigma_2(e)/c$. Since $1 \leq p = |\,\{\sigma_2(e_1)\} \cup \{\sigma_2(e_2)\}\,| \leq 2$, and the color set $C^* = C \setminus (C_1^1 \cup C_1^2 \cup C_1^3 \cup C_2^1 \cup C_2^2)$ is of size at least $10 - 4 \cdot 2 \geq p$, there is a p-element set $\{c_j : j \in [1, p]\} \subseteq C^*$. Now recolor G_2 using the permutation π of C that induces the permutation of color classes of σ_2, under which the color classes $C(\sigma_2(e_j))$ and $C[c_j]$ are interchanged for each $j \in [1, p]$, and all remaining color classes are fixed. It is easy to see that the common extension of σ_1 and $\pi \circ \sigma_2$ is a 3-FEC of G with the color set C, a contradiction. □

From Lemma 5.4.1 we can infer that G contains no pendant vertices.

Corollary 5.4.2. *The minimum degree of G is at least 2.*

In addition, using Lemma 5.4.1, we can show that G contains no loops.

Lemma 5.4.3. *G is loopless.*

Proof. Suppose, to the contrary, that there is a loop e in G. If e bounds a 1-face, then it is 3-facially adjacent to at most 6 edges in G and thus we obtain a 3-FEC with at most 10 colors of G by removing e, coloring the obtained graph, and finally coloring e with one of at least 4 available colors. On the other hand, if e does not bound a 1-face, then its unique endvertex is a cut vertex in G, a contradiction to Lemma 5.4.1. □

It follows that in G every k-face is incident with k distinct vertices and with k distinct edges. In the rest, we mainly state properties regarding 2-vertices and small faces in G.

Lemma 5.4.4. *A 4-vertex in G has at most three 2-neighbors.*

Proof. Suppose the contrary and let v be a 4-vertex adjacent to four 2-vertices v_1, v_2, v_3 and v_4 in a clockwise order. Let v_{i+4} be the other neighbor of v_i, for $1 \leq i \leq 4$. Let G' be the graph obtained from G by deleting the vertices v, v_1, v_2, v_3 and v_4. By the minimality of G, there exists a 3-FEC coloring σ of G' with at most 10 colors. Notice that each of the edges vv_i and v_iv_{i+4} has at least 4 available colors. Let X_j, $1 \leq j \leq 8$ be a variable associated with the edge vv_j if $j \leq 4$ and the edge $v_{j-4}v_j$ otherwise. Let us now define the following polynomial, simulating the conflicts between the non-colored edges:

$$\begin{aligned} F(X_1,\ldots,X_8) =& (X_1 - X_2)(X_1 - X_4)(X_1 - X_5)(X_1 - X_6)(X_1 - X_8) \\ &\cdot (X_2 - X_3)(X_2 - X_5)(X_2 - X_6)(X_2 - X_7)(X_3 - X_4) \\ &\cdot (X_3 - X_6)(X_3 - X_7)(X_3 - X_8)(X_4 - X_5)(X_4 - X_7) \\ &\cdot (X_4 - X_8)(X_5 - X_6)(X_5 - X_8)(X_6 - X_7)(X_7 - X_8). \end{aligned}$$

The coefficient of the monomial $X_1^3X_2^3X_3^3X_4^3X_5^2X_6^2X_7^2X_8^2$ in $F(X_1,\ldots,X_8)$ is equal to 6[1], and thus by Theorem 3.2.3 we can extend the coloring σ to the coloring of G using at most 10 colors. $\square$

Let us now prove a lemma that we will require in the proofs later on.

Lemma 5.4.5. *There is no separating cycle of length at most* 7 *in* G.

Proof. Suppose the contrary and let C be a separating cycle of length at most 7. Let G_1 be the subgraph of G induced by the vertex set $V(\text{int}(C)) \cup V(C)$ and let G_2 be the subgraph of G induced by the vertex set $V(\text{ext}(C)) \cup V(C)$. By the minimality of G, there exists a 3-FEC σ_1 and a 3-FEC σ_2 of G_1 and G_2, respectively, using the same set of at most 10 colors. Notice that, since the length of C is at most 7, every edge of C is 3-facially adjacent to all the other edges of C in both G_1 and G_2. Thus, all the edges of C receive distinct colors in both σ_1 and σ_2. Hence, permuting the colors in σ_1 such that the colors of the edges of C coincide in σ_1 and in σ_2, results in a 3-FEC of G with at most 10 colors. $\square$

The next properties we are interested in are the absence of small faces and faces of length 8. In fact, using the same approach as in the paper, it is easy to show that, for every $\ell \geq 1$, in G every face is of length at least $\ell + 2$ and contains no face of length $2\ell + 2$.

Lemma 5.4.6. *Every face in* G *is of length at least* 5.

Proof. Suppose the contrary and let α be a face of G of length at most 4. Let $G' = G/\alpha$ and let, by the minimality of G, σ be a 3-FEC of G' using at most 10 colors. Next, observe that each edge of α is 3-facially adjacent to at most six edges of G' in G. Thus, each edge of α has at least 4 available colors. By Theorem 3.2.1, we can therefore extend the coloring σ to obtain a 3-FEC of G using at most 10 colors. $\square$

The way we prove the following Lemma 5.4.7 and several other lemmas is by identifying two edges of the same face α which are not in conflict in G such that the resulting graph remains planar.

[1] We verified the values of the coefficients with a computer program (also in the other proofs).

Lemma 5.4.7. *There are no* 8*-faces in* G.

Proof. Suppose the contrary and let α be an 8-face in G and e and f be two edges at facial-distance 4 on α. Let G' be the graph obtained from G by identifying the edges e and f and let σ be a 3-FEC of G' using at most 10 colors. Observe that the edges e and f are not 3-facially adjacent in G, otherwise G would contain either a separating cycle of length at most 5 (contradicting Lemma 5.4.5) or a 3-face (contradicting Lemma 5.4.6). Therefore, after we uncolor every edge of α distinct from e and f, σ induces a partial 3-FEC of G in which the edges e and f receive the same color.

To extend the coloring σ to a coloring of G, notice that all six non-colored edges of α have at least 3 available colors. Furthermore, among those edges there are exactly three distinct pairs of edges at facial-distance 4. If we can color any such pair with the same color, then the remaining four edges will each have at least 2 available colors. Furthermore, each of them is at facial-distance at most 3 from exactly two other non-colored edges. Applying Theorem 3.2.1, we obtain a 3-FEC using at most 10 colors. Therefore, we may assume that the union of available colors of any such pair is of size at least 6, with each edge having at least 3 available colors. Thus, we can extend the coloring σ to a 3-FEC of G by Theorem 3.2.2. □

With the following two lemmas, we present properties of 2-vertices in G.

Lemma 5.4.8. *Every* 2*-vertex in* G *has at least one* 3^+*-neighbor.*

Proof. Suppose to the contrary that v is a 2-vertex with neighbors u_1 and u_2, both being 2-vertices. Let $G' = G/u_1v$ and let, by the minimality of G, σ be a 3-FEC of G' using at most 10 colors. Notice that facial-distances between the edges in G are at least the distances between them in G', and thus the coloring σ induces a partial 3-FEC of G in which only the edge u_1v is non-colored. However, there are only nine edges in the 3-facial-neighborhood of u_1v, and therefore at least one color is available for u_1v (to extend σ to a 3-FEC of G), a contradiction. □

Before continuing, let us recall that $A(e)$ denotes the set of available colors for the edge e.

Lemma 5.4.9. *Let* (u, v) *be a* 2*-thread in* G *incident with an* 8^+*-face* α. *Then, within facial-distance* 3 *on the face* α, *except from* u, v *is adjacent only to* 3^+*-vertices.*

Proof. Suppose the contrary and let a 2-thread (u, v) be 3-facially adjacent to a 2-vertex $w \in \{v_2, v_3\}$ of α. We use the labeling of vertices as depicted in Figure 5.6.

Let $G' = G/\{uu_1, uv, vv_1, v_1v_2, v_2v_3\}$ and let σ be a 3-FEC of G'. In the coloring of G induced by σ, regardless which is w, we have $|A(v_2v_3)| \geq 2$, $|A(v_1v_2)| \geq 2$, $|A(vv_1)| \geq 4$, $|A(uv)| \geq 4$, and $|A(uu_1)| \geq 3$. If $A(uu_1) \cap A(v_2v_3) \neq \emptyset$, then we color uu_1 and v_2v_3 with the same color

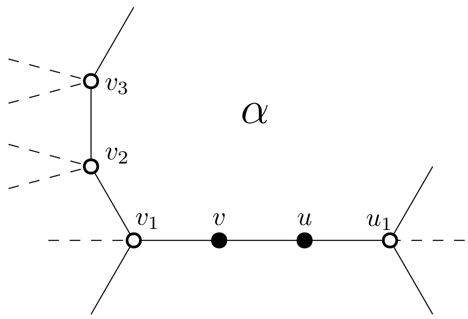


Figure 5.6: A reducible configuration with a 2-thread and a 2-vertex $w \in \{v_2, v_3\}$.

(recall that they are not 3-facially adjacent since α is an 8^+-face), and color the remaining three edges by Theorem 3.2.2.

On the other hand, if $A(uu_1) \cap A(v_2v_3) = \emptyset$, then in the union of available colors of the five non-colored edges we have at least 5 colors, and it is easy to see that again Theorem 3.2.2 can be applied to color all the edges of G, a contradiction. □

We now define $n_2^t(\alpha)$ to be the number of 2-vertices incident with a face α which belong to 2-threads. Let a *k-path* of a face α, $k \in \{2, 3^+\}$, be a maximal facial path in α composed of k-vertices. If we have $n_2(\alpha) > 0$, then we can partition the set $V(\alpha)$ of the vertices incident with α into sets $\{V^i \ : \ i = 1, \ldots, 2p\}$, for some positive integer p, such that each set V^{2i-1} induces a 2-path P^{2i-1} of α, and the set V^{2i} induces a 3^+-path P^{2i} of α that follows P^{2i-1} in the clockwise orientation of α for each $i = 1, \ldots, p$ (and P^1 follows P^{2p}). A *section* of α is a pair (V^{2i-1}, V^{2i}), $i = 1, \ldots, p$; the pair (V^{2i-1}, V^{2i}) is a *j-section* of α if $|V^{2i-1}| = j \in \{1, 2\}$ (see Lemma 5.4.8). Let $S_j(\alpha)$ denote the set of j-sections of α, $j = 1, 2$.

Corollary 5.4.10. *For a k-face α of G, where $k \geq 8$ and $n_2(\alpha) > 0$, we have*

$$n_2(\alpha) \leq \left\lfloor \frac{k}{2} \right\rfloor \qquad \text{and} \qquad |S_2(\alpha)| \leq \left\lfloor \frac{k - 2 \cdot |S_1(\alpha)|}{5} \right\rfloor .$$

Moreover, if $k = 11$ and $|S_2(\alpha)| > 0$, then $n_2(\alpha) \leq 4$.

Proof. Let $\{V^i \ : \ i = 1, \ldots, 2p\}$ be the partition of $V(\alpha)$ as defined in the above paragraph. If $(V^{2i-1}, V^{2i}) \in S_1(\alpha)$, then $|V^{2i}| \geq 1$. On the other hand, if $(V^{2i-1}, V^{2i}) \in S_2(\alpha)$, then, by Lemma 5.4.9, $|V^{2i}| \geq 3$. Therefore,

$$\begin{aligned} k &= \sum_{i=1}^{p} \left(|V^{2i-1}| + |V^{2i}|\right) \\ &\geq 2|S_1(\alpha)| + 5|S_2(\alpha)| \\ &= 2\big(|S_1(\alpha)| + 2|S_2(\alpha)|\big) + |S_2(\alpha)| . \end{aligned}$$

From above we infer that

$$n_2(\alpha) = |S_1(\alpha)| + 2|S_2(\alpha)| \le \frac{1}{2}\big(k - |S_2(\alpha)|\big) \le \frac{k}{2}\,,$$

implying

$$n_2(\alpha) \le \left\lfloor \frac{k}{2} \right\rfloor,$$

as well as

$$|S_2(\alpha)| \le \frac{1}{5}\big(k - 2|S_1(\alpha)|\big)\,,$$

implying

$$|S_2(\alpha)| \le \left\lfloor \frac{k - 2\cdot|S_1(\alpha)|}{5} \right\rfloor.$$

Now suppose that $k = 11$ and $|S_2(\alpha)| > 0$, which implies $|S_2(\alpha)| = q \in \{1, 2\}$, since otherwise $n_2(\alpha) \ge 2q \ge 6$ contradicting that $n_2(\alpha) \le \lfloor \frac{11}{2} \rfloor = 5$.

If $q = 1$, then the number of 3^+-vertices of α, that are in α within facial-distance 3 from a vertex of the 2-thread of α, is 6 (by Lemma 5.4.9). At most two of the three remaining vertices of α are 2-vertices (by Lemma 5.4.8), hence $|S_1(\alpha)| \le 2$ and $n_2(\alpha) \le 2 + 2 \cdot 1 = 4$.

If $q = 2$, consider an arbitrary vertex x of α that is not a part of a 2-thread of α. The vertex x is in α within facial-distance 2 from a vertex of at least one of the two 2-threads of α, and so, by Lemma 5.4.9, x is a 3^+-vertex. This reasoning leads to $n_2(\alpha) = 2 \cdot 2 = 4$. □

The next lemma was used in the proofs of several later lemmas. Lemma shows that the presence of 3-vertices may, in some cases, enable us to recolor certain edges.

Lemma 5.4.11. *Let uv be an edge with $d(u) = 3$, and let uu_1, uu_2 be the other two edges incident with u. Consider a partial 3-facial edge coloring of G, in which the edge uv is, and the edges uu_1, uu_2 are not colored. If $|A(uu_1) \cap A(uu_2)| = k$ for some $k \ge 3$, then there are at least $k-2$ colors in $A(uu_1) \cap A(uu_2)$ such that each can be used to recolor the edge uv in such a way that the result is again a partial 3-facial edge coloring of G.*

Proof. In the 3-facial-neighborhood of uv, there are at most two edges which are not 3-facially adjacent to uu_1 or uu_2, which means that there are at least $k - 2$ available colors for uv from the intersection $A(uu_1) \cap A(uu_2)$. □

We now continue by stating several properties regarding small vertices incident with small faces.

Lemma 5.4.12. *Every 5-face in G is incident only with 4^+-vertices.*

Proof. Suppose the contrary and let α be a 5-face of G incident with a 3-vertex v_1, where the vertices are labeled as in Figure 5.7. (Note that G is not the 5-cycle C_5.)

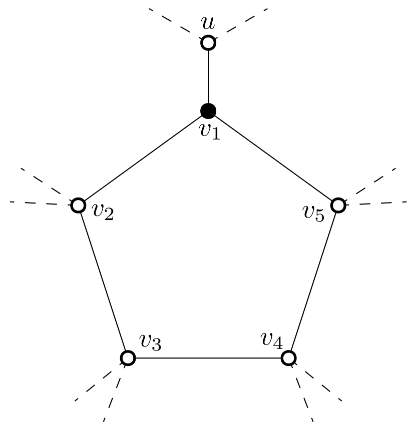


Figure 5.7: A reducible 5-face incident with a 3-vertex.

Let σ be a 3-FEC of $G' = G/\alpha$. It induces a partial 3-FEC of G with the five edges of α being non-colored. Each of the non-colored edges has at least 4 available colors. By Theorem 3.2.2, if the union of the five sets of available colors contains at least 5 distinct colors, then σ can be extended to G. Therefore, we may assume that $A(e)$ is the same for every $e \in E(\alpha)$, say $A(e) = [1, 4]$. In such a case, the face α is incident with 3^+-vertices only: if $d(v_i) = 2$ for some $i \in [2, 5]$, then both edges incident with v_i have at least 5 available colors. Now, we recolor the edge uv_1 with a color $j \in A(v_1v_2) \cap A(v_1v_5) = [1, 4]$. (By Lemma 5.4.11, there are at least 2 possibilities for the choice of j.) Recall that the 2-connected graph G contains neither 3-faces nor separating cycles of length at most 5. Thus, the edges v_2v_3, v_3v_4, and v_4v_5 are not within facial-distance 3 from the edge uv_1, and they retain $[1, 4]$ as the set of available colors. Furthermore, $\sigma(uv_1) \notin [1, 4]$ replaces j in the set of available colors for the edges v_1v_2 and v_1v_5. So, the coloring of G' can be extended to G using Theorem 3.2.2, a contradiction. □

Lemma 5.4.13. *Both neighbors of a* 2*-vertex incident with a* 6*-face in* G *are* 4^+*-vertices.*

Proof. We divide the proof in two parts. First, we show that a 2-vertex does not have a 2-neighbor, i.e., there is no 2-thread on a 6-face. Suppose the contrary and let α be a 6-face with an incident 2-thread (u, v) and let $G' = G/\alpha$. Then, G' admits a 3-FEC σ with at most 10 colors, which induces a 3-FEC of G with only the edges of α being non-colored. The three edges incident with the vertices u and v have at least 6 available colors, and the other edges of α have at least 4 available colors. It is easy to see that we can extend σ to all edges of G by applying Theorem 3.2.2, a contradiction.

Second, suppose that a 2-vertex v of a 6-face α is adjacent to a 3-vertex u. Let u_1 be the neighbor of u, distinct from v, which is incident with α, and u_2 the third neighbor of u. Again, consider $G' = G/\alpha$ and a 3-FEC σ of G' using at most 10 colors. In the coloring of G induced by σ, only the edges of α are non-colored. Every non-colored edge has at least 4 available colors, while the two edges incident with v have at least 5 available colors. Hence, if the set $A(\alpha)$ contains at least 6 colors, then we can apply Theorem 3.2.2 and we are done.

Thus, we may assume that $A(\alpha)$ contains precisely 5 colors, say $[1, 5]$. Notice that the sets of available colors on the edges of α not incident with v are not necessarily the same. However, since $|A(uv)| = 5$, the intersection of $A(e) \cap A(uv)$, for any $e \in E(\alpha)$, contains at least 4 colors. Therefore, from among at least 2 colors that can be used to recolor the edge uu_2 by Lemma 5.4.11, at least one, say j, appears in $A(e)$ for some $e \in E(\alpha) \setminus \{uu_1, uv\}$. Then, after recoloring the edge uu_2 with the color j, the new set of available colors for $E(\alpha)$ is of size 6, and we can apply Theorem 3.2.2 to find a 3-FEC of G with at most 10 colors, a contradiction. □

In addition, from Lemmas 5.4.6, 5.4.12, and 5.4.13 we may show the following two properties.

Corollary 5.4.14. *No 2-thread in G is incident with a 6^--face.*

Lemma 5.4.15. *A 2-vertex in G is incident with at least one 7^+-face.*

Proof. We again proceed by contradiction. Since 2-vertices are not incident with 5^--faces by Lemmas 5.4.6 and 5.4.12, suppose that there is a 2-vertex v in G incident with two 6-faces. Let $G' = G - v$. By the minimality, there is a 3-FEC σ of G' using at most 10 colors. Consider now the coloring of G induced by σ, in which only the two edges incident with v remain non-colored. Each of the two edges has at least 2 available colors, so we can color them, and thus extend σ to all edges of G, a contradiction. □

It turned out that 7-faces are the most important obstruction to deal with in order to obtain the final proof. We now state several structural properties from perspective of 7^+-faces and their incidence with 2-vertices.

Lemma 5.4.16. *Every 2-thread incident with a 7-face in G has at least one 4^+-neighbor.*

Proof. Suppose the contrary and let α be a 7-face incident with a 2-thread (v_2, v_3), where the other neighbors of v_2 and v_3 (v_1 and v_4, respectively) are both 3-vertices. We label the vertices as depicted in Figure 5.8.

Let $G' = G/\alpha$ and let σ be a 3-FEC of G'. In the partial coloring of G induced by σ, only the edges of α are non-colored, and the number of available colors is at least 4 for arbitrary non-colored edge, while it is at least 6 for the three edges incident with v_2 and/or v_3. It is easy to verify that if the set $A(\alpha)$ of available colors contains at least 7 colors, then

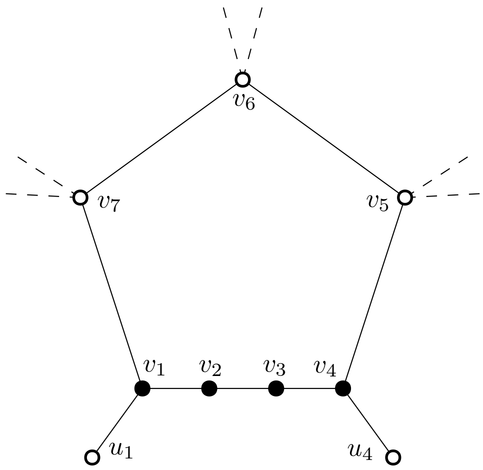


Figure 5.8: A reducible 7-face incident with a 2-thread with two 3-neighbors.

we can complete the coloring by Theorem 3.2.2. Thus we may assume that $|A(\alpha)| = 6$, say $A(\alpha) = [1, 6]$. Additionally, we may assume that $\sigma(u_1v_1) = 7$.

So, $|A(v_1v_2) \cap A(v_1v_7)| \geq 4$, and we can recolor u_1v_1 with a color from $I = A(v_1v_2) \cap A(v_1v_7) \cap A(u_1v_1)$, since $|I| \geq 2$ by Lemma 5.4.11. If there is an edge e of α which is not 3-facially adjacent to u_1v_1 and $A(e)$ contains a color from I, say 1, then we can recolor u_1v_1 with 1. The new set of available colors for $E(\alpha)$ is then $[1, 7]$, and hence we can apply Theorem 3.2.2 to find a contradictory 3-FEC of G.

Note that if $|I| \geq 3$, then we can always find a suitable edge e. Therefore, we may assume $|I| = 2$, say $I = \{1, 2\}$, hence $d(v_7) \geq 3$, and, by symmetry, $d(v_5) \geq 3$. Therefore, by Lemmas 5.4.6 and 5.4.8, there is no edge in the set $\{v_4v_5, v_5v_6, v_6v_7\}$ that is 3-facially adjacent to u_1v_1; thus, we have, say, $A(v_1v_7) = [1, 4]$ and $A(v_4v_5) = A(v_5v_6) = A(v_6v_7) = [3, 6]$. Analogously as above, at least two colors from $A(v_3v_4) \cap A(v_4v_5)$ can be used to recolor u_4v_4 by Lemma 5.4.11. Since the color of $A(\alpha)$ involved in the recoloring is still available for v_6v_7, the new set of available colors for $E(\alpha)$, namely $[1, 6] \cup \{\sigma(u_4v_4)\}$, is of size 7, a contradiction. □

Lemma 5.4.17. *A 2-thread in G is incident with at most one 7-face.*

Proof. Suppose the contrary and let (v_1, v_2) be a 2-thread incident with two 7-faces α and α'. Let $G' = G \setminus \{v_1, v_2\}$. By Lemma 5.4.7, there is a 3-FEC σ of G' using at most 10 colors such that two edges of the face in G' corresponding to the faces α and α' in G have the same color assigned. This means that in the coloring of G induced by σ, each of the three non-colored edges (the edges incident with the 2-thread) have at least 3 available colors, and therefore we can extend σ to all edges of G, a contradiction. □

Lemma 5.4.18. *Let α be a 7-face in G with a 2-thread (v_2, v_3) and at* *least one 2-vertex v distinct from v_2 and v_3. Then, every 2-vertex incident with α has a 2-neighbor and a 4^+-neighbor or two 4^+-neighbors.*

Proof. Suppose the contrary and let α be a 7-face with the vertices labeled as in Figure 5.9, with a 2-vertex incident with a 3-vertex. We present three possibilities (up to symmetry) for a neighboring 2-vertex and a 3-vertex; namely, in the case (a), there is a 3-neighbor of a 2-thread, and in the cases (b) and (c) a 3-neighbor of a 2-vertex v, which is not a part of the 2-thread (v_2, v_3). By Lemma 5.4.8, we may assume that $v \in \{v_5, v_6, v_7\}$.

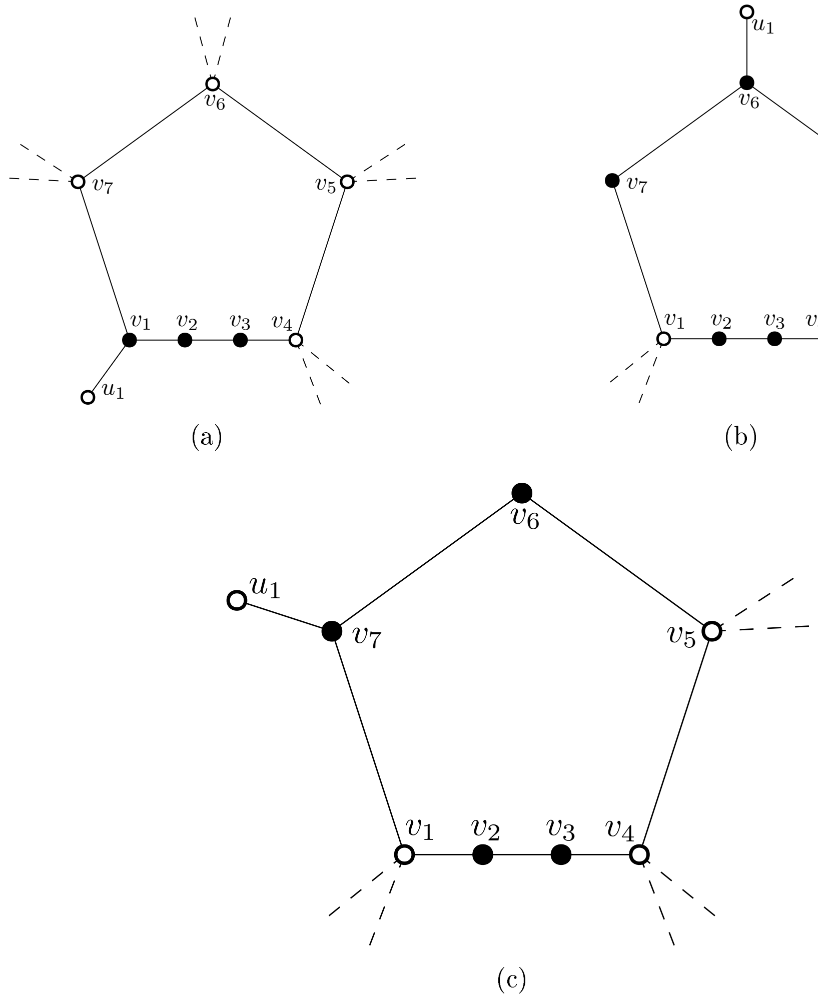


Figure 5.9: The three possible configurations of a 7-face incident with a 2-thread, a 2-vertex, and a 3-vertex.

We prove the lemma for all three cases at once. Suppose to the contrary that α (one of the three possible ones) exists in G. Let $G' = G/\alpha$ and let σ be a 3-FEC of G' with at most 10 colors. In the coloring of G induced by σ, only the edges of α are non-colored. Notice that the three edges incident with the 2-thread (v_2, v_3) have at least 6 available colors, the two edges incident with v have at least 5, and the remaining two edges have at least 4. From this it follows that if $|A(\alpha)| \geq 7$, then Theorem 3.2.2 applies, and we can color all the edges of α with a different color, hence extending σ to G, a contradiction.

So, we may assume that $|A(\alpha)| = 6$. Denote by v' the 3-vertex adjacent to u_1 (hence, $v' \in \{v_1, v_6, v_7\}$), and let v_1', v_2' be the two neighbors of v' on α. We claim that there exists an edge e' in α, which is not 3-facially adjacent to u_1v', such that $|A(v'v_1') \cap A(v'v_2') \cap A(e')| \geq 3$. Note first that by the above argument on the number of available colors, the intersection of available colors of any two edges of α, where at least one of them is incident with a 2-vertex, is at least of size 3. If v' is not v_1, then $e' = v_1v_2$ is not 3-facially adjacent to u_1v' by Lemma 5.4.5, and since $|A(v_1v_2)| = 6$, the claim follows. Otherwise, if $v' = v_1$, we may assume $v_1' = v_2$, and we choose $e' \in \{v_4v_5, v_6v_7\}$ in such a way that e' is incident with a 2-vertex. Similarly as above, since $|A(v_1v_2)| = 6$ and $|A(v'v_2') \cap A(e')| \geq 3$, the claim follows. Now, by Lemma 5.4.11, recoloring u_1v' with a color $c \in A(v'v_1') \cap A(v'v_2') \cap A(e')$ introduces the color $\sigma(u_1v') \notin A(\alpha)$ to the set of available colors for $E(\alpha)$. Since the color c is still available for e', the new 3-FEC of G' can be extended to G by Theorem 3.2.2, a contradiction. □

Let us remark that Lemma 5.4.18 does not forbid the existance of two 2-threads incident with a common 7-face.

Lemma 5.4.19. *If a 7-face α in G is incident with at least two 2-vertices but no 2-thread, then every 2-vertex incident with α has at least one 4^+-neighbor.*

Proof. Suppose the contrary and let α be a 7-face in G incident with at least two 2-vertices, where one of them, call it v_1, has two 3-neighbors. Note that by symmetry we may also assume that either v_3 or v_4 is a

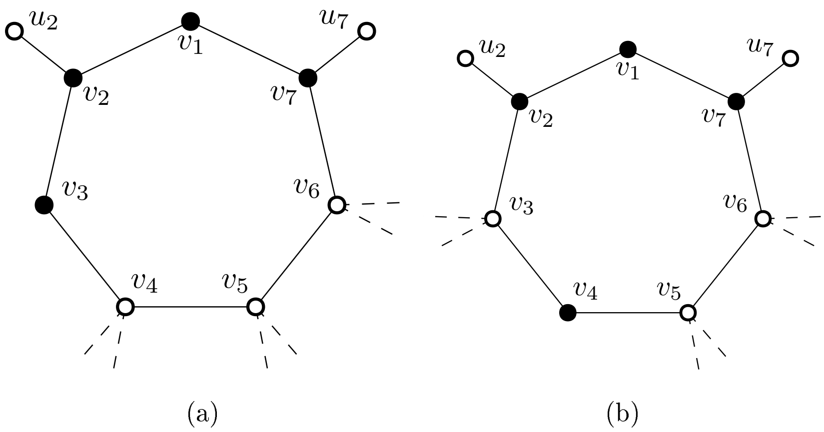


Figure 5.10: A 7-face with at least two incident 2-vertices, where one of them has two 3-neighbors.

2-vertex, hence there are two possibilities as depicted in Figure 5.10. Moreover, by Lemmas 5.4.5 and 5.4.12, v_4v_5 is 3-facially adjacent neither to u_2v_2 nor to u_7v_7, and so recoloring u_2v_2 and/or u_7v_7 does not change the set of available colors for v_4v_5.

Consider a 3-FEC σ of G/α using at most 10 colors. In G, σ induces a coloring with only the edges of α being non-colored. Every non-colored edge incident with a 2-vertex has at least 5 available colors and every other edge has at least 4 available colors. Moreover, for every two edges e_1 and e_2 of α which are both incident with the same 2-vertex, we have that $|A(e_1) \cap A(e_2)| \geq 4$. By assumption, there are at least two 2-vertices in α and thus at least four edges have at least 5 available colors. This implies that the union of available colors of every subset of k edges is of size at least k, for $k \leq 5$. We divide the proof into three cases regarding the number of colors in the union $A(\alpha)$.

Case (1): Suppose first that $|A(\alpha)| = 5$, say $A(\alpha) = [1,5]$. Then $A(v_1v_2) = A(v_1v_7) = A(\alpha)$. We may also assume that $\sigma(u_2v_2) = 6$ and $\sigma(u_7v_7) = 7$. We intend to recolor the edges u_2v_2 and u_7v_7 with two colors c_1 and c_2 from $A(\alpha)$ such that after recoloring, c_1 and c_2 will still be available colors for some edges of α, and so the colors of $[1,7]$ will be available for $E(\alpha)$.

By Lemma 5.4.11, u_2v_2 can be recolored with at least two colors from $A(v_1v_2) \cap A(v_2v_3)$. Since $|A(v_1v_2) \cap A(v_2v_3) \cap A(v_4v_5)| \geq 4$ in both cases depicted in Figure 5.10, we can recolor u_2v_2 with a color $c_1 \in A(v_1v_2) \cap A(v_2v_3) \cap A(v_4v_5)$, and thus make the color 6 available for $E(\alpha)$. Next, we recolor u_7v_7 with (possibly the only) color from $A(v_1v_7) \cap A(v_6v_7)$, and thus make the color 7 available for $E(\alpha)$. Note that then c_1 is still available for v_4v_5, c_2 is still available for v_3v_4, hence all colors of $[1,7]$ are available for $E(\alpha)$.

It remains to show that the union of available colors of any six edges of α contains at least 6 colors. Suppose this is not true, and there is $e' \in E(\alpha)$ such that the set A of available colors for $E(\alpha) \setminus \{e'\}$ is of size 5. Then, the set of available colors for e' contains exactly two colors $c_1', c_2' \in [1,7] \setminus A$, and it is easy to see that $\{c_1', c_2'\} = \{c_1, c_2\}$. However, u_2v_2 is colored with $c_1 \in A(v_4v_5)$, and u_7v_7 is colored with $c_2 \in A(v_3v_4)$, therefore e' can be neither v_4v_5 nor v_3v_4, a contradiction. Thus, by Theorem 3.2.2, we can color each non-colored edge with a distinct color from the set $[1,7]$, which provides a 3-FEC of G with at most 10 colors.

Case (2): Now, suppose that $|A(\alpha)| = 6$, say $A(\alpha) = [1,6]$. First, note that at most one of the edges u_2v_2 and u_7v_7 is colored with a color from $A(\alpha)$, otherwise $|A(\alpha)| \geq |A(v_1v_2)| + 2 \geq 7$. Observe that we can proceed in this way, since the cases **(3)**, when $|A(\alpha)| \geq 7$, and **(2)**, when $|A(\alpha)| = 6$, are analyzed independently from each other. So, we may assume that $\sigma(u_2v_2) = 7$ or $\sigma(u_7v_7) = 7$. We suppose the former, i.e., $\sigma(u_2v_2) = 7$, and note that the proof for the second case proceeds similarly, although not completely symmetrically, due to the assumption that one of the vertices v_3 and v_4 is a 2-vertex. We consider two cases regarding the color of u_7v_7.

Case (2.1): Suppose first that u_7v_7 is colored with a color from $A(\alpha)$, say $\sigma(u_7v_7) = 6$. Then, $A(v_1v_2) = A(v_1v_7) = A(\alpha) \setminus \{6\} = [1,5]$. We split this case further into two subcases, regarding which of the vertices v_3

and v_4 is a 2-vertex (recall that, by symmetry, we know precisely one of v_3, v_4 is of degree 2).

Case (2.1.1): If v_3 is a 2-vertex, then $|A(v_1v_2) \cap A(v_2v_3)| \geq 4$ and $|A(v_1v_2) \cap A(v_2v_3) \cap A(v_6v_7)| \geq 3$. Thus, by Lemma 5.4.11, we can recolor u_2v_2 with a color c_1 from $A(v_1v_2) \cap A(v_2v_3) \cap A(v_6v_7)$. By Lemmas 5.4.5 and 5.4.12, the set of available colors for v_6v_7 does not change. Therefore, the set of available colors for $E(\alpha)$ changes to $[1,7]$, and it only remains to show that any set $E \subseteq E(\alpha)$ with $|E| = 6$ has its set of available colors of size at least 6. So, suppose the contrary, and let $e \in E(\alpha)$ be such that the set of available colors for $E(\alpha) \setminus \{e\}$ is of size 5. This means that there are two colors in $[1,7]$ that are available only for e. Note that all colors of $[1,5] \setminus \{c_1\}$ are available for v_1v_2 and v_1v_7, and color 7 is available for v_1v_2, v_2v_3, v_3v_4, and v_1v_7. So, the above two colors must be c_1 and 6. However, since c_1 is available for v_6v_7, while 6 is not (recall that u_7v_7 is colored with 6), no edge $e \in E(\alpha)$ can have the required property, a contradiction. Hence, we can apply Theorem 3.2.2 to extend the present coloring of G/α to G.

Case (2.1.2): If v_4 is a 2-vertex, then v_3 is a 3^+-vertex. If there is a color c_1 from $A(v_1v_2) \cap A(v_2v_3)$ with which we can recolor u_2v_2 and c_1 is also in the set of available colors of some edge that is not 3-facially adjacent to u_2v_2, then we proceed as in the case **(2.1.1)**. So we may assume that u_2v_2 can only be recolored with a unique color, say, 1, meaning that $|A(v_1v_2) \cap A(v_2v_3)| = 3$, and, without loss of generality, $A(v_2v_3) = \{1,2,3,6\}$, $A(v_3v_4) = A(v_4v_5) = [2,6]$, $A(v_5v_6) \subset [2,6]$, and $A(v_6v_7) = [2,5]$. Now, by Lemma 5.4.11, there are at least two colors from $A(v_1v_7) \cap A(v_6v_7) \subseteq A(v_3v_4)$ to recolor u_7v_7, and we do it with color c_2. By Lemma 5.4.1, the edge u_2v_2 is incident with two distinct faces; let α_1 be that incident with v_1v_2, and let α_2 be the other one. Next, consider e_i^j, the j-th edge following u_2v_2 in the direction from v_2 to u_2 in α_i, for $i = 1,2$ and $j = 1,2,3$. Note that from $|A(v_2v_3)| = 4$ it follows that $d(u_2) \geq 3$, hence, by Lemma 5.4.1, $\{e_1^1, e_1^2, e_1^3\} \cap \{e_2^1, e_2^2, e_2^3\} = \emptyset$. As a consequence of $\sigma(u_2v_2) = 7$, $\sigma(u_7v_7) = 6$, and $A(v_1v_2) = [1,5]$, we have $\{\sigma(e_1^1), \sigma(e_1^2)\} \subseteq [8,10]$. Moreover, $6 \in A(v_2v_3)$, and so $6 \notin \{\sigma(e_2^1), \sigma(e_2^2)\}$. Finally, u_2v_2 can be recolored neither with 2 nor with 3; therefore, $\{\sigma(e_1^3), \sigma(e_2^3)\} = \{2,3\}$. The above reasoning shows that after recoloring u_7v_7 with c_2, we can recolor u_2v_2 with 6, which transforms the set of available colors for $E(\alpha)$ to $[1,7]$. As in the previous case, it remains to verify that any set $E \subseteq E(\alpha)$ with $|E| = 6$ has its set of available colors of size at least 6. This is true, since c_2 and 6 are available for v_3v_4 and v_4v_5, 7 is available for v_1v_2, v_2v_3, and v_1v_7, and each color of $[1,5] \setminus \{c_2\}$ is available for v_1v_2 and v_1v_7. Thus, again by Theorem 3.2.2, we can find a required 3-FEC of G.

Case (2.2): Now, suppose that u_7v_7 is colored with a color not in $A(\alpha)$, say $\sigma(u_7v_7) = 8$. We proceed as in the previous cases. If there is a color c_1 from $A(v_1v_2) \cap A(v_2v_3)$ with which we can recolor u_2v_2 and c_1 appears as an available color of a non-colored edge that is not 3-facially adjacent to u_2v_2, then we are done. Otherwise, we may assume that all 4 colors of $A(v_6v_7)$ are available for the edges v_3v_4, v_4v_5, and v_5v_6. Then recoloring u_7v_7 with a color from $A(v_1v_7) \cap A(v_6v_7)$ (at least one color for such a recoloring is guaranteed by Lemma 5.4.11) increases the size of the set of available colors for $E(\alpha)$ to 7. Again, every color, that is available for $E(\alpha)$, is available for at least two edges of α; thus, we can use Theorem 3.2.2 to obtain a contradiction as above.

Case (3): Finally, suppose that $|A(\alpha)| \geq 7$. To apply Theorem 3.2.2, we only need to show that any subset of $E(\alpha)$ of size 6 has the set of available colors of size at least 6. If this is not the case, there is a set $E \subseteq E(\alpha)$ of size 6 such that $|A(E)| = 5$, hence $E(\alpha) \setminus E = \{e\}$ implies $|A(e) \setminus A(E)| \geq 2$. Clearly, we have $|A(v_1v_2) \cap A(v_2v_3) \cap A(e')| \geq 3$ for every $e' \in E$. Pick an edge $e'' \in E$ that is not 3-facially adjacent to u_2v_2. By Lemma 5.4.11, at least one color from $A(v_1v_2) \cap A(v_2v_3) \cap A(e'')$ can be used to recolor u_2v_2. In this way, the size of the set of colors available for E increases to 6. Besides that, at least one color of $A(e) \setminus A(E)$ remains available for e, and so the set of available colors is of size at least 6 for any subset of $E(\alpha)$ of size 6 that contains e. Thus we can apply Theorem 3.2.2 again. □

As, by Lemma 5.4.7, there are no 8-faces in G, we can now consider 9- and 10-faces, respectively.

Lemma 5.4.20. *No* 9*-face in* G *is incident with a* 2*-vertex.*

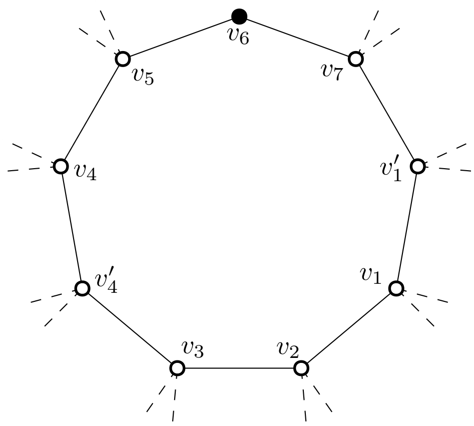


Figure 5.11: A reducible 9-face with an incident 2-vertex.

Proof. Suppose the contrary and let α be a 9-face incident with a 2-vertex. We label the vertices as depicted in Figure 5.11. Let G' be the graph obtained by identifying the edges v_1v_1' and v_4v_4', and let σ' be a 3-FEC of G' using at most 10 colors. From the coloring of G induced by σ' we create the coloring σ by uncoloring all edges of $E(\alpha) \setminus \{v_1v_1', v_4v_4'\}$. Observe that the edges v_1v_1' and v_4v_4' are not 3-facially adjacent in G, otherwise G would contain a separating cycle of length at most 7, or a 5-face with an incident

2-vertex, contradicting Lemma 5.4.5 or Lemma 5.4.12. Therefore, σ is a partial 3-FEC of G, in which the edges v_1v_1' and v_4v_4' receive the same color. Note that each of the edges $v_1v_2, v_2v_3, v_3v_4', v_4v_5, v_7v_1'$ has at least 3 available colors, while the two edges v_5v_6 and v_6v_7, incident with the 2-vertex v_6, have at least 4 available colors. Next, we associate with each edge of α distinct from v_1v_1' and v_4v_4' a variable X_i, $i \in \{1, \dots, 7\}$, in clockwise order starting from v_1v_2. To apply Theorem 3.2.3, we define the following polynomial:

$$\begin{aligned} F(X_1, \dots, X_7) =&(X_1 - X_2)(X_1 - X_3)(X_1 - X_6)(X_1 - X_7)(X_2 - X_3) \\ &\cdot(X_2 - X_4)(X_2 - X_7)(X_3 - X_4)(X_3 - X_5)(X_4 - X_5) \\ &\cdot(X_4 - X_6)(X_4 - X_7)(X_5 - X_6)(X_5 - X_7)(X_6 - X_7). \end{aligned}$$

The coefficient of the monomial $X_1^2X_2^2X_3^2X_4^2X_5^2X_6^3X_7^2$ in $F(X_1, \dots X_7)$ is equal to -3, thus by Theorem 3.2.3, we can extend the coloring σ to the 3-FEC of G using at most 10 colors. □

Lemma 5.4.21. *Every* 10*-face in* G *is incident with at most two* 2*-vertices.*

Proof. Suppose the contrary and let α be a 10-face in G incident with at least three 2-vertices. Let the vertices of α be labeled as depicted in Figure 5.12. We prove the lemma by considering three cases regarding the distances between 2-vertices. Namely, it suffices to show that the facial-distance in the face α between two 2-vertices does not belong to the set $\{1, 3, 4\}$. We do it by using Theorem 3.2.3, in which the variable X_i is associated with the edge v_iv_{i+1} for every $i \in [1, 9]$, and the variable X_{10} is associated with the edge v_1v_{10}.

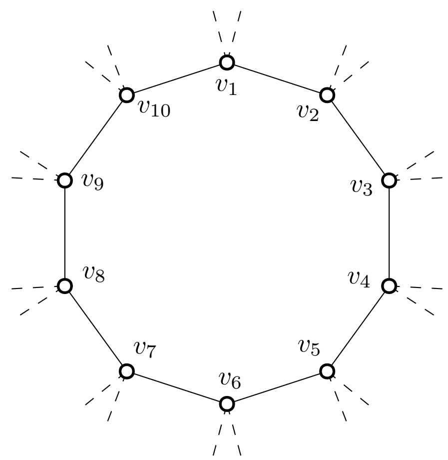


Figure 5.12: Labeling of the 10-face α.

Case (1): Suppose first that there are two adjacent 2-vertices in α, say v_1 and v_2. Consider the graph G_1' obtained from G by identifying the edges v_4v_5 and v_8v_9. It admits a 3-FEC σ_1' using at most 10 colors.

The coloring of G induced by σ_1' is not necessarily a 3-FEC. However, by Lemmas 5.4.5 and 5.4.8, v_4v_5 and v_8v_9 are not 3-facially adjacent in G, hence uncoloring the edges of $E(\alpha)\setminus\{v_4v_5, v_8v_9\}$ yields a partial 3-FEC σ_1 of G with $\sigma_1(v_4v_5) = \sigma_1(v_8v_9)$. Note that in this setting, the edges v_1v_2, v_2v_3, and v_1v_{10} have at least 5 available colors, and the other five edges of α have at least 3 available colors. Now, we define the polynomial:

$$\begin{aligned}F_1(X_1,\dots,X_{10}) =&(X_1-X_2)(X_1-X_3)(X_1-X_9)(X_1-X_{10})\\ &\cdot(X_2-X_3)(X_2-X_5)(X_2-X_9)(X_2-X_{10})\\ &\cdot(X_3-X_5)(X_3-X_6)(X_3-X_{10})(X_5-X_6)(X_5-X_7)\\ &\cdot(X_6-X_7)(X_6-X_9)(X_7-X_9)(X_7-X_{10})(X_9-X_{10})\,.\end{aligned}$$

Expanding it, we see that the coefficient of the monomial $X_1^4X_2^4X_3^2X_5^2X_6^1X_7^2X_{10}^3$ in $F_1(X_1,\dots,X_{10})$ is 1, and thus, by Theorem 3.2.3, we can extend σ_1 to G, a contradiction.

Case (2): Suppose now that there are 2-vertices at distance 3 in α, say v_1 and v_4. Consider the graph G_2' obtained from G by identifying the edges v_5v_6 and v_9v_{10}. It admits a 3-FEC σ_2' using at most 10 colors. By Lemmas 5.4.5 and 5.4.8, v_5v_6 and v_9v_{10} are not 3-facially adjacent, and so by uncoloring the edges of $E(\alpha)\setminus\{v_5v_6, v_9v_{10}\}$, we obtain a partial 3-FEC σ_2 of G with $\sigma_2(v_5v_6) = \sigma_2(v_9v_{10})$. Note that in this setting the edges v_1v_2, v_1v_{10}, v_3v_4, and v_4v_5 have at least 4 available colors, and the other four edges of α have at least 3 available colors. We define the polynomial:

$$\begin{aligned}F_2(X_1,\dots,X_{10}) =&(X_1-X_2)(X_1-X_3)(X_1-X_4)(X_1-X_8)(X_1-X_{10})\\ &\cdot(X_2-X_3)(X_2-X_4)(X_2-X_{10})\\ &\cdot(X_3-X_4)(X_3-X_6)(X_3-X_{10})(X_4-X_6)(X_4-X_7)\\ &\cdot(X_6-X_7)(X_6-X_8)(X_7-X_8)(X_7-X_{10})(X_8-X_{10})\,.\end{aligned}$$

Realizing that the coefficient of the monomial $X_1^3X_2^2X_3^2X_4^3X_6^2X_7^2X_8^1X_{10}^3$ in $F_2(X_1,\dots,X_{10})$ is -1, we infer that σ_2 can be extended to G by Theorem 3.2.3, a contradiction.

Case (3): Suppose now that there are 2-vertices at distance 4 in α, say v_1 and v_5. Note that the argument of the case **(2)** is also valid here, since the only difference is that the edge v_3v_4 may now have only 3 available colors. This is sufficient for applying Theorem 3.2.3 to extend σ_2 to G, since the exponent of X_3 in the above monomial of $F_2(X_1,\dots,X_{10})$ is 2. $\square$

Finally, we need to consider close 6- and 7-faces before we are ready to give the discharging proof.

Lemma 5.4.22. *If a 6-face α_1 and a 7-face α_2 of G share a 2-vertex v, and $u \neq v$ is a vertex of α_2, then $d(u) \geq 3$.*

Proof. Suppose the contrary and let $u \neq v$ be a 2-vertex of α_2. Observe that by Lemma 5.4.13, v is the only 2-vertex incident with both α_1 and α_2, thus u is either at facial-distance 2 or at facial-distance 3 from v. Consider now the graph $G' = G - v$. Note that the remaining edges incident with either α_1 or α_2 form a 9-face in G'. Label the edges of G according to Figure 5.11 with $v_6 = u$, and let σ' be a 3-FEC with at most 10 colors of the graph obtained from G' by identifying the edges $e = v_1v_1'$ and $e' = v_4v_4'$ (as in the proof of Lemma 5.4.20). One can easily observe that in any case, one of the edges e and e' is incident with α_1, while the other is incident with α_2. Thus the edges e and e' are not incident with a common face in G. It follows that the only conflict of the coloring σ of $G-v$ induced by σ' vanishes when the vertex v with its incident edges is added back to $G-v$. Finally, since at most 8 colors appear on the edges incident with α_1 and α_2, the two non-colored edges incident with v both have at least 2 available colors. Hence, we can extend σ to all edges of G, a contradiction. □

Lemma 5.4.23. *Let α_1 and α_2 be distinct 7-faces of G with a common 2-vertex v that has a 3-neighbor u and a 4^+-neighbor w. Furthermore, let u_1 and w_1 be the vertices of α_1 adjacent to u and w, respectively. Finally, let either $d(u_1) \geq 3$ and $e_1 \in E(\alpha_1) \setminus \{uu_1, uv, vw\}$ or $d(u_1) = 2$ and $e_1 = ww_1$, and let $e_2 \in E(\alpha_2) \setminus \{uv, vw\}$. Then, the edge e_1 is not 3-facially adjacent to the edge e_2.*

Proof. Suppose to the contrary that e_1 is within facial-distance 3 from e_2. First realize that the faces α_1 and α_2 share the vertices u, v, and w only (use Lemmas 5.4.6 and 5.4.12).

Let α be a face incident with both α_1 and α_2. Consider a facial path P of length $\ell \leq 4$ in α having the first edge e_1 and the last edge e_2. Note that $e_1 \in E(\alpha_1) \setminus E(\alpha_2)$ and $e_2 \in E(\alpha_2) \setminus E(\alpha_1)$, hence α is unique and $\ell \geq 2$. From $e_1 \neq e_2$ we infer that $|e_1 \cap e_2| \leq 1$. Moreover, $|e_1 \cap e_2| = 1$ yields $e_1 \cap e_2 = \{w\}$, which in turn means that the edges e_1 and e_2 are not facially adjacent to each other (since w is a 4^+-vertex). So, $e_1 \cap e_2 = \emptyset$ and $\ell \in \{3, 4\}$.

If $\ell = 3$, then the second edge of P is x_1x_2, where x_1 is a vertex of α_1, x_2 is a vertex of α_2, and the requirements on e_1 imply $u \notin \{x_1, x_2\}$. Moreover, from $d(w) \geq 4$ it follows that x_1x_2 is incident neither with α_1 nor with α_2. The faces α_1 and α_2 create in $G-v$ a 10-face $\alpha_{1,2}$ incident with ten distinct vertices and ten distinct edges. From the two facial paths joining x_1 to x_2 in $\alpha_{1,2}$, one does, and the other does not contain the vertex u; let P^+ be the former and P^- the latter one. Denote by ℓ^+ and ℓ^- the length of P^+ and P^-, respectively, and so $\ell^+ + \ell^- = 10$. Use P^+, P^-, and the edge x_2x_1 to construct cycles $C^+ = P^+x_1$ and $C^- = P^-x_1$. The sum of lengths of C^+ and C^- is $(\ell^+ + 1) + (\ell^- + 1) = 12$. We have $\min(\ell^+, \ell^-) = \ell^* \leq 5$, where $* \in \{+, -\}$. If C^* is a separating cycle, its existence contradicts Lemma 5.4.5. On the other hand, if C^* is not separating, it either bounds a face in G contradicting one of Lemmas 5.4.6, 5.4.12, 5.4.13, or it has at least one chord, which ultimately yields a contradiction to Lemma 5.4.6.

If $\ell = 4$, then $P = y_1x_1zx_2y_2$, where $e_1 = y_1x_1$, $e_2 = x_2y_2$, and $u \notin \{x_1, x_2\}$. Let P^+, P^-, ℓ^+, ℓ^-, and ℓ^* be defined as in the case $\ell = 3$, and let $C^+ = P^+zx_1$, $C^- = P^-zx_1$. Now, the sum of lengths of C^+ and C^- is $(\ell^+ + 2) + (\ell^- + 2) = 14$. Again by Lemma 5.4.5, the cycle C^* is not separating. If $\ell^* \leq 4$, a contradiction is reached as above. Finally, assume that $\ell^* = 5 = l^+ = l^-$. Since C^+ is not a separating cycle (we can choose $* = +$), and does not have a chord (this would contradict Lemma 5.4.6), it bounds a face in G, which is impossible by Lemmas 5.4.8 and 5.4.17. $\square$

Lemma 5.4.24. *Let α_1 and α_2 be two 7-faces in G that have a common 2-vertex v. If α_1 and α_2 have at least two incident 2-vertices each, then v has two 4^+-neighbors.*

Proof. Suppose the contrary and let u and w be distinct neighbors of v. By Lemmas 5.4.17 and 5.4.19, we may assume, without loss of generality, that $d(u) = 3$ and $d(w) \geq 4$. For both $i \in \{1, 2\}$ there is in the face α_i a neighbor $u_i \neq v$ and $w_i \neq v$ of u and w, respectively. Again without loss of generality, we may assume that $d(u_1) \geq d(u_2)$. Furthermore, let $\alpha_{1,2}$ be the 10-face in $G - v$ created from α_1 and α_2 (cf. the proof of Lemma 5.4.23).

Case (1): If $d(u_2) \geq 3$, there exist vertices $x \neq v$ and $y \neq v$ incident with α_1 and α_2, respectively, such that $d(x) = d(y) = 2$. Let G' be the graph obtained from $G - v$ by identifying the two edges incident with x and the two edges incident with u_2. By the minimality of G, there exists a 3-FEC σ' of G' using at most 10 colors. From the coloring of $G - v$ induced by σ' we obtain a coloring σ by uncoloring all edges of $\alpha_{1,2}$ that are incident neither with x nor with u_2. By Lemma 5.4.23, σ is a partial 3-FEC of G. Let E_i be the set of (all) three non-colored edges incident in G with the face α_i, for $i = 1, 2$. We can color the edges of E_1 and E_2 separately, i.e., when coloring the edges of E_i, we suppose that the edges of E_{3-i} are still non-colored, for $i = 1, 2$. For that purpose, note that the edge $uu_1 \in E_1$ has at least 3 available colors, and there is an edge $e_2 \in E_2$ incident with y such that e_2 has at least 3 available colors as well. Furthermore, at least 2 colors are available for any other edge in $E_1 \cup E_2$. Therefore, by Theorem 3.2.2, the mentioned separate coloring of edges in $E_1 \cup E_2$ is possible, and, by Lemma 5.4.23, results in a 3-FEC of $G - v$, in which edges incident with $\alpha_{1,2}$ use at most 8 colors. Two of the remaining colors then suffice to color the edges uv and vw.

Case (2): If $d(u_1) \geq 3$ and $d(u_2) = 2$, there exists a 2-vertex x incident with α_1. Let G' be the graph constructed from $G - v$ by identifying the two edges incident with x and the two edges incident with w_2. By the minimality of G, there exists a 3-FEC σ' of G' using at most 10 colors. From the coloring of $G - v$ induced by σ' we obtain a coloring σ by uncoloring all edges of $\alpha_{1,2}$ that are incident neither with x nor with w_2. By Lemma 5.4.23, σ is a partial 3-FEC of G. Let the edge sets E_1 and E_2 be defined as in the case **(1)**. Color first the edges of E_1 by Theorem 3.2.2

noting that the number of available colors is at least 4 for the edge uu_1 and at least 2 for the remaining two edges. Next, color the three edges of E_2, again by Theorem 3.2.2, having in mind that the number of available colors is now at least 3 for the two edges incident with u_2 and at least 2 for the last edge. The edges uv and vw are then colored as before.

Case (3): If $d(u_1) = d(u_2) = 2$, let G' be created from $G - v$ by identifying the edges ww_1 and ww_2. By the minimality of G, there exists a 3-FEC σ' of G' using at most 10 colors. From the coloring of $G - v$ induced by σ' we obtain a coloring σ by uncoloring all edges of $\alpha_{1,2}$ not incident with w. By Lemma 5.4.23, σ is a partial 3-FEC of G, and, without loss of generality, we may assume that $\sigma(ww_1) = \sigma(ww_2) = 1$. Denote by E_i^+/E_i^- the set of non-colored edges incident in G with the face α_i that are/are not incident with u_i, for $i = 1, 2$. Notice that the number of available colors is at least 6 for any edge of $E_1^+ \cup E_2^+$ and at least 3 for any edge of $E_1^- \cup E_2^-$.

Suppose now that we are able to use the same color for an edge of E_1^+ and an edge of E_2^-. Then, by Theorem 3.2.1, σ is extendable to $G - v$. A similar extension is possible if the same color can be used either for an edge of E_1^- and an edge of E_2^+, or for the edge of E_1^- incident with w_1 and the edge of E_2^- incident with w_2. The final extension of the coloring of $G - v$ to G works as in the case **(1)**.

Thus, we may assume, without loss of generality, that the set of available colors is $[2, 4]$ for each edge in E_1^-, $[5, 10]$ for each edge in E_2^+, $[5, 7]$ for each edge in E_2^- and $[2, 4] \cup [8, 10]$ for each edge in E_1^+. This, however, leads to a contradiction: since in the facial path $u_2uu_1z_1z_2$ (where z_2 is necessarily not incident with α_1) the edge z_1z_2 has a color from $[5, 7]$, the set of available colors for $uu_2 \in E_2^+$ is not $[5, 10]$. □

Lemma 5.4.25. *Let α_1 and α_2 be two 7-faces in G that have a common 2-vertex v. If α_1 has at least three incident 2-vertices, then v is the only 2-vertex incident with α_2.*

Proof. Suppose the contrary and let v be a 2-vertex incident with 7-faces α_1 and α_2, where $n_2(\alpha_1) \geq 3$ and $n_2(\alpha_2) \geq 2$. By Lemma 5.4.24, both neighbors of v, v_1 and v_2, are 4^+-vertices. This implies that every pair of edges $e_1 \in E(\alpha_1)$ and $e_2 \in E(\alpha_2)$, which are not incident with v, are not 3-facially adjacent by Lemmas 5.4.5 and 5.4.8. Furthermore, by Lemma 5.4.8, we also have that there exist vertices u_1 and u_2 of α_1 and α_2, respectively, such that $u_1, u_2 \notin \{v, v_1, v_2\}$ and $d(u_1), d(u_2) \geq 3$.

Denote by $\alpha_{1,2}$ the face of the graph $G - v$ created from the faces α_1 and α_2. Let G' be the graph obtained from $G - v$ by identifying the two edges incident with u_1 and the two edges incident with u_2. By the minimality of G, there exists a 3-FEC σ' of G' using at most 10 colors. From the coloring of $G - v$ induced by σ' we obtain a coloring σ by uncoloring all edges of $\alpha_{1,2}$ that are incident neither with u_1 nor with u_2. By mimicking the proof of Lemma 5.4.23, we show that if e_i is any edge of $\alpha_{1,2}$ incident in G with the face α_i, for $i = 1, 2$, then e_1 is not 3-facially adjacent (in G) to e_2: since $d(v_1) \geq 4$ and $d(v_2) \geq 4$, the cycle C^* of length at most 7 from the mentioned proof either is separating or has a chord, in both cases we obtain a contradiction. So, σ is a partial 3-FEC of $G - v$.

Let us extend σ to a 3-FEC of $G-v$. For that purpose consider in the face $\alpha_{1,2}$ a 2-vertex $w_1 \neq v$ and a 2-vertex $w_2 \neq v$ that is in G incident with the face α_1 and α_2, respectively. If the sets of edges E_1 and E_2 are defined as in the proof of Lemma 5.4.24, case **(1)**, the edges of E_1 and those of E_2 can be colored separately (using Theorem 3.2.2). Indeed, $|E_i| = 3$, while the number of available colors is at least 3 for any (at least one) edge of E_i incident with w_i and at least 2 for any of the remaining edges of E_i, for $i = 1, 2$.

The number of available colors is now at least 2 for both non-colored edges vv_1, vv_2 of $G - v$, hence there is a 3-FEC of G using at most 10 colors, a contradiction. □

Discharging proof

We are now able to give a complete proof using the discharging procedure to show that there are no plane graphs satisfying all of the above stated properties. We begin by assigning an initial charge (denoted by ch_0) to all the vertices and faces of G as follows. For every vertex $v \in V(G)$, we set

$$\mathrm{ch}_0(v) = 2d(v) - 6\,,$$

and for every face $\alpha \in F(G)$, we set

$$\mathrm{ch}_0(\alpha) = \ell(\alpha) - 6\,.$$

By Euler's Formula the sum of all initial charges, is

$$\begin{aligned}\sum_{v\in V(G)} \mathrm{ch}_0(v) + \sum_{\alpha\in F(G)} \mathrm{ch}_0(\alpha) &= \sum_{v\in V(G)} \big(2d(v) - 6\big) + \sum_{\alpha\in F(G)} \big(\ell(\alpha) - 6\big) \\ &= -12\,.\end{aligned}$$

Next, we apply the following discharging rules to redistribute the charges between the vertices and faces of G.

R_1 Every 4^+-vertex sends $\frac{1}{5}$ to every incident 5-face.

R_2 For each pair v and u, where v is a 4^+-vertex and u is a 2-vertex adjacent to v and incident with faces α_1 and α_2 (note that $\alpha_1 \neq \alpha_2$ by 2-connectivity of G), a charge is sent according to the following (without loss of generality, we may assume that $\ell(\alpha_1) \leq \ell(\alpha_2)$ and if $\ell(\alpha_1) = \ell(\alpha_2)$, then $n_2(\alpha_1) \geq n_2(\alpha_2)$):

(*a*) If $\ell(\alpha_1) = 6$, then v sends $\frac{2}{3}$ to α_1.

(b) If $\ell(\alpha_1) = \ell(\alpha_2) = 7$ and $n_2(\alpha_1) = n_2(\alpha_2) = 2$, then v sends $\frac{1}{3}$ to α_1 and $\frac{1}{3}$ to α_2.

(c) If $\ell(\alpha_1) = \ell(\alpha_2) = 7$, $n_2(\alpha_1) \geq 2$, and $n_2(\alpha_2) = 1$, then v sends $\frac{2}{3}$ to α_1.

(d) If $\ell(\alpha_1) = 7$ and $\ell(\alpha_2) \geq 8$, then v sends $\frac{2}{3}$ to α_1.

R_3 Every face sends 1 to every incident 2-vertex that is not a part of a 2-thread.

R_4 Every 7-face sends $\frac{5}{6}$ to every incident 2-vertex that is a part of a 2-thread.

R_5 Every 8^+-face sends $\frac{7}{6}$ to every incident 2-vertex that is a part of a 2-thread.

We are now ready to complete our proof of Theorem 5.3.3.

Proof of Theorem 5.3.3. Clearly, the redistribution of charges does not change the total charge of G. So,

$$\sum_{v \in V(G)} \mathrm{ch_f}(v) + \sum_{\alpha \in F(G)} \mathrm{ch_f}(\alpha) = -12\,, \tag{5.1}$$

where $\mathrm{ch_f}(v)/\mathrm{ch_f}(\alpha)$ stands for the final charge (the "local" result of the charge redistribution) of a vertex v/a face α of G. We are going to show that final charges of vertices and faces of G are all nonnegative. This will mean that the total final charge of G is nonnegative too in contradiction to (5.1).

We first show that each vertex $v \in V(G)$ has a nonnegative final charge. In particular, since by Lemma 5.4.2 there are no 1-vertices in G, and 3-vertices have initial charge 0 while not sending any charge, we only consider 2-vertices and 4^+-vertices.

• Suppose first that v is a 2-vertex in G, incident with faces α_1 and α_2. Without loss of generality, we assume $\ell(\alpha_1) \leq \ell(\alpha_2)$. If v is not a part of a 2-thread, then it receives 1 from each of α_1 and α_2 by R_3. Hence, $\mathrm{ch_f}(v) = 2d(v) - 6 + 2 \cdot 1 = 0$. If v is a part of a 2-thread, then $\ell(\alpha_1) \geq 7$ by Corollary 5.4.14. Moreover, by Lemma 5.4.17, $\ell(\alpha_2) \geq 8$, and thus by R_5, v receives $\frac{7}{6}$ from α_2. On the other hand, v receives at least $\frac{5}{6}$ from α_1 by R_4 or R_5. Hence, $\mathrm{ch_f}(v) \geq 2d(v) - 6 + \frac{5}{6} + \frac{7}{6} = 0$.

• Now, suppose that v is a 4^+-vertex. Note that, by Lemma 5.4.12, $i_5(v) + n_2(v) \leq d(v)$, where $i_5(v)$ is the number of 5-faces incident with v. Moreover, if $d(v) = 4$, then $n_2(v) \leq 3$ by Lemma 5.4.4, and if additionally $n_2(v) = 3$, then v is not incident with a 5-face by Lemma 5.4.12. Thus, if $d(v) = 4$, then v sends at most $3 \cdot \frac{2}{3}$ of charge by R_1 and/or R_2, and so $\mathrm{ch_f}(v) \geq 2d(v) - 6 - 3 \cdot \frac{2}{3} = 0$. If $d(v) \geq 5$, then v sends at most $\frac{2}{3}$ of charge for each of at most $d(v)$ adjacent 2-vertices by R_1 and/or R_2, and so $\mathrm{ch_f}(v) \geq 2d(v) - 6 - d(v) \cdot \frac{2}{3} > 0$. So, after redistribution of charges, all vertices in G have nonnegative final charges.

Next, we show that each face $\alpha \in F(G)$ has a nonnegative final charge. Again, we consider several cases, regarding the length of α. Recall that by Lemma 5.4.6, α is of length at least 5.

- Suppose that α is a 5-face in G. By Lemma 5.4.12, it is incident only with 4^+-vertices, and so it receives $5 \cdot \frac{1}{5}$ by R_1. Moreover, it does not send any charge, thus $\mathrm{ch_f}(\alpha) = \ell(\alpha) - 6 + 5 \cdot \frac{1}{5} = 0$.

- Suppose that α is a 6-face in G. By Lemma 5.4.13, every 2-vertex incident with α is adjacent to two 4^+-vertices. Thus, for every adjacent 2-vertex, α receives $2 \cdot \frac{2}{3}$ by $R_2(a)$, and sends 1 by R_3. Altogether, its final charge is $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 + 2n_2(\alpha) \cdot \frac{2}{3} - n_2(\alpha) = \frac{1}{3}n_2(\alpha) \geq 0$.

- Suppose that α is a 7-face in G. It sends charge to incident 2-vertices by R_3 and R_4, and it receives charge from incident 4^+-vertices by R_2. We consider the cases regarding incident 2-vertices. If $n_2(\alpha) \leq 1$, then, by R_3, $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 - n_2(\alpha) = 1 - n_2(\alpha) \geq 0$.

Now, suppose that α is incident with two 2-vertices v_1 and v_2, and let α_1 and α_2 be the faces incident with v_1 and v_2, respectively, that are distinct from α (possibly, $\alpha_1 = \alpha_2$). Then, by Lemma 5.4.22, none of these 2-vertices is incident with a 6-face. If v_1 and v_2 form a 2-thread, then, by Lemma 5.4.17, they are also incident with an 8^+-face. By Lemma 5.4.16, at least one of v_1 and v_2 has a 4^+-neighbor which sends $\frac{2}{3}$ to α by $R_2(d)$. On the other hand, α sends $\frac{5}{6}$ to each of v_1 and v_2 by R_4. Hence, $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 + \frac{2}{3} - 2 \cdot \frac{5}{6} = 0$. Thus, we may assume that v_1 and v_2 are not adjacent, and by Lemma 5.4.19, each of them has at least one 4^+-neighbor. If $i \in \{1, 2\}$ and $\ell(\alpha_i) = 7$, then, by Lemma 5.4.25, $n_2(\alpha_i) \leq 2$; if, moreover, $n_2(\alpha_i) = 2$, then, by Lemma 5.4.24, v_i has two 4^+-neighbors. Therefore, α receives at least $2 \cdot \frac{2}{3}$ by $R_2(b)$, $R_2(c)$, or $R_2(d)$, and sends $2 \cdot 1$ by R_3. Hence, $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 + 2 \cdot \frac{2}{3} - 2 \cdot 1 = \frac{1}{3}$.

Next, if α is incident with three 2-vertices, we distinguish two subcases. Suppose first that α is incident with a 2-thread. Then, by Lemma 5.4.18, each of the incident 2-vertices has at least one 4^+-neighbor, and by $R_2(c)$ and $R_2(d)$, α receives at least $3 \cdot \frac{2}{3}$ of charge (note that by Lemma 5.4.25, if $\ell(\alpha_1) = 7$, then α receives charge by $R_2(c)$). It sends 1 by R_3 and $2 \cdot \frac{5}{6}$ by R_4. Hence, $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 + 3 \cdot \frac{2}{3} - 1 - 2 \cdot \frac{5}{6} = \frac{1}{3}$. Similarly, if α is not incident with a 2-thread, then, by Lemma 5.4.19, each of the incident 2-vertices has at least one 4^+-neighbor, and by $R_2(c)$ and $R_2(d)$, α receives at least $3 \cdot \frac{2}{3}$ of charge. Since α sends $3 \cdot 1$ by R_3, its final charge is $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 + 3 \cdot \frac{2}{3} - 3 \cdot 1 = 0$.

Finally, suppose that α is incident with four 2-vertices. In this case, α

is incident with at least one 2-thread. Then, by Lemma 5.4.18, each 2-vertex incident with α has a 4^+-neighbor, and any 4^+-vertex incident with α sends $\frac{2}{3}$ to α by $R_2(c)$ or $R_2(d)$ for each of its 2-neighbors. Since α sends at most $2 \cdot \frac{5}{6}$ and $2 \cdot 1$ by R_4 and R_3, its final charge is $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 + 4 \cdot \frac{2}{3} - 2 \cdot \frac{5}{6} - 2 \cdot 1 = 0$.

- By Lemma 5.4.7, we can skip the assumption that α is an 8-face in G.

- Suppose that α is a 9-face in G. Then, by Lemma 5.4.20, α is incident with no 2-vertex, and hence $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 = 3$.

- Suppose that α is a 10-face in G. By Lemma 5.4.21, α is incident with at most two 2-vertices, and so it sends at most $2 \cdot \frac{7}{6}$ charge by R_3 or R_5. So, $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 - 2 \cdot \frac{7}{6} = \frac{5}{3}$.

- Suppose that α is an 11-face in G. Then, by Corollary 5.4.10, α is incident with at most five 2-vertices. If $n_2^t(\alpha) = 0$, then it sends charge only by R_3. Thus, $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 - 5 = 0$. If $n_2^t(\alpha) \geq 1$, then, by Corollary 5.4.10, $n_2(\alpha) \leq 4$. The charge from α is sent by R_3 and/or R_5, thus $\mathrm{ch_f}(\alpha) \geq \ell(\alpha) - 6 - 4 \cdot \frac{7}{6} = \frac{1}{3}$.

- Suppose that α is a k-face in G, $k \geq 12$. If $n_2^t(\alpha) = 0$, then α sends charge only by R_3, and so, by Corollary 5.4.10, $\mathrm{ch_f}(\alpha) \geq k - 6 - \lfloor k/2 \rfloor \geq \frac{k-12}{2} \geq 0$. If $n_2^t(\alpha) > 0$, then α sends charge by R_3 and/or R_5, in total at most, again by Corollary 5.4.10,

$$
\begin{aligned}
|S_1(\alpha)| + 2|S_2(\alpha)| \cdot \frac{7}{6} &\leq |S_1(\alpha)| + \frac{7}{3} \cdot \left\lfloor \frac{k - 2|S_1(\alpha)|}{5} \right\rfloor \\
&\leq |S_1(\alpha)| + \frac{1}{15}\big(7k - 14|S_1(\alpha)|\big) \\
&= \frac{1}{15}\big(7k + |S_1(\alpha)|\big) \\
&\leq \frac{1}{15}\left(7k + \left\lfloor \frac{k}{2} \right\rfloor\right) \\
&\leq \frac{k}{2}.
\end{aligned}
$$

Thus, for any value of $n_2^t(\alpha)$, $\mathrm{ch_f}(\alpha) \geq k - 6 - \frac{k}{2} = \frac{k-12}{2} \geq 0$.

As all the vertices and faces of G end up with a nonnegative final charge we obtain that the total charge in G is nonnegative, a contradition. $\square$

Chapter 6

Facial-Parity Colorings of Plane Graphs

$\mathcal{I}$n this chapter, we will consider facial-parity colorings of the vertices and edges of plane graphs, defined in their respective sections later on. One of the main motivation are parity colorings which were first introduced by Bunde, Milans, West, and Wu [48, 49] and are defined as follows. For an edge coloring of a simple graph, a *parity walk* is such a walk along which every color is appears an even number of times. A *parity edge coloring* is an edge coloring with no parity path. In other words, for every path P there exists a color c such that c appears an odd number of times on the edges of P. Similarly, a *strong parity edge coloring* is an edge coloring with no open parity walk. Note that even though it is not explicitly stated, every parity edge coloring is a proper edge coloring, since two edges of the same color sharing a vertex would already form a parity path of length 2. It is also clear, by definition, that every strong parity edge coloring is also a parity edge coloring.

Another motivation to study facial-parity colorings comes from the study of subgraph covering problems. We say that the graphs $H_1, \dots, H_k$ *cover* a graph G if for all $1 \le i \le k$, H_i is a subgraph of G and $\bigcup_{1 \le i \le k} E(H_i) = E(G)$. Recall that a graph G is even (odd) if every vertex of G has even degree (odd degree).

In 1978, Matthews [167] showed that the following holds.

Theorem 6.0.1 (Matthews [167]). *Every bridgeless graph G can be covered by at most 3 even subgraphs.*

As a counterpart to the result on even subgraph cover, Pyber [173], in 1991, showed the following result in case of covering a graph with odd subgraphs.

Theorem 6.0.2 (Pyber [173]). *Every simple graph G can be covered by at most 4 edge-disjoint odd subgraphs. Moreover, if $|V(G)|$ is even, then 3 odd subgraphs are sufficient.*

In addition, Pyber noted that the graph achieving the upper bound is the graph W_4, a wheel with four spokes. Later, Mátrai [166] showed that every simple graph can be covered by at most 3 (not necessarily edge-disjoint) odd subgraphs.

Theorem 6.0.3 (Mátrai [166])**.** *Every simple graph G can be covered by at most* 3 *odd subgraphs.*

Moreover, Mátrai provided an infinite family of simple connected graphs for which there exists no cover with 3 edge-disjoint odd subgraphs. This answered a question by Pyber whether there exists an infinite family of simple graphs for which the upper bound of Theorem 6.0.2 is achieved. The construction is as follows: take an even number of copies of the wheel graph W_4 and one additional vertex v. From each copy of W_4, remove a single edge from that copy and connect its two end-vertices with the vertex v (see Figure 6.1).

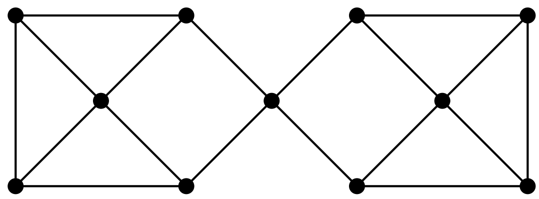


Figure 6.1: A construction of Mátrai with two copies of W_4 having no cover with 3 edge-disjoint odd subgraphs.

Finally, in 2015, Lužar, Petruševski, and Škrekovski [163] proved that 6 edge-disjoint odd subgraphs are enough to cover any loopless connected multigraph, where the upper bound is achieved by Shannon triangles of even type. A *Shannon triangle* is a loopless multigraph G on three pairwise adjacent vertices. A Shannon triangle is of *type* (p, q, r), with $p \geq q \geq r$, if for the three pairs of vertices, they are connected by p, q, and r parallel edges, respectively (see Figure 6.2). A Shannon triangle is of *even type* if p, q, and r are all even. Figure 6.2 depicts four types of Shannon triangles, namely, the types $(1, 1, 1)$, $(2, 1, 1)$, $(2, 2, 1)$, or $(2, 2, 2)$. Note that any other type of Shannon triangle can be reduced to one of the above four with respect to odd covers.

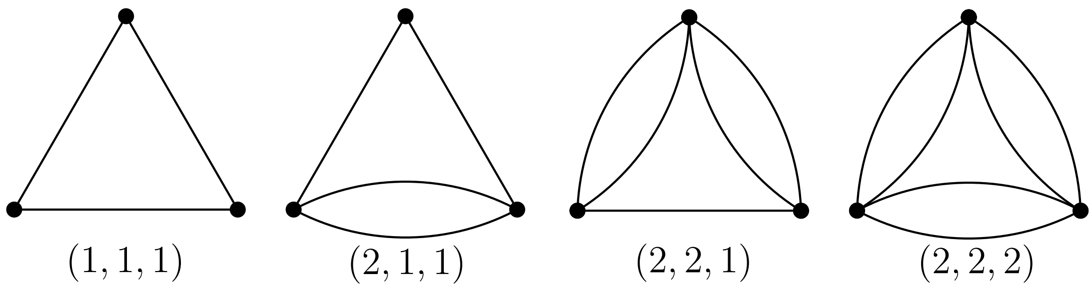


Figure 6.2: The four irreducible types of Shannon triangles.

Theorem 6.0.4 (Lužar et al. [163]). *Every loopless connected multigraph G has an edge-disjoint cover with at most 6 odd subgraphs.*

Note that it is enough to consider only loopless multigraphs as loops count twice for the degree of the vertex which does not change the parity of its degree.

In the case of planar graphs, Theorem 6.0.2 implies that the edges of every 3-connected plane graph G can be colored with at most 4 colors in such a way that for every face α and for every color c, either no edge or an odd number of edges incident with α are colored with color c. This follows from the fact that a dual graph G^* of a 3-connected plane graph is a simple plane graph. However, note that such a coloring may not be facially-proper.

6.1 Facial-parity edge coloring

Motivated by the parity colorings and the odd covers of graphs presented above, Czap, Jendroľ, and Kardoš [72] defined a *facial-parity edge coloring* (also known as *odd edge coloring*, see [68]) of a connected bridgeless plane graph G, i.e., a facially-proper coloring of the edges of G such that for every face α and every color c, there is either a zero or an odd number of occurrences of color c on the boundary walk of α. The minimum number of colors needed for such a coloring is called the *facial-parity chromatic index* and denoted by $\chi'_{\mathrm{fp}}(G)$. Note that we require G to be bridgeless, as every bridge appears twice on the boundary walk of the face incident with it (e.g., paths do not admit a facial-parity edge coloring). However, one can easily extend the definition to all plane graphs by requiring that each face is incident with zero or an odd number of edges of each color, which is equivalent to the previous definition in the case of connected bridgeless plane graphs.

Note also that every facial-parity edge coloring is also a 1-facial edge coloring, but the converse does not hold as can be seen already by the graphs C_4 and C_5, whose facial-parity chromatic index is 4 and 5, respectively. Czap et al. [72] proved that a constant upper bound exists by showing that every connected bridgeless plane graph admitsa facial-parity edge coloring with at most 92 colors. Later, Czap, Jendroľ, Kardoš, and Soták [69] improved the bound to 20 colors. The final improvement on the general upper bound for connected bridgeless plane graphs was done by Lužar and Škrekovski [164] in 2013.

Theorem 6.1.1 (Lužar and Škrekovski [164]). *Let G be a connected bridgeless plane graph. Then,*

$$\chi'_{\mathrm{fp}}(G) \le 16\,.$$

To this day the bound from Theorem 6.1.1 is the best known. In the case of outerplane graphs even better results are known. In 2012, Czap [67] proved the following.

Theorem 6.1.2 (Czap [67]). *Let G be a connected bridgeless outerplane graph. Then,*

$$\chi'_{\mathrm{fp}}(G) \leq 15\,.$$

The result of Czap was further improved in 2015 by Bálint and Czap [17] who proved that with the exception of the graph G_1 from Figure 6.3, every connected bridgeless outerplane graph requires at most 9 colors.

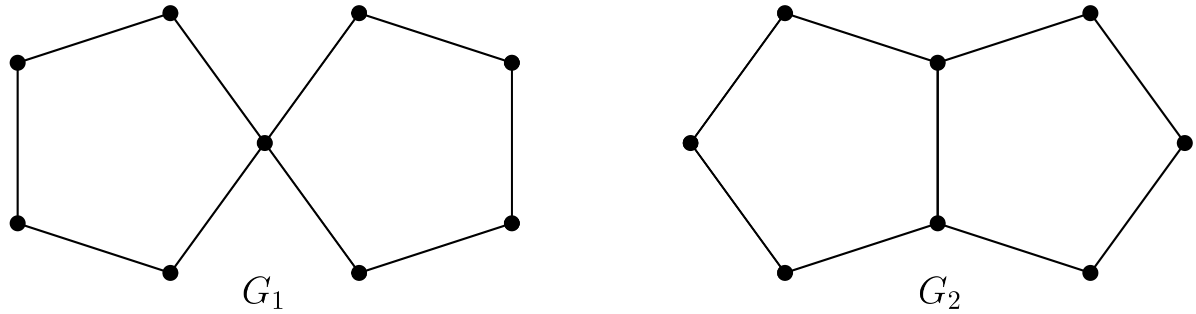


Figure 6.3: Two outerplane graphs requiring 10 and 9 colors, respectively, in any facial-parity edge coloring.

Theorem 6.1.3 (Bálint and Czap [17]). *Let G be a connected bridgeless outerplane graph distinct from G_1 depicted in Figure 6.3. Then,*

$$\chi'_{\mathrm{fp}}(G) \leq 9\,.$$

It is known that there are outerplane graphs requiring 9 and even 10 colors (see Figure 6.3) which implies that the correct upper bound in general is somewhere between 10 and 16 colors.

Building upon the work on facial-parity edge coloring, we show that the correct upper bound is between 12 and 16 colors by providing an infinite family of connected bridgeless plane graphs G (even 2-connected) for which $\chi'_{\mathrm{fp}}(G) = 12$.

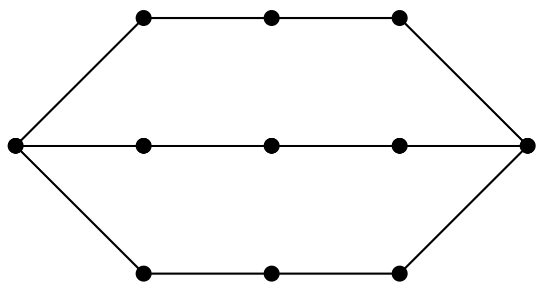


Figure 6.4: The graph $\Theta_{4,4,4}$ with 12 edges and $\chi'_{\mathrm{fp}}(G) = 12$.

Theorem 6.1.4 (Theorem 2 in [198]). *For any integer $k \geq 3$, there exists a 2-connected plane graph G with $4k$ vertices and $\chi'_{\mathrm{fp}}(G) = 12$.*

Proof. Let G be a Theta graph. Fix some plane embedding of G (e.g., see Fig 6.4). Clearly, G is 2-connected and it can be edge decomposed into three internally vertex-disjoint paths P_1, P_2, and P_3, where P_i and P_j, $1 \leq i < j \leq 3$, are both incident to the unique face α_{ij}. Let $f : E(G) \to \mathbb{N}$ be any facial-parity edge-coloring of G. First suppose that some color c appears an even number of times on the edges of some path P_i. Without loss of generality, we can assume that $i = 1$. Since P_1 is incident to both α_{12} and α_{13}, it follows that the color c must appear an odd number of times on the edges of both P_2 and P_3, but then it appears an even number of times on the edges incident with α_{23}, a contradiction. It follows directly that no color can appear on two distinct paths P_i and P_j at the same time. Therefore, the number of colors needed to color the edges of G is the sum of the number of colors needed to color the edges of each P_i individually. Let us consider again a single path $P \in \{P_1, P_2, P_3\}$ and let the length of P be ℓ. In the case when $\ell = 1$, it is easy to see that we need exactly 1 color to color the single edge of P. Therefore, we need to consider the following remaining four cases:
Case 1: If $\ell = 2m$ for some $m \in \mathbb{N}$, where m is odd, then we can properly color the edges of P with exactly two colors c_1 and c_2, each appearing m times on P.
Case 2: If $\ell = 2m + 1$ for some $m \in \mathbb{N}$, where m is even, then we can color the edges of P with exactly three colors c_1, c_2 and c_3, where each of the colors c_1 and c_2 appears $m - 1$ times on P and the color c_3 appears 3 times on P.
Case 3: If $\ell = 2m + 1$ for some $m \in \mathbb{N}$, where m is odd, then we can color the edges of P with exactly three colors c_1, c_2 and c_3, where each of the colors c_1 and c_2 appears m times on P and the color c_3 appears only once on P.
Case 4: If $\ell = 4m$ for some $m \in \mathbb{N}$, then we can color the edges of P with exactly four colors c_1, c_2, c_3 and c_4, where each of the colors c_1 and c_2 appears $2m - 1$ times on P and each of the colors c_3 and c_4 appears only once on P.
It follows that if each of the paths P_i has length divisible by 4, then $\chi'_{\mathrm{fp}}(G) = 12$, thus proving the theorem. The smallest such case is depicted in Fig. 6.4, where all three paths are of length 4 and G has 12 edges. □

In fact, from the proof it directly follows that for any integer $6 \leq t \leq 12$, there exist an infinite family of connected bridgeless plane graphs G with $\chi'_{\mathrm{fp}} = t$. The smallest example with $\chi'_{\mathrm{fp}}(G) = 12$ is depicted in Figure 6.4.

A question remains whether 12 colors are always sufficient and the best affirmative result in this direction is by Czap et al. [69] from 2012, which states that this holds in the case of 3-edge-connected plane graphs.

Theorem 6.1.5 (Czap et al. [69]). *Let G be a 3-edge-connected plane graph. Then,*

$$\chi'_{\mathrm{fp}}(G) \leq 12\,.$$

6.2 Facial-parity vertex coloring

In 2009, Czap and Jendroľ [70] introduced the notion of a *strong parity vertex coloring* (also known as a *strong odd coloring*), i.e., a coloring of the vertices of a 2-connected plane graph in which for every face α and every color c, there is either a zero or an odd number of occurrences of color c on the boundary walk of α. The minimum number of colors needed for a strong parity vertex coloring is called the *strong parity chromatic number* and is denoted by χ_{sp}. Note that the strong parity vertex coloring need not be proper. In addition, 2-connectedness is required as there exist connected, but not 2-connected, graphs for which there exists no strong parity vertex coloring, i.e., two triangles identified at a single vertex. However, similarly as in the case of the edge version, one can easily extend the definition to all plane graphs by requiring that each face is incident with zero or an odd number of vertices of each color which is equivalent to the previous definition in the case of 2-connected plane graphs.

If we add the condition that the coloring must be facially-proper (which is the same as proper), then we get a *facial-parity vertex coloring* of a 2-connected plane graph G, i.e., a proper coloring of the vertices of G such that for every face α and every color c, α is incident with either an odd or zero vertices of color c. The minimum number of colors needed for such a coloring is called the *facial-parity chromatic number* and denoted by $\chi_{\mathrm{fp}}(G)$.

In 2011, Czap, Jendroľ, and Voigt [73] showed the existence of a constant upper bound for the facial-parity chromatic number by proving that every 2-connected plane graph admits a facial-parity vertex coloring with at most 118. Three years later, Kaiser et al. [143] improved the upper bound.

Theorem 6.2.1 (Kaiser et al. [143]). *Let G be a 2-connected plane graph. Then,*

$$\chi_{\mathrm{fp}}(G) \leq 97\,.$$

Currently, the upper bound from Theorem 6.2.1 is the best known result.

In the case of outerplane graphs, however, much more is known. In 2011, Czap [66] considered the facial-parity chromatic number of outerplane graphs and showed that, in general, there always exists a facial-parity vertex coloring of an outerplane graph with at most 12 colors and provided an example of an outerplane graph with $\chi_{\mathrm{fp}}(G) = 10$ (see G_1 from Figure 6.5).

Theorem 6.2.2 (Czap [66]). *Let G be a 2-connected outerplane graph. Then,*

$$\chi_{\mathrm{fp}}(G) \leq 12\,.$$

A year later, Wang, Finbow, and Wang [203] managed to improve the results of Czap by showing that 10 colors are always enough. In fact, they showed that one can always find a facial-parity vertex coloring of

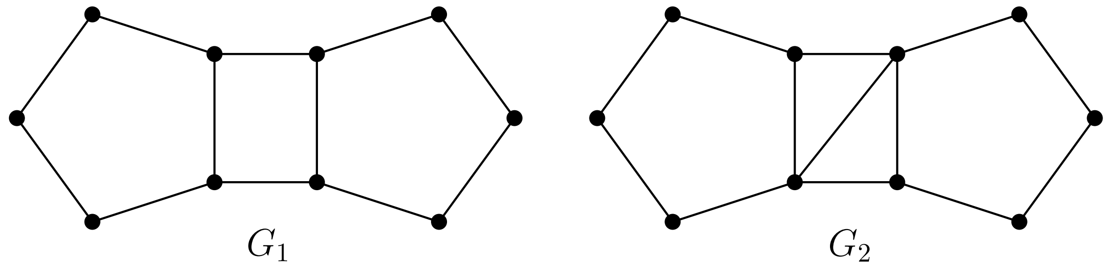


Figure 6.5: Two outerplane graphs requiring 10 colors in any facial-parity vertex coloring.

a 2-connected outerplane graph with 9 colors with the exception of two outerplane graphs which require 10 colors (see Figure 6.5).

Theorem 6.2.3 (Wang et al. [203]). *Let G be a 2-connected outerplane. Then,*

$$\chi_{\mathrm{fp}}(G) \leq 10, .$$

Moreover,

$$\chi_{\mathrm{fp}}(G) \leq 9$$

if and only if G is distinct from G_1 and G_2 depicted in Figure 6.5.

Going back to the case of 2-connected plane graphs, we show that there exists an infinite family of 2-connected plane graphs G for which $\chi_{\mathrm{fp}}(G) = 12$.

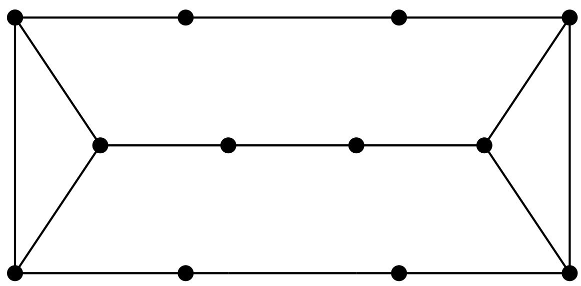


Figure 6.6: The line graph of the graph $\Theta_{4,4,4}$ which has 12 vertices and $\chi_{\mathrm{fp}}(G) = 12$.

Theorem 6.2.4 (Theorem 4 in [198]). *For any integer $k \geq 3$, there exists a 2-connected plane graph G with $4k$ vertices and $\chi_{\mathrm{fp}}(G) = 12$.*

Theorem 6.2.4 follows directly from Theorem 6.1.4 by taking line graphs of $\Theta_{p,q,r}$ graphs. Similarly, it follows from the proof of Theorem 6.1.4 that for any integer $6 \leq t \leq 12$, there exist an infinite family of 2-connected plane graphs G with $\chi_{\mathrm{fp}} = t$. The smallest example with $\chi_{\mathrm{fp}}(G) = 12$ is depicted in Figure 6.6.

It remains an open problem to find the correct upper bound for the facial-parity chromatic number of 2-connected plane graphs. Current results show that the correct value is between 12 and 97, although, results point towards the lower end of this interval.

Chapter 7

Final Remarks to Part I

Let us now give some remarks to the first part of this thesis.

7.1 Remarks on proper colorings of planar graphs

In Chapter 4, we considered proper 3-coloring of (mostly) planar graphs. We will now turn our attention to the adynamic coloring and for that we will also define the dynamic coloring. A *dynamic coloring* is a proper coloring of the vertices of a graph G such that for every vertex v of degree at least 2, there exist two distinct vertices $u, w \in N(v)$ with distinct colors. On the other hand, an *adynamic coloring* is a proper coloring of the vertices of a graph G such that there exists a vertex v of degree at least 2 for which all the vertices in $N(v)$ are colored with the same color. Clearly, a dynamic coloring always exists, while in order for a graph to admit an adynamic coloring there must exist a vertex of degree at least 2 with an independent neighborhood, i.e., is not incident to a triangle. Such a vertex is called a *mono-vertex*. The smallest number of colors needed for a dynamic coloring is called the dynamic chromatic number, denoted by $\chi_{\mathrm{d}}(G)$, and similarly, the smallest number of colors needed for an adynamic coloring is called the adynamic chromatic number, denoted by $\chi_{\mathrm{m}}(G)$.

Let us denote by $\mathcal{M}$ the class of graphs containing a mono-vertex. It is easy to see that for every graph G in the complement of $\mathcal{M}$, $\chi(G) = \chi_{\mathrm{d}}(G)$. On the other hand, for every graph G in $\mathcal{M}$, $\chi(G) \in \{\chi_{\mathrm{d}}(G), \chi_{\mathrm{m}}(G)\}$, see [199]. In the same paper, the authors also gave the following strengthening of the Four-Color-Theorem.

Theorem 7.1.1 (Šurimová et al. [199]). *Let $G \in \mathcal{M}$ be a planar graph with $\chi(G) = 4$. Then $\chi_{\mathrm{m}}(G) = 4$.*

Although every planar graph is adynamically 4-colorable, there is no one-to-one correspondence with the 4-colorable planar graphs as there exist 3-colorable planar graphs which are adynamically 4-colorable (see Figure 7.1). Thus, it is interesting to study when does a 3-colorable planar graph in $\mathcal{M}$ admit an adynamic 3-coloring. In [199], the authors proved the following.

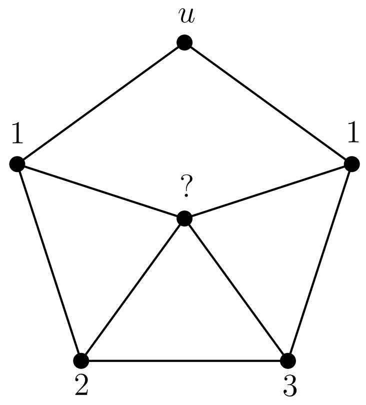


Figure 7.1: A wheel graph W_5 without a spoke which is not adynamically 3-colorable with u being the only mono-vertex.

Theorem 7.1.2 (Šurimová et al. [199]). *Every triangle-free planar graph in $\mathcal{M}$ admits an adynamic* 3*-coloring.*

Note also that this result is a corollary of Theorem 4.1.4 and the fact that every triangle-free planar graph has a vertex of degree at most 3. However, in the case of planar graphs with at least two triangles, there exist examples that need 4 colors in any adynamic coloring (see, e.g., Figure 4.5(b)). As a result, Šurimová et al. [199] posed a conjecture that every planar graph G in $\mathcal{M}$ with at most one triangle has $\chi_{\mathrm{m}}(G) \leq 3$, which we answer in affirmative.

Theorem 7.1.3 (Theorem 4.1 in [157]). *Every planar graph in $\mathcal{M}$ with at most one triangle is adynamically* 3*-colorable.*

Proof. We proceed by contradiction. Let G be a minimal counterexample in terms of the number of vertices with some fixed embedding. By Theorem 4.1.4, G has exactly one triangle T. Suppose first that there is a 2-vertex v in G and let $N(v) = \{v_1, v_2\}$. The graph G' obtained by splitting v into two adjacent vertices both connected to v_1 and v_2 is planar with at most three triangles and thus 3-colorable by Theorem 4.1.1. Any coloring of G' induces a coloring of G in which v_1 and v_2 receive the same color, a contradiction.

Therefore, $\delta(G) \geq 3$. Moreover, by the Handshaking Lemma and the Euler's Formula, there are at least nine 3-vertices in G, and so at least six 3-vertices are not incident with T. Hence, by Theorem 4.2.6, G contains a subgraph D isomorphic to K_4'. Since $\delta(G) \geq 3$, there is a 5-face α of D that is not a face in G.

The graph induced by the interior of α in G and $V(\alpha)$ is a triangle-free plane graph, which we can 3-color adynamically. This coloring gives us a coloring of the vertices of α and fixes also the color of the vertex of T that is not incident with α. It remains to color (eventual) interiors of the other two 5-faces of D in G and the interior of T. All can be colored by Theorem 4.1.6. This completes the proof. □

7.2 Remarks on ℓ-facial edge coloring

In Chapter 5, we proved that Conjecture 5.3.1 holds for $\ell = 3$ and mentioned that for $\ell \geq 4$ it remains open. The main difficulty in proving Conjecture 5.3.1 in general lies in 2-vertices. In particular, in long threads. Note that several structural results regarding a minimal counterexample to Conjecture 5.3.1 for the case when $\ell = 3$ presented in Chapter 5 can be adapted also for the general case. As an example, let us prove the following observation which is a generalization of Lemma 5.4.6.

Observation 7.2.1. Let ℓ be a positive integer and let G be plane graph such that G is a minimal counterexample to Conjecture 5.3.1. Then G contains no face of length at most $(\ell + 1)$.

Proof. Suppose to the contrary and let α be any face of G of length at most $(\ell+1)$. Let G' be the graph obtained from G by contracting all the edges of α. By the minimality of G, there exists an ℓ-facial edge coloring σ of G with at most $3\ell + 1$ colors. Next, observe that each edge of α is ℓ-facially adjacent to at most 2ℓ distinct edges of G'. Thus, each edge e of α has the set of available colors $A(e)$ of size at least $\ell + 1$. By Theorem 3.2.1, we can therefore extend the coloring σ to obtain an ℓ-facial edge coloring of G using at most $3\ell + 1$ colors. □

Part II

Chapter 8

Preliminaries to Part II

In this chapter, we define notions and present auxiliary results that we will use in Part II of this thesis. For the sake of compactness, we also present several new results in some sections of this chapter.

Let G be a graph and let S be a cutset in G. Then G has a partition (A, B, S) for some non-empty sets $A, B \subseteq V(G)$ such that there are no edges in $G - S$ with one endpoint in A and one endpoint in B. Such a partition is called a *cut-partition*. Let u and v be two non-adjacent vertices in G. A cutset $S \subseteq V(G)$ is called a (u,v)-*separator* if u and v belong to different components of $G - S$. A (u,v)-separator S is *minimal* if no proper subset S' of S is a (u,v)-separator. A *minimal separator* in G is a set $S \subseteq V(G)$ that is a minimal (u,v)-separator for some pair of non-adjacent vertices u and v. A well-known characterization of minimal separators (see, e.g., [112]) is that a set $S \subseteq V(G)$ is a minimal separator if and only if the graph $G - S$ contains at least two S-*full components*, i.e., two components such that every vertex in S has a neighbor in each of them.

A *hole* in a graph is an induced cycle C_k with $k \geq 4$. A graph G is *chordal* if it does not contain any hole, i.e., G is $\{C_4, C_5, \ldots\}$-free. From the definitions of chordal graphs, induced minors, and induced topological minors, we can immediately observe the following.

Observation 8.0.1. Let G be a graph. Then, the following conditions are equivalent:

1. G is a chordal graph.
2. G is C_4-induced-minor-free.
3. G is C_4-induced-topological-minor-free.

A graph H with $V(H) = V(G)$ is a *triangulation* of a graph G if H is chordal and G is a subgraph of H. A triangulation H of G is *minimal* if there exists no triangulation H' of G such that H' is a proper subgraph of H. Note that, by definition, every minimal triangulation of a chordal graph G is the graph G itself.

Let G be a graph and let H be a minor of G. A *minor model* of H in G is a collection $\{X_u \mid u \in V(H)\}$ of pairwise disjoint subsets of $V(G)$ called *branch sets*, which we will refer to also as bags, such that each bag X_u induces a connected subgraph of G and for every two adjacent vertices $u, v \in V(H)$, there is an edge in G with one endpoint in X_u and one endpoint in X_v. Similarly, for an induced minor H of G, an *induced minor model* of H in G is a minor model of H in G with an additional property that if there exists an edge in G between a vertex of X_u and a vertex of X_v then $uv \in E(H)$.

8.1 Ramsey's theorem

In 1929, Ramsey, proved that in every sufficiently large graph there exists a clique of size k or an independent set of size ℓ. This result is known as Ramsey's theorem.

Theorem 8.1.1 (Ramsey [174]). *For every pair of positive integers k and ℓ, there exists the least positive integer $R(k, \ell)$ such that every graph with at least $R(k, \ell)$ vertices contains either a clique of size k or an independent set of size ℓ.*

The numbers $R(k, \ell)$ are called the Ramsey numbers. The following known upper bound on $R(k, \ell)$ is easy to show.

Lemma 8.1.2. *For any integers $k, \ell \geq 1$,*

$$R(k, \ell) \leq \binom{k+\ell-2}{k-1}.$$

Proof. If $\ell = 1$, then $\binom{k+\ell-2}{k-1} = \binom{k-1}{k-1} = 1$ and if $k = 1$, then $\binom{k+\ell-2}{k-1} = \binom{\ell-1}{0} = 1$. In both cases, $R(k, \ell) = 1$ and the inequality holds.

Let G be an n-vertex graph with $n \geq R(k-1, \ell) + R(k, \ell-1)$, where $k, \ell \geq 2$. Let v be any vertex of G. Suppose first that $|N(v)| \geq R(k-1, \ell)$. Let G' be the subgraph of G induced on the vertices of $N(v)$. By induction, G' contains an independent set of size ℓ, in which case G contains an independent set of size ℓ, or G' contains a clique K of size $k-1$, in which case K, together with v, forms a clique of size k in G.

We may thus assume that $|N(v)| < R(k-1, \ell)$, but then v has at least $R(k, \ell-1)$ non-neighbors. In this case, let G' be the subgraph of G induced on the non-neighbors of v. By induction, G' contains a clique of size k, in which case G contains a clique of size k, or G' contains an independent set I of size $\ell-1$, in which case I, together with v, forms an independent set of size ℓ in G.

Finally, an induction on k and ℓ, together with Pascal's formula for binomials, shows that

$$R(k, \ell) \leq R(k-1, \ell) + R(k, \ell-1)$$

$$\leq \binom{k+\ell-3}{k-2} + \binom{k+\ell-3}{k-1}$$
$$= \binom{k+\ell-2}{k-1}. \qquad \square$$

8.2 Tree decompositions and treewidth

A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ where T is a tree and every node t of T is assigned a vertex subset $X_t \subseteq V(G)$ called a bag such that the following conditions hold:

(1) every vertex belongs to some bag;

(2) for every edge $e \in E(G)$ there exists a node $t \in V(T)$ such that the bag X_t contains both endpoints of e; and

(3) for every vertex $v \in V(G)$ the subgraph T_u of T induced by the nodes from the set $\{t \in V(T) \mid v \in X_t\}$ is a tree.

A tree decomposition is called *trivial* if it contains a single bag, i.e., the unique bag contains all the vertices of G. The *width* of a tree decomposition $\mathcal{T}$, denoted by width($\mathcal{T}$), is defined as follows: $\mathrm{width}(\mathcal{T}) = \max\{|X_t| - 1 \mid t \in V(T)\}$. The *treewidth* of a graph G, denoted by tw(G), is defined as the minimum width of a tree decomposition of G, taken over all tree decompositions. A graph class $\mathcal{G}$ has *bounded treewidth* if there exists a constant c such that $\mathrm{tw}(G) \leq c$ for all graphs $G \in \mathcal{G}$ and *unbounded treewidth* otherwise. The *size* of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$, denoted by $|\mathcal{T}|$, is defined as

$$|\mathcal{T}| = |V(T)| + \sum_{t\in V(T)} |X_t|\,.$$

Among various equivalent characterizations of treewidth, in 1986 Robertson and Seymour [176] gave the following characterization in terms of graph triangulations.

Theorem 8.2.1 (Robertson and Seymour [176]). *For every graph G, we have*

$$\mathrm{tw}(G) = \min\{\omega(H) - 1 : H \textit{ is a triangulation of } G\}\,.$$

An important property of tree decompositions states that every clique must be contained in some bag.

Lemma 8.2.2 (Scheffler [180], Bodlaender and Möhring [30]). *Let G be a graph, let $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ be a tree decomposition of G, and let C be a clique in G. Then there exists a bag X_t such that $C \subseteq X_t$.*

Since the only minimal triangulation of a chordal graph G is the graph G itself, Theorem 8.2.1 and Lemma 8.2.2 imply the following.

Theorem 8.2.3 (see, e.g., [28]). *For every graph G we have that* $\mathrm{tw}(G) \geq \omega(G) - 1$, *with equality for all induced subgraphs if and only if G is chordal.*

In addition, the following result of a similar flavor as Lemma 8.2.2 holds.

Lemma 8.2.4 (Lemma 2.3 in [77]). *Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of a graph G. Then for every set $S \subseteq V(G)$ such that every pair of vertices in S is contained in a bag of $\mathcal{T}$, there exists a bag X_t such that $S \subseteq X_t$.*

Proof. Suppose that every pair of vertices of S is contained in a bag of $\mathcal{T}$. For each $v \in S$, the set of nodes of T labeled by bags containing v induces a subtree T_v of T. Since every two vertices in S are contained in a bag of $\mathcal{T}$, every two of the subtrees in $\{T_v : v \in S\}$ have a node in common. It is known (and easy to see) that for any family of node sets of subtrees of a tree there exists a node $t \in V(T)$ common to all the trees T_v, $v \in S$. We infer that $v \in X_t$ for all $v \in S$, that is, $S \subseteq X_t$. □

The following result implies the fact that for every graph G, $\mathrm{tw}(G) \geq \delta(G)$ (see, e.g., [29]).

Lemma 8.2.5 (Lemma 2.6 in [77]). *Let G be a graph and let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of G. Then there exists a vertex $v \in V(G)$ and a node $t \in V(T)$ such that $N[v] \subseteq X_t$.*

In order to prove Lemma 8.2.5, we require a few more definitions which help give further characterizations of chordal graphs. A vertex v in a graph G is called *simplicial* if $N(v)$ is a clique. A *clique tree* of a graph G is a tree decomposition of G such that the bags are exactly the maximal cliques of G (see, e.g., [26]). Given a collection $\{T_1, \ldots, T_n\}$ of subtrees in a tree T, the *intersection graph* of $\{T_1, \ldots, T_n\}$ is the graph with the vertex set $\{1, \ldots, n\}$, in which two distinct vertices i and j are adjacent if and only if T_i and T_j have a vertex in common. In the following theorem we combine results from several sources [26, 50, 85, 108, 201].

Theorem 8.2.6. *Let G be a graph. Then, the following conditions are equivalent:*

1. *G is a chordal graph.*
2. *G has a clique tree.*
3. *G is the intersection graph of subtrees in a tree.*
4. *Every non-null induced subgraph of G has a simplicial vertex.*

We are now ready to prove Lemma 8.2.5.

Proof of Lemma 8.2.5. Let G' be the graph with the vertex set $V(G)$ such that two distinct vertices u and v are adjacent in G' if and only if there exists a bag X_t of $\mathcal{T}$ with $u, v \in X_t$. Note that for every vertex $v \in V(G)$ it holds that $N_G[v] \subseteq N_{G'}[v]$. Recall that for each vertex $v \in V(G')$, the set of nodes $t \in V(T)$ such that $v \in X_t$ induces a subtree T_v of T. Thus, two distinct vertices u and v of G' are adjacent if and only if the corresponding trees T_u and T_v have a node in common. This means that G' is the intersection graph of subtrees in a tree and hence, by Theorem 8.2.6, G' is a chordal graph. Observe that $\mathcal{T}$ is also a tree decomposition of G'. By Theorem 8.2.6, G' has a simplicial vertex v, and hence by Lemma 8.2.2, there must exist a node $t \in V(T)$ such that $N_{G'}[v] \subseteq X_t$. Therefore, $N_G[v] \subseteq X_t$, which concludes the proof. □

8.3 Potential maximal cliques

A *potential maximal clique* in a graph G is a set $X \subseteq V(G)$ such that X is a maximal clique in some minimal triangulation of G. Buchitté and Todinca gave the following characterization of potential maximal cliques.

Theorem 8.3.1 (Buchitté and Todinca [43]). *Let G be a graph and let $X \subseteq V(G)$. Then X is a potential maximal clique in G if and only if the following two conditions hold:*

1. *For every component C of $G-X$, some vertex of X has no neighbors in C.*
2. *For every two non-adjacent vertices $u, v \in X$ there exists a component C of $G - X$ in which both u and v have a neighbor.*

Furthermore, if X is a potential maximal clique, then the family of the neighborhoods, in G, of the components of $G - X$ are exactly the family of minimal separators S in G such that $S \subset X$.

Let X be a potential maximal clique in a graph G, let C be a component of $G - X$, and let $S = N_G(V(C))$. We say that S is an *active separator for X* if there exist two non-adjacent vertices $u, v \in S$ such that C is the only component of $G - X$ such that C contains both a neighbor of u, as well as a neighbor of v. Note that this definition is slightly different in [44, Definition 13]. Due to Theorem 8.3.1, however, both definitions are equivalent. In addition, note that it is possible for a potential maximal clique to not have an active separator. However, if they do, then, as a corollary of [44, Theorem 15], the following result holds.

Theorem 8.3.2 (Theorem 3.5 in [78]). *Let X be a potential maximal clique in a graph G and let S be an active minimal separator for X. Then there exists a minimal separator T of G such that $X \subseteq S \cup T$.*

Additionally, the following result, which is a part of the statement of [44, Proposition 19], is also true.

Proposition 8.3.3. *Let $a \in V(G)$ and let X be a potential maximal clique of G such that $a \notin X$. Let C_a be the connected component of $G-X$ containing a and let $S = N_G(V(C_a))$. Then either X is a potential maximal clique of $G - a$ or S is an active separator for X.*

In order to prove the next theorem, we require the following property of minimal separators.

Lemma 8.3.4 (Lemma 3.7 in [78]). *Let G be a graph, let G' be an induced subgraph of G, and let S' be a minimal separator in G'. Then there exists a minimal separator S in G such that $S' \subseteq S$.*

Proof. Let C' and D' be two S'-full components of the graph $G' - S'$. Fix two vertices $u \in V(C')$ and $v \in V(D')$. Consider the set $\tilde{S} = S' \cup (V(G) \setminus V(G'))$. This set is a u,v-separator in G, and therefore contains a minimal u,v-separator S in G. Since the set $\tilde{S}$ is disjoint from $V(C') \cup V(D')$ and $S \subseteq \tilde{S}$, the set S is also disjoint from $V(C') \cup V(D')$. Furthermore, since every vertex in S' has a neighbor in both C' and D', we must have $S' \subseteq S$ as otherwise S would not be a u,v-separator in G. □

From Theorem 8.3.2, we get that every potential maximal clique containing an active minimal separator can be covered by two minimal separators in G. This result can be further refined as follows.

Theorem 8.3.5 (Theorem 3.8 in [78]). *Let G be a graph and let X be a potential maximal clique in G. Then X is either a clique in G or there exist two minimal separators S and T in G such that $X \subseteq S \cup T$.*

Proof. We use induction on the number of vertices of G. If G is chordal, then the only minimal triangulation of G is G itself, and hence the potential maximal cliques of G coincide with its maximal cliques. So we may assume that G is not chordal.

Consider a potential maximal clique X of G and suppose that X is not a clique. If X contains an active minimal separator S, then by Theorem 8.3.2, there exists a minimal separator T of G such that $X \subseteq S \cup T$.

Assume now that X does not contain any active minimal separator. First, we show that $X \neq V(G)$. Suppose that $X = V(G)$. Then there exists a minimal triangulation H of G in which $X = V(H)$ is a maximal clique. This implies that H is a complete graph. Since G is not chordal, it is not complete, and thus has two distinct non-adjacent vertices u and v. But now, deleting the edge uv from H produces a minimal triangulation of G properly contained in H, a contradiction. Thus, $X \neq V(G)$.

Since $X \neq V(G)$, there exists a vertex $a \in V(G) \setminus X$. Let $G' = G - a$, let C_a be the connected component of $G - X$ containing a, and let $S = N_G(V(C_a))$. Since S is not an active separator for X, Proposition 8.3.3 implies that X is a potential maximal clique of G'. By the induction hypothesis, there exist two minimal separators S' and T' in G' such that $X \subseteq S' \cup T'$. By Lemma 8.3.4, there exist two minimal separators S and T in G such that $S' \subseteq S$ and $T' \subseteq T$. Thus, $X \subseteq S \cup T$. □

8.4 Block-cutpoint trees

Recall that a *block* of a graph G is a maximal connected subgraph of G without a cut-vertex. Every connected graph G has a unique *block-cutpoint* tree, that is, a tree T with $V(T) = \mathcal{B}(G) \cup \mathcal{C}(G)$, where $\mathcal{B}(G)$ is the set of blocks of G and $\mathcal{C}(G)$ is the set of cut-vertices of G, such that every edge of T has an endpoint in $\mathcal{B}(G)$ and the other in $\mathcal{C}(G)$, with $B \in \mathcal{B}(G)$ adjacent to $v \in \mathcal{C}(G)$ if and only if $v \in V(B)$. Every leaf of the block-cutpoint tree is a block of G, called a *leaf block* of G. As shown by Hopcroft and Tarjan [134], a block-cutpoint tree of a connected graph G can be computed in linear time. It is easy to see that a block-cutpoint tree is also a valid tree decomposition of a graph. Block-cutpoint trees can be used in algorithms to provide an efficient reduction of the problem of computing a certain invariant of a given graph to the same (or a similar) problem on the blocks of the graph (see, e.g., [61]).

8.5 SPQR trees

The SPQR trees are one of the classical tools for decomposing 2-connected graphs into triconnected components, see [193, 194]. *Triconnected components* form a system of smaller graphs that describe the 2-vertex cuts in the graph. The SPQR trees were formally introduced in 1990 by Di Battista and Tamassia [79] (see also [80, 116, 135]).

We use the definition of SPQR trees as used by Dujmović et al. [89], which is slightly different than the original definition (see also [77] for the following definition). An *SPQR tree* of a 2-connected graph G is a tree S in which every node (vertex of S) is of one of the three types: a P-node, an R-node, or an S-node. Each node a is assigned with a set $X_a \subseteq V(G)$. If a is a P-node, then X_a is a 2-cutset in G. If a is an R-node or an S-node, then X_a is the vertex set of a graph G_a also assigned to node a. For every S-node a, the graph G_a is a cycle, while for every R-node a, the graph G_a is 3-connected. Every edge in S has exactly one endpoint that is a P-node. For every two adjacent nodes a and b of S such that b is a P-node, the two vertices in X_b are adjacent in G_a. An edge uv in G_a such that $uv \notin E(G)$ is called a *virtual edge*; otherwise, it is called a *real edge*. For every virtual edge uv in G_a there exists a P-node b adjacent to a such that $X_b = \{u, v\}$. No two P-nodes are associated with the same 2-cutset of G. We denote the sets of all P-nodes, R-nodes, and S-nodes of S by $N_{\mathrm{P}}(S)$, $N_{\mathrm{R}}(S)$, and $N_{\mathrm{S}}(S)$, or simply by N_{P}, N_{R}, and N_{S}, respectively, if the SPQR-tree S is clear from the context. The graphs G_a for each R-node or S-node a are called the *triconnected components* of G. Thus, the SPQR tree S represents G as a collection of triconnected components that are joined at 2-cutsets (P-nodes).

Note that an SPQR tree S of a graph G naturally defines a tree decomposition $\mathcal{T}$ of G simply by assigning to each node $a \in V(S)$ the set X_a as the corresponding bag.

Let us now show the following lemma.

Lemma 8.5.1 (Lemma 4.4 in [78]). *Let G be a 2-connected graph and S an SPQR tree of G. For each node a of S that is either an* R*-node or an* S*-node, the graph G_a is an induced topological minor of G.*

Proof. Fix any node a of S that is either an R-node or an S-node. Note that the graph G'_a obtained from G_a by deleting the virtual edges is an induced subgraph of G. Thus, we only need to show that for every virtual edge uv of G_a, there exists a path P in G with endpoints u and v such that u and v are the only two vertices contained in G_a. In addition, we require that no two such paths intersect, except possibly in one of their endpoints.

Let us fix any virtual edge uv. Since uv is a virtual edge, there exists a node b such that b is a P-node containing both u and v, where u and v are non-adjacent. It follows, that u and v form a 2-cut in G. Let c be the node of S, distinct from a, containing both u and v. Since G is 2-connected, it follows that the connected component H of $G \setminus \{u, v\}$ containing $V(G_c \setminus \{u, v\})$ is connected and thus, there exists a minimal path P between u and v in G containing only the vertices from H. Therefore, P contains no vertices of G_a except for its endpoints. Since no two virtual edges of G_a are the same, no two P-nodes corresponding to the virtual edges of G_a are the same, thus proving that none of the obtained paths intersect in more than a single endpoint. $\square$

Let G be a graph. The *Hadwiger number* of G, denoted by $\eta(G)$, is the maximum integer p such that G contains K_p as a minor (see [152]). Since the Hadwiger number is monotone under taking minors, we readily obtain the following inequality (which we will use in the following sections).

Corollary 8.5.2 (Corollary 4.5 in [78]). *Let G be a graph, S an SPQR tree of G, and a an* R*-node or an* S*-node of S. Then $\eta(G_a) \leq \eta(G)$.*

As shown by Gutwenger and Mutzel [116], an SPQR tree of a graph G can be computed in linear time. Although the algorithm of Gutwenger and Mutzel computes a slightly different tree, the same holds in the case of our definition as well. Additionally, as was shown by Di Battista and Tamassia [80], any SPQR tree of a graph G has $\mathcal{O}(|V(G)|)$ nodes. Furthermore, for any SPQR trees, the following bounds on the number of vertices in each set corresponding to R-nodes and S-nodes can be shown, see [78] for the proof.

Lemma 8.5.3 (Lemma 4.6 in [78]). *Let G be a 2-connected n-vertex graph and S an SPQR tree of G. Then*

$$\sum_{a \in N_R \cup N_S} |X_a| \leq 3n - 6 \qquad \text{and} \qquad \sum_{a \in N_R} |X_a| \leq 2n - 4\,.$$

Additionally, we use the following bound on the total number of edges in the graphs G_a.

Lemma 8.5.4 (Hopcroft and Tarjan [135]). *Let G be a 2-connected graph and S an SPQR tree of G. Then*

$$\sum_{a \in N_R \cup N_S} |E(G_a)| \leq 3|E(G)| - 6\,.$$

Chapter 9

Bounding Treewidth by a Function of the Clique Number

Let us recall from the previous chapter that $\mathrm{tw}(G) \geq \omega(G) - 1$ for all graphs G (see Theorem 8.2.3), with equality if and only if G is a chordal graph. It is thus natural to ask whether for which graph classes can we give an upper bound to treewidth in terms of a clique number. A graph class $\mathcal{G}$ is said to be (tw, ω)-*bounded* if it admits a (tw, ω)-*binding function*, that is, a function f such that for every graph G in the class and any induced subgraph G' of G, the treewidth of G' is at most $f(\omega(G'))$. Furthermore, $\mathcal{G}$ is said to be *polynomially* (tw, ω)-*bounded* if it admits a polynomial (tw, ω)-binding function. One of the simplest such classes is the class of chordal graphs where for every graph G in the class it holds that $\mathrm{tw}(G) = \omega(G) - 1$. An interesting question is thus to understand what other classes of graphs are (tw, ω)-bounded. To this end, we study when forbidding a single graph with respect to one of the six well known graph containment relations, namely, subgraph, topological minor and minor relation, as well as their induced counterparts, results in a (tw, ω)-bounded graph class.

Besides the class of chordal graphs, other simple examples of (tw, ω)-bounded graph classes are classes of graphs of bounded independence number, as a consequence of Ramsey's Theorem.

Lemma 9.0.1 (Lemma 2.6 in [76]). *Let H be an edgeless graph. Then the class of H-free graphs is* (tw, ω)*-bounded, with a binding function $f(k) = R(k+1, |V(H)|) - 2$.*

Proof. Let H be an edgeless graph. Let $k \in \mathbb{N}$ and let G be an H-free graph such that $\omega(G) = k$. Since H is edgeless, G contains no independent set of size $|V(H)|$. By Ramsey's theorem (Theorem 8.1.1) we get that the number of vertices in G is thus strictly smaller than the Ramsey number $R(k+1, |V(H)|)$. In fact, since $\mathrm{tw}(G) \leq |V(G)| - 1$ (due to the fact that a single bag containing all the vertices of a graph is a valid tree decomposition), we have that

$$\mathrm{tw}(G) \leq |V(G)| - 1 \leq R(k+1, |V(H)|) - 2\,. \quad \square$$

Before continuing our analysis of graphs H for which the class of graphs excluding H, with respect to one of the six considered relations, is (tw, ω)-bounded, we need to understand some simple graph classes that are (tw, ω)-unbounded. A graph class is (tw, ω)-*unbounded* if it is not (tw, ω)-bounded. The following definition of an elementary wall is from [60] (up to symmetry). To obtain an *elementary wall* (or simply *wall*) of height h and width r, we start with a grid of height h and width $2r$. Let $C_1, \ldots, C_{2r}$ denote the columns of the grid ordered from left to right. For each column C_j, we let $e_1^j, \ldots, e_{h-1}^j$ be the edges in j-th column from top to bottom. Then, if j is odd, we delete all edges e_i^j where i is odd and if j is even, we delete all edges e_i^j where i is even. Finally, we remove all vertices of degree 1. Let q be a non-negative integer. A q-*subdivided wall* is a graph obtained from an elementary wall by subdividing each edge q times. See Figure 9.1 for an example of an elementary wall and a 1-subdivided wall.

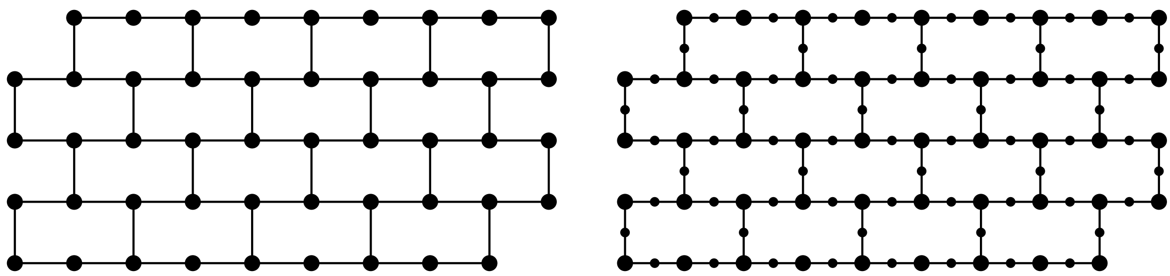


Figure 9.1: An example of an elementary wall graph (of height 5 and width 10) on the left and a 1-subdivided wall graph on the right.

First, let us state two properties of treewidth that we will need later on. The first is the fact that treewidth does not increase under taking minors (i.e., treewidth is *monotone* under taking minors).

Lemma 9.0.2 (See, e.g., [28]). *Let G be a graph and let H be a minor of G. Then* $\mathrm{tw}(H) \leq \mathrm{tw}(G)$.

The second property is due to Harvey and Wood [123] and gives a connection between treewidth of a graph G and that of its line graph.

Theorem 9.0.3 (Harvey and Wood [123]). *Let G be any graph and let $L(G)$ be its line graph. Then* $\mathrm{tw}(L(G)) \geq \frac{1}{2}(\mathrm{tw}(G) + 1) - 1$.

We are now ready to prove the following result.

Lemma 9.0.4 (Lemma 2.7 in [76]). *The class of balanced complete bipartite graphs and, for all $q \geq 0$, the class of q-subdivided walls and the class of their line graphs, are (tw, ω)-unbounded.*

Proof. By Lemma 8.2.5 we have that $\mathrm{tw}(K_{n,n}) \geq n$ and, since $K_{n,n}$ is bipartite, we have $\omega(K_{n,n}) = 2$. Thus, we get that the class of balanced complete bipartite graphs is (tw, ω)-unbounded.

Next, it is known that the class of elementary walls has unbounded treewidth (see, e.g., [60]). Thus, since elementary walls do not contain any cliques of size 3, we obtain that the class of elementary walls is also (tw, ω)-unbounded. Furthermore, using Lemma 9.0.2, the class of q-subdivided walls for any $q \geq 0$ is also (tw, ω)-unbounded.

Finally, by Theorem 9.0.3 and the fact that elementary walls are subcubic, we get that the class of line graphs of q-subdivided walls has unbounded treewidth and the clique number of each graph in these classes is bounded by 3. Hence all these classes are also (tw, ω)-unbounded. □

From Lemma 9.0.4 we get the following immediate corollary.

Corollary 9.0.5 (Corollary 2.8 in [76]). *Let $\subseteq$ be any graph containment relation and let H be a graph such that the class of graphs excluding H with respect to relation $\subseteq$ is* (tw, ω)*-bounded. Then H is in relation $\subseteq$ with some balanced complete bipartite graph, with some q-subdivided wall, for each $q \geq 0$, and with the line graph of some q'-subdivided wall, for each $q' \geq 0$.*

In the coming sections we present a complete characterization of the graphs H for which the class of graphs excluding H with respect to one of the six containment relations (subgraph, topological minor, minor, induced subgraph, induced topological minor, and induced minor) is (tw, ω)-bounded as presented in [76]. The results are summarized in Table 9.1. With $\mathcal{S}$ we denote the class of graphs in which every connected component is either a path or a subdivided claw. With W_n, $n \geq 3$, we denote the wheel graph with n spokes (i.e, a graph obtained from the cycle C_n by adding a single universal vertex), with K_p^-, $p \geq 2$, we denote the complete graph K_p with a single edge deleted, and with $K_{2,q}^+$, $q \geq 1$, we denote the complete bipartite graph $K_{2,q}$ by adding an edge between the two vertices in the part of size 2 (see Figure 9.2 for graphs W_4, K_5^-, and $K_{2,3}^+$).

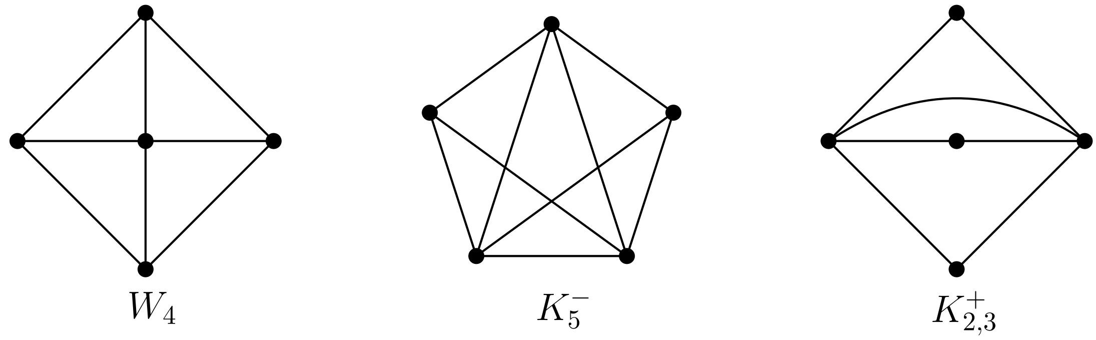


Figure 9.2: The graphs W_4, K_5^-, and $K_{2,3}^+$.

	General	Induced
Subgraph	$H \in \mathcal{S}$	$H \subseteq_{\text{is}} P_3$ or H is edgeless
Topological minor	H is subcubic and planar	$H \subseteq_{\text{is}} C_3$, $H \subseteq_{\text{is}} C_4$, $H \cong K_4^-$, or H is edgeless
Minor	H is planar	$H \subseteq_{\text{is}} W_4$, $H \subseteq_{\text{is}} K_5^-$, $H \subseteq_{\text{is}} K_{2,q}$, or $H \subseteq_{\text{is}} K_{2,q}^+$ for some $q \in \mathbb{N}$

Table 9.1: Summary of (tw, ω)-bounded graph classes excluding a fixed graph H for six graph containment relations [76].

9.1 Forbidding a subgraph, topological minor, or minor

We start by considering the three non-induced variants subgraph, topological minor and minor relations. To this end, we rely on several known results on treewidth and graph minors theory. First, let us prove the following result.

Lemma 9.1.1 (Lemma 5.1 in [76]). *Let H be a graph and let $\mathcal{G}$ be a graph class contained in the class of H-subgraph-free graphs. Then $\mathcal{G}$ is (tw, ω)-bounded if and only if $\mathcal{G}$ has bounded treewidth.*

Proof. Note that if in a graph class treewidth is bounded, then the graph class is also (tw, ω)-bounded. Thus, we only need to show the converse direction. Assume that the graph class $\mathcal{G}$ is (tw, ω)-bounded, with a binding function f. Note that no graph $G \in \mathcal{G}$ can have a clique of size $|V(H)|$, since otherwise G would not be H-subgraph-free. Hence, for every graph $G \in \mathcal{G}$ we have $\omega(G) \leq k$, where $k = |V(H)| - 1$. It follows that $\text{tw}(G) \leq \max\{f(1), \ldots, f(k)\}$. □

Let us now state an important result due to Robertson and Seymour [177] from their series on graph minors.

Theorem 9.1.2 (Robertson and Seymour [177]). *For every planar graph H, the class of H-minor-free graphs has bounded treewidth.*

We also need the following result on the graph from the class $\mathcal{S}$.

Lemma 9.1.3 (Golovach et al. [110]). *For every graph $H \in \mathcal{S}$, a graph G is H-subgraph-free if and only if it is H-minor-free.*

We are now ready to characterize the graphs H for which the class of H-subgraph-free graphs is (tw, ω)-bounded.

Theorem 9.1.4 (Theorem 5.4 in [76]). *For every graph H, the following conditions are equivalent.*

1. *The class of H-subgraph-free graphs is (tw, ω)-bounded.*
2. *The class of H-subgraph-free graphs has bounded treewidth.*

3. *$H \in \mathcal{S}$.*

Proof. Equivalence between Statements 1 and 2 follows from Lemma 9.1.1.

Suppose now that the class of H-subgraph-free graphs has bounded treewidth. By Corollary 9.0.5, H must be a subgraph of some elementary wall. It follows that H is subcubic. Suppose next that H contains a connected component with two vertices u and v of degree 3 and let ℓ be the distance between u and v. Then the class of ℓ-subdivided walls is a subclass of the class of H-subgraph-free graphs, a contradiction with Corollary 9.0.5. Thus, every connected component of H has at most one vertex of degree 3. By a similar reasoning, we infer that H is also acyclic, and thus $H \in \mathcal{S}$.

Finally, suppose that $H \in \mathcal{S}$. Then following Lemma 9.1.3 every H-subgraph-free graph is also H-minor-free. Hence, by Theorem 9.1.2, the class of H-subgraph-free graphs has bounded treewidth. □

Let us now consider the topological minor relation. We first state the following known result.

Lemma 9.1.5 (see, e.g., Diestel [81]). *A subcubic graph H is a minor of a graph G if and only if H is a topological minor of G.*

We are now ready to use a similar approach as in the case of subgraph relation.

Theorem 9.1.6 (Theorem 5.6 in [76]). *For every graph H, the following conditions are equivalent.*

1. *The class of H-topological-minor-free graphs is* (tw, ω)*-bounded.*
2. *The class of H-topological-minor-free graphs has bounded treewidth.*
3. *H is subcubic and planar.*

Proof. Since every H-topological-minor-free graph is also H-subgraph-free, the equivalence between Statements 1 and 2 is implied by Lemma 9.1.1.

Next, suppose that the class of H-topological-minor-free graphs has bounded treewidth. Due to Corollary 9.0.5, H is a topological minor of some elementary wall. Thus, since every elementary wall is both subcubic and planar, H must also be subcubic and planar.

Finally, suppose that H is subcubic and planar. Since H is subcubic, by Lemma 9.1.5 we obtain that every H-topological-minor-free graph is also H-minor-free. Furthermore, since H is planar, by Theorem 9.1.2, the class of H-topological-minor-free graphs has bounded treewidth. □

To conclude this section we now state a similar theorem in terms of the minor relation.

Theorem 9.1.7 (Theorem 5.7 in [76]). *For every graph H, the following conditions are equivalent.*

1. *The class of H-minor-free graphs is* (tw, ω)*-bounded.*
2. *The class of H-minor-free graphs has bounded treewidth.*
3. *H is planar.*

Proof. Since every H-minor-free graph is also H-subgraph-free, we can again use Lemma 9.1.1 to infer that Statements 1 and 2 are equivalent.

Next, suppose that the class of H-minor-free graphs has bounded treewidth. By Corollary 9.0.5, H is a minor of some elementary wall. Thus, H is planar.

Finally, suppose that H is planar. Then Theorem 9.1.2 implies that the class of H-minor-free graphs has bounded treewidth. □

9.2 Forbidding an induced subgraph or an induced topological minor

We next consider graph classes excluding a graph H as in induced subgraph or as an induced topological minor. Due to Lemma 9.0.1 and Corollary 9.0.5 we can immediately get the following characterization of (tw, ω)-bounded graph classes when excluding a single forbidden induced subgraph.

Theorem 9.2.1 (Theorem 3.1 in [76]). *Let H a graph. Then, the class of H-free graphs is* (tw, ω)*-bounded if and only if one of the following conditions holds.*

1. *$H \subseteq_{\mathrm{is}} P_3$, with a binding function $f(k) = k - 1$.*
2. *H is edgeless, with a binding function $f(k) = R(k + 1, |V(H)|) - 2$.*

Proof. If H is edgeless, then we can apply Lemma 9.0.1. Suppose that $H \subseteq_{\mathrm{is}} P_3$. Then every H-free graph G is also P_3-free and G is a disjoint union of complete graphs. Thus, $\mathrm{tw}(G) = \omega(G) - 1$.

Suppose now that H is neither edgeless nor an induced subgraph of P_3, and that the class of H-free graphs is (tw, ω)-bounded. By Corollary 9.0.5, H is an induced subgraph of some complete bipartite graph and also an induced subgraph of the line graph of some elementary wall. In particular, H must be isomorphic to a complete bipartite graph $K_{p,q}$, with $1 \le p \le q$ (note that we must have $p, q \ge 1$ since H is not edgeless). Furthermore, since line graphs of elementary walls are $\{\text{claw}, C_4\}$-free (or, equivalently, $\{K_{1,3}, K_{2,2}\}$-free), we infer that H must be isomorphic to either $K_{1,1}$ or $K_{1,2}$. Thus, H is an induced subgraph of P_3, a contradiction. □

Let us now consider the induced topological minor relation. Towards this end we need the following definition. A *block-cactus graph* is a graph in which every block is either a cycle or a complete graph. Hartinger, in her PhD thesis [121], proved the following result.

Lemma 9.2.2 (Hartinger [121])**.** *A graph G is a block-cactus graph if and only if G is K_4^--induced-minor-free.*

In fact, the approach described in the thesis can also be used to show that the two properties are also equivalent to excluding K_4^- as an induced topological minor.

Lemma 9.2.3 (Lemma 3.2 in [76])**.** *Let G be a graph. Then, the following conditions are equivalent:*

1. *G is K_4^--induced-minor-free.*
2. *G is K_4^--induced-topological-minor-free.*
3. *G is a block-cactus graph.*

Proof. Since every induced topological minor in G is also an induced minor, G is K_4^--induced-topological-minor-free if it is K_4^--induced-minor-free.

Suppose that G is K_4^--induced-topological-minor-free and that G is not a block-cactus graph. We first show that G contains a hole. Suppose not. Then G is a chordal K_4^--free graph, and thus a block graph (see [148]), that is, a graph every block of which is a complete graph. But then, G is a block-cactus graph, a contradiction. Hence, G must contain a hole C, and in particular there exists some block B of G such that $V(C) \subseteq V(B)$. Since B is connected but not a cycle, there exists a vertex $x \in V(B)\setminus V(C)$ with a neighbor in $V(C)$. If $|N(x) \cap V(C)| \geq 2$, it is easy to see that G contains a subdivision of K_4^- as an induced subgraph, a contradiction. Thus, $|N(x) \cap V(C)| = 1$ and every vertex in $V(B) \setminus V(C)$ has at most one neighbor in C. Now, take a vertex $z \in V(B) \setminus V(C)$ which has a neighbor $v \in V(C)$ such that z minimizes the length of a shortest path P between z and C not containing v. We know that P must exist since B has no cut-vertex. Also, we may assume that v has no other neighbor in P, otherwise we could replace z with this vertex and get a shorter path. Let $v' \in V(C)$ be the vertex of P in $V(C) \setminus \{v\}$ and z' be the neighbor of v' in P. Recall that z' has only one neighbor in $V(C)$. Using a similar argument as for v, we may assume that v' has no other neighbor in P. The minimality of P implies that the internal vertices of P do not have a neighbor in C. Hence, $G[V(C) \cup V(P)]$ is a subdivision of K_4^-, a contradiction. This shows that every K_4^--induced-topological-minor-free graph is a block-cactus graph.

Finally, let G be a block-cactus graph and let H be an induced minor of G. It is not difficult to see that the class of block-cactus graphs is closed under vertex deletions and edge contractions. Therefore, H is also a block-cactus graph. Since K_4^- is not a block-cactus graph, H cannot be isomorphic to K_4^-. Therefore, G is K_4^--induced-minor-free. □

In the case of block-cactus graphs we can easily derive the following result.

Lemma 9.2.4 (Lemma 3.3 in [76]). *The class of block-cactus graphs is* (tw, ω)*-bounded, with a binding function* $f(k) = \max\{k - 1, 2\}$.

Proof. The treewidth of a graph G is the maximum treewidth of its blocks (see, e.g., [28]). Since the treewidth of a complete graph of order k is $k-1$ and the treewidth of a cycle is two, the result follows. □

We can now state the following characterization of (tw, ω)-bounded graph classes when excluding a single forbidden induced topological minor.

Theorem 9.2.5 (Theorem 3.4 in [76]). *Let H be a graph. Then, the class of H-induced-topological-minor-free graphs is* (tw, ω)*-bounded if and only if one of the following conditions holds.*

1. $H \subseteq_{\mathrm{is}} C_3$ *or* $H \subseteq_{\mathrm{is}} C_4$*, in which case the binding function is* $f(k) = k - 1$.
2. $H \cong K_4^-$*, in which case a binding function is* $f(k) = \max\{k - 1, 2\}$.
3. H *is edgeless, in which case a binding function is* $f(k) = R(k + 1, |V(H)|) - 2$.

Proof. If H is edgeless, then Lemma 9.0.1 applies. If $H \subseteq_{\mathrm{is}} C_3$ or $H \subseteq_{\mathrm{is}} C_4$, then $H \subseteq_{\mathrm{itm}} C_4$. Hence, by Observation 8.0.1, the class of H-induced-topological-minor-free graphs is a subclass of the class of chordal graphs, and is thus (tw, ω)-bounded. If $H \cong K_4^-$, then according to Lemma 9.2.3 the class of H-induced-topological-minor-free graphs is the class of block-cactus graphs, and Lemma 9.2.4 applies.

For the converse direction, suppose that $H \not\subseteq_{\mathrm{is}} C_3$, $H \not\subseteq_{\mathrm{is}} C_4$, $H \not\cong K_4^-$, H is not edgeless, and that the class of H-induced-topological-minor-free graphs is (tw, ω)-bounded. By Corollary 9.0.5, H is an induced topological minor of some complete bipartite graph and an induced topological minor of the line graph of some 1-subdivided wall. Since the line graph of every 1-subdivided wall is planar, subcubic, and claw-free, H must also be planar, subcubic, and claw-free. Furthermore, since H is an induced topological minor of some complete bipartite graph, we must have that $H \subseteq_{\mathrm{itm}} K_{2,3}$, since otherwise either H would not be planar or it would not be subcubic. Finally, claw-freeness implies that $H \in \{P_2, P_3, C_3, C_4, K_4^-\}$, a contradiction. □

9.3 Forbidding an induced minor

Finally, we turn to graph classes excluding a single graph H as an induced minor. First, we will show that the two classes of K_5^-- and W_4-induced-minor-free graphs are (tw, ω)-bounded.

In 2018, Belmonte et al. [21] and Brešar et al. [46] observed that the following result can be derived from the proof of Theorem 9 in [132].

Theorem 9.3.1. *For every graph F and every planar graph H, the class of graphs that are both F-minor-free and H-induced-minor-free has bounded treewidth.*

Notice that a complete graph is excluded as a minor if and only if it is excluded as an induced minor. Thus, we get the following corollary to Theorem 9.3.1.

Corollary 9.3.2 (Corollary 4.4 in [76])**.** *For every positive integer p and every planar graph H, the class of $\{K_p, H\}$-induced-minor-free graphs has bounded treewidth.*

Let us recall that the Hadwiger number $\eta(G)$ is the largest value p such that K_p is a minor of G. From the definition it follows that no graph G contains $K_{\eta(G)+1}$ as a minor or as an induced minor. We can thus prove the following result.

Corollary 9.3.3 (Corollary 4.5 in [76])**.** *Let H be a planar graph. The class of H-induced-minor-free graphs is (η, ω)-bounded if and only if it is (tw, ω)-bounded.*

Proof. Suppose that the class of H-induced-minor-free graphs is (η, ω)-bounded and let f be an (η, ω)-binding function for the class. Let $k \in \mathbb{N}$ and let G be an H-induced-minor-free graph with $\omega(G) = k$. Then $\eta(G) \leq f(k)$, that is, G is $K_{f(k)+1}$-induced-minor-free. By Corollary 9.3.2, the treewidth of G can be bounded from above by some constant $g(k)$ depending only on k. Thus, g is a (tw, ω)-binding function for the class.

The converse direction holds due to the fact that $\eta(G) \leq \mathrm{tw}(G) + 1$ which follows from Lemma 9.0.2. □

Since both K_5^- and W_4 are planar, by Corollary 9.3.3, it suffices to show that the class of K_5^--induced-minor-free graphs and the class of W_4-induced-minor-free graphs are (η, ω)-bounded. We begin with the former.

Theorem 9.3.4 (Theorem 4.1 in [76])**.** *For each $p \geq 2$, the class of K_p^--induced-minor-free graphs is (η, ω)-bounded, with a binding function $f(k) = \max\{2p - 4, k\}$.*

Proof. Fix $p \geq 2$ and $k \in \mathbb{N}$, and let G be a K_p^--induced-minor-free graph with $\omega(G) = k$. Let $q = \max\{2p - 4, k\} + 1$. We want to show that G contains no K_q as a minor. Suppose for a contradiction that G contains K_q as a minor. Fix a minor model $M = (X_u : u \in V(K_q))$ of K_q in G such that the total number of vertices in the bags, that is, the sum $\sum_{u \in V(K_q)} |X_u|$, is minimized.

If for all $u \in V(K_q)$ we have $|X_u| = 1$, then the set $\bigcup_{u \in V(K_q)} X_u$ is a clique in G, implying that $\omega(G) \geq |V(K_q)| = q \geq k + 1$, a contradiction. Therefore, there exists some $u \in V(K_q)$ such that $|X_u| \geq 2$. Furthermore, note that for every vertex $y \in X_u$ there exists a vertex $v(y)$ of $K_q - u$ such that y has no neighbors in $X_{v(y)}$, since otherwise replacing the bag X_u with $\{y\}$ would result in a minor model of K_q smaller than M. Since $|X_u| \geq 2$ and the subgraph of G induced by X_u is connected, there exists a vertex $x \in X_u$ such that the subgraph of G induced by $X_u \setminus \{x\}$ is connected. (For example, take x to be a leaf of a spanning tree of $G[X_u]$.)

Let Z be the set of vertices $z \in V(K_q) \setminus \{u\}$ such that x has a neighbor in X_z. Suppose first that $|Z| \geq (q-1)/2$. Recall that $X_{v(x)}$ is a bag in which x has no neighbor. In particular, $v(x) \neq u$ and $v(x) \notin Z$. Then, the bags from $(X_z : z \in Z)$ along with $\{x\}$ and $X_{v(x)}$ form an induced minor model of $K^-_{|Z|+2}$. Since $|Z| + 2 \geq (q-1)/2 + 2 \geq (2p-4)/2 + 2 = p$, we obtain a contradiction with the fact that G is K^-_p-induced-minor-free.

Finally, suppose that $|Z| < (q-1)/2$. The minimality of M implies that Z is non-empty and for some $w \in Z$ we have $\left(\bigcup_{v \in X_w} N(v)\right) \cap X_u = \{x\}$. Let $Z' = V(K_q) \setminus (Z \cup \{u\})$. Note that for every vertex $z \in Z'$ there exists an edge from X_z to $X_u \setminus \{x\}$. Since $|Z| + |Z'| = q-1$ and $|Z| < (q-1)/2$, we have $|Z'| \geq (q-1)/2$. Furthermore, $w \in Z$ and hence $w \notin Z'$. Thus, the bags from $(X_z : z \in Z')$ along with $X_u \setminus \{x\}$ and X_w form an induced minor model of $K^-_{|Z'|+2}$, leading again to a contradiction with the fact that G is K^-_p-induced-minor-free. $\square$

Using similar techniques but with more involved arguments we can now show that also the class of W_4-induced-minor-free graphs is (η, ω)-bounded.

Theorem 9.3.5 (Theorem 4.2 in [76])**.** *The class of W_4-induced-minor-free graphs is (η, ω)-bounded, with a binding function $f(k) = k + 5$.*

Proof. Fix a positive integer k and let G be a W_4-induced-minor-free graph with $\omega(G) = k$. Let $q = k + 5$ (note that $q \geq 6$) and F be the graph K^-_q. We claim that G does not contain F as an induced minor. We denote by $U \subset V(F)$ the set of universal vertices in F. To derive a contradiction, suppose that G contains F as an induced minor and fix an induced minor model $M = (X_u : u \in V(F))$ of F in G such that the size of $\bigcup_{u \in U} X_u$ is minimized. We will refer to this condition as *property* $(\star)$. We denote by x and y the two non-adjacent vertices in F. It is clear that if for all $u \in U$ we have $|X_u| = 1$, then the set $\bigcup_{u \in U} X_u$ is a clique in G, a contradiction since $|U| = q - 2 > k = \omega(G)$. Hence, there exists a vertex $u \in U$ such that $|X_u| \geq 2$.

Partition the bag X_u arbitrarily into two non-empty bags X_{u_1} and X_{u_2}, both inducing a connected subgraph in G. (For example, we can take ℓ to be a leaf of a spanning tree of $G[X_u]$ and set $X_{u_1} = \{\ell\}$ and $X_{u_2} = X_u \setminus \{\ell\}$.) Let M' be the collection of bags obtained from M by removing the bag X_u and adding the bags X_{u_1} and X_{u_2}. Let F' be the graph obtained from the subgraph of G induced by the union of bags in M' by contracting each of the bags in M' into a single vertex. Note that the vertex set of F' is $(V(F) \setminus \{u\}) \cup \{u_1, u_2\}$ and that M' is an induced minor model of F' in G. In particular, F' is an induced minor of G. Notice that u_1 and u_2 are adjacent in F'. Let $U' = U \setminus \{u\}$ and observe that $U' \subseteq V(F')$. Note that $d_{F'}(u_1) \geq 2$, otherwise u_1 would only be adjacent to u_2, and thus we could replace X_u with X_{u_2} in M to obtain an induced minor model of F in G that would contradict the fact that M satisfies property ($\star$). For the same reason, $d_{F'}(u_2) \geq 2$.

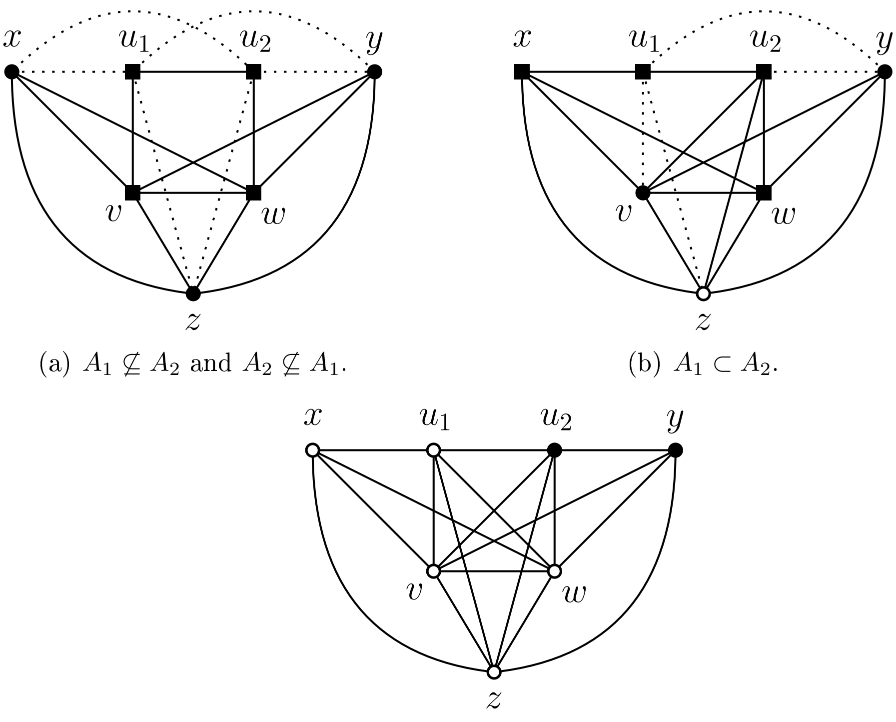
Suppose first that $d_{F'}(u_1) = 2$. Let v be the neighbor of u_1 different from u_2. If $v = x$, then we could redefine $X_u = X_{u_2}$ and $X_x = X_x \cup X_{u_1}$ in M to obtain an induced minor model of F in G showing that M does not respect property ($\star$). Thus, $v \neq x$. Similarly, $v \neq y$. Consequently, $v \in U'$. The fact that M satisfies property ($\star$) also implies that v is not adjacent to u_2. Since $|U'| = |U| - 1 = q - 3 > 2$, there is a vertex $w \in U' \setminus \{v\}$. Note that w is not adjacent to u_1, and hence must be adjacent to u_2. We obtain that $\{x, u_2, y, v\}$ induces a C_4 in F' and $\{x, u_2, y, v\} \subseteq N(w)$. Therefore, G contains W_4 as an induced minor, a contradiction. Thus, we have $d_{F'}(u_1) \geq 3$. By symmetry, we also have $d_{F'}(u_2) \geq 3$.

For $i \in \{1, 2\}$, let A_i be the set of vertices in U' adjacent to u_i. By symmetry, it suffices to consider the following two cases depending on A_1 and A_2.

Case (1): $A_1 \nsubseteq A_2$ and $A_2 \nsubseteq A_1$.
Let $v \in A_1 \setminus A_2$ and $w \in A_2 \setminus A_1$. Notice that v and w are adjacent, and therefore $\{v, u_1, u_2, w\}$ induces a C_4 in F'. Suppose first that u_1 is adjacent to neither x nor y. Then u_2 is adjacent to both x and y. Furthermore, since $d_{F'}(u_1) \geq 3$, vertex u_1 must have a neighbor $z \in U' \setminus \{v, w\}$. Hence, every vertex in $\{v, u_1, u_2, w\}$ has a neighbor in the set $\{x, y, z\}$. Since $\{x, y, z\}$ induces a connected subgraph of F', we infer that W_4 is an induced minor of F', and thus of G, a contradiction. A similar conclusion is obtained if u_2 is adjacent to neither x nor y. We may thus assume that u_1 is adjacent to either x or y, and the same for u_2. Since $|U'| = |U| - 1 = q - 3 \geq 3$, there is a vertex $z \in U' \setminus \{v, w\}$. Again, since $\{x, y, z\}$ induces a connected subgraph of F', we conclude that W_4 is an induced minor of F, and thus of G, a contradiction. See Figure 9.3(a) for an illustration.

Case (2): $A_1 \subseteq A_2$.
Necessarily, $A_2 = U'$, and hence u_2 cannot be adjacent to both x and y, otherwise M would not satisfy property ($\star$). Without loss of generality, assume that u_2 is not adjacent to x. Then u_1 is adjacent to x.

Suppose first that A_1 is a proper subset of A_2. Then there exists a vertex $w \in A_2 \setminus A_1$. Note that the vertices $\{x, u_1, u_2, w\}$ induce a C_4 in F'. We claim that $A_1 = \emptyset$. Indeed, suppose for a contradiction that there exists a vertex $v \in A_1$. Then $v \neq w$ and v is universal in F'. Therefore, F' contains an induced copy of W_4 in F' with vertex set $\{x, u_1, u_2, w, v\}$; in particular, W_4 is an induced minor of G, a contradiction. Thus, $A_1 = \emptyset$, as claimed. This means that u_1 does not have any neighbors in U', and hence using the inequality $d_{F'}(u_1) \geq 3$ we infer that $N_{F'}(u_1) = \{u_2, x, y\}$. Recall that $|U'| \geq 3$. Choose any vertex $z \in U' \setminus \{w\}$. Note that every vertex in F'



(a) $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$.

(b) $A_1 \subset A_2$.

(c) $A_1 = A_2 = U$.

Figure 9.3: Representation of the different cases considered in the proof of Theorem 9.3.5. The induced minor contains all plain edges and is a subgraph of the graph induced by plain and dotted edges. Black squared vertices induce a C_4 and black round vertices are contracted into a single vertex (see [76]).

is adjacent to either y or z. In particular, since $\{y, z\} \cap \{x, u_1, u_2, w\} = \emptyset$ and $\{y, z\}$ induces a connected subgraph of F', we conclude that W_4 is an induced minor of F, and thus of G, a contradiction. See Figure 9.3(b) for an illustration.

We may thus assume that $A_1 = A_2$. The fact that M satisfies property $(\star)$ implies that u_1 cannot be adjacent to both x and y. Thus, since u_1 is adjacent to x, it is not adjacent to y. Consequently, u_2 is adjacent to y. Now, observe that the graph obtained by contracting the edge $\{u_2, y\}$ in F' is isomorphic to F. Hence, we can modify M by redefining $X_u = X_{u_1}$ and $X_y := X_y \cup X_{u_2}$ and get a minor model of F. However, this implies that M does not respect property $(\star)$, a contradiction. See Figure 9.3(c) for an illustration.

We conclude that G is K_q^--induced-minor-free, and following Theorem 9.3.4 we obtain that $\eta(G) \leq \max\{2q - 5, k\} = 2k + 1$. $\square$

As a consequence of Theorem 9.3.4 and Theorem 9.3.5, together with Corollary 9.3.3 we obtain the following two results.

Corollary 9.3.6 (Corollary 4.7 in [76]). *The class of W_4-induced-minor-free graphs is (tw, ω)-bounded.*

Corollary 9.3.7 (Corollary 4.6 in [76]). *The class of K_5^--induced-minor-*

free graphs is (tw, ω)*-bounded.*

Finally, we finish the complete dichotomy in terms of one forbidden graph by showing that the class of $K_{2,q}$-induced-minor-free graphs is (tw, ω)-bounded for which we need several definitions from the beginning of Chapter 8. We make use of the following well known result due to Skodinis [184].

Theorem 9.3.8 (Skodinis [184]). *Let s be a positive integer and let $\mathcal{G}$ be the class of graphs in which all minimal separators have size at most s. Then, $\mathcal{G}$ is (tw, ω)-bounded, with a binding function $f(k) = \max\{k, 2s\} - 1$.*

Using Theorem 9.3.8, we can now prove the following result.

Lemma 9.3.9 (Lemma 4.10 in [76]). *For every $q \in \mathbb{N}$, the class of $K_{2,q}$-induced-minor-free graphs is (tw, ω)-bounded, with a binding function*

$$f(k) = \max\{k, 2R(k+1, q) - 2\} - 1\,.$$

Proof. Fix two positive integers q and k, and let G be a $K_{2,q}$-induced-minor-free graph with $\omega(G) = k$. We claim that every minimal separator in G has size at most $R(k+1, q) - 1$. Suppose this is not the case, and let u and v be two non-adjacent vertices in G such that $|S| \geq R(k+1, q)$ for some minimal u,v-separator S in G. Since $|S| \geq R(k+1, q)$, Ramsey's theorem implies that $G[S]$ contains either a clique of size $k+1$ or an independent set of size q. Since $\omega(G[S]) \leq \omega(G) = k$, we infer that $G[S]$ contains an independent set I of size q. Let C_u and C_v denote the connected components of $G - S$ containing u and v, respectively. By the minimality of S, every vertex in S has a neighbor in C_u and a neighbor in C_v. But now, the sets $V(C_u)$, $V(C_v)$, and $\{x\}$ for all $x \in I$ form the bags of an induced minor model of $K_{2,q}$ in G, a contradiction. Therefore, every minimal separator in G has size at most $R(k+1, q) - 1$. Using Theorem 9.3.8, we obtain that $\mathrm{tw}(G) \leq \max\{k, 2R(k+1, q) - 2\} - 1$. $\square$

Due to Lemma 9.0.4 and Corollaries 9.3.6, and 9.3.7 we can now state the following characterization.

Theorem 9.3.10 (Theorem 4.13 in [76]). *Let H be a graph. Then, the class of H-induced-minor-free graphs is (tw, ω)-bounded if and only if one of the following conditions holds: $H \subseteq_{\mathrm{is}} W_4$, $H \subseteq_{\mathrm{is}} K_5^-$, $H \subseteq_{\mathrm{is}} K_{2,q}$ for some $q \in \mathbb{N}$, or $H \subseteq_{\mathrm{is}} K_{2,q}^+$ for some $q \in \mathbb{N}$.*

Proof. Suppose that the class of H-induced-minor-free graphs is (tw, ω)-bounded. Since, by Lemma 9.0.4, the class of balanced complete bipartite graphs is (tw, ω)-unbounded, H must be an induced minor of some complete bipartite graph $K_{n,n}$. Let $M = (X_u : u \in H)$ be an induced minor model of H in $K_{n,n}$. We define two types of bags in M: the *tiny bags* containing a single vertex and the *large bags* containing at least 2 vertices. It is clear that the set of large bags corresponds to a clique in H, while the union of the tiny bags induces a complete bipartite subgraph of $K_{n,n}$.

Hence, $H \cong K_{p,q} * K_r$ for some $p, q, r \geq 0$ where $*$ represents the join of the two graphs, that is, the addition of all possible edges between vertices in $K_{p,q}$ and vertices in K_r. Without loss of generality, we assume that $p \leq q$. Observe that H needs to be planar, otherwise the class of H-induced-minor-free graphs would contain the class of elementary walls, which by Lemma 9.0.4 is (tw, ω)-unbounded. Hence, we can analyze the possible values for p, q, and r that allow H to be planar. Let us first notice that $p \leq 2$, as otherwise H would contain $K_{3,3}$ as a subgraph and would thus not be planar. Similarly, $r \leq 4$ since otherwise H would contain K_5 as a subgraph. Also, it is easily observed that if $p = 1$, then $H \cong K_{0,q} * K_{r+1}$, and similarly if $q = 1$, then $H \cong K_{p,0} * K_{r+1}$. Hence, we may assume that $p \in \{0, 2\}$ and $q \neq 1$. Consider the following cases:

- Case $r = 4$: Then $p = q = 0$, otherwise $K_5 \subseteq_{\mathrm{s}} H$. Hence, $H \cong K_4$.
- Case $r = 3$: Then $p = 0$, otherwise $K_{3,3} \subseteq_{\mathrm{s}} H$. If $q \geq 3$, then $K_{3,3} \subseteq_{\mathrm{s}} H$, and thus $q \leq 2$. If $q = 0$, then $H \cong K_3$, and if $q = 2$, then $H \cong K_5^-$.
- Case $r = 2$: Then $p = 0$, otherwise $K_{3,3} \subseteq_{\mathrm{s}} H$. This implies that $H \cong K_{2,q}^+$.
- Case $r = 1$: If $p = 2$, then $q = 2$ (since otherwise $K_{3,3} \subseteq_{\mathrm{s}} H$) and $H \cong W_4$. If $p = 0$, then $H \cong K_{1,q}$.
- Case $r = 0$: Then H is edgeless or $H \cong K_{2,q}$.

Thus, $H \subseteq_{\mathrm{is}} W_4$, $H \subseteq_{\mathrm{is}} K_5^-$, $H \subseteq_{\mathrm{is}} K_{2,q}$, or $H \subseteq_{\mathrm{is}} K_{2,q}^+$, for some $q \in \mathbb{N}$, as desired.

For the converse, suppose first that $H \subseteq_{\mathrm{is}} K_{2,q}$ or $H \subseteq_{\mathrm{is}} K_{2,q}^+$ for some $q \in \mathbb{N}$. It is not difficult to notice that $K_{2,q}^+$ is an induced minor of $K_{2,q+1}$, obtained by contracting one edge. From Lemma 9.3.9 it then follows that the class of H-induced-minor-free graphs is (tw, ω)-bounded. If $H \subseteq_{\mathrm{is}} W_4$ or $H \subseteq_{\mathrm{is}} K_5^-$, then Corollaries 9.3.6 and 9.3.7 apply. □

Chapter 10

Tree Decompositions with Bounded Independence Number

Let G be a graph and let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of G. The *independence number* of $\mathcal{T}$, denoted by $\alpha(\mathcal{T})$, is defined as the maximum independence number over all bags X_t of a graph $G[X_t]$ induced on the vertices of X_t. The *tree-independence number* of G, denoted by $\mathsf{tree}\text{-}\alpha(G)$, is then the minimum independence number among all possible tree decompositions of G. This parameter was initially introduced by Yolov [209] (under the name α-treewidth) and rediscovered independently in [77, 78]. Note that, unlike in treewidth, we are not interested in tree decompositions of smallest possible width, but rather those with smallest possible independence number.

Our main motivation to study the tree-independence number is as follows. Consider a tree decomposition of constant width. Then, for a given problem, each bag of the tree decomposition interacts with an optimal solution to the problem only in a bounded number of ways, all of which can be enumerated efficiently. Using this fact we may often use dynamic programming approach which leads to efficient algorithms for the given problem. To generalize the idea, we may, instead of constant width, require that for each bag there is only a polynomial number of ways in which an optimal solution can interact with the bag, all of which can be enumerated efficiently. A particular case is when for each bag, the number of vertices that interact with an optimal solution is bounded. In recent years, this approach was independently suggested also by Maria Chudnovsky in several of her talks [54, 55, 56, 57]. To this end, note that bounded tree-independence number implies that the number of vertices that an optimal solution to the Max Weight Independent Set problem can contain from any given bag is bounded. Given a graph G together with vertex weights for each vertex of G, Max Weight Independent Set problem asks to find an independent set in G with the largest possible weight sum.

10.1 Tree-independence number: basic properties

With Theorem 8.2.6 we have already mentioned a couple possible characterizations of chordal graphs. We now show that chordal graphs can also be characterized using the tree-independence number.

Theorem 10.1.1 (Theorem 3.3 in [77]). *Let G be a graph. Then* $\mathsf{tree}\text{-}\alpha(G) \leq 1$ *if and only if G is chordal.*

Proof. Suppose that G is a chordal graph. Then, by Theorem 8.2.6, G has a clique tree and thus $\mathsf{tree}\text{-}\alpha(G) \leq 1$. Conversely, if G has a tree decomposition $\mathcal{T}$ with independence number at most one, then every bag of $\mathcal{T}$ is a clique in G. Since $\mathcal{T}$ is a tree decomposition, for every vertex u of G the subgraph T_u of T induced by the set $\{t \in V(T) : u \in X_t\}$ is a tree. Furthermore, since each bag of $\mathcal{T}$ is a clique, two distinct vertices u and v of G are adjacent if and only if they belong to a same bag, which is in turn equivalent to the condition that T_u and T_v have a vertex in common. Thus, G is the intersection graph of the collection of subtrees $\{T_u : u \in V(G)\}$. Applying Theorem 8.2.6, we conclude that G is chordal. □

In particular we have that $\mathsf{tree}\text{-}\alpha(K_n) = 1$ for all n. We now prove that the tree-independence number of all cycles (with the exception of C_3) is equal to 2.

Theorem 10.1.2 (Lemma 4.2 in [78]). *For every integer $n \geq 4$, the tree-independence number of the cycle C_n is exactly 2.*

Proof. Since C_n is not a chordal graph, Theorem 10.1.1 implies that $\mathsf{tree}\text{-}\alpha(C_n) \geq 2$. Let $v_1, \dots, v_n$ be an order of the vertices of C_n along the cycle. We construct a tree decomposition $\mathcal{T} = (T, \{X_t : t \in V(T)\})$ of C_n as follows. The tree T is a path on $(n-2)$ vertices $(t_1, \dots, t_{n-2})$. For each $i \in \{1, \dots, n-2\}$, the bag X_{t_i} consists of vertices $\{v_i, v_{i+1}, v_n\}$. Note that in every bag X of $\mathcal{T}$ there exist two consecutive vertices of C_n, and hence $G[X]$ contains at least one edge. Furthermore, each bag X contains exactly 3 vertices. This implies that $\alpha(G[X]) \leq |X| - 1 = 2$. Hence, $\alpha(\mathcal{T}) \leq 2$ and consequently $\mathsf{tree}\text{-}\alpha(C_n) \leq 2$. □

Recall that a block-cactus graphs is a graph in which every block is a cycle or a complete graph. Theorems 10.1.1 and 10.1.2 imply the following.

Corollary 10.1.3 (Corollary 4.3 in [78]). *The tree-independence number of any block-cactus graph is at most 2.*

Next, observe that the trivial tree decomposition of any graph G (i.e., all the vertices are contained in the unique bag) has independence number equal to $\alpha(G)$. Therefore, the following observation immediately follows.

Observation 10.1.4 (Observation 3.7 in [77]). For every graph G we have $\mathsf{tree}\text{-}\alpha(G) \leq \alpha(G)$.

Let us now prove the following lemma, which we will also use in order to prove that the tree-independence number is NP-hard to compute.

Lemma 10.1.5 (Lemma 3.4 in [77]). *Let G be a graph and let G' be the graph obtained from two disjoint copies of G by adding all possible edges between them. Then* $\mathsf{tree}\text{-}\alpha(G') = \alpha(G)$.

Proof. Let us denote by G_1 and G_2 the two disjoint copies of G such that $V(G') = V(G_1) \cup V(G_2)$. Observe that every independent set in G' is entirely contained in either G_1 or G_2 and hence, we have $\alpha(G') = \alpha(G)$. Thus, the trivial tree decomposition of G' has independence number equal to $\alpha(G)$ and consequently $\mathsf{tree}\text{-}\alpha(G') \leq \alpha(G)$.

For the converse direction, let us consider an arbitrary tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G'. By Lemma 8.2.5, there exists a vertex $v \in V(G)$ and a node $t \in V(T)$ such that $N[v] \subseteq X_t$. Assume without loss of generality that $v \in V(G_1)$. Then $V(G_2) \subseteq N(v) \subseteq X_t$, and therefore $\alpha(G'[X_t]) \geq \alpha(G_2) = \alpha(G)$. Thus, every tree decomposition of G' contains a bag inducing a subgraph with independence number at least $\alpha(G)$. This shows that $\mathsf{tree}\text{-}\alpha(G') \geq \alpha(G)$. Therefore, equality must hold. $\square$

As a consequence of Lemma 10.1.5, there exists an infinite family of graphs attaining the equality from Observation 10.1.4, namely, the family of complete balanced bipartite graphs $K_{n,n}$.

Corollary 10.1.6 (Corollary 3.6 in [77]). *For every positive integer n, we have*

$$\mathsf{tree}\text{-}\alpha(K_{n,n}) = n\,.$$

On the other hand, the gap between tree-independence number and independence number can be arbitrarily large, as can be seen already by trees. Moreover, since computing the independence number of a graph is NP-hard (see [147]), the hardness result follows directly from Lemma 10.1.5.

Theorem 10.1.7 (Theorem 3.5 in [77]). *Computing the tree-independence number of a given graph is NP-hard.*

Another hardness result was obtained by Dallard et al. [75] where the authors solved the problem of recognizing graphs with tree-independence number at most k when $k \geq 4$.

Theorem 10.1.8 (Dallard et al. [75]). *For every constant $k \geq 4$, it is NP-complete to decide whether* $\mathsf{tree}\text{-}\alpha(G) \leq k$ *for a given graph G.*

Another way of bounding tree-independence number is by bounding treewidth as is shown with the following theorem.

Theorem 10.1.9 (Theorem 3.8 in [77]). *For every graph G,* $\mathsf{tree}\text{-}\alpha(G) \leq \mathrm{tw}(G) + 1$*, and this bound is sharp: for every integer $k \neq 2$, there exists a graph G such that* $\mathsf{tree}\text{-}\alpha(G) = k$ *and* $\mathrm{tw}(G) = k - 1$.

Proof. From the definitions it directly follows that the independence number of any tree decomposition is at most its width plus one. Thus, taking $\mathcal{T}$

to be a tree decomposition of G with minimum possible width, we obtain $\mathsf{tree}\text{-}\alpha(G) \leq \alpha(\mathcal{T}) \leq \mathit{width}(\mathcal{T}) + 1 = \mathrm{tw}(G) + 1$.

For $k = 1$, the graph K_1 satisfies $\mathsf{tree}\text{-}\alpha(K_1) = k$ and $\mathrm{tw}(K_1) = k - 1$.

Fix $k \geq 3$, let $S = \{1, \ldots, k\}$, and let G be the graph obtained from a complete graph with vertex set S by replacing each of its edges ij with k paths of length two connecting i and j. We claim that $\mathrm{tw}(G) = k - 1$ and $\mathsf{tree}\text{-}\alpha(G) = k$. Since $\mathsf{tree}\text{-}\alpha(G) \leq \mathrm{tw}(G) + 1$, it suffices to show that $\mathrm{tw}(G) \leq k - 1$ and that $\mathsf{tree}\text{-}\alpha(G) \geq k$.

The inequality $\mathrm{tw}(G) \leq k-1$ follows from Theorem 8.2.1 and the observation that the graph G' obtained from G by adding to it all edges between vertices in the set S is a chordal graph with clique number k.

It remains to show that $\mathsf{tree}\text{-}\alpha(G) \geq k$. Consider an arbitrary tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G. We claim that there exists a bag of $\mathcal{T}$ having independence number at least k. This is clearly the case if there exists a bag X_t such that $S \subseteq X_t$. Thus, we may assume that no bag of $\mathcal{T}$ contains S. By Lemma 8.2.4, there exist two distinct vertices $i, j \in S$ that are not contained in the same bag of $\mathcal{T}$. Since i and j are not contained in a same bag of $\mathcal{T}$, the subtrees T_i and T_j are disjoint. Note that, by construction of G, the vertices i and j have k common neighbors. Let $u \in N(i) \cap N(j)$ be such a vertex. Then u belongs to a bag in $\mathcal{T}$ containing i, and similarly for j. In other words, T_u intersects T_i and T_j, which implies that $T_i \cup T_u \cup T_j$ is connected. Let P be the path in T connecting T_i and T_j. Clearly, P is a subgraph of T_u, and thus for any node $t \in P$, the bag X_t of $\mathcal{T}$ contains u. Since this holds for every common neighbor of i and j, every such bag contains $N(i) \cap N(j)$, which is an independent set of size k. Thus, as $\mathcal{T}$ can be any tree decomposition of G, we conclude that $\mathsf{tree}\text{-}\alpha(G) \geq k$. □

Note also that in Theorem 10.1.9 we require that $k \neq 2$, otherwise, if we would have $k = 2$, then that would imply that $\mathsf{tree}\text{-}\alpha(G) = 2$ and $\mathrm{tw}(G) = 1$. However, $\mathrm{tw}(G) = 1$ implies that G is acyclic. In particular, this implies that G is chordal, which is a contradiction with Theorem 10.1.1.

We next show that tree-independence number behaves well under induced minors. Observe that deleting edges can increase the tree-independence number of a graph, thus we may not consider monotonicity under minors.

Proposition 10.1.10 (Proposition 3.9 in [77]). *Let G be a graph and G' an induced minor of G. Then* $\mathsf{tree}\text{-}\alpha(G') \leq \mathsf{tree}\text{-}\alpha(G)$.

Proof. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be an arbitrary tree decomposition of G. First, we show that the deletion of a vertex does not increase the tree-independence number. Let v be a vertex of G. Let $\mathcal{T}'$ be the tree decomposition obtained from $\mathcal{T}$ by removing v from all of the bags that contain it. Observe that $\mathcal{T}'$ is a tree decomposition of $G - v$. Clearly, $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$, and hence $\mathsf{tree}\text{-}\alpha(G - v) \leq \mathsf{tree}\text{-}\alpha(G)$.

Now, we show that the contraction of an edge does not increase the tree-independence number. Let $e = uv$ be an edge of G and G/e denote the graph obtained from G after contracting the edge e. We denote by w the vertex of G/e that corresponds to the contracted edge. We construct a tree decomposition $\mathcal{T}' = (T, \{X'_t\}_{t \in V(T)})$ (where T is the same tree as in $\mathcal{T}$) of G/e as follows. For each node t of $\mathcal{T}'$, we have two cases: if X_t contains neither u nor v, then we set $X'_t = X_t$; otherwise, we set $X'_t = (X_t \setminus \{u, v\}) \cup \{w\}$. We claim that $\mathcal{T}'$ is a tree decomposition of G/e such that $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$. First, observe that $\mathcal{T}'$ is a tree decomposition of G/e, as it satisfies all the defining conditions of a tree decomposition. To verify that $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$, fix a bag X'_t of $\mathcal{T}'$ and let $I \subseteq X'_t$ be an independent set in G/e. If $w \notin I$, then I also corresponds to an independent set in $G[X_t]$, and hence $|I| \leq \alpha(G[X_t])$. On the other hand, if $w \in I$, then either $(I \setminus \{w\}) \cup \{u\})$ or $(I \setminus \{w\}) \cup \{v\})$ is an independent set in $G[X_t]$. In particular, we again get that $|I| \leq \alpha(G[X_t])$. Thus, we have in both cases that $|I| \leq \alpha(\mathcal{T})$. It follows that $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$, and hence $\textsf{tree-}\alpha(G/e) \leq \textsf{tree-}\alpha(G)$.

If G' is an induced minor of G, then G' can be obtained from G by a sequence of vertex deletions and edge contractions, which implies that $\textsf{tree-}\alpha(G') \leq \textsf{tree-}\alpha(G)$. □

Finally, we now show that bounded tree-independence number implies (tw, ω)-boundedness and the existence of a polynomial (tw, ω)-binding function.

Lemma 10.1.11 (Lemma 3.2 in [77]). *For every positive integer k, the class of graphs with tree-independence number at most k is* (tw, ω)*-bounded, with a binding function $f(p) = R(p + 1, k + 1) - 2$, which is a polynomial of degree k.*

Proof. By Lemma 8.1.2 $R(p, k) \leq \binom{p+k-2}{k-1}$ for all positive integers p and k. For fixed k, this is a polynomial in p of degree $k - 1$.

Let us now fix $p \in \mathbb{Z}_+$ and let G be a graph such that $\omega(G) = p$ and $\textsf{tree-}\alpha(G) \leq k$. Fix a tree decomposition $\mathcal{T}$ of G with independence number at most k. Note that every bag of $\mathcal{T}$ induces a subgraph of G with independence number at most k and clique number at most p. Thus, for every bag X of $\mathcal{T}$, Ramsey's theorem implies that $|X| \leq R(p+1, k+1) - 1$. It follows that $\mathrm{tw}(G) \leq R(p + 1, k + 1) - 2$, as claimed. □

10.2 Tree-independence number: forbidding a structure

In the previous chapter we gave a complete characterization of (tw, ω)-bounded graph classes in terms of a single forbidden graph with respect to one of the six considered relations (subgraph, topological minor, minor, and their induced variants). In this and the following sections, we in fact prove that the same characterization in terms of a single forbidden graph holds for classes of graphs of bounded tree-independence number.

First, let us prove that the same characterization holds for the subgraph, topological minor, and minor relations.

Theorem 10.2.1 (Theorem 7.3: parts 1-3 in [78]). *For every graph H, the following statements hold.*

1. *The class of H-subgraph-free graphs has bounded tree-independence number if and only if $H \in \mathcal{S}$.*
2. *The class of H-topological-minor-free graphs has bounded tree-independence number if and only if H is subcubic and planar.*
3. *The class of H-minor-free graphs has bounded tree-independence number if and only if H is planar.*

Proof. Fix a graph H and one of the three graph containment relations. Let $\mathcal{G}$ be the class of graphs excluding H with respect to this relation. Assume first that $\mathcal{G}$ has bounded tree-independence number. Then, by Lemma 10.1.11, $\mathcal{G}$ is (tw, ω)-bounded. Thus, for each of the three graph containment relations, the forward implication holds due to Theorems 9.1.4, 9.1.6, and 9.1.7.

Conversely, Theorems 9.1.4, 9.1.6, and 9.1.7 also imply that $\mathcal{G}$ has bounded treewidth. Then, by Theorem 10.1.9, the tree-independence number is bounded for the class $\mathcal{G}$. □

For the remaining three graph containment relations, let us first prove the following, which is a part of the results of Theorem 7.3: parts 4-6 in [78].

Theorem 10.2.2. *Let H be an edgeless graph and let $\mathcal{G}$ be the class of H-free graphs. Then $\mathcal{G}$ has bounded tree-independence number.*

Proof. Let H be an edgeless graph with n vertices and let $\mathcal{G}$ be H-free. Then every graph in $\mathcal{G}$ has independence number at most $n-1$ and thus, by Observation 10.1.4, $\mathcal{G}$ has bounded tree-independence number. □

We now give a characterization of graph classes with bounded tree-independence number in terms of a single forbidden induced subgraph.

Theorem 10.2.3 (Theorem 7.3: 4 in [78]). *Let H be a graph. The class of H-free graphs has bounded tree-independence number if and only if H is either an induced subgraph of P_3 or an edgeless graph.*

Proof. Fix a graph H and let $\mathcal{G}$ be the class of H-free graphs. Assume that $\mathcal{G}$ has bounded tree-independence number. By Lemma 10.1.11, $\mathcal{G}$ is (tw, ω)-bounded. Then, by Theorem 9.2.1, the forward implication holds.

Conversely, if H is an edgeless graph, then, by Theorem 10.2.2, $\mathcal{G}$ has bounded tree-independence number by $|V(H)| - 1$. We may thus assume that H is an induced subgraph of P_3. In that case, for every graph G in

$\mathcal{G}$, every connected component of G is a complete graph and thus chordal. By Theorem 10.1.1, G has tree-independence number 1. It follows that $\mathcal{G}$ has bounded tree-independence number by 1. □

Next, we give a characterization of graph classes with bounded tree-independence number in terms of a single forbidden induced topological minor.

Theorem 10.2.4 (Theorem 7.3: 5 in [78]). *Let H be a graph. The class of H-itm-free graphs has bounded tree-independence number if and only if H is either an induced topological minor of C_4 or K_4^-, or H is edgeless.*

Proof. Fix a graph H and let $\mathcal{G}$ be the class of H-itm-free graphs. Assume that $\mathcal{G}$ has bounded tree-independence number. By Lemma 10.1.11, $\mathcal{G}$ is (tw, ω)-bounded. Then, by Theorem 9.2.5, the forward implication holds.

Conversely, if H is edgeless, then, again by Theorem 10.2.2, $\mathcal{G}$ has bounded tree-independence number by $|V(H)| - 1$. Next, assume that H is an induced topological minor of C_3. Then, $\mathcal{G}$ is a subclass of the class of C_4-itm-free graphs. By Observation 8.0.1, $\mathcal{G}$ is a subclass of chordal graphs. By Theorem 10.1.1, $\mathcal{G}$ has bounded tree-independence number by 1. Finally, we may assume that H is an induced topological minor of K_4^-. Then, by Lemma 9.2.3, $\mathcal{G}$ is a subclass of the class of block-cactus graph. By Corollary 10.1.3, $\mathcal{G}$ has bounded tree-independence number by 2. □

The cases when H is an induced minor are somewhat more complicated and will be considered in the following sections.

10.3 Tree-independence number: $K_{2,q}$-im-free graphs

Let G be a graph. The *pmc-independence number* of G, denoted by $\alpha_{pmc}(G)$, is the maximum independence number of a subgraph of G induced by some potential maximal clique in G (see Section 8.3 for the definition). Let μ denote the *matrix multiplication exponent*, i.e., the smallest real number such that two $n \times n$ binary matrices can be multiplied in time $\mathcal{O}(n^{\mu+\epsilon})$ for all $\epsilon > 0$. A result from [9] shows that $\mu < 2.37286$. We can now prove that the pmc-independence number is an upper bound on tree-independence number.

Lemma 10.3.1 (Lemma 3.3 in [78]). *Let G be a graph and n be the number of vertices of G. Then $\mathsf{tree}\text{-}\alpha(G) \leq \alpha_{pmc}(G)$. Moreover, a tree decomposition of G with at most n nodes and independence number at most $\alpha_{pmc}(G)$ can be computed in time $\mathcal{O}(n^{\mu} \log n)$.*

Proof. First, we compute a minimal triangulation G' of G in time $\mathcal{O}(n^{\mu} \log n)$ due to [130]. Since every minimal triangulation is a chordal graph, we then use an algorithm due to Berry and Simonet [25] to compute in time $\mathcal{O}(|V(G')| + |E(G')|) = \mathcal{O}(n^2)$ a clique tree $\mathcal{T} = (T, \{X_t : t \in V(T)\})$ of G' with at most $|V(G')| = n$ nodes. Note that every bag of

$\mathcal{T}$ is a maximal clique of G', and hence a potential maximal clique in G. Furthermore, clique tree $\mathcal{T}$ is also a tree decomposition of G. Since the independence number of this tree decomposition is at most $\alpha_{pmc}(G)$, this gives an upper bound on the tree-independence number of G. $\square$

Let us now consider the class of $K_{2,q}$-im-free graphs. Note that for $q = 1$, $K_{2,1} \cong P_3$ and the class of $K_{2,1}$-im-free graphs is a subclass of the class of chordal graphs and so the tree-independence number of the class is bounded by 1. Note that for any $q \geq 2$, the class of $K_{2,q}$-im-free graphs contains the class of chordal graphs (see Observation 8.0.1), in particular, if $q \geq 3$, the class of chordal graphs is properly contained in the class of $K_{2,q}$-im-free graphs.

Lemma 10.3.2 (Lemma 3.2 in [78]). *Let q be a positive integer. A graph G is $K_{2,q}$-induced-minor-free if and only if every minimal separator in G induces a subgraph with independence number less than q.*

Proof. Fix $q \geq 1$ and a graph G. Let S be a minimal separator in G, and let C and D be two S-full components of $G - S$. Note that for any independent set I of G contained in S, deleting from G all vertices in $V(G) \setminus (V(C) \cup V(D) \cup I)$ and then contracting all edges fully contained within C or D yields a graph isomorphic to $K_{2,|I|}$, showing that G contains $K_{2,|I|}$ as an induced minor. Thus, if G is $K_{2,q}$-induced-minor-free, then $\alpha(G[S]) < q$.

Suppose now that G contains $K_{2,q}$ as an induced minor. Fix a bipartition $\{A, B\}$ of $K_{2,q}$ such that $A = \{a_1, a_2\}$ and $B = \{b_1, \ldots, b_q\}$, and let $M = (X_u : u \in V(K_{2,q}))$ be an induced minor model of $K_{2,q}$ in G that minimizes the sum $\sum_{b \in B} |X_b|$. Since each vertex $b \in B$ has degree two in $K_{2,q}$, the minimality of M implies that $|X_b| = 1$ for all $b \in B$. Let $I = \cup_{b \in B} X_b$ and $W = I \cup X_{a_1} \cup X_{a_2}$. Since M is an induced minor model of $K_{2,q}$ in G, the set I is independent in G, while the sets X_{a_1} and X_{a_2} induce connected subgraphs of G with no edges between them. In particular, since every vertex in I has a neighbor in G in both X_{a_1} and X_{a_2}, we infer that I is a minimal separator in the subgraph of G induced by W. Furthermore, since $I \cup (V(G) \setminus W)$ separates X_{a_1} from X_{a_2} in G, there exists a minimal separator S in G such that $I \subseteq S$. Consequently, $\alpha(G[S]) \geq |I| = |B| = q$. $\square$

Observe that Theorem 8.3.5 and Lemma 10.3.2 directly imply the following.

Lemma 10.3.3 (Lemma 3.9 in [78]). *For every integer $q \geq 2$ and every $K_{2,q}$-induced-minor-free graph G, the pmc-independence number of G is at most $2q - 2$.*

Note that in the case when $q = 2$, the upper bound given by Lemma 10.3.3 is not sharp as chordal graphs have pmc-independence number equal to 1 due to the fact that the potential maximal cliques coincide with the maximal cliques. Hence, for every chordal graph G, $\alpha_{pmc}(G) \leq 1$. The bound can also be improved for the case $q = 3$.

Lemma 10.3.4 (Lemma 3.10 in [78]). *Let G be a $K_{2,3}$-induced-minor-free graph. Then $\alpha_{pmc}(G) \leq 3$.*

Proof. Suppose for a contradiction that G is a $K_{2,3}$-induced-minor-free graph that contains an independent set I of size 4 that is contained in a potential maximal clique X. By Theorem 8.3.1, every two non-adjacent vertices in X are in the neighborhood of some component of $G - X$. We can thus fix, for any two distinct vertices $x, y \in I$, a component $C(x, y)$ of $G - X$ in which both x and y have a neighbor. Next, we show that these components are pairwise distinct. Suppose that this is not the case. Then there exists a component C of $G - X$ with at least three neighbors in I. By Theorem 8.3.1, the neighborhood of $V(C)$ is a minimal separator in G. However, by Lemma 10.3.2 no minimal separator in G contains three pairwise non-adjacent vertices, which implies that $|N(V(C)) \cap I| \leq 2$, a contradiction. To complete the proof, let us fix, for any two distinct vertices $x, y \in I$, an arbitrary induced x, y-path $P(x, y)$ in G such that all internal vertices of $P(x, y)$ belong to $C(x, y)$. Writing $I = \{x, y, u, v\}$, we now see that x, y, the path $P(x, y)$, the path formed by taking $P(x, u)$ together with $P(u, y)$, and the path formed by taking $P(x, v)$ together with $P(v, y)$ form an induced subgraph of G isomorphic to a subdivision of $K_{2,3}$. This contradicts the fact that G is $K_{2,3}$-induced-minor-free. $\square$

For $K_{2,3}$-im-free graphs, the bound given by Lemma 10.3.4 is achieved by the 6-cycle. This follows from the fact that $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6)$ has a tree decomposition with exactly two bags $X_1 = \{v_1, v_3, v_5\}$ and $X_2 = \{v_2, v_4, v_6\}$, formed by two disjoint maximum independent sets. Thus, $\alpha_{pmc}(C_6) = 3$.

Using Lemmas 10.3.3 and 10.3.1, we immediately get the following result.

Theorem 10.3.5 (Theorem 3.11 in [78]). *For every integer $q \geq 2$ and every*
$K_{2,q}$-induced-minor-free graph G with n vertices, the tree-independence number of G is at most $2q - 2$. Moreover, a tree decomposition of G with at most n nodes and with independence number at most $2q - 2$ can be computed in time $\mathcal{O}(n^{\mu} \log n)$, where $\mu < 2.37286$ is the matrix multiplication exponent.

Recall that the tree-independence number of the complete bipartite graph $K_{q-1,q-1}$ is equal to $q - 1$ (see Corollary 10.1.6) which gives a lower bound on the tree-independence number of $K_{2,q}$-im-free graphs, while Theorem 10.3.5 gives the upper bound of $2q-2$ for such graphs. Thus, for every graph G in the class of $K_{2,q}$-im-free graphs, $q - 1 \leq \textsf{tree-}\alpha(G) \leq 2q - 2$.

10.4 Tree-independence number: refinements

In order to prove that the tree-independence number is bounded in the classes of W_4-im-free graphs and K_5^--im-free graphs we will use slightly different definitions. Although, we could apply a very similar approach as will be described in the remaining sections of this chapter to prove that the tree-independence number is bounded in the above mentioned graph classes, this will allow us to obtain better algorithmic results presented in the next chapter.

Definition 10.4.1 (Definition 4.1 in [77]). Given a non-negative integer ℓ, an *ℓ-refined tree decomposition* of a graph G is a pair $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)_{t \in V(T)}\})$ such that $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is a tree decomposition of G, and for every $t \in V(T)$ we have $U_t \subseteq X_t$ and $|U_t| \leq \ell$. We refer to $\mathcal{T}$ as the *underlying tree decomposition* of $\widehat{\mathcal{T}}$.

Note that any concept defined for tree decompositions is naturally defined also for the ℓ-refined tree decompositions by considering the underlying tree decomposition.

Using the definition of the ℓ-refined tree decompositions, we can now use the sets U_t to give the following refined definition of the tree-independence number.

Definition 10.4.2 (Definition 4.2 in [77]). Given a non-negative integer ℓ, the *residual independence number* of an ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ of a graph G, denoted by $\widehat{\alpha}(\widehat{\mathcal{T}})$, is defined as

$$\widehat{\alpha}(\widehat{\mathcal{T}}) = \max_{t \in V(T)} \alpha(G[X_t \setminus U_t]) \,.$$

The *ℓ-refined tree-independence number* of a graph G is defined as the minimum residual independence number of an ℓ-refined tree decomposition of G, and denoted by $\ell\text{-}\mathsf{tree}\text{-}\alpha(G)$.

Observe that each ℓ-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)_{t \in V(T)}\})$ of a graph G is also an $(\ell+1)$-refined tree decomposition. It follows that for all graphs G, $(\ell+1)\text{-}\mathsf{tree}\text{-}\alpha(G) \leq \ell\text{-}\mathsf{tree}\text{-}\alpha(G)$ for all $\ell \geq 0$. Furthermore, setting $U_t = \emptyset$ for all $t \in V(T)$ we get that for each $\ell \geq 0$, $\widehat{\alpha}(\widehat{\mathcal{T}}) = \alpha(\mathcal{T})$, where $\mathcal{T}$ is the underlying tree decomposition of $\widehat{\mathcal{T}}$. Thus, the following observation immediately follows.

Observation 10.4.3 (Observation 4.3 in [77]). For every graph G and every integer $\ell \geq 0$, we have

$$\ell\text{-}\mathsf{tree}\text{-}\alpha(G) \leq \mathsf{tree}\text{-}\alpha(G) \leq \ell\text{-}\mathsf{tree}\text{-}\alpha(G) + \ell \,.$$

In particular, equalities hold when $\ell = 0$, i.e., $0\text{-}\mathsf{tree}\text{-}\alpha(G) = \mathsf{tree}\text{-}\alpha(G)$.

Using Observation 10.4.3 together with Lemma 10.1.11, we can now state the following result, which is a generalization of Lemma 10.1.11 in terms of the ℓ-refined tree-independence number.

Lemma 10.4.4 (Lemma 2.11 in [78]). *For every two non-negative integers k and ℓ, the class of graphs with ℓ-refined tree-independence number at most k is* (tw, ω)*-bounded, with a binding function $f(p) = R(p+1, k+1) + \ell - 2$, which is a polynomial of degree k. In particular, for every positive integer k, the class of graphs with tree-independence number at most k is* (tw, ω)*-bounded, with a binding function $f(p) = R(p+1, k+1) - 2$.*

Observe that Theorem 10.1.1 states that a graph G has a 0-refined tree decomposition with residual independence number at most 1 (also independence number 1) if and only if G is chordal. In addition, Theorem 8.2.3 states that a chordal graph satisfies $\mathrm{tw}(G) = \omega(G) - 1$. We now generalize this result for every integer $\ell \geq 0$.

Proposition 10.4.5 (Proposition 4.5 in [77]). *Let ℓ be a non-negative integer and let G be a graph admitting an ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ with residual independence number at most* 1. *Then $width(\widehat{\mathcal{T}}) \leq \mathrm{tw}(G) + \ell$ and $\mathrm{tw}(G) \leq \omega(G) - 1 + \ell$.*

Proof. Let $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)_{t \in V(T)}\})$ be an ℓ-refined tree decomposition and let X_t be a largest bag of the underlying tree decomposition. Note that since $\widehat{\mathcal{T}}$ has residual independence number at most 1, we get that $X_t \setminus U_t$ induces a clique in G. Using Theorem 8.2.3, we have that

$$width(\widehat{\mathcal{T}}) = |X_t| - 1 = |U_t| + |X_t \setminus U_t| - 1 \leq \ell + \omega(G) - 1 \leq \ell + \mathrm{tw}(G)\,.$$

In addition, since each bag induced a clique in G after removing at most ℓ vertices, it follows that $\mathrm{tw}(G) \leq \omega(G) - 1 + \ell$. □

We will now give some algorithmic results that will be used in the next chapter. We start by showing that, for a clique cutset C with a cut-partition (A, B, C), if we are given the ℓ-refined tree decompositions of the graphs $G[A \cup C]$ and $G[B \cup C]$, then we can combine the two in order to obtain an ℓ-refined tree decomposition of the whole graph in linear time with respect to the size of the starting ℓ-refined tree decompositions.

Proposition 10.4.6 (Proposition 4.6 in [77]). *Let C be a clique cutset in a graph G and let (A, B, C) be a cut-partition of G. Let $G_A = G[A \cup C]$ and $G_B = G[B \cup C]$, and let $\widehat{\mathcal{T}}_A$ and $\widehat{\mathcal{T}}_B$ be ℓ-refined tree decompositions of G_A and G_B, respectively. Then we can compute in time $\mathcal{O}(|\widehat{\mathcal{T}}_A| + |\widehat{\mathcal{T}}_B|)$ an ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ of G such that $\widehat{\alpha}(\widehat{\mathcal{T}}) = \max\{\widehat{\alpha}(\widehat{\mathcal{T}}_A), \widehat{\alpha}(\widehat{\mathcal{T}}_B)\}$.*

Proof. Since C is a clique in G_A and in G_B, by Lemma 8.2.2 there exists a bag X_A of $\widehat{\mathcal{T}}_A$ such that $C \subseteq X_A$, and a bag X_B in $\widehat{\mathcal{T}}_B$ such that $C \subseteq X_B$. Take the disjoint union of the tree T_A of $\widehat{\mathcal{T}}_A$ with the tree T_B of $\widehat{\mathcal{T}}_B$, and add an edge connecting the nodes corresponding to X_A and X_B. This results in a tree such that, if we combine the two assignments of the pairs

(X_t, U_t) to the nodes of the trees of $\widehat{\mathcal{T}}_A$ and $\widehat{\mathcal{T}}_B$ into one, we obtain an ℓ-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)\}_{t \in V(T)})$ of G. Indeed, since every vertex of G is a vertex of G_A or G_B, every vertex of G is in at least one bag of $\widehat{\mathcal{T}}$. A similar argument shows that for every edge of G, both endpoints belong to a common bag. Finally, let us verify that for every vertex $u \in V(G)$ the subgraph of T induced by the set of bags containing u is connected. If $u \in A \cup B$, then this follows from the corresponding properties of $\widehat{\mathcal{T}}_A$ and $\widehat{\mathcal{T}}_B$. Suppose now that $u \in C$. Then the subgraph $T_A(u)$ of T_A induced by the set of bags of $\widehat{\mathcal{T}}_A$ containing u is connected and contains the node corresponding to X_A. Similarly, the subgraph $T_B(u)$ of T_B induced by the set of bags of $\widehat{\mathcal{T}}_B$ containing u is connected and contains the node corresponding to X_B. Thus, the subgraph $T(u)$ of T induced by the set of bags of $\mathcal{T}$ containing u is isomorphic to the graph obtained from the disjoint union of $T_A(u)$ and $T_B(u)$ together with the edge between X_A and X_B, and hence $T(u)$ is connected. By construction, $\widehat{\mathcal{T}}$ is an ℓ-refined tree decomposition with residual independence number $\widehat{\alpha}(\widehat{\mathcal{T}}) = \max\{\widehat{\alpha}(\widehat{\mathcal{T}}_A), \widehat{\alpha}(\widehat{\mathcal{T}}_B)\}$. Finding a bag X_A such that $C \subseteq X_A$ can be done in time $\mathcal{O}(|\widehat{\mathcal{T}}_A|)$, and similarly for X_B. Thus, the time complexity of the above procedure is $\mathcal{O}(|\widehat{\mathcal{T}}_A| + |\widehat{\mathcal{T}}_B|)$. □

Taking $\ell = 0$, as a corollary, we get the following.

Proposition 10.4.7 (Proposition 3.10 in [77]). *Let C be a clique cutset in a graph G and let (A, B, C) be a cut-partition of G. Let $G_A = G[A \cup C]$ and $G_B = G[B \cup C]$, and let $\mathcal{T}_A$ and $\mathcal{T}_B$ be tree decompositions of G_A and G_B, respectively. Then we can compute in time $\mathcal{O}(|\mathcal{T}_A| + |\mathcal{T}_B|)$ a tree decomposition $\mathcal{T}$ of G such that $\alpha(\mathcal{T}) = \max\{\alpha(\mathcal{T}_A), \alpha(\mathcal{T}_B)\}$.*

Moreover, Proposition 10.4.7 further implies the following result.

Corollary 10.4.8 (Corollary 3.11 in [77]). *Let C be a clique cutset in a graph G, let (A, B, C) be a cut-partition of G, and let $G_A = G[A \cup C]$ and $G_B = G[B \cup C]$. Then*

$$\textsf{tree-}\alpha(G) = \max\{\textsf{tree-}\alpha(G_A), \textsf{tree-}\alpha(G_B)\}\,.$$

Proof. To justify that $\textsf{tree-}\alpha(G) = \max\{\textsf{tree-}\alpha(G_A), \textsf{tree-}\alpha(G_B)\}$, note first that G_A and G_B are induced subgraphs of G and hence

$$\max\{\textsf{tree-}\alpha(G_A), \textsf{tree-}\alpha(G_B)\} \leq \textsf{tree-}\alpha(G)$$

by Proposition 10.1.10. For the converse inequality, let $\mathcal{T}_A$ and $\mathcal{T}_B$ be tree decompositions of G_A and G_B, respectively, such that $\alpha(\mathcal{T}_A) = \textsf{tree-}\alpha(G_A)$ and $\alpha(\mathcal{T}_B) = \textsf{tree-}\alpha(G_B)$. Proposition 10.4.7 implies the existence of a tree decomposition $\mathcal{T}$ of G such that $\alpha(\mathcal{T}) = \max\{\alpha(\mathcal{T}_A), \alpha(\mathcal{T}_B)\}$. It follows that

$$\textsf{tree-}\alpha(G) \leq \alpha(\mathcal{T})$$

$$
\begin{aligned}
&= \max\{\alpha(\mathcal{T}_A), \alpha(\mathcal{T}_B)\} \\
&= \max\{\mathsf{tree}\text{-}\alpha(G_A), \mathsf{tree}\text{-}\alpha(G_B)\}\,. \qquad \square
\end{aligned}
$$

10.5 Tree-independence number: reductions

We will now use block-cutpoint trees and SPQR trees in order to reduce the problem of computing ℓ-refined tree decompositions with small residual independence number to triconnected components.

Let $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ be a function. We say that f is *superadditive* if the inequality

$$f(x_1 + x_2, y_1 + y_2) \geq f(x_1, y_1) + f(x_2, y_2)$$

holds for all $x_1, x_2, y_1, y_2 \in \mathbb{Z}^+$. Note that any superadditive function is non-decreasing with respect to both coordinates. Indeed, for all $x_1, x_2, y \in \mathbb{Z}^+$ with $x_1 \leq x_2$, we have $f(x_1, y) \leq f(x_1, y) + f(x_2 - x_1, 0) \leq f(x_2, y)$, which shows that f is non-decreasing in the first coordinate. A similar argument shows that f is non-decreasing in the second coordinate.

10.5.1 Reduction to 2-connected graphs

First, we apply Proposition 10.4.6 to clique cutsets of size at most one in order to reduce the problem to the case of 2-connected graphs in a hereditary graph class $\mathcal{G}$.

Proposition 10.5.1 (Proposition 4.1 in [78]). *Let $\mathcal{G}$ be a hereditary graph class for which there exist non-negative integers k and ℓ such that for each 2-connected graph in $\mathcal{G}$, with n vertices and m edges, one can compute in time $f(n, m)$ an ℓ-refined tree decomposition with at most $g(n, m)$ nodes and residual independence number at most k, where f and g are superadditive functions. Then, for any graph G in $\mathcal{G}$ with $n \geq 1$ vertices and m edges, one can compute in time $\mathcal{O}(n + m + f(2n, m))$ an ℓ-refined tree decomposition of G with $\mathcal{O}(n + g(2n, m))$ nodes and residual independence number at most* $\max\{2 - \ell, k\}$.

Proof. Let $G \in \mathcal{G}$ be a graph with $n \geq 1$ vertices and m edges. Using Breadth-First Search and the Hopcroft-Tarjan algorithm [134], we compute in time $\mathcal{O}(n + m)$ the connected components $G_1, \ldots, G_p$ of G and the corresponding block-cutpoint trees $T_1, \ldots, T_p$. Let $\mathcal{B}_2(G_i)$ and $\mathcal{B}_3(G_i)$ denote the sets of blocks of G_i with exactly two, resp. at least three, vertices. For each component G_i we compute an ℓ-refined tree decomposition $\widehat{\mathcal{T}}_{G_i}$ of G_i recursively as follows.

At each step of the recursion, let G_i' be the current graph and B be a leaf block of G_i'. Initially, we take $G_i' = G_i$.

1. First, if $G_i' = B$, then we consider the following two cases.

If $B \in \mathcal{B}_2(G_i)$, then B is a complete graph of order at most two. In this case we compute the ℓ-refined tree decomposition $\widehat{\mathcal{T}}_{G_i'}$ of G_i' consisting of a single node t whose bag X_t is $V(G_i')$ and the set U_t to be any subset of $V(G_i')$ with $\min\{\ell, 2\}$ vertices. We can compute $\widehat{\mathcal{T}}_{G_i'}$ in constant time; note that this ℓ-refined tree decomposition of G_i' satisfies

$$\begin{aligned}\widehat{\alpha}(\widehat{\mathcal{T}}_{G_i'}) &= \alpha(G_i'[X_t \setminus U_t]) \\ &\leq |X_t| - |U_t| \\ &= 2 - \min\{\ell, 2\} \\ &= \max\{2 - \ell, 0\} \\ &\leq \max\{2 - \ell, k\}.\end{aligned}$$

Otherwise, $B \in \mathcal{B}_3(G_i)$ is 2-connected, and we compute in time $f(|V(G_i')|, |E(G_i')|)$ an ℓ-refined tree decomposition $\widehat{\mathcal{T}}_{G_i'}$ of G_i' with at most $f(|V(G_i')|, |E(G_i')|)$ nodes and residual independence number at most $k \leq \max\{2 - \ell, k\}$.

2. Second, if $G_i' \neq B$, let (X, Y, Z) be a cut-partition where $X = V(B) \setminus \{v\}$, $Y = V(G_i') \setminus V(B)$, and $Z = \{v\}$ for the unique cut-vertex v of G_i' contained in B. We then compute an ℓ-refined tree decomposition $\widehat{\mathcal{T}}_{G_i'}$ of G_i' by recursively computing ℓ-refined tree decompositions of the block B and the graph $G_i'[Y \cup Z]$, and applying Proposition 10.4.6 to the cut-partition (X, Y, Z).

This recursive procedure takes time

$$\mathcal{O}\left(|\mathcal{B}_2(G_i)|\right) + \sum_{B \in \mathcal{B}_3(G_i)} f(|V(B)|, |E(B)|)$$

$$= \mathcal{O}\left(|V(T_i)| + \sum_{B \in \mathcal{B}(G_i)} f(|V(B)|, |E(B)|) \right).$$

The total number of nodes of $\widehat{\mathcal{T}}_{G_i}$ is upper bounded by

$$|\mathcal{B}_2(G_i)| + \sum_{B \in \mathcal{B}_3(G_i)} g(|V(B)|, |E(B)|)$$

$$\leq \quad |V(T_i)| + \sum_{B \in \mathcal{B}(G_i)} g(|V(B)|, |E(B)|).$$

Finally, we create an ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ of G with residual independence number at most $\max\{2 - \ell, k\}$ by combining the ℓ-refined tree decompositions $\widehat{\mathcal{T}}_{G_1}, \ldots, \widehat{\mathcal{T}}_{G_p}$ of its connected components in the obvious way (for example, by iteratively applying Proposition 10.4.6 with respect to a sequence of cut-partitions involving the empty clique cutset). The number of nodes of $\widehat{\mathcal{T}}$ is the sum of the numbers of nodes of $\widehat{\mathcal{T}}_{G_i}$ over all $i \in \{1, \ldots, p\}$. The overall running time of the algorithm is

$$\mathcal{O}\left(\sum_{i=1}^{p} |V(T_i)| + \sum_{i=1}^{p} \sum_{B \in \mathcal{B}(G_i)} f(|V(B)|, |E(B)|)\right).$$

An inductive argument based on the number of blocks shows that for every connected graph H with at least one cut-vertex (and thus with at least two blocks), we have $|\mathcal{C}(H)| < |\mathcal{B}(H)| \leq |V(H)| - 1$. Thus, for all $i \in \{1, \ldots, p\}$, we have $|V(T_i)| = |\mathcal{B}(G_i)| + |\mathcal{C}(G_i)| \leq 2|V(G_i)| - 3$, which implies $\sum_{i=1}^{p} |V(T_i)| = \mathcal{O}(n)$. Furthermore, for all $i \in \{1, \ldots, p\}$, we have

$$\begin{aligned}
\sum_{B \in \mathcal{B}(G_i)} |V(B)| &= |V(G_i)| + \sum_{v \in \mathcal{C}(G_i)} (d_{T_i}(v) - 1) \\
&= |V(G_i)| + |E(T_i)| - |\mathcal{C}(G_i)| \\
&= |V(G_i)| + |V(T_i)| - 1 - |\mathcal{C}(G_i)| \\
&= |V(G_i)| + |\mathcal{B}(G_i)| - 1 \\
&\leq 2|V(G_i)| - 2
\end{aligned}$$

and

$$\sum_{B \in \mathcal{B}(G_i)} |E(B)| \leq \sum_{B \in \mathcal{B}(G_i)} |E(B)| = |E(G_i)|.$$

Since f is superadditive and non-decreasing in the first coordinate, we have

$$\begin{aligned}
\sum_{B \in \mathcal{B}(G_i)} f(|V(B)|, |E(B)|) &\leq f\left(\sum_{B \in \mathcal{B}(G_i)} |V(B)|, \sum_{B \in \mathcal{B}(G_i)} |E(B)|\right) \\
&\leq f\left(2|V(G_i)| - 2, |E(G_i)|\right).
\end{aligned}$$

Applying the superadditivity and monotonicity properties of f once again, we obtain

$$\begin{aligned}
\sum_{i=1}^{p} f\left(2|V(G_i)| - 2, |E(G_i)|\right) &\leq f\left(2\sum_{i=1}^{p} |V(G_i)| - 2p, \sum_{i=1}^{p} |E(G_i)|\right) \\
&\leq f\left(2n, m\right),
\end{aligned}$$

and the running time of the algorithm is $\mathcal{O}(n + m + f(2n, m))$, as claimed.

The number of nodes of $\widehat{\mathcal{T}}$ is at most

$$\sum_{i=1}^{p}\left(|V(T_i)| + \sum_{B\in\mathcal{B}(G_i)} g(|V(B)|, |E(B)|)\right).$$

Applying the same arguments as we did above for the function f to the function g leads to an upper bound for the number of nodes of $\widehat{\mathcal{T}}$ given by $\mathcal{O}(n + g(2n, m))$. □

10.5.2 Reduction to triconnected components

Let G a 2-connected graph and S be an SPQR-tree of G. Recall that every R-node a of S corresponds to the triconnected component G_a of G. As already observed in Lemma 8.5.1, G_a is an induced topological minor of G. Moreover, the subgraph H of G induced by $V(G_a)$ is a spanning subgraph of G_a and every edge in G_a that is not an edge of G is a virtual edge. Suppose that G_a has bounded tree-independence number. In order to infer that H has bounded tree-independence number, then, for the set F of virtual edges within G_a, we must be able to find a vertex cover S of F with a bounded number ℓ of vertices. Note that this is in line with the notion of ℓ-refined tree decompositions, as deleting every vertex of S from G_a results in an induced subgraph of G.

An easy example shows that the independence number of triconnected components can be small while the tree-independence number is large (see also Example 4.8 in [78]). Consider a graph G obtained from the complete graph K_n, $n \geq 4$, by subdividing each edge once. Note that the unique triconnected component G_a corresponding to the unique R-node a of G is formed by the original vertices of K_n, thus forming a clique in G_a. It follows that G_a has independence number 1 while G contains a $K_{t,t}$ as an induced topological minor, where $t = \lfloor \frac{n}{2} \rfloor$. Thus, by Corollary 10.1.6 and Proposition 10.1.10, we obtain that $\mathsf{tree}\text{-}\alpha(G) \geq \mathsf{tree}\text{-}\alpha(K_{t,t}) = \lfloor \frac{n}{2} \rfloor$.

We will now give several definitions that will help us reduce the problem of computing ℓ-refined tree decompositions with small residual independence number from 2-connected graphs to triconnected components.

Definition 10.5.2 (Definition 4.9 in [78])**.** Let G be a graph, $F^* \subseteq E(G)$ and G' a subgraph of G. We say that a set $U \subseteq V(G')$ is an F^*-*cover* of G' if U contains at least one endpoint of every edge in $E(G') \cap F^*$.

Definition 10.5.3 (Definition 4.10 in [78])**.** Let G be graph and $F^* \subseteq E(G)$. An ℓ-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)_{t\in V(T)}\})$ of G is said to be F^*-*covering* if for every node $t \in V(T)$ the set U_t is an F^*-cover of $G[X_t]$.

Definition 10.5.4 (Definition 4.11 in [78])**.** Let $\mathcal{G}$ be a graph class, let G be a graph in $\mathcal{G}$, and let $F^* \subseteq E(G)$. We say that F^* is a $\mathcal{G}$-*safe* set of edges of G if deleting from G any subset of F^* results in a graph that belongs to $\mathcal{G}$.

Definition 10.5.5 (Definition 4.12 in [78])**.** Let G be a graph, $F \subseteq E(G)$, and $\mathcal{T}$ a tree decomposition (either usual or ℓ-refined) of G. An *F-mapping* of $\mathcal{T}$ is a mapping with domain F assigning to every edge $e \in F$ a node $t_e \in V(T)$ such that $e \subseteq X_{t_e}$.

Note that for any graph G, any set $F \subseteq E(G)$, and any (usual or ℓ-refined) tree decomposition $\mathcal{T}$ of G, there exists an F-mapping of $\mathcal{T}$, since the endpoints of every edge in G are contained in some bag of $\mathcal{T}$. Furthermore, it is clear that an F-mapping of $\mathcal{T}$ can be computed in polynomial time. However, for a more efficient computation of an F-mapping of $\mathcal{T}$, it may be best to compute it together with $\mathcal{T}$. To this end, we define the following property of a graph class and also a problem associated with it.

Definition 10.5.6 (Definition 4.13 in [78])**.** Given two non-negative integers k and ℓ, a graph class $\mathcal{G}$ is said to be *(k,ℓ)-tree decomposable* if for every 3-connected graph $G \in \mathcal{G}$ and any $\mathcal{G}$-safe set F^* of edges of G, there exists an F^*-covering ℓ-refined tree decomposition of G with residual independence number at most k.

(k,ℓ)-TREE DECOMPOSITION$(\mathcal{G})$
Input: A 3-connected graph $G \in \mathcal{G}$ and two sets of edges $F^* \subseteq F \subseteq E(G)$ such that F^* is $\mathcal{G}$-safe.
Output: An F^*-covering ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ of G with residual independence number at most k and an F-mapping of $\widehat{\mathcal{T}}$.

For a (k,ℓ)-tree decomposable graph class $\mathcal{G}$ closed under induced topological minors, the following key lemma reduces the problem of computing ℓ-refined tree decompositions with bounded residual independence number of 2-connected graphs in $\mathcal{G}$ to the (k,ℓ)-TREE DECOMPOSITION$(\mathcal{G})$ problem.

Lemma 10.5.7 (Lemma 4.14 in [78])**.** *Let $\mathcal{G}$ be a graph class closed under induced topological minors for which there exist non-negative integers k and ℓ such that $\mathcal{G}$ is (k,ℓ)-tree decomposable and (k,ℓ)-TREE DECOMPOSITION$(\mathcal{G})$ can be solved in time $f(n,m)$ on graphs with n vertices and m edges so that the resulting ℓ-refined tree decomposition has $g(n,m)$ nodes, where f and g are superadditive functions. Then, for any 2-connected graph G in $\mathcal{G}$, with n vertices and m edges, one can compute in time $\mathcal{O}(m+f(2n,3m))$ an ℓ-refined tree decomposition of G with $\mathcal{O}(n+g(2n,3m))$ nodes and residual independence number at most $\max\{3-\ell,k\}$.*

Proof. Let G be a 2-connected graph in $\mathcal{G}$ with n vertices and m edges. We compute in time $O(n+m)$ an SPQR-tree S of G. Let $\widehat{\mathcal{T}} = (T,\{(X_t,U_t) : t \in V(T)\})$ be the 0-refined tree decomposition of G corresponding to the SPQR-tree S. Note that $\widehat{\mathcal{T}}$ is ℓ-refined. Our goal is to obtain an ℓ-refined tree decomposition of G of bounded residual independence number by updating $\widehat{\mathcal{T}}$ iteratively, as follows.

First, we iterate over all P-nodes b in S and set U_b to be any subset of X_b with cardinality $\min\{\ell, 2\}$. Since the number of P-nodes is $\mathcal{O}(n)$, this modification takes time $\mathcal{O}(n)$. Furthermore, for each P-node b we then have

$$\alpha(G[X_b \backslash U_b]) \leq |X_b| - |U_b| = 2 - \min\{\ell, 2\} = \max\{2-\ell, 0\} \leq \max\{3-\ell, k\}\,.$$

Then, for each R-node or S-node a in S we compute an $\widehat{\alpha}$-bounded ℓ-refined tree decomposition $\widehat{\mathcal{T}_a}$ of G_a (using the assumption of the lemma in the case of R-nodes) and replace the corresponding node in $\widehat{\mathcal{T}}$ with $\widehat{\mathcal{T}_a}$. Let us describe the update procedure in detail. For each node a of S that is an R-node or an S-node, we perform the following three steps.

Step 1. We compute two sets of edges F_a^* and F_a such that $F_a^* \subseteq F_a \subseteq E(G_a)$. Recall that node a is adjacent in S to P-nodes only, and for every neighbor b of a in S, the set X_b corresponding to the node b is a 2-cutset in G such that the two vertices in X_b are adjacent in G_a. We define $F_a = \{X_b : b \text{ is adjacent to } a \text{ in } S\}$ and $F_a^* = \{e \in F_a : e \text{ is a virtual edge in } G_a\}$. Clearly, F_a and F_a^* can be obtained in time $O(d_S(a))$.

The second step relies on the following property of the sets F_a^*.

Claim 10.5.8. For each R-node or S-node a of S, the set F_a^* is a $\mathcal{G}$-safe set of edges of G_a.

Proof of Claim 10.5.8. By Lemma 8.5.1, there exists a subdivision G_a' of G_a that is an induced subgraph of G. We want to show that deleting from G_a any subset of edges in F_a^* results in a graph in $\mathcal{G}$. Since every edge $e \in F_a^*$ is a virtual edge of G_a, the two endpoints of e are non-adjacent in G. Thus, the edges in F_a^* correspond to a collection of internally vertex-disjoint paths $\{P_e : e \in F_a^*\}$ such that each P_e is an induced path of length at least two in H_a' (and thus in G) connecting the endpoints of e. Consider an arbitrary set $F \subseteq F_a^*$. Since G_a' is an induced subgraph of G, each real edge in G_a is also an edge in G_a'. It follows that the graph $G_a - F$ can be obtained from G_a' by deleting the internal vertices of P_e for all $e \in F$ and by contracting to a single edge each path P_e for all $e \in F_a^* \setminus F$. Hence, $G_a - F$ is an induced topological minor of G_a' and thus of G. Since $G \in \mathcal{G}$ and $\mathcal{G}$ is closed under induced topological minors, we have $G_a - F \in \mathcal{G}$, as claimed. ■

Step 2. We compute an ℓ-refined tree decomposition $\widehat{\mathcal{T}_a} = (T_a, \{(X_t^a, U_t^a) : t \in V(T_a)\})$ of $G[X_a]$ with residual independence number at most $\max\{3-\ell, k\}$ and an F_a-mapping of $\widehat{\mathcal{T}_a}$. We do so by computing an ℓ-refined tree decomposition of G_a. Since $G[X_a]$ is a subgraph of G_a, the decomposition $\widehat{\mathcal{T}_a}$ is also an ℓ-refined tree decomposition of $G[X_a]$. We consider two cases depending on whether a is an S-node or an R-node of S.

- If a is an S-node of S, then G_a is a cycle and we can apply a similar approach as in Theorem 10.1.2. In time $\mathcal{O}(|V(G_a)|)$ we compute a cyclic order $v_1, \dots, v_h$ of the vertices of the cycle. We construct the desired ℓ-refined tree decomposition $\widehat{\mathcal{T}_a}$ of G_a as follows. The tree T_a is an $(h-2)$-vertex path $(t_1, \dots, t_{h-2})$; for each $i \in \{1, \dots, h-2\}$, the bag $X^a_{t_i}$ consists of vertices $\{v_i, v_{i+1}, v_h\}$, and $U^a_{t_i}$ is any subset of $X^a_{t_i}$ with exactly $\min\{\ell, |X^a_{t_i}|\}$ vertices. Note that every bag X^a_t of $\widehat{\mathcal{T}_a}$ has size 3 and hence
$$\alpha(G[X^a_t \setminus U^a_t]) \le |X^a_t| - |U^a_t| \le \max\{3-\ell, 0\} \le \max\{3-\ell, k\}\,.$$
The corresponding F_a-mapping of $\widehat{\mathcal{T}_a}$ can be obtained as follows. For each edge $e = v_iv_{i+1} \in F_a$ with $1 \le i \le h$ (indices modulo h), we do the following: if $1 \le i \le h-2$, we map e to $t_i \in V(T_a)$; if $i = h-1$, we map e to $t_{h-2} \in V(T_a)$; and if $i = h$, we map e to $t_1 \in V(T_a)$. Clearly, both $\widehat{\mathcal{T}_a}$ and the F_a-mapping can be obtained in $O(|V(G_a)|)$ time.
- Otherwise, a is an R-node of S. Recall that $\mathcal{G}$ is closed under induced topological minors. By Lemma 8.5.1, G_a is an induced topological minor of $G \in \mathcal{G}$, and thus G_a belongs to $\mathcal{G}$. By Claim 10.5.8, F^*_a is a $\mathcal{G}$-safe set of edges of G_a. Since G_a is 3-connected, by the assumption of the lemma, we can compute an F^*_a-covering ℓ-refined tree decomposition $\widehat{\mathcal{T}_a}$ of G_a with residual independence number at most k and an F_a-mapping of $\widehat{\mathcal{T}_a}$ in time $f(|V(G_a)|, |E(G_a)|)$. Let t be a node of $\widehat{\mathcal{T}_a}$. Since U^a_t is an F^*_a-cover of $G_a[X^a_t]$, every virtual edge in $G_a[X^a_t]$ has an endpoint in U^a_t. We infer that the subgraphs of G and G_a induced by the set $X^a_t \setminus U^a_t$ are the same and thus,
$$\alpha(G[X^a_t \setminus U^a_t]) = \alpha(G_a[X^a_t \setminus U^a_t]) \le \widehat{\alpha}(\widehat{\mathcal{T}_a}) \le k \le \max\{3-\ell, k\}\,.$$

Step 3. This step consists in, informally speaking, replacing the node a and the bag X_a in $\widehat{\mathcal{T}}$ by the newly computed ℓ-refined tree decomposition $\widehat{\mathcal{T}_a}$ of G_a. We first compute the forest $T' = (T-a)+T_a$ and then make T' connected by iterating over all neighbors b of a in S. Since every edge of S has exactly one endpoint which is a P-node and a is not a P-node of S, we infer that b must be a P-node of S, and thus X_b contains exactly two vertices in G, which are adjacent in G_a. Let $e \in E(G_a)$ be the edge with endpoints in X_b. By the definition of F_a, the edge e belongs to F_a. Recall that an F_a-mapping of $\widehat{\mathcal{T}_a}$ has been computed in Step 2. Hence, given the edge e, the F_a-mapping of $\widehat{\mathcal{T}_a}$ returns, in constant time, a node c of the tree T_a whose bag X_c contains the endpoints of e. We connect b to c in T'.

Notice that once all the neighbors of a have been considered, T' becomes a tree. We now set $\widehat{\mathcal{T}} = (T', \{(X_t, U_t) : t \in V(T-a)\} \cup \{(X_t^a, U_t^a) : u \in V(T_a)\})$. It is not difficult to verify that after this modification, $\widehat{\mathcal{T}}$ remains an ℓ-refined tree decomposition of G. The time complexity of this step is proportional to $\mathcal{O}(d_S(a) + |\widehat{\mathcal{T}}_a|)$.

We now reason about the overall time complexity of Steps 1–3 for a fixed R-node or S-node a of S. Recall that N_{R} and N_{S} denote the sets of R-nodes and S-nodes of S, respectively. Assume first that $a \in N_{\mathrm{S}}$. Then the complexity is $\mathcal{O}(d_S(a) + |V(G_a)| + |\widehat{\mathcal{T}}_a|)$. Since G_a is a cycle and our construction of $\widehat{\mathcal{T}}_a$ implies that $|\widehat{\mathcal{T}}_a| = O(|V(G_a)|)$, this simplifies to $\mathcal{O}(d_S(a) + |V(G_a)|)$. Assume now that $a \in N_{\mathrm{R}}$. Then the complexity is $\mathcal{O}(d_S(a) + f(|V(G_a)|, |E(G_a)|) + |\widehat{\mathcal{T}}_a|)$, which simplifies to $\mathcal{O}(d_S(a) + f(|V(G_a)|, |E(G_a)|))$ since $|\widehat{\mathcal{T}}_a| \leq f(|V(G_a)|, |E(G_a)|)$.

By construction, the final ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ has residual independence number at most $\max\{3-\ell, k\}$. The overall time complexity of the algorithm is

$$O\Big(n + m + \sum_{a \in N_{\mathrm{S}}} (d_S(a) + |V(G_a)|) + \sum_{a \in N_{\mathrm{R}}} (d_S(a) + f(|V(G_a)|, |E(G_a)|))\Big)\,,$$

or, equivalently,

$$\mathcal{O}\Big(n + m + \sum_{a \in N_{\mathrm{R}} \cup N_{\mathrm{S}}} d_S(a) + \sum_{a \in N_{\mathrm{S}}} |V(G_a)| + \sum_{a \in N_{\mathrm{R}}} f(|V(G_a)|, |E(G_a)|)\Big)\,.$$

Lemma 8.5.3 implies that $\sum_{a \in N_{\mathrm{S}}} |V(G_a)| \leq \sum_{a \in N_{\mathrm{R}} \cup N_{\mathrm{S}}} |X_a| \leq 3n-6$. Furthermore, we know that $\sum_{a \in N_{\mathrm{R}} \cup N_{\mathrm{S}}} d_S(a) = |E(S)| = O(m)$ due to [116]. Since the function f is superadditive, we have

$$\sum_{a \in N_{\mathrm{R}}} f(|V(G_a)|, |E(G_a)|) \leq f\Big(\sum_{a \in N_{\mathrm{R}}} |V(G_a)|, \sum_{a \in N_{\mathrm{R}}} |E(G_a)|\Big)\,.$$

By Lemma 8.5.3, we also have that $\sum_{a \in N_{\mathrm{R}}} |V(G_a)| \leq 2n$. Additionally, by Lemma 8.5.4 we get that $\sum_{a \in N_{\mathrm{R}}} |E(G_a)| \leq 3m$. Since f is non-decreasing in each coordinate and $m \geq n$ as G is 2-connected, the running time simplifies to $\mathcal{O}\left(m + f(2n, 3m)\right)$.

The number of nodes of $\widehat{\mathcal{T}}$ is at most

$$|N_{\mathrm{P}}| + \sum_{a \in N_{\mathrm{S}}} (|V(G_a)| - 2) + \sum_{a \in N_{\mathrm{R}}} g(|V(G_a)|, |E(G_a)|)\,.$$

Following the fact that $|N_{\mathrm{P}}| \leq |V(S)| = \mathcal{O}(n)$ and by Lemma 8.5.3, we get that $|N_{\mathrm{P}}| + \sum_{a \in N_{\mathrm{S}}}(|V(G_a)| - 2) = \mathcal{O}(n)$. Applying the same arguments as we did for the function f to the function g shows that $\sum_{a \in N_{\mathrm{R}}} g(|V(G_a)|, |E(G_a)|) \leq g(2n, 3m)$. Thus, the number of nodes of $\widehat{\mathcal{T}}$ is of the order $\mathcal{O}(n + g(2n, 3m))$. This completes the proof. $\square$

We can now combine the reduction using block-cutpoint trees (see Proposition 10.5.1) with the reduction using SPQR trees (see Lemma 10.5.7) to obtain the following result.

Theorem 10.5.9 (Theorem 4.16 in [78]). *Let $\mathcal{G}$ be a graph class closed under induced topological minors for which there exist non-negative integers k and ℓ such that $\mathcal{G}$ is (k, ℓ)-tree decomposable and (k, ℓ)-*TREE DECOMPOSITION($\mathcal{G}$) *can be solved in time $f(n, m)$ on graphs with n vertices and m edges so that the resulting ℓ-refined tree decomposition has $g(n, m)$ nodes, where f and g are superadditive functions. Then, for any graph G in $\mathcal{G}$ with $n \geq 1$ vertices and m edges, one can compute in time $\mathcal{O}(n + m + f(4n, 3m))$ an ℓ-refined tree decomposition of G with $\mathcal{O}(n + g(4n, 3m))$ nodes and with residual independence number at most* $\max\{3 - \ell, k\}$.

Proof. By Lemma 10.5.7, there exist positive integers c and d such that for any 2-connected graph $G \in \mathcal{G}$ with n vertices and m edges, one can compute in time $c \cdot (m + f(2n, 3m))$ an ℓ-refined tree decomposition of G with residual independence number at most $\max\{3 - \ell, k\}$ and with $d \cdot (n + g(2n, 3m))$ nodes. Let $\hat{k} = \max\{3 - \ell, k\}$ and let us define two functions, $\hat{f} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$ and $\hat{g} : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{Z}_+$, as follows: for each $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, we set $\hat{f}(x, y) = c \cdot (y + f(2x, 3y))$ and $\hat{g}(x, y) = d \cdot (x + g(2x, 3y))$. Thus, $\mathcal{G}$ is a hereditary graph class such that for each 2-connected graph G in $\mathcal{G}$ with n vertices and m edges, one can compute in time $\hat{f}(n, m)$ an ℓ-refined tree decomposition $\widehat{\mathcal{T}}$ with $\hat{g}(n, m)$ nodes and with residual independence number at most $\hat{k}$. Furthermore, the fact that f and g are superadditive implies that $\hat{f}$ and $\hat{g}$ are superadditive. By Proposition 10.5.1, for any graph G in $\mathcal{G}$ with $n \geq 1$ vertices and m edges, one can compute in time $\mathcal{O}(n + m + \hat{f}(2n, m))$ an ℓ-refined tree decomposition of G with $\mathcal{O}(n + \hat{g}(2n, m))$ nodes and with residual independence number at most $\max\{2 - \ell, \hat{k}\} = \max\{3 - \ell, k\}$. Since $\hat{f}(2n, m) = c \cdot (m + f(4n, 3m))$ and $\hat{g}(2n, m) = c \cdot (n + g(4n, 3m))$, the theorem follows. $\square$

10.6 Tree-independence number: W_4-im-free graphs

In this section and the next one, we apply Theorem 10.5.9 in order to prove that if $\mathcal{G}$ is the class of H-induced-minor-free graphs for $H \in \{W_4, K_5^-\}$, then $\mathcal{G}$ has bounded tree-independence number. In particular, we show that $\mathcal{G}$ is $(1,3)$-tree decomposable and also develop an algorithm running in polynomial time for the $(1,3)$-TREE DECOMPOSITION($\mathcal{G}$) problem. If we combine this result with Theorem 11.1.2 we are able to get an $\mathcal{O}(|V(G)|^3)$ algorithm for the MAX WEIGHT INDEPENDENT SET problem in case of vertex-weighted graphs G from the class $\mathcal{G}$. In a similar way, applying Theorem 11.2.6 leads to an algorithm running in polynomial time for the MAX WEIGHT INDEPENDENT PACKING problem.

Since the (k,ℓ)-TREE DECOMPOSITION($\mathcal{G}$) problem deals with 3-connected graphs, we first need to characterize the 3-connected graphs in $\mathcal{G}$. Let us start with the class of W_4-induced-minor-free graphs.

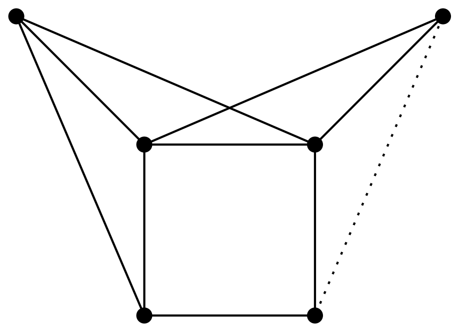


Figure 10.1: A graph containing W_4 as an induced minor obtained by contracting the dotted edge [78].

Lemma 10.6.1 (Lemma 5.1 in [78]). *Let G be a 3-connected graph. Then G is W_4-induced-minor-free if and only if G is chordal.*

Proof. First, let us assume that G is a 3-connected W_4-induced-minor-free graph. Suppose that there exists an induced cycle C of length at least 4 in G. Since G is 3-connected, it follows that every component of $G - V(C)$ has at least three distinct neighbors among the vertices of C. Additionally, every component of $G - V(C)$ must have at most three distinct neighbors in C, otherwise we could contract the component with at least four distinct neighbors in C into a single vertex, delete all the other components and contract the edges of C to obtain a W_4 as an induced minor. It follows that $G - V(C)$ must have at least two distinct components. Let H be a component of $G - V(C)$, with vertices u, v, and w being the three distinct neighbors of H in C. Observe, that there must exist a component H' of $G - V(C)$, such that H' has neighbors x, y, and z in C, where x is a vertex in the middle of the u,w-path in C containing v (notice that x may be equal to v), y is a vertex in the middle of the u,w-path in C not containing v, and z may be any vertex of C distinct from vertices x and y. If not, then $\{u, w\}$ is a cutset in G of size two. Next observe, that since y is distinct from the vertices u,v, and w and at least one of the vertices u or w is distinct from the vertices x, y, and z, we can always contract the edges of C into a cycle of length four, to obtain an induced minor isomorphic to the graph depicted in Figure 10.1. Finally, contracting the dotted edge as shown in Figure 10.1, we obtain W_4 as an induced minor, a contradiction.

For the other direction, assume that G is a chordal graph. Suppose that G contains W_4 as an induced minor. It follows that G also contains C_4 as an induced minor. But then, G contains some cycle of length at least four as an induced subgraph which is a contradiction with the assumption that G is chordal. $\square$

By Lemma 10.6.1, we have that every 3-connected graph in the class $\mathcal{G}$ consisting of W_4-induced-minor-free graphs is chordal. In order to show that $\mathcal{G}$ is $(1,3)$-tree decomposable and to develop a linear-time algorithm for the $(1,3)$-TREE DECOMPOSITION($\mathcal{G}$) problem (Lemma 10.6.3), we need to define the following concepts closely related to chordal graphs.

A *total order* of a set V of elements is a permutation $(v_1,\dots,v_n)$ of V such that any two elements are comparable, i.e., for any pair of elements x and y from the set V, $x \leq y$ or $y \leq x$, where the relation $\leq$ is antisymmetric, i.e., if $x \leq y$ and $y \leq x$, then $x = y$. A *vertex ordering* of a graph G is a total order $(v_1,\dots,v_n)$ of its vertices. A *module* in G is a set $M \subseteq V(G)$ such that every vertex not in M that has a neighbor in M is adjacent to all the vertices in M. A *moplex* in G is an inclusion-maximal module $M \subseteq V(G)$ that is a clique and its neighborhood $N(M)$ is either empty or a minimal separator in G. A *perfect moplex partition* of G is an ordered partition $(M_1,\dots,M_k)$ of $V(G)$ such that for all $i \in \{1,\dots,k\}$, M_i is a moplex in the subgraph of G induced by $\cup_{j=i}^{k} M_j$. Given an ordered partition $\pi = (Z_1,\dots,Z_k)$ of $V(G)$ and a vertex ordering $\sigma = (v_1,\dots,v_n)$ of G, we say that σ is *compatible with* π if $v_i \in Z_p$, $v_j \in Z_q$, and $p < q$ imply $i < j$. It was shown by Berry and Bordat [24] that every graph has a moplex. It follows that every graph has a perfect moplex partition. A *perfect moplex ordering* of G is a vertex ordering compatible with a perfect moplex partition. Using graph search algorithms such as Lexicographic Breadth-First Search (LexBFS) [178] or Maximum Cardinality Search (MCS) [189], one can compute in linear time a perfect moplex ordering of a given graph by reversing the ordering returned by LexBFS or MCS (see [24, 26, 23]).

Lemma 10.6.2 (Lemma 5.2 in [78]). *Let G be a chordal graph with n vertices and m edges and $F \subseteq E(G)$. Then, one can compute in time $\mathcal{O}(n+m)$ a clique tree $\mathcal{T}$ of G with at most n nodes and an F-mapping of $\mathcal{T}$.*

Proof. Using LexBFS or MCS, we compute in linear time a perfect moplex ordering $(v_1,\dots,v_n)$ of G. Berry and Simonet gave in [25] a linear-time algorithm that takes as input a connected chordal graph G and a perfect moplex ordering of G, and computes a clique tree $\mathcal{T}$ of G. We explain the idea of their algorithm in terms of the perfect moplex partition $(M_1,\dots,M_k)$ of G corresponding to the given perfect moplex ordering. The bags of the computed clique tree $\mathcal{T}$ are the maximal cliques of G, which are exactly the sets $X_1,\dots,X_k$ of the form $X_i = N_{G_i}[M_i]$ for all $i \in \{1,\dots,k\}$ where G_i is the subgraph of G induced by $\cup_{j=i}^{k} M_j$. The algorithm processes the moplexes in order from M_k to M_1. It starts with a clique tree $\mathcal{T}_k$ of G_k with a tree containing a unique node k and the corresponding bag $X_k = M_k$.

Then, for each $i = k-1, \dots, 1$, the algorithm computes a clique tree $\mathcal{T}_i$ of G_i from the clique tree $\mathcal{T}_{i+1}$ of G_{i+1} by adding to the tree of $\mathcal{T}_{i+1}$ a new node i associated with bag X_i, and an edge (i, j) where j is the smallest number in $\{i+1, \dots, k\}$ such that there is an edge in G from M_i to M_j. The final clique tree of $G = G_1$ is given by $\mathcal{T} = \mathcal{T}_1$.

We compute an F-mapping of $\mathcal{T}$ as follows. For each edge $v_iv_j \in F$ such that $i < j$, we assign the edge v_iv_j to the unique node $p \in \{1, \dots, k\}$ of $\mathcal{T}$ such that $v_i \in M_p$. This can be done in time $\mathcal{O}(|F|) = \mathcal{O}(|E(G)|)$.

Let us justify that the so-defined mapping is indeed an F-mapping of $\mathcal{T}$. Consider an edge $v_iv_j \in F$ with $i < j$, and let p and q be the unique nodes of $\mathcal{T}$ such that $v_i \in M_p$ and $v_j \in M_q$, respectively. Since the vertex ordering $(v_1, \dots, v_n)$ is compatible with the perfect moplex partition $(M_1, \dots, M_k)$, we have $p \le q$. We need to show that v_i and v_j both belong to $X_p = N_{G_p}[M_p]$. First, we have that $v_i \in M_p \subseteq X_p$. Second, since $p \le q$, we have $v_j \in V(G_p)$ and thus v_j is adjacent to v_i in G_p. In particular, we have $v_j \in N_{G_p}[M_p]$. Thus, $v_iv_j \subseteq X_p$, which is what we wanted to show.

Let now G be an arbitrary chordal graph with n vertices and m edges. The above approach can be extended in a straightforward way to the case when G is not connected, by computing the connected components and applying the above algorithm to each component. The resulting forest of clique trees of the components can be turned into a clique tree of G by adding the appropriate number of edges between the clique trees of the components. By construction, the obtained clique tree $\mathcal{T}$ of G has at most n nodes. □

Lemma 10.6.3 (Lemma 5.3 in [78]). *Let $\mathcal{G}$ be the class of W_4-induced-minor-free graphs. Then $\mathcal{G}$ is $(1,3)$-tree decomposable and $(1,3)$-*Tree Decomposition*$(\mathcal{G})$ can be solved on graphs with n vertices and m edges in time $\mathcal{O}(n \cdot m)$ so that the resulting 3-refined tree decomposition with residual independence number at most 1 has at most n nodes.*

Proof. Let G be a 3-connected W_4-induced-minor-free graph with n vertices and m edges and let $F^* \subseteq F \subseteq E(G)$ such that F^* is $\mathcal{G}$-safe. We want to compute an F^*-covering 3-refined tree decomposition $\widehat{\mathcal{T}}$ of G with $\widehat{\alpha}(\widehat{\mathcal{T}}) \le 1$, together with an F-mapping of $\widehat{\mathcal{T}}$.

By Lemma 10.6.1, G is chordal. By Lemma 10.6.2, one can compute in linear time a 0-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)\}_{t \in V(T)})$ of G with residual independence number 1 having at most n nodes and an F-mapping of $\widehat{\mathcal{T}}$. Note that $\widehat{\mathcal{T}}$ already satisfies all the desired properties, except that it may fail to be F^*-covering. To fix this, we next redefine the sets U_t for all $t \in V(T)$.

Let $t \in V(T)$. If $|X_t| \le 4$, then we set U_t to be any subset of $\min\{|X_t|, 3\}$ vertices of X_t. This in particular implies that $|X_t \setminus U_t| \le 1$, and hence U_t is an F^*-cover of $G[X_t]$ of size at most 3.

Now suppose that $|X_t| \geq 5$. Consider the set L of all edges in F^* that have both endpoints in X_t. We claim that any two edges in L have a common endpoint. Suppose that this is not the case, and let $e = uv$ and $f = xy$ be two disjoint edges in L. As $|X_t| \geq 5$, there exists a vertex z in X_t distinct from any of u, v, x, y. Since X_t is a clique in G, the subgraph H of G induced by $\{u, v, x, y, z\}$ is isomorphic to K_5, and thus $H - \{e, f\}$ is isomorphic to W_4. This implies that $G - \{e, f\}$ is not W_4-induced-minor-free. However, since $\{e, f\} \subseteq L \subseteq F^*$, this is a contradiction with the fact that F^* is $\mathcal{G}$-safe.

If $L = \emptyset$, we set $U_t = \emptyset$. Otherwise, we choose an arbitrary edge $e = xy \in L$ and set $U_t = \{x, y\}$. Such a set U_t can be computed in time $\mathcal{O}(|X_t| \cdot |F^*|) = \mathcal{O}(n \cdot m)$, by iterating over the edges in F^* and checking whether their endpoints both belong to X_t. Note that since any two edges in L have a common endpoint, all edges of the graph $G[X_t \setminus U_t]$ are in $E(G) \setminus F^*$, and thus U_t is an F^*-cover of $G[X_t]$ of size at most 2. □

Theorem 10.5.9 and Lemma 10.6.3 imply the following.

Theorem 10.6.4 (Theorem 5.4 in [78]). *For any W_4-induced-minor-free graph G with n vertices and m edges, one can compute in time $\mathcal{O}(n \cdot m)$ a 3-refined tree decomposition of G with $\mathcal{O}(n)$ nodes and residual independence number at most* 1.

Proof. The class of W_4-induced-minor-free graphs is closed under induced topological minors, and Lemma 10.6.3 shows that Theorem 10.5.9 is satisfied with $k = 1$ and $\ell = 3$, $f(n, m) = \mathcal{O}(n \cdot m)$, and $g(n, m) = n$. Therefore, given a W_4-induced-minor-free graph G with n vertices and m edges, Theorem 10.5.9 implies that one can compute an ℓ-refined tree decomposition of G with $\mathcal{O}(n)$ nodes and with residual independence number at most $\max\{3 - \ell, k\} = 1$ in time $\mathcal{O}(n + m + f(4n, 3m))$, which is $\mathcal{O}(n \cdot m)$. □

Note that Lemma 10.4.4 and Theorem 10.6.4 imply that the class of W_4-induced-minor-free graphs admits a linear (tw, ω)-binding function $f(p) = R(p + 1, 2) + 3 - 2 = p + 2$. Furthermore, the approach used in to prove Theorem 10.6.4 can be used to show that if G is a W_4-induced-minor-free graph, then the inequality $\mathrm{tw}(G) \geq \eta(G) - 1$, which holds for all graphs, is in fact satisfied with equality. In order to see this, let us assume that G is 2-connected. Then the proof of Theorem 10.6.4 provides a tree decomposition of G obtained from tree decompositions of graphs G_a over all R- and S-nodes a of a fixed SPQR tree of G. Using the structure of the corresponding graphs G_a, we can then verify that the constructed tree decompositions have only bags with at most $\eta(G_a)$ vertices. Thus, applying Corollary 8.5.2 we can show that the whole tree decomposition of G contains only bags with at most $\eta(G)$ vertices.

Proposition 10.6.5 (Proposition 5.6 in [78]). *If G is a W_4-induced-minor-free graph, then* $\mathrm{tw}(G) = \eta(G) - 1$.

Remark 10.6.6. We remark that for general classes of H-induced-minor-free graphs where H is planar, only the existence of a function bounding the treewidth in terms of the Hadwiger number is known (see Corollary 9.3.3; see also [21, 46, 132]).

Next, by Theorem 10.6.4, we have that every W_4-induced-minor-free graph G satisfies the inequality $3\text{-}\mathsf{tree}\text{-}\alpha(G) \leq 1$. We can thus use Observation 10.4.3 to obtain the following result.

Corollary 10.6.7 (Corollary 5.7 in [78]). *The tree-independence number of any W_4-induced-minor-free graph is at most* 4.

Remark 10.6.8 (Remark 5.8 in [78]). The bound on the tree-independence number given by Corollary 10.6.7 is sharp: there exist arbitrarily large 2-connected W_4-induced minor-free graphs with tree-independence number 4. Take an integer $q \geq 4$ and let F_q be the graph obtained from a complete graph with vertex set $S = \{1, 2, 3, 4\}$ by replacing each of its edges ij with q paths of length two connecting i and j. Note that F_q has exactly 4 vertices of degree more than two. Neither deleting vertices nor contracting edges having an endpoint of degree at most two can increase the number of vertices of degree more than two. It follows that every induced minor of F_q has at most 4 vertices of degree more than two. Since the graph W_4 has 5 vertices and minimum degree 3, we infer that F_q is W_4-induced-minor-free, and hence $\mathsf{tree}\text{-}\alpha(F_q) \leq 4$ by Theorem 10.6.4. To see that the inequality is satisfied with equality, it suffices to show that $\mathsf{tree}\text{-}\alpha(F_4) \geq 4$. But this was already observed in the proof of Theorem 10.1.9.

10.7 Tree-independence number: K_5^--im-free graphs

Let us now apply the same approach as was described at the beginning of Section 10.6 to the class of K_5^--induced-minor-free graphs. Similarly, we begin with the characterization of the 3-connected graphs in the class. Let us first state the following useful result regarding 3-connected graphs.

Theorem 10.7.1 (Tutte [192]). *Every* 3*-connected graph with at least* 5 *vertices has an edge whose contraction results in a* 3*-connected graph.*

Theorem 10.7.2 (Theorem 6.2 in [78]). *For every graph G, the following statements are equivalent.*

1. *G is* 3*-connected and K_5^--induced-minor-free.*
2. *G is either a complete graph with at least four vertices, a wheel, a $K_{3,3}$, or a $\overline{C_6}$.*

Proof. It is straightforward to verify that if G is either a complete graph K_n with $n \geq 4$, a wheel, $G \cong K_{3,3}$, or $G \cong \overline{C_6}$, then G is 3-connected and K_5^--induced-minor-free.

We prove the converse direction using induction on $n = |V(G)|$. So let G be a 3-connected K_5^--induced-minor-free graph. Since G is 3-connected, $n \geq 4$, and if $n = 4$, then G is complete since otherwise it would contain a vertex of degree at most two. Suppose that $n \geq 5$. By Theorem 10.7.1, G has an edge $e = uv$ whose contraction results in a 3-connected graph G'. Since G' is also K_5^--induced-minor-free, the induction hypothesis implies that G' is either a complete graph, a wheel, $G' \cong K_{3,3}$, or $G' \cong \overline{C_6}$. We analyze each of the four cases separately.

Case (1): G' is a complete graph.
In this case, $N_G(u) \cup N_G(v) = V(G) \setminus \{u, v\}$. Let $a = |N_G(u) \setminus N_G(v)|$, $b = |N_G(u) \cap N_G(v)|$, and $c = |N_G(v) \setminus N_G(u)|$. If $a = c = 0$, then G is complete. We may thus assume by symmetry that $a > 0$. If $b \geq 2$, then any two vertices from $N_G(u) \cap N_G(v)$, along with u, v, and a vertex from $N_G(u) \setminus N_G(v)$, would induce a subgraph of G isomorphic to K_5^-, a contradiction. Thus $b \leq 1$. Since G is 3-connected, we have $d_G(v) \geq 3$ and hence $c = d_G(v) - (b + 1) \geq 2 - b > 0$. If $a = c = 1$, then the 3-connectedness of G implies that $b = 1$, and G is isomorphic to W_4. We may thus assume by symmetry that $a \geq 2$. If $b = 1$, then any two vertices from $N_G(u) \setminus N_G(v)$, along with u, a vertex from $N_G(u) \cap N_G(v)$, and a vertex from $N_G(v) \setminus N_G(u)$, form an induced subgraph of G isomorphic to K_5^-, a contradiction. Thus $b = 0$ and consequently $c = d_G(v) - 1 \geq 2$. But now, the graph obtained from the subgraph H of G induced by u, v, any two vertices from $N_G(u) \setminus N_G(v)$, and any two vertices from $N_G(v) \setminus N_G(u)$ by contracting an edge from u to one of the vertices in $V(H) \cap (N_G(u) \setminus N_G(v))$ is isomorphic to K_5^-, a contradiction.

Case (2): G' is a wheel W_{n-2} with $n \geq 6$.
Let x be the universal vertex in G' and let C be the cycle $G' - x$, with a cyclic order of the vertices $v_1, \ldots, v_{n-2}$. We analyze two subcases depending on whether the vertex w of G' to which the edge uv was contracted corresponds to x, the central vertex of the wheel, or not.

Case (2.1): $w = x$.
In this case, $V(C) \subseteq N_G(u) \cup N_G(v)$. Since G is 3-connected, each of u and v has at least two neighbors on C. Suppose first that each of u and v has only two neighbors on C. Since C has at least four vertices, the neighborhoods of u and v on C are disjoint. Thus $|V(C)| = 4$ and G is either $\overline{C_6}$ or $K_{3,3}$, depending on whether the two neighbors of u on C are adjacent or not. We may thus assume that one of u and v, say v, has at least three neighbors on C. If u and v are both adjacent to three consecutive vertices on C, say v_j, v_{j+1}, v_{j+2} (indices modulo $n - 2$), then the subgraph of G induced by $\{u, v, v_j, v_{j+1}, v_{j+2}\}$ is isomorphic to K_5^-, a contradiction. Thus, we may assume in particular that u is not adjacent to v_1, and, consequently, v is adjacent to v_1. Let v_i and v_j be the two neighbors of u on C such that i is as small as possible and j is as large as possible. Then $1 < i < j \leq n - 2$. Now, if v has a neighbor in

$\{v_2, \ldots, v_{j-1}\}$ and a neighbor in $\{v_j, \ldots, v_{n-2}\}$, then contracting in G all the edges of the paths $(v_2, v_3, \ldots, v_{j-1})$ and $(v_j, v_{j+1}, \ldots, v_{n-2})$ results in a graph isomorphic to K_5^-, a contradiction. Similarly, if v has a neighbor in $\{v_2, \ldots, v_i\}$ and a neighbor in $\{v_{i+1}, \ldots, v_{n-2}\}$, then contracting in G all the edges of the paths $(v_2, v_3, \ldots, v_i)$ and $(v_{i+1}, v_{i+2}, \ldots, v_{n-2})$ results in a graph isomorphic to K_5^-, a contradiction. Next, if v has at least two neighbors in $\{v_i, \ldots, v_j\}$, say v_{k_1} and v_{k_2}, with $k_1 < k_2$, then contracting in G all the edges of the paths $(v_2, v_3, \ldots, v_{k_1})$ and $(v_{k_1+1}, v_{k_2+2}, \ldots, v_{n-2})$ results in a graph isomorphic to K_5^-, a contradiction. Thus, all the neighbors of v in $\{v_2, \ldots, v_{n-2}\}$ are in $\{v_2, \ldots, v_{i-1}\}$ or in $\{v_{j+1}, \ldots, v_{n-2}\}$. We may assume without loss of generality that all the neighbors of v are in $\{v_2, \ldots, v_{i-1}\}$. Consequently, $i > 3$ and the vertices v_2 and v_3 are adjacent to v and non-adjacent to u. It follows that contracting in G all the edges of the paths $(v_3, v_4, \ldots, v_{j-1})$ and $(v_j, v_{j+1}, \ldots, v_{n-2}, v)$ results in a graph isomorphic to K_5^-, a contradiction.

Case (2.2): $w \neq x$.
We may assume without loss of generality that $w = v_1$. Thus, $N_G(u) \cup N_G(v) = \{u, v, v_{n-2}, x, v_2\}$. Since G is 3-connected, each of the vertices v_2 and v_{n-2} must have a neighbor in the set $\{u, v\}$. If the edges in G having one endpoint in $\{v_2, v_{n-2}\}$ and the other one in $\{u, v\}$ form a matching of size two, then u and v must both be adjacent to x since G is 3-connected, and hence G is a wheel, W_{n-1}, in this case. We may thus assume without loss of generality that u is adjacent to both v_2 and v_{n-2}. Furthermore, since $d_G(v) \geq 3$, we may assume that v is adjacent to v_{n-2}. Suppose that v is not adjacent to x. Then v is adjacent to v_2 and u is adjacent to x, and contracting in G the edge vv_2 and all the edges of the path $(v_3, \ldots, v_{n-3})$ results in a graph isomorphic to K_5^-, a contradiction. Hence, v is adjacent to x. But now, contracting in G the edge uv_2 and all the edges of the path $(v_3, \ldots, v_{n-3})$ results in a graph isomorphic to K_5^-, a contradiction.

Case (3): G' is isomorphic to $K_{3,3}$.
Let w be the vertex of G' to which the edge uv was contracted and let $A = \{u_1, u_2, w\}$ and $B = \{v_1, v_2, v_3\}$ be the two independent sets partitioning $V(G')$. Since A forms an independent set in G', it follows that both u and v are non-adjacent with the vertices u_1 and u_2 in G. On the other hand, $B \subseteq N_G(u) \cup N_G(v)$, and, since G is 3-connected, each of u and v has at least two neighbors in B. We may assume without loss of generality that u and v are both adjacent to v_2. Since each vertex in the set $\{u, v\}$ has a neighbor in the set $\{v_1, v_3\}$ and vice versa, we may assume without loss of generality that u is adjacent to v_1 and v is adjacent to v_3. It follows that contracting in G the edges uv_1 and vv_3 results in a graph isomorphic to K_5^-, a contradiction.

Case (4): G' is isomorphic to $\overline{C_6}$.
Let $v_1, \ldots, v_6$ be a cyclic order of the vertices of the C_6. It follows that

in G', the three odd-indexed vertices form a clique, and the same is true for the even-indexed vertices. Additionally, in G' we also have the edges v_1v_4, v_3v_6, and v_2v_5. We may assume that v_1 is the vertex obtained by contracting the edge uv. Therefore, $N_G(u) \cup N_G(v) = \{u, v, v_3, v_4, v_5\}$ and, since G is 3-connected, each of the vertices u and v has at least two neighbors among the vertices $\{v_3, v_4, v_5\}$. Since each vertex in the set $\{u, v\}$ has a neighbor in the set $\{v_3, v_5\}$ and vice versa, we may assume without loss of generality that u is adjacent to v_3 and v is adjacent to v_5. Furthermore, we may assume without loss of generality that u is adjacent to v_4. Now, if v is adjacent to v_3, then contracting the edges uv_4 and v_2v_5 results in a graph isomorphic to K_5^-, a contradiction. On the other hand, if v is adjacent to v_4, then contracting the edges vv_5 and v_3v_6 results in a graph isomorphic to K_5^-, again a contradiction. □

We can now use Theorem 10.7.2 in order to derive the following algorithmic result which will help us apply Theorem 10.5.9 to the class of K_5^--induced-minor-free graphs.

Lemma 10.7.3 (Lemma 6.3 in [78]). *Let $\mathcal{G}$ be the class of K_5^--induced-minor-free graphs. Then $\mathcal{G}$ is $(1, 3)$-tree decomposable and $(1, 3)$-*Tree Decomposition*$(\mathcal{G})$ can be solved on graphs with n vertices and m edges in time $\mathcal{O}(n + m)$ so that the resulting tree decomposition has at most $n - 3$ nodes.*

Proof. Let G be a 3-connected K_5^--induced-minor-free graph, let $n = |V(G)|$, and let $F^* \subseteq F \subseteq E(G)$ such that F^* is $\mathcal{G}$-safe. We want to compute an F^*-covering 3-refined tree decomposition $\widehat{\mathcal{T}}$ of G with residual independence number at most 1, together with an F-mapping of $\widehat{\mathcal{T}}$.

By Theorem 10.7.2, G is either a complete graph with at least four vertices, a wheel, $K_{3,3}$, or $\overline{C_6}$. In constant time we check if $G \cong K_{3,3}$ or $G \cong \overline{C_6}$. If this is not the case, then G is either a complete graph or a wheel. One can distinguish among these two cases in constant time using vertex degrees. We consider each of the four cases independently.

Case (1): G is a complete graph.
In this case, G admits a trivial 0-refined tree decomposition $\widehat{\mathcal{T}}$ with a tree T consisting of only one node t and a unique bag $X_t = V(G)$. The corresponding F-mapping of $\widehat{\mathcal{T}}$ maps each edge $e \in F$ to t. Note that $|X_t| \geq 4$ since G is 3-connected. We set U_t to be any subset of X_t of cardinality 3 and claim that U_t is an F^*-cover of $G[X_t] = G$. Suppose this is not the case and let $e = xy$ be an edge in F^* such that $\{x, y\} \cap U_t = \emptyset$. Then the subgraph of $G - e$ induced by $U_t \cup \{x, y\}$ is isomorphic to K_5^- and thus $G - e \notin \mathcal{G}$. However, this contradicts the assumption that F^* is $\mathcal{G}$-safe. Hence, $\widehat{\mathcal{T}}$ is an F^*-covering 3-refined tree decomposition of G with residual independence number at most 1.

Case (2): G is a wheel W_{n-1} with $n \geq 5$.

In linear time we identify the universal vertex v_0 in G and compute a cyclic order $v_1, \ldots, v_{n-1}$ of the vertices of the cycle $G - v_0$. We construct the desired 3-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)\}_{t \in V(T)})$ of G as follows. The tree T is an $(n-3)$-vertex path $(t_1, \ldots, t_{n-3})$; for each $i \in \{1, \ldots, n-3\}$, the bag X_{t_i} consists of the vertices $\{v_0, v_i, v_{i+1}, v_{n-1}\}$, and we set U_{t_i} to an arbitrary subset of X_{t_i} with cardinality 3. Following the fact that $|X_t \setminus U_t| \leq 1$, we get that $\widehat{\mathcal{T}}$ is an F^*-covering 3-refined tree decomposition of G with residual independence number at most 1. The corresponding F-mapping of $\widehat{\mathcal{T}}$ can be obtained as follows. For each edge $e = v_i v_j \in F$ with $0 \leq i < j \leq n-1$, we map e to $t_e \in V(T)$ where

$$t_e = \begin{cases} t_i & \text{if } 1 \leq i < j \leq n-2 \text{ or } (i,j) = (1, n-1)\,, \\ t_j & \text{if } i = 0 \text{ and } j \leq n-3\,, \\ t_{n-3} & \text{otherwise}\,. \end{cases}$$

Note that both $\widehat{\mathcal{T}}$ and the F-mapping can be obtained in linear time.

Case (3): G is $K_{3,3}$.
Let A and B be the two independent sets partitioning $V(G)$. Then G has a tree decomposition such that every bag X_t contains all the vertices of A and exactly one vertex of B, and thus $|X_t| \leq 4$. This tree decomposition can be turned into an F^*-covering 3-refined tree decomposition $\widehat{\mathcal{T}}$ with residual independence number at most one by defining, for instance, $U_t = A$ for all nodes t. In this case, both the 3-refined tree decomposition $\widehat{\mathcal{T}}$ and an arbitrary F-mapping of $\widehat{\mathcal{T}}$ can be obtained in constant time.

Case (4): G is $\overline{C_6}$.
Let u and v be two non-adjacent vertices of G and let (u_1, u_2, u_3, u_4) be a 4-vertex path formed by the vertices of G other than u and v. Labeling the vertices of a 3-vertex path with the sets $\{u_1, u_2, u, v\}$, $\{u_2, u_3, u, v\}$, $\{u_3, u_4, u, v\}$ in order yields a tree decomposition of G with bags of size 4. Again, the tree decomposition can be turned into an F^*-covering 3-refined tree decomposition $\widehat{\mathcal{T}}$ with residual independence number at most one by taking, for each bag X_t, the set U_t to be an arbitrary subset of X_t with cardinality 3. This 3-refined tree decomposition $\widehat{\mathcal{T}}$ and an arbitrary F-mapping of $\widehat{\mathcal{T}}$ can be obtained in constant time.

In each case, the computed 3-refined tree decomposition has at most $n-3$ nodes. $\square$

Theorem 10.5.9 and Lemma 10.7.3 imply the following.

Theorem 10.7.4 (Theorem 6.4 in [78]). *For any K_5^--induced-minor-free n-vertex graph G, one can compute in linear time a 3-refined tree decomposition of G with $\mathcal{O}(n)$ nodes and residual independence number at most 1.*

Proof. The class of K_5^--induced-minor-free graphs is closed under induced topological minors, and Lemma 10.7.3 shows that Theorem 10.5.9 is satisfied with $k = 1$, $\ell = 3$, $f(n, m) = \mathcal{O}(n + m)$, and $g(n, m) = n$. Therefore, given a K_5^--induced-minor-free graph G with n vertices and m edges, Theorem 10.5.9 implies that one can compute a 3-refined tree decomposition of G with $\mathcal{O}(n)$ nodes and residual independence number at most $\max\{3-\ell, k\} = 1$ in time $\mathcal{O}(n+m+f(4n, 3m))$, which is $\mathcal{O}(n+m)$. □

Similarly as for the class of W_4-induced-minor-free graphs, Lemma 10.4.4 and Theorem 10.7.4 imply the existence of a linear (tw, ω)-binding function $f(p) = p + 2$ for the class of K_5^--induced-minor-free graphs. Moreover, in a similar way as mentioned before Proposition 10.6.5, our approach that leads to the proof of Theorem 10.7.4 can be used to show the following.

Proposition 10.7.5 (Proposition 6.6 in [78]). *If G is a K_5^--induced-minor-free graph, then* $\mathrm{tw}(G) = \eta(G) - 1$.

Theorem 10.7.4 and Observation 10.4.3 imply the following.

Corollary 10.7.6 (Corollary 6.7 in [78]). *The tree-independence number of any K_5^--induced-minor-free graph is at most* 4.

Remark 10.7.7 (Remark 6.10 in [78]). Consider again the family $\{F_q\}_{q\geq 4}$ of graphs from Remark 10.6.8. The same arguments as used therein to show that graphs F_q are W_4-induced-minor-free also show that these graphs are K_5^--induced-minor-free. Thus, the bound on the tree-independence number given by Corollary 10.6.7 is sharp; there exist arbitrarily large 2-connected K_5^--induced minor-free graphs with tree-independence number 4.

Chapter 11

Maximum Weight Independent Set and Generalizations

$\mathcal{I}$n the previous chapter we set up some important notions and results that will be useful in this chapter. In what follows, we will prove that the weighted version of the INDEPENDENT SET problem is efficiently solvable in all (tw, ω)-bounded graph classes that we discussed before (see Table 9.1 on p.95 for a quick summary). Let us now give the definition of the problem.

MAXIMUM WEIGHT INDEPENDENT SET (MWIS)
Input: A graph G and a weight function $w : V(G) \mapsto \mathbb{Q}^+$.
Output: An independent set I in G of maximum possible weight $w(I)$, where $w(I) = \sum_{v \in I} w(x)$.

In what follows, we will discuss solvability of the MWIS problem in (tw, ω)-bounded graph classes excluding a single structure (see also Section 8 in [76]).

Theorem 11.0.1. *For every graph H and every graph class $\mathcal{G}$ the following holds.*

- *If $H \in \mathcal{S}$ and $\mathcal{G}$ is the class of H-subgraph-free graphs, or*
- *if H is a subcubic planar graph and $\mathcal{G}$ is the class of H-topological-minor-free graphs, or*
- *if H is a planar graph and $\mathcal{G}$ is the class of H-minor-free graphs*

then MAXIMUM WEIGHT INDEPENDENT SET *problem is solvable in linear time for the class $\mathcal{G}$.*

Proof. Due to Theorems 9.1.4, 9.1.6, and 9.1.7, in all three cases the class $\mathcal{G}$ has bounded treewidth. We can thus use the linear-time algorithm of Bodlaender [27] to compute a tree decomposition of the input graph G of constant width. Then we use the approach of Arnborg, Lagergren, and

Seese [15] in order to compute a maximum weight independent set in linear time with the help of previously computed tree decomposition. □

Theorem 11.0.2. *Let $H \subseteq_{\text{is}} P_3$ or H is edgeless and let $\mathcal{G}$ be the class of H-free graphs. Then* Maximum Weight Independent Set *problem is solvable in polynomial time for the class $\mathcal{G}$.*

Proof. Suppose first that $H \subseteq_{\text{is}} P_3$. Then every H-free graph is a disjoint union of complete graphs. Thus, in order to find a maximum weight independent set in any H-free graph G, it is enough to find a vertex of maximum weight in each of the complete graphs. This can be done in linear time by first computing the connected components of G using a Breadth First Search algorithm and then for each connected component (which is a clique in G) iterate over the vertices of it to find one with maximum weight.

Finally, if H is edgeless, then every H-free graph contains independent sets of size at most $|V(H)| - 1$. Thus, we can enumerate all independent sets of G and in time $\mathcal{O}(|V(G)|^{|V(H)|-1})$ compute one with a maximum weight. □

Note that the proof of Theorem 11.0.2 actually gives a linear time algorithm in the case when $H \subseteq_{\text{is}} P_3$.

In the proof of the next theorem, we will require the following definition. The *clique-width* of a graph G is the minimum number of labels needed to construct G using the following four operations:

- Creation of a new vertex v with label i.
- Disjoint union of two labeled graphs G and H.
- Joining by an edge each vertex with label i to each vertex with label j.
- Renaming label i to j.

Theorem 11.0.3. *Let $H \subseteq_{\text{itm}} C_4$, or $H \subseteq_{\text{itm}} K_4^-$, or H is edgeless and let $\mathcal{G}$ be the class of H-itm-free graphs. Then* Maximum Weight Independent Set *problem is solvable in polynomial time for the class $\mathcal{G}$.*

Proof. Suppose first that $H \subseteq_{\text{itm}} C_4$. Then, by Observation 8.0.1, $\mathcal{G}$ is a subclass of the class of chordal graphs, for which a linear time algorithm for the MWIS problem is known [104].

Next, assume that $H \subseteq_{\text{itm}} K_4^-$. Then, by Lemma 9.2.3, $\mathcal{G}$ is a subclass of the class of block-cactus graphs. From the fact that the clique-width of every block-cactus graphs is at most 6 (see [144]) and the algorithm of Courcelle et al. [63], it follows that the MWIS problem is solvable in polynomial time for the class $\mathcal{G}$.

Finally, if H is edgeless, then every H-itm-free graph is also H-free and hence, from the proof of Theorem 11.0.2 it follows that the maximum weight independent set can be computed in time $\mathcal{O}(|V(G)|^{|V(H)|-1})$. $\square$

Theorems 11.0.1, 11.0.2, and 11.0.3 settle all the cases of (tw, ω)-bounded graph classes excluding a single graph H with respect to one of the six discussed graph containment relations, with the exception of the induced minor relation. To this end, we will first show that for every $k \geq 1$, the Maximum Weight Independent Set problem is solvable in polynomial time provided that the input graph is given along with a tree decomposition with independence number at most k. Next, we will generalize this result by showing polynomial-time solvability of a more general problem, called the Maximum Weight Independent Packing problem under the same setting.

11.1 Maximum Weight Independent Set problem

A tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of a graph G is said to be *rooted* if we distinguish one node r of T, called a *root node*, which we take as the root of T. Rooting a tree decomposition naturally gives a parent-child relations in the tree T. A *leaf* of a rooted tree T is a node with no children. We follow [65] for the following definition. We say that a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ is *nice* if it is rooted and the following conditions are satisfied:

(a) If $t \in V(T)$ is the root or a leaf of T, then $X_t = \emptyset$;

(b) Every non-leaf node t of T is one of the following three types:

- **Introduce node:** a node t with exactly one child t' such that $X_t = X_{t'} \cup \{v\}$ for some vertex $v \in V(G) \setminus X_{t'}$;
- **Forget node:** a node t with exactly one child t' such that $X_t = X_{t'} \setminus \{v\}$ for some vertex $v \in X_{t'}$;
- **Join node:** a node t with exactly two children t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.

In what follows, we assume that both introduce and forget nodes are also labeled with the unique vertex v which is introduced or forgotten. We say that an ℓ-refined tree decomposition is *nice* if its underlying tree decomposition is nice. Given a graph G and a tree decomposition $\mathcal{T}$ of G with width at most k, one can compute a nice tree decomposition of G with width at most k in polynomial time (see, e.g., [65]). We use the same approach in order to prove the following lemma.

Lemma 11.1.1 (Lemma 5.1 in [77]). *Given an ℓ-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)\}_{t \in V(T)})$ with width k of a graph G, one can compute in time $\mathcal{O}(k^2 \cdot |V(T)|)$ a nice ℓ-refined tree decomposition $\widehat{\mathcal{T}}' = (T', \{(X_{t'}, U_{t'})\}_{t' \in V(T')})$ of G that has at most $\mathcal{O}(k \cdot |V(T)|)$ nodes* *and such that for every node $t' \in V(T')$ there exists a node $t \in V(T)$ such that $X_{t'} \subseteq X_t$ and $U_{t'} = U_t \cap X_{t'}$. In particular, $\widehat{\alpha}(\widehat{\mathcal{T}}') \leq \widehat{\alpha}(\widehat{\mathcal{T}})$.*

Proof. Consider the following algorithm.

1. We traverse the tree T and check for every two adjacent nodes t and t' in T if the bags X_t and $X_{t'}$ are comparable (i.e., if $X_t \subseteq X_{t'}$ or $X_{t'} \subseteq X_t$). If, say, $X_t \subseteq X_{t'}$, then we contract the edge tt' and label the resulting node with the pair $(X_{t'}, U_{t'})$. Hence, we now assume that no two adjacent nodes of T have comparable bags.
2. We choose a node of T with degree at most one as its root r and compute the corresponding parent-child relationship in T.
3. We assure that each node has at most two children, by replacing each node t of T with $d \geq 3$ children $c_1, \ldots, c_d$ with a path consisting of d nodes $t_1, \ldots, t_d$, each associated with the same pair (X_t, U_t), making t_1 a child of the parent of t, and for all $j \in \{1, \ldots, d\}$, making c_j a child of t_j (note that if $j < d$, then t_{j+1} is also a child of t_j).
4. For every node t of T with exactly two children t_1 and t_2, we label t as a join node. For $i \in \{1, 2\}$, if $X_t \neq X_{t_i}$, then we subdivide the edge tt_i and associate the pair (X_t, U_t) to the new node.
5. For each leaf node t of T, we add a new node t' associated with the pair $(\emptyset, \emptyset)$ and make t' a child of t.
6. We add a new node r' associated with the pair $(\emptyset, \emptyset)$ and make r a child of r' (that is, r' becomes the new root).
7. For every node t of T that is not already labeled as a join node there is a unique child t' of t. We replace the edge $t't$ with a path $(t' = t_0, \ldots, t_p, t_{p+1}, \ldots, t_{p+q} = t)$ of length $p+q$ where $p = |X_{t'} \setminus X_t|$ and $q = |X_t \setminus X_{t'}|$. For all $i \in \{1, \ldots, p\}$, the node t_i is a forget node labeled with (X_{t_i}, U_{t_i}) where X_{t_i} forgets one vertex from $X_{t_{i-1}} \setminus X_t$ and $U_{t_i} = U_{t'} \cap X_{t_i}$. Similarly, for all $j \in \{1, \ldots, q-1\}$, the node t_{p+j} is an introduce node labeled with $(X_{t_{p+j}}, U_{t_{p+j}})$ where $X_{t_{p+j}}$ introduces one vertex from $X_t \setminus X_{t_{p+j-1}}$, and $U_{t_{p+j}} = U_t \cap X_{t_{p+j}}$. The last node of the path is the node t, which is an introduce node.

Note that each of the above steps modifies an ℓ-refined tree decomposition into another one. Let us denote by $\widehat{\mathcal{T}}' = (T', \{(X_{t'}, U_{t'})\}_{t' \in V(T')})$ the final ℓ-refined tree decomposition. By construction, every non-leaf node of T' has a unique label (an introduce node, a forget node, or a join node) and $\widehat{\mathcal{T}}'$ is a nice ℓ-refined tree decomposition of G such that for every node t' of T', there exists a node t in T such that $X_{t'} \subseteq X_t$ and $U_{t'} = U_t \cap X_{t'}$. Consider a node t' of T' such that $\widehat{\alpha}(\widehat{\mathcal{T}}') = \alpha(G[X_{t'} \setminus U_{t'}])$ and let t be a node of T such that $X_{t'} \subseteq X_t$ and $U_{t'} = U_t \cap X_{t'}$. Then,

$$X_{t'} \setminus U_{t'} = X_{t'} \setminus (U_t \cap X_{t'}) = X_{t'} \setminus U_t \subseteq X_t \setminus U_t$$

and hence

$$\widehat{\alpha}(\widehat{\mathcal{T}}') = \alpha(G[X_{t'} \setminus U_{t'}]) \leq \alpha(G[X_t \setminus U_t]) \leq \widehat{\alpha}(\widehat{\mathcal{T}}) .$$

We now reason about the complexity of obtaining $\widehat{\mathcal{T}}'$ by considering the complexity of each aforementioned step. Step 1 takes $\mathcal{O}(k^2)$ time for every edge of T, and thus $\mathcal{O}(k^2 \cdot |V(T)|)$ overall; the resulting tree T' has $\mathcal{O}(|V(T)|)$ nodes. Step 2 can be done in time $\mathcal{O}(|V(T)|)$. Step 3 can be done in time $\mathcal{O}(|V(T)|)$ and results in a tree T' with $\mathcal{O}(|V(T)|)$ nodes. Step 4 can be done in time $\mathcal{O}(k^2 \cdot |V(T)|)$ and Step 5 in time $\mathcal{O}(|V(T)|)$; both steps yield a tree with $\mathcal{O}(|V(T)|)$ nodes. Step 6 can be done in constant time. Finally, Step 7 can be done in time $\mathcal{O}(k^2 \cdot |V(T)|)$ and results in an ℓ-refined tree decomposition with $\mathcal{O}(k \cdot |V(T)|)$ nodes. □

We can now use Lemma 11.1.1, as well as adapt the dynamic programming approach for solving the Max Weight Independent Set problem in graph classes of bounded treewidth (see, e.g., [65]) to graph classes with bounded ℓ-refined tree independence number, for some integer $\ell \geq 0$.

Theorem 11.1.2 (Theorem 5.2 in [77]). *For every integer $k \geq 1$,* Max Weight Independent Set *is solvable in time $\mathcal{O}(2^\ell \cdot |V(G)|^{k+1} \cdot |V(T)|)$ if the input vertex-weighted graph G is given with an ℓ-refined tree decomposition $\widehat{\mathcal{T}} = (T, \{(X_t, U_t)\}_{t \in V(T)})$ with residual independence number at most k.*

Proof. Let $n = |V(G)|$ and $w : V(G) \to \mathbb{Q}_+$ be the weight function. We first apply Lemma 11.1.1 and compute in time $\mathcal{O}(n^2 \cdot |V(T)|)$ a nice ℓ-refined tree decomposition $\widehat{\mathcal{T}}' = (T', \{(X_t, U_t)\}_{t \in V(T')})$ of G with $\mathcal{O}(n \cdot |V(T)|)$ bags and such that

$$\widehat{\alpha}(\widehat{\mathcal{T}}') \leq \widehat{\alpha}(\widehat{\mathcal{T}}) \leq k .$$

Recall that, by definition, $\widehat{\mathcal{T}}'$ is rooted at some node r of T'. For every node $t \in V(T')$, we denote by V_t the union of all bags $X_{t'}$ such that $t' \in V(T')$ is a (not necessarily proper) descendant of t in T'.

For each node $t \in V(T')$, we compute the family $\mathcal{S}_t$ of all sets $S \subseteq X_t$ that are independent in G. Note that each set $S \in \mathcal{S}_t$ is the disjoint union of sets S_1 and S_2 where $S_1 = S \cap U_t$ and $S_2 = S \cap (X_t \setminus U_t)$. Since $\widehat{\mathcal{T}}'$ is an ℓ-refined tree decomposition with residual independence number at most k, we have that $|S_1| \leq |U_t| \leq \ell$ and $|S_2| \leq \alpha(G[X_t \setminus U_t]) \leq k$. It follows that the family $\mathcal{S}_t$ can be computed in time $\mathcal{O}(2^{|U_t|} \cdot |X_t \setminus U_t|^k) = \mathcal{O}(2^\ell \cdot n^k)$ by enumerating all $\mathcal{O}(2^\ell)$ candidate sets for S_1, all $\mathcal{O}(n^k)$ candidate sets for S_2, and verifying if the union $S_1 \cup S_2$ is independent in G. This can be done in constant time if a set S with $|S| \leq k + \ell$ is independent in G, since $k + \ell$ is constant and we assume that G is represented with an adjacency matrix. If it is not represented with an adjacency matrix, we can first compute such a representation from the adjacency lists in time $\mathcal{O}(n^2)$. We then traverse the tree T' bottom-up and use a dynamic programming approach to compute, for every node $t \in V(T')$ and every set $S \in \mathcal{S}_t$, the value of $c[t, S]$, defined as the maximum weight of an independent set I in the graph $G[V_t]$ such that $I \cap X_t = S$.

Since $\widehat{\mathcal{T}}'$ is nice, we have $X_r = \emptyset$; in particular, the only independent set S with $S \subseteq X_r$ is the empty set. Furthermore, $V_r = V(G)$, and hence $c[r, \emptyset]$ corresponds to the maximum weight of an independent set in G, which is what we want to compute.

We consider various cases depending on the type of a node $t \in V(T')$. For each type we give a formula for computing the value $c[t, S]$ for all $S \in \mathcal{S}_t$ from the already computed values of $c[t', S']$ where t' is a child of t in T' and $S' \in \mathcal{S}_{t'}$.

Leaf node. By the definition of a nice tree decomposition it follows that $X_t = \emptyset$. Thus, we have $\mathcal{S}_t = \{\emptyset\}$, $V_t = \emptyset$, and $c[t, \emptyset] = 0$.

Introduce node. By definition, t has exactly one child t' and $X_t = X_{t'} \cup \{v\}$ holds for some vertex $v \in V(G) \setminus X_{t'}$. For an arbitrary set $S \in \mathcal{S}_t$, we have

$$c[t, S] = \begin{cases} c[t', S] & \text{if } v \notin S\,, \\ c[t', S \setminus \{v\}] + w(v) & \text{otherwise.} \end{cases}$$

Forget node. By definition, t has exactly one child t' in T' and $X_t = X_{t'} \setminus \{v\}$ holds for some vertex $v \in X_{t'}$. Note that for a set $S \in \mathcal{S}_t$, the set $S \cup \{v\}$ belongs to $\mathcal{S}_{t'}$ if and only if it is independent in G, that is, if no vertex in S is adjacent to v. For an arbitrary set $S \in \mathcal{S}_t$, we have

$$c[t, S] = \begin{cases} \max\{c[t', S], c[t', S \cup \{v\}]\} & \text{if no vertex in } S \text{ is adjacent to } v, \\ c[t', S] & \text{otherwise.} \end{cases}$$

Join node. By definition, t has exactly two children t_1 and t_2 in T' and it holds that $X_t = X_{t_1} = X_{t_2}$. For an arbitrary set $S \in \mathcal{S}_t$, we have

$$c[t, S] = c[t_1, S] + c[t_2, S] - w(S)\,.$$

The only way in which our algorithm differs from the standard one (see [65]) is that we compute $c[t, S]$ only for sets S in the family $\mathcal{S}_t$, and not for all subsets of the bag X_t. We therefore omit the description of the recurrence relations leading to the dynamic programming algorithm and the proof of correctness.

It remains to estimate the time complexity. We need time $\mathcal{O}(n^2 \cdot |V(T)|)$ to compute $\widehat{\mathcal{T}}'$. At each of the $\mathcal{O}(n \cdot |V(T)|)$ nodes $t \in V(T')$, we perform a constant-time computation for each set $S \in \mathcal{S}_t$, resulting in an overall time complexity of $\mathcal{O}(2^\ell \cdot n^k)$ per node. Thus, the total time complexity of the algorithm is $\mathcal{O}(n^2 \cdot |V(T)|) + \mathcal{O}(n \cdot |V(T)| \cdot 2^\ell \cdot n^k) = \mathcal{O}(2^\ell \cdot n^{k+1} \cdot |V(T)|)$, as claimed. $\square$

In the case when $\ell = 0$, we can use Observation 10.4.3 to immediately get the following corollary of Theorem 11.1.2.

Corollary 11.1.3 (Corollary 5.3 in [77]). *For every $k \geq 1$,* Max Weight Independent Set *is solvable in time $\mathcal{O}(|V(G)|^{k+1} \cdot |V(T)|)$ if the input vertex-weighted graph G is given with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ with independence number at most k.*

11.2 Maximum Weight Independent Packing problem

In order to state what a Maximum Weight Independent Packing problem is, we will first describe one construction and give some related results.

Let G be a graph and let $\mathcal{H} = \{H_j\}_{j \in J}$ be a family of connected subgraphs of G. By $G(\mathcal{H})$, we denote the graph with vertex set J, in which two distinct elements $i, j \in J$ are adjacent if and only if H_i and H_j either have a vertex in common or there is an edge in G connecting them. The above construction was considered by Cameron and Hell in [52]. In particular, they focused on the following case. Let G be a graph and let $\mathcal{F}$ be a (finite or infinite) set of connected graphs. By $\mathcal{H}(G, \mathcal{F})$ we denote the family of all subgraphs of G isomorphic to a member of $\mathcal{F}$. In particular, we have that:

- for $\mathcal{H} = \mathcal{H}(G, \{K_1\})$, we get that $G(\mathcal{H}) \cong G$, and
- for $\mathcal{H} = \mathcal{H}(G, \{K_2\})$, we get that $G(\mathcal{H})$ is isomorphic to the square of the line graph of G.

A *square graph* of a graph G is the graph obtained from G by adding to it all the non-edges uv of G such that u and v have a common neighbor in G. A construction similar to $G(\mathcal{H})$ was studied by Duchet [88]. More recently, Gartland et al. [107] considered the special case when $\mathcal{H}$ consists of all connected induced subgraphs of G, referring in this case to the derived graph $G(\mathcal{H})$ as the *blob graph* of G.

Cameron and Hell [52] also proved that for any chordal graph G, any set $\mathcal{F}$ of connected graphs, and $\mathcal{H} = \mathcal{H}(G, \mathcal{F})$, the graph $G(\mathcal{H})$ is chordal, which is a generalization of an analogous result due to Cameron [51] in the case when $\mathcal{F} = \{K_2\}$. Due to Theorem 10.1.1, the result of Cameron and Hell states that $\mathsf{tree}\text{-}\alpha(G(\mathcal{H})) \leq 1$ whenever $\mathsf{tree}\text{-}\alpha(G) \leq 1$. We now give a generalization of this result in the following way. We show that mapping any graph G to the graph $G(\mathcal{H})$, where $\mathcal{H}$ is an arbitrary collection of non-null connected subgraphs of G, cannot increase the tree-independence number. That implies that for any graph class $\mathcal{G}$ with bounded tree-independence number and any set $\mathcal{F}$ of connected non-null graphs, also the class $\{G(\mathcal{H}) : G \in \mathcal{G}, \mathcal{H} = \mathcal{H}(G, \mathcal{F})\}$ has bounded tree-independence number.

Lemma 11.2.1 (Lemma 6.1 in [77]). *Let G be a graph, let $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ be a tree decomposition of G, and let $\mathcal{H} = \{H_j\}_{j\in J}$ be a finite family of connected non-null subgraphs of G. Then $\mathcal{T}' = \big(T, \{X'_t\}_{t\in V(T)}\big)$ with $X'_t = \{j \in J : V(H_j) \cap X_t \neq \emptyset\}$ for all $t \in V(T)$ is a tree decomposition of $G(\mathcal{H})$ such that $\alpha(\mathcal{T}') \leq \alpha(\mathcal{T})$.*

Proof. Let us first show that $\mathcal{T}'$ is a tree decomposition of $G(\mathcal{H})$. First, note that since $V(G(\mathcal{H})) = J$, for each $t \in V(T)$ the set X'_t is indeed a subset of $V(G(\mathcal{H}))$.

Let $j \in J$ be a vertex of $G(\mathcal{H})$. Fix a vertex $v \in V(H_j)$ and consider any bag X_t of $\mathcal{T}$ such that $v \in X_t$. Then $v \in V(H_j) \cap X_t$ and hence $j \in X'_t$. Thus, every vertex of $G(\mathcal{H})$ belongs to a bag of $\mathcal{T}'$.

Let $\{i, j\}$ be an edge of $G(\mathcal{H})$. Assume first that the subgraphs H_i and H_j have a vertex in common, say v. Since $\mathcal{T}$ is a tree decomposition of G, there exists some $t \in V(T)$ such that $v \in X_t$. The fact that $v \in V(H_i) \cap X_t$ implies that $i \in X'_t$, and similarly, $j \in X'_t$. Assume now that there exist vertices $u \in V(H_i)$ and $v \in V(H_j)$ such that $uv \in E(G)$. Since $\mathcal{T}$ is a tree decomposition of G, there exists some $t \in V(T)$ such that $\{u, v\} \subseteq X_t$. The fact that $u \in V(H_i) \cap X_t$ implies that $i \in X'_t$. Similarly, the fact that $v \in V(H_j) \cap X_t$ implies that $j \in X'_t$. Hence, for every edge of $G(\mathcal{H})$ there exists a bag of $\mathcal{T}'$ containing both endpoints of the edge.

Next, consider an arbitrary vertex $j \in J$ of $G(\mathcal{H})$. We need to show that the set of nodes $t \in V(T)$ such that $j \in X'_t$ induces a connected subgraph of T. Let us denote for each vertex $v \in V(H_j)$ by T_v the subgraph of T induced by the nodes $t \in V(T)$ such that $v \in X_t$. Since $\mathcal{T}$ is a tree decomposition of G, each T_v is a connected subgraph of T, that is, a subtree. For a node $t \in V(T)$, the condition $j \in X'_t$ is equivalent to the condition $V(H_j) \cap X_t \neq \emptyset$, that is, there exists a vertex $v \in V(H_j)$ such that $v \in X_t$. Therefore, $j \in X'_t$ if and only if there exists a vertex $v \in V(H_j)$ such that t belongs to the tree T_v. It thus suffices to show that the union T_j of the trees T_v over all vertices $v \in V(H_j)$ forms a connected graph. Suppose for a contradiction that T_j is not connected and fix a component C of T_j. Let us denote by U the set of vertices $u \in V(H_j)$ such that $V(T_u) \subseteq V(C)$. Since $V(C) \neq V(T_j)$, we have $U \neq V(H_j)$. By the connectedness of H_j, there is an edge $uv \in E(H_j)$ such that $u \in U$ and $v \in V(H_j) \setminus U$. Let t be a node of T such that $\{u, v\} \subseteq X_t$. The trees T_u and T_v both contain node t, and hence the connected component C of T_j contains both T_u and T_v. This implies that $v \in U$, a contradiction.

It remains to show that $\alpha(\mathcal{T}) \geq \alpha(\mathcal{T}')$. Let t be a node of T maximizing the independence number of the subgraph of $G(\mathcal{H})$ induced by X'_t. Let k be this independence number and let $I \subseteq J$ be an independent set of cardinality k in the subgraph of $G(\mathcal{H})$ induced by X'_t. Then for any two distinct elements $i, j \in I$ the graphs H_i and H_j are vertex-disjoint subgraphs in $\mathcal{H}$ such that no edge of G has one endpoint in H_i and the other one in H_j. Each element $i \in I$ belongs to the bag X'_t, which implies that $V(H_i) \cap X_t \neq \emptyset$, and hence there exists a vertex u_i in $V(H_i) \cap X_t$. Since any two distinct vertices u_i and u_j belong to the subgraphs H_i and H_j of G, which are vertex-disjoint and with no edges between them, the set $\{u_i : i \in I\}$ is an independent set of cardinality k in the subgraph of G induced by X_t. This shows that $\alpha(\mathcal{T}) \geq k = \alpha(\mathcal{T}')$, as claimed. □

Lemma 11.2.1 immediately gives the following result.

Theorem 11.2.2 (Theorem 6.2 in [77]). *Let G be a graph and let $\mathcal{H}$ be a finite family of connected non-null subgraphs of G. Then*

$$\mathsf{tree}\text{-}\alpha(G(\mathcal{H})) \leq \mathsf{tree}\text{-}\alpha(G)\,.$$

In particular, Theorem 11.2.2 implies that for any graph G, the tree-independence number of its blob graph is bounded by $\mathsf{tree}\text{-}\alpha(G)$.

We also prove the algorithmic version of Lemma 11.2.1 which we will use later on. Before we state it, we will state the following standard lemma (see, e.g., [112]) that will be necessary for the analysis of the time complexity.

Lemma 11.2.3 (Lemma 6.3 in [77]). *Let $V = \{v_1, \ldots, v_n\}$ be a set and let $\mathcal{S} = \{S_i\}_{i \in I}$ be a finite family of subsets of V. Then there exists an algorithm running in time $\mathcal{O}(|V| + |I| + \sum_{i \in I} |S_i|)$ that sorts each set S_i with respect to the ordering $v_1, \ldots, v_n$ of V.*

Proof. Let B be the bipartite incidence graph of the family $\mathcal{S}$, that is, B has vertex set $V \cup I$, and edge set $\{\{v, i\} : v \in V, i \in I, v \in S_i\}$. We can compute the adjacency lists of the graph B in time $\mathcal{O}(|V| + |I| + \sum_{i \in I} |S_i|)$ as follows. We fix an ordering of the set I, say $I = \{i_1, \ldots, i_m\}$. For each $i \in I$, the set S_i already gives the adjacency list of i. We initialize the adjacency lists for each $v \in V$ to the empty lists. For all $j = 1, \ldots, m$, we iterate over the elements v of S_{i_j} and add i_j to the end of the adjacency list of vertex v. We now have the adjacency lists of all vertices of B; those for $v \in V$ are already sorted, while those for $i \in I$ need not be.

To sort the adjacency lists for vertices $i \in I$, we iterate over the adjacency lists of vertices $v \in V$ in a similar way as we did above for $i \in I$. We reset the adjacency lists for all $i \in I$ to the empty lists. For all $i = 1, \ldots, n$, we iterate over the elements $j \in I$ of the adjacency list of v_i and add v_i to the end of the adjacency list for vertex $j \in I$. At the end of this procedure, the adjacency list of each $i \in I$ will contain exactly the elements of S_i, sorted with respect to the ordering $v_1, \ldots, v_n$ of V. The total time complexity of the procedure is proportional to the number of vertices and edges of the graph B, that is, $\mathcal{O}(|V| + |I| + \sum_{i \in I} |S_i|)$. □

We are now ready to state the algorithmic version of Lemma 11.2.1.

Corollary 11.2.4 (Corollary 6.4 in [77]). *There exists an algorithm that takes as input a graph G, a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G, and a tree decomposition $\mathcal{T} = \big(T, \{X_t\}_{t\in V(T)}\big)$ of G, and computes in time $\mathcal{O}(|J| \cdot ((|J| + |V(T)|) \cdot |V(G)| + |E(G)|))$ the graph $G(\mathcal{H})$ and a tree decomposition $\mathcal{T}' = \big(T, \{X'_t\}_{t\in V(T)}\big)$ of $G(\mathcal{H})$ such that $\alpha(\mathcal{T}') \le \alpha(\mathcal{T})$.*

Proof. Fix an arbitrary ordering of the vertex set of G. Using Lemma 11.2.3, we first sort the vertex set of each of the graphs H_j, $j \in J$, as well as each bag X_t, $t \in V(T)$, with respect to the fixed ordering of $V(G)$, in time

$$\mathcal{O}\left(|V(G)| + |J| + \sum_{j\in J} |V(H_j)|\right) + \mathcal{O}\left(|V(G)| + |V(T)| + \sum_{t\in V(T)} |X_t|\right) = \\ = \mathcal{O}\Big((|J| + |V(T)|) \cdot |V(G)|\Big)\,.$$

Note that using this sortedness assumption, we can compute the union and the intersection of any two sorted sets $X, Y \subseteq V(G)$ in time $\mathcal{O}(|V(G)|)$. To compute the graph $G(\mathcal{H})$, we only need to explain how to compute its edge set, since the vertex set is J. For each $j \in J$, we perform a BFS traversal up to distance two from a new vertex v_j added to G which we make adjacent to all the vertices of H_j. Let R_j be the set of vertices of G reached this way. Then, for all $i \in J \setminus \{j\}$, the graph H_i is adjacent to H_j in $G(\mathcal{H})$ if and only if at least one vertex of H_i belongs to R_j. This can be tested in time $\mathcal{O}(|V(G)|)$ by first sorting the set R_j and then computing the intersection $V(H_i) \cap R_j$. Hence, this procedure over all $j \in J$ can be carried out in time $\mathcal{O}(|J| \cdot (|V(G)| + |E(G)|)) + \mathcal{O}(|J|^2 \cdot |V(G)|) = \mathcal{O}(|J|^2 \cdot |V(G)| + |J| \cdot |E(G)|)$.

To compute $\mathcal{T}'$, we need to compute for each $t \in V(T)$ the bag X'_t consisting of all vertices $j \in J$ of $G(\mathcal{H})$ such that $V(H_j) \cap X_t \ne \emptyset$. All the intersections $V(H_j) \cap X_t$ can be computed in time $\mathcal{O}(|J| \cdot |V(T)| \cdot |V(G)|)$. Thus, the total time complexity of the algorithm is $\mathcal{O}(|J|^2 \cdot |V(G)| + |J| \cdot |E(G)| + |J| \cdot |V(T)| \cdot |V(G)|) = \mathcal{O}(|J| \cdot ((|J| + |V(T)|) \cdot |V(G)| + |E(G)|))$. □

We are now ready to give the definition of the MAXIMUM WEIGHT INDEPENDENT PACKING problem. Let G be a graph and let $\mathcal{H} = \{H_j\}_{j\in J}$ be a family of connected subgraphs of G. A subfamily $\mathcal{H}'$ of $\mathcal{H}$ is an *independent $\mathcal{H}$-packing* in G if for every pair of graphs H_i and H_j in $\mathcal{H}'$, H_i and H_j are vertex-disjoint ($V(H_i) \cap V(H_j) = \emptyset$) and there is no edge connecting H_i and H_j in G, i.e., $\mathcal{H}'$ is an independent set in the graph $G(\mathcal{H})$. Assume that the subgraphs in $\mathcal{H}$ are equipped with a weight function $w : J \to \mathbb{Q}^+$

assigning weight w_j to each subgraph H_j. For a set $I \subseteq J$, the *weight* of the family $\mathcal{H}' = \{H_i\}_{i \in I}$ is defined as the sum $\sum_{i \in I} w_i$.

MAXIMUM WEIGHT INDEPENDENT PACKING
Input: A graph G, a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G, and a weight function $w : J \to \mathbb{Q}^+$ on the subgraphs in $\mathcal{H}$.
Output: An independent $\mathcal{H}$-packing in G of maximum weight.

The MAXIMUM WEIGHT INDEPENDENT PACKING problem is a common generalization of several problems studied in the literature, including:

- The MAXIMUM WEIGHT INDEPENDENT $\mathcal{F}$-PACKING problem, which is a special case when $\mathcal{F}$ is a fixed finite family of connected graphs and $\mathcal{H} = \mathcal{H}(G, \mathcal{F})$ is the set of all subgraphs of G isomorphic to a member of $\mathcal{F}$.
- The INDEPENDENT $\mathcal{F}$-PACKING problem (see [52]), which corresponds to the unweighted case.
- The MAXIMUM WEIGHT INDEPENDENT SET problem, which corresponds to the case $\mathcal{F} = \{K_1\}$.
- The MAX WEIGHT INDUCED MATCHING problem (see, e.g., [3, 171]), which corresponds to the case $\mathcal{F} = \{K_2\}$.
- The DISSOCIATION SET problem (see, e.g., [170, 208, 210]), which corresponds to the case when $\mathcal{F} = \{K_1, K_2\}$ and the weight function assigns to each subgraph H_j, $j \in J$, the weight equal to $|V(H_j)|$.
- The k-SEPARATOR problem (see, e.g., [22, 159]), which corresponds to the case when $\mathcal{F}$ contains all connected graphs with at most k vertices, the graph G is equipped with a vertex weight function $w : V(G) \to \mathbb{Q}^+$, and the weight function on $\mathcal{H}$ assigns to each subgraph H_j, $j \in J$, the weight equal to $\sum_{x \in V(H_j)} w(x)$.

In order to reduce the MAXIMUM WEIGHT INDEPENDENT PACKING problem to the MAXIMUM WEIGHT INDEPENDENT SET problem in polynomial time, we can use the following observation.

Observation 11.2.5 (Observation 7.1 in [77]). Let G be a graph, let $\mathcal{H} = \{H_j\}_{j \in J}$ be a finite family of connected non-null subgraphs of G, and a let $w : J \to \mathbb{Q}^+$ be a weight function on the subgraphs in $\mathcal{H}$. Let I be an independent set in $G(\mathcal{H})$ of maximum weight with respect to the weight function w. Then I is an independent $\mathcal{H}$-packing in G of maximum weight.

We can now use Corollaries 11.1.3 and 11.2.4, and Observation 11.2.5, to obtain an analogous result to Corollary 11.1.3 for the MAXIMUM WEIGHT INDEPENDENT PACKING problem.

Theorem 11.2.6 (Theorem 7.2 in [77])**.** *Let k be a positive integer. Then, given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G, the* Maximum Weight Independent Packing *problem can be solved in time $\mathcal{O}(|J| \cdot ((|J| + |V(T)|) \cdot |V(G)| + |E(G)| + |J|^k \cdot |V(T)|)$ if G is given together with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ with independence number at most k.*

Proof. Let G be a graph, let $\mathcal{H} = \{H_j\}_{j\in J}$ be a finite family of connected non-null subgraphs of G, let $w : J \to \mathbb{Q}^+$ be a weight function on the subgraphs in $\mathcal{H}$, and let $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ be a tree decomposition of G with independence number at most k. By Corollary 11.2.4, we can compute in time $\mathcal{O}(|J| \cdot ((|J| + |V(T)|) \cdot |V(G)| + |E(G)|))$ the graph $G(\mathcal{H})$ and a tree decomposition $\mathcal{T}' = \big(T, \{X'_t\}_{t\in V(T)}\big)$ of $G(\mathcal{H})$ with independence number at most k. Using Corollary 11.1.3, we now compute in time $\mathcal{O}(|J|^{k+1} \cdot |V(T)|)$ an independent set I in $G(\mathcal{H})$ of maximum weight with respect to the weight function w. By Observation 11.2.5, I is a maximum-weight independent $\mathcal{H}$-packing in G. The claimed running time follows. □

The case when the subgraphs in $\mathcal{H}$ have bounded order generalizes the Maximum Weight Independent $\mathcal{F}$-Packing problem. In particular, for this case, the time complexity can be slightly improved compared to the one obtained by directly applying Theorem 11.2.6.

Theorem 11.2.7 (Theorem 7.3 in [77])**.** *Let k and r be two positive integers. Then, given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq r$ for all $j \in J$, the* Maximum Weight Independent Packing *problem can be solved in time $\mathcal{O}\left(|V(G)|^{r(k+1)} \cdot |V(T)|\right)$ if G is given together with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ with independence number at most k.*

Proof. We assume that G is represented with an adjacency matrix, since otherwise we can first compute such a representation from the adjacency lists in time $\mathcal{O}(|V(G)|^2)$. Note that $|V(G(\mathcal{H}))| = |J| = \mathcal{O}(|V(G)|^r)$, since by assumption each graph in $\mathcal{H}$ has at most r vertices, and for any such set of vertices we have at most $2^{r(r-1)/2} = \mathcal{O}(1)$ choices for the edge set. We compute the edge set of $G(\mathcal{H})$ in time $\mathcal{O}(|V(G(\mathcal{H}))|^2) = \mathcal{O}(|V(G)|^{2r})$, as follows. For every two distinct $i, j \in J$, we check in time $\mathcal{O}(\max\{|V(H_i)|, |V(H_j)|\}) = \mathcal{O}(1)$ if H_i and H_j have a vertex in common. If this is the case, then we add $\{i, j\}$ to the edge set of $G(\mathcal{H})$. If this is not the case, then we check in time $\mathcal{O}(|V(H_i)| \cdot |V(H_j)|) = \mathcal{O}(1)$ if there is an edge in G connecting a vertex of H_i with a vertex of H_j. If this is the case, then we add $\{i, j\}$ to the edge set of $G(\mathcal{H})$.

For the rest of the proof, we use the same approach as in the proof of Theorem 11.2.6. In particular, we compute the tree decomposition $\mathcal{T}'$ of $G(\mathcal{H})$ in time $\mathcal{O}(|J| \cdot |V(T)| \cdot |V(G)|) = \mathcal{O}(|V(G)|^{r+1} \cdot |V(T)|)$, and a maximum-weight independent set I in $G(\mathcal{H})$ in time $\mathcal{O}(|J|^{k+1} \cdot |V(T)|) = \mathcal{O}(|V(G)|^{r(k+1)} \cdot |V(T)|)$. The total time complexity of the algorithm is $\mathcal{O}(|V(G)|^{2r} + |V(G)|^{r+1} \cdot |V(T)| + |V(G)|^{r(k+1)} \cdot |V(T)|)$, which simplifies to $\mathcal{O}(|V(G)|^{r(k+1)} \cdot |V(T)|)$, as claimed. □

As an immediate corollary we get the following.

Corollary 11.2.8 (Corollary 7.4 in [77]). *Let $\mathcal{F}$ be a nonempty finite set of connected nonnull graphs and let r be the maximum number of vertices of a graph in $\mathcal{F}$. Then, for every $k \geq 1$, the* Max Weight Independent $\mathcal{F}$-Packing *problem is solvable in time $\mathcal{O}\left(|V(G)|^{r(k+1)} \cdot |V(T)|\right)$ if the input graph G is given with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ with independence number at most k.*

As a remark, note that we did not derive a result generalizing Corollary 11.2.8 with the use of the ℓ-refined tree decompositions of bounded residual independence number. The reason behind this is the fact that Lemma 11.2.1 does not seem to generalize to ℓ-refined tree decompositions in a way that the the residual independence number would be preserved. To see this, let us consider the following example.

Example 11.2.9 (Remark 7.5 in [77]). Fix two positive integers ℓ and p and let G be the tree consisting of a vertex a adjacent to ℓ other vertices forming a set $B = \{b_1, \ldots, b_\ell\}$ such that each b_i is also adjacent to p vertices of degree one, forming a set C_i. Let $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ be a tree decomposition of G such that T is the graph $K_{1,p\ell}$, the high-degree node of T is labeled with the bag $\{a\} \cup B$, and the $p\ell$ leaves of T are labeled with bags corresponding to the edges in G containing a vertex of degree one. To make this tree decomposition ℓ-refined, we set $U_t = B$ for the high-degree node t of T and $U_t = \emptyset$ for all the other nodes. The residual independence number of this ℓ-refined tree decomposition is 1. Let $\mathcal{H} = \mathcal{H}(G, \{K_2\})$ be the family of all connected subgraphs of G of order two. Then, the graph $G(\mathcal{H})$ is isomorphic to the graph obtained from the graph $K_{1,\ell}$ by substituting a clique of size $\ell+1$ into the vertex of degree ℓ and a clique of size p into each vertex of degree one (see, e.g., [58] for the definition of substitution). Let $\mathcal{T}' = (T, \{X'_t\}_{t\in V(T)})$ be the tree decomposition of $G(\mathcal{H})$ as defined in Lemma 11.2.1, that is, $X'_t = \{H \in V(G(\mathcal{H})) : V(H) \cap X_t \neq \emptyset\}$ for all $t \in V(T)$. Then, there is no way to turn $\mathcal{T}'$ into an $f(\ell)$-refined tree decomposition of $G(\mathcal{H})$ with residual independence number 1. Indeed, consider the bag $X_t = \{a\} \cup B$ of $\mathcal{T}$ labeling the high-degree node of T. Since every edge of G has an endpoint in X_t, the bag of $\mathcal{T}'$ corresponding to node t is $X'_t = V(G(\mathcal{H}))$. Using the structure of the graph $G(\mathcal{H})$, we see that the smallest subset U'_t of X'_t such that the independence number of the subgraph of $G(\mathcal{H})$ induced by $X'_t \setminus U'_t$ is 1 has size $(\ell - 1)p$, which cannot be bounded from above by any function depending only on ℓ.

11.3 The classes of $K_{2,q}$-, W_4-, and K_5^--im-free graphs

In this section we give the proof to the following theorem.

Theorem 11.3.1. *Let $H \subseteq_{\text{im}} K_{2,q}$, for some $q \geq 2$, or $H \subseteq_{\text{im}} W_4$, or $H \subseteq_{\text{im}} K_5^-$, or H is edgeless and let $\mathcal{G}$ be the class of H-im-free graphs. Then* Maximum Weight Independent Set *problem is solvable in polynomial time for the class $\mathcal{G}$.*

Note that the edgeless case is the same as in the proof of Theorem 11.0.2. For the remaining three cases see Theorems 11.3.3, 11.3.6, and 11.3.9, respectively.

11.3.1 $K_{2,q}$-induced-minor-free graphs

Theorems 10.3.5 and 11.2.6 have the following consequence for the Maximum Weight Independent Packing problem.

Theorem 11.3.2 (Theorem 3.14 in [78])**.** *For every integer $q \geq 2$, given a $K_{2,q}$-induced-minor-free graph G and a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected nonnull subgraphs of G, the* Maximum Weight Independent Packing *problem can be solved in time $\mathcal{O}(|J| \cdot |V(G)| \cdot (|V(G)| + |J|^{2q-2}))$.*

Proof. Let G be an n-vertex $K_{2,q}$-induced-minor-free graph, $\mathcal{H} = \{H_j\}_{j \in J}$ a finite family of connected non-null subgraphs of G, and $w : J \to \mathbb{Q}^+$ a weight function. By Theorem 10.3.5, we can compute in time $\mathcal{O}(n^\mu \log n)$ a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of G with $\mathcal{O}(n)$ nodes and with independence number at most $2q-2$. Thus, by Theorem 11.2.6, Maximum Weight Independent Packing problem can be solved on G in time $\mathcal{O}(|J| \cdot n \cdot (n + |J|^{2q-2}))$. □

Applying Theorem 11.3.2 to the case when $\mathcal{H}$ corresponds to the set of all 1-vertex subgraphs of the input graph G, we obtain the following.

Corollary 11.3.3 (Corollary 3.15 in [78])**.** *For every integer $q \geq 2$, the* Maximum Weight Independent Set *problem is solvable in time $\mathcal{O}(n^{2q})$ on n-vertex $K_{2,q}$-induced-minor-free graphs.*

For the case $q = 3$ the bounds on the running time of the algorithm proving Theorem 11.3.2 can be improved by using Lemma 10.3.4 instead of Lemma 10.3.3, which, by Lemma 10.3.1, improves the upper bound on the tree-independence number of $K_{2,3}$-induced-minor-free graphs from 4 to 3. We thus obtain the following.

Theorem 11.3.4 (Theorem 3.16 in [78])**.** *Let G be an n-vertex $K_{2,3}$-induced-minor-free graph and let $\mathcal{H} = \{H_j\}_{j \in J}$ be a finite family of connected non-null subgraphs of G. Then the* Maximum Weight Independent Packing *problem can be solved in time $\mathcal{O}(|J| \cdot |V(G)| \cdot (|V(G)| + |J|^3))$. In particular, the* Maximum Weight

INDEPENDENT SET *problem is solvable in time* $\mathcal{O}(n^5)$ *on* n*-vertex* $K_{2,3}$*-induced-minor-free graphs.*

Let us remark that Theorem 11.3.4 gives an answer to the question of Beisegel et al. in [19, 20], regarding the complexity of the MAX INDEPENDENT SET problem in the class of 1-perfectly orientable graphs. We say that a graph G is 1-*perfectly orientable* if its edges can be oriented in such a way that no vertex has a pair of non-adjacent out-neighbors, i.e., for a directed edge (u, v), u is the in-neighbor of v and v is the out-neighbor of u. The class of 1-perfectly orientable graphs was introduced in 1982 by Skrien [185] and studied by Bang-Jensen et al. [18] and, more recently, by Hartinger and Milanič [122] and by Brešar et al. [46]. A known result states that every 1-perfectly orientable graph is $K_{2,3}$-induced-minor-free (see [122]), and hence, applying Theorem 11.3.4 we get the following result.

Corollary 11.3.5 (Corollary 3.17 in [78]). *The* MAXIMUM WEIGHT INDEPENDENT SET *problem is solvable in time* $\mathcal{O}(n^5)$ *on* n*-vertex* 1*-perfectly orientable graphs.*

11.3.2 W_4-induced-minor-free graphs

Corollary 10.6.4 and Theorem 11.1.2 imply the existence of a polynomial-time algorithm for the MAXIMUM WEIGHT INDEPENDENT SET problem in the class of W_4-induced-minor-free graphs.

Corollary 11.3.6 (Corollary 5.9 in [78]). *The* MAXIMUM WEIGHT INDEPENDENT SET *problem can be solved in time* $\mathcal{O}(n^3)$ *for* n*-vertex* W_4*-induced-minor-free graphs.*

Remark 11.3.7 (Remark 5.10 in [78]). The improvement in the running time when using Theorem 11.1.2 instead of Corollary 11.1.3 is significant: $\mathcal{O}(n^3)$ instead of $\mathcal{O}(n^6)$.

Furthermore, since a 3-refined tree decomposition with residual independence number at most 1 has independence number at most 4, Corollary 10.6.4 and Theorem 11.2.6 imply the existence of a polynomial-time algorithm for the MAXIMUM WEIGHT INDEPENDENT PACKING problems in the class of W_4-induced-minor-free graphs.

Theorem 11.3.8 (Theorem 5.11 in [78]). *Given a* W_4*-induced-minor-free graph* G *and a finite family* $\mathcal{H} = \{H_j\}_{j\in J}$ *of connected non-null subgraphs of* G*, the* MAXIMUM WEIGHT INDEPENDENT PACKING *problem can be solved in time*

$$\mathcal{O}(|J| \cdot |V(G)| \cdot (|V(G)| + |J|^4)) .$$

Proof. Let G be an n-vertex W_4-induced-minor-free graph, given along with a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs and a weight function $w : J \to \mathbb{Q}^+$. By Corollary 10.6.4 and Observation 10.4.3, we can compute in time $\mathcal{O}(n^3)$ a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$

of G with $\mathcal{O}(n)$ nodes and with independence number at most 4. Thus, by Theorem 11.2.6, the Maximum Weight Independent Packing problem can be solved on G in time $\mathcal{O}(|J| \cdot |V(G)| \cdot (|V(G)| + |J|^4))$. $\square$

11.3.3 K_5^--induced-minor-free graphs

Corollary 10.7.4 and Theorem 11.1.2 have the following algorithmic consequence for the Maximum Weight Independent Set problem in the class of K_5^--induced-minor-free graphs.

Corollary 11.3.9 (Corollary 6.9 in [78]). *The* Maximum Weight Independent Set *problem can be solved in time $\mathcal{O}(n^3)$ for n-vertex K_5^--induced-minor-free graphs.*

Remark 11.3.10 (Remark 6.10 in [78]). Similarly as for the class of W_4-induced-minor-free graphs (see Remark 11.3.7), the improvement in the running time when using Theorem 11.1.2 instead of Corollary 11.1.3 is significant: $\mathcal{O}(n^3)$ instead of $\mathcal{O}(n^6)$.

Corollary 10.7.4 and Theorem 11.2.6 imply the existence of a polynomial-time algorithm for the Maximum Weight Independent Packing problems in the class of K_5^--induced-minor-free graphs.

Theorem 11.3.11 (Theorem 6.11 in [78]). *Given a K_5^--induced-minor-free graph G and a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G, the* Maximum Weight Independent Packing *problem can be solved in time*

$$\mathcal{O}(|J| \cdot |V(G)| \cdot (|V(G)| + |J|^4))\,.$$

Proof. Let G be an n-vertex K_5^--induced-minor-free graph, given along with a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs and a weight function $w : J \to \mathbb{Q}^+$. By Corollary 10.7.4 and Observation 10.4.3, we can compute in linear time a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of G with $\mathcal{O}(n)$ nodes and independence number at most 4. Thus, by Theorem 11.2.6, the Maximum Weight Independent Packing problem can be solved on G in time $\mathcal{O}(|J| \cdot |V(G)| \cdot (|V(G)| + |J|^4))$. $\square$

Chapter 12

Final Remarks to Part II

Let us now give some remarks to the second part of this thesis.

12.1 Remarks on (tw, ω)-boundedness and open questions

In Chapter 9, we showed for which graphs H with respect to one of the six graph containment relations is the class of graphs $\mathcal{G}$ excluding H with respect to the relation (tw, ω)-bounded. Then in Chapter 10, we showed that when forbidding a single graph H with respect to one of the six graph containment relations, (tw, ω)-boundedness is equivalent to having bounded tree-independence number. Moreover, Lemma 10.1.11 shows that a graph class having bounded tree-independence number is polynomially (tw, ω)-bounded, in particular, it is (tw, ω)-bounded. Let us now state these observations as a corollary.

Corollary 12.1.1 (Corollary 7.4 in [78]). *For every graph H and each of the six graph containment relations (the subgraph, topological minor, and minor relations, and their induced variants), the following statements are equivalent for the class $\mathcal{G}$ of graphs excluding H with respect to the relation.*

1. *$\mathcal{G}$ is (tw, ω)-bounded.*
2. *$\mathcal{G}$ is polynomially (tw, ω)-bounded.*
3. *$\mathcal{G}$ has bounded tree-independence number.*

Furthermore, whenever the above conditions are satisfied, there is a polynomial-time algorithm for computing a tree decomposition with bounded independence number of a graph in $\mathcal{G}$.

This opens up many interesting questions with respect to (tw, ω)-boundedness. Among those, it would be interesting to understand (tw, ω)-bounded graph classes defined by finitely many forbidden structures.

Question 12.1.2 (Question 9.1 in [76]). *Which graph classes defined by larger finite sets of forbidden structures (with respect to various graph containment relations) are* (tw, ω)*-bounded?*

In particular, it would be interesting to understand which (tw, ω)-bounded graph classes defined by finitely many forbidden induced subgraphs have bounded tree-independence number. As discussed in [76], in the case of subgraph relation, at least one excluded structure would have to be from $\mathcal{S}$, in the case of topological minor relation, at least one excluded structure would have to be a subcubic planar graph, and in the case of minor relation, at least one excluded structure would have to be planar. Each of those would give a subclass of some (tw, ω)-bounded graph class obtained by excluding a single structure.

In case of forbidden induced subgraphs, however, Lozin and Razgon [160] proved that the following holds.

Theorem 12.1.3 (Lozin and Razgon [160]). *For any graphs $H_1, \ldots, H_p$, the class of $\{H_1, \ldots, H_p\}$-free graphs has bounded treewidth if and only if the set $\{H_1, \ldots, H_p\}$ contains a complete graph, a complete bipartite graph, a graph from $\mathcal{S}$, and the line graph of a graph from $\mathcal{S}$.*

As a direct consequence of Theorem 12.1.3, we obtain the following result characterizing (tw, ω)-bounded graph classes in terms of finitely many forbidden induced subgraphs.

Corollary 12.1.4 (Corollary 9.3 in [76]). *For any graphs $H_1, \ldots, H_p$, the following conditions are equivalent.*

1. *The class of $\{H_1, \ldots, H_p\}$-free graphs is (tw, ω)-bounded.*
2. *The class of $\{K_4, H_1, \ldots, H_p\}$-free graphs has bounded treewidth.*
3. *The set $\{H_1, \ldots, H_p\}$ contains a complete bipartite graph, a graph from $\mathcal{S}$, and the line graph of a graph from $\mathcal{S}$.*

Proof. If the class of $\{H_1, \ldots, H_p\}$-free graphs has a (tw, ω)-binding function f, then the treewidth of any $\{K_4, H_1, \ldots, H_p\}$-free graph is at most $f(3)$. Thus, (1) implies (2). By Theorem 12.1.3, (2) implies (3). Finally, if the set $\{H_1, \ldots, H_p\}$ contains a complete bipartite graph, a graph from $\mathcal{S}$, and the line graph of a graph from $\mathcal{S}$, then by Theorem 12.1.3 for every positive integer k there exists a constant $f(k)$ such that every $\{K_{k+1}, H_1, \ldots, H_p\}$-free graph has treewidth at most $f(k)$. Thus, the class of $\{H_1, \ldots, H_p\}$-free graphs is (tw, ω)-bounded and (3) implies (1). $\square$

For the induced topological minor and induced minor relations, Question 12.1.2 remains open.

Another possible question is the following (see also Question 9.4 in [76]).

Question 12.1.5 (Question 8.4 in [78]). *Is every (tw, ω)-bounded graph class polynomially (tw, ω)-bounded?*

In some cases (see Theorems 9.1.4, 9.1.6, 9.1.7, 9.2.1, and 9.2.5) we even obtained linear (tw, ω)-binding functions. Thus, one may ask for necessary and/or sufficient conditions for a graph class to be linearly (tw, ω)-bounded.

Question 12.1.6 (Question 9.5 in [76]). *Which graph classes have a linear* (tw, ω)*-binding function?*

Going back to Question 12.1.5, in [78], we posed the following conjecture.

Conjecture 12.1.7 (Conjecture 8.5 in [78]). Let $\mathcal{G}$ be a hereditary graph class. Then $\mathcal{G}$ is (tw, ω)-bounded if and only if $\mathcal{G}$ has bounded tree-independence number.

Note that, thanks to Lemma 10.1.11, a positive answer to Conjecture 12.1.7 would give a positive answer to Question 12.1.5.

Recently, Abrishami et al. [2] proved that the class $\mathcal{G}$ of (even hole, diamond, pyramid)-free graphs is (tw, ω)-bounded. To complement this result, Abrishami et al. [1], proved that the class $\mathcal{G}$ has bounded tree-independence number, thus providing another positive result towards Conjecture 12.1.7. In fact, both results were obtained for a superclass $\mathcal{C}$ of (even hole, diamond, pyramid)-free graphs, namely, for (C_4, diamond, theta, pyramid, prism, even wheel)-free graphs. As explained in [1], the fact that $\mathcal{C}$ is a superclass of $\mathcal{G}$ follows from the fact that (even-hole)-free graphs are (theta, prism, even wheel)-free. Their result thus proves that the Maximum Weight Independent Set problem is solvable in polynomial time in the class $\mathcal{C}$. Moreover, in [1], they posed the following conjecture.

Conjecture 12.1.8 (Abrishami et al. [1]). The class of (even hole, diamond)-free graphs has bounded tree-independence number.

12.2 Further algorithmic results

In [77], we asked whether the Maximum Weight Independent Set problem is solvable in polynomial time in any class of graphs with bounded tree-independence number. A result of Yolov (see [209], Theorem 4.5) resolves this question in affirmative. Later, Dallard et al. [75], gave the following algorithm with better running time and approximation for the tree-independence number.

Theorem 12.2.1 (Dallard et al. [75]). *There exists an algorithm that, given an n-vertex graph G and an integer k, in time $2^{O(k^2)} \cdot n^{O(k)}$ either outputs a tree decomposition of G with independence number oat most $8k$, or concludes that the tree-independence number of G is larger than k.*

Combining their work with the results from Chapter 11, this gives a polynomial-time solvability for the Maximum Weight Independent Packing problem, which is a generalization of the MWIS problem, in any graph class with bounded tree-independence number.

Further work in this direction was done by Milanič and Rzążewski [168] as follows. Given a positive integer d, a *distance-d independent set* in a graph G is a set of vertices at pairwise distance at least d. The Distance-d Independent Set problem is defined as follows.

Distance-d Independent Set
Input: A graph G and a weight function $w : V(G) \mapsto \mathbb{Q}^+$.
Output: A distance-d independent set I in G of maximum possible weight $w(I)$, where $w(I) = \sum_{v \in I} w(x)$.

Note that in the case when $d = 2$ we obtain the classical Maximum Weight Independent Set problem.

Given a positive integer d, a graph G and a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G, a *distance-d $\mathcal{H}$-packing* in G is a subfamily $\mathcal{H}' = \{H_i\}_{i \in I}$ of subgraphs from $\mathcal{H}$ (i.e., $I \subseteq J$) that are at pairwise distance at least d. We can thus define the following problem.

Maximum Weight Distance-d Packing
Input: A graph G, a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G, and a weight function $w : J \to \mathbb{Q}^+$ on the subgraphs in $\mathcal{H}$.
Output: A distance-d $\mathcal{H}$-packing in G of maximum weight.

Note that the Maximum Weight Distance-d Packing problem generalizes the problems discussed in Chapter 11. Given the above definitions, Milanič and Rzążewski [168] proved the following result.

Theorem 12.2.2 (Milanič and Rzążewski [168]). *For every positive integer d and every $k \geq 1$, given a graph G, a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G, and a weight function $w : J \to \mathbb{Q}^+$ on the subgraphs in $\mathcal{H}$, the* Maximum Weight Distance-d Packing *problem is solvable in time*

$$O\big((|V(G)| + |V(T)|) \cdot |E(G)| + \\ + |J| \cdot |V(G)| \cdot ((|J| + |V(T)| + |V(G)|) + |J|^{k+1} \cdot |V(T)|)\big)$$

if G is given with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ with independence number at most k.

In particular, Theorem 12.2.2 generalizes the result of Theorem 11.2.6.

12.3 Other related work

The concept of (tw, ω)-boundedness is part of the following more general framework. Given two graph invariants ρ and σ and a graph class $\mathcal{G}$, we say that $\mathcal{G}$ is (ρ, σ)-*bounded* if there exists a (ρ, σ)-*binding function* for $\mathcal{G}$, that is, a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ and every induced subgraph H of G, we have $\rho(H) \leq f(\sigma(H))$.

One of the examples is (β, χ)-boundedness, where β denotes the coloring number of a graph, i.e., the largest integer k for which there exists a subgraph of G with minimum degree $k-1$. This property was studied by Jensen and Toft [142] in their book on graph coloring problems where they referred to (β, χ)-bounded graph families as *color-bound*. Next, Gyárfás and Zaker [119] studied (δ, χ)-bounded graph classes. Hermelin et al. [131] considered classes of intersection graphs of arithmetic progressions with bounded jumps and proved that they are (pw, ω)-bounded, where pw denotes the pathwidth of the graph. Note that every path decomposition is a tree decomposition, but not vice versa. In addition, several other pairs of invariants were studied in terms of (ρ, σ)-boundedness. Perhaps one of the most well-known and extensively studied is the case where $(\rho, \sigma) = (\chi, \omega)$ (see, e.g., [31, 165, 211]). Such graph classes are usually called *χ-bounded* and were introduced by Gyárfás in the late 1980s to generalize perfection [118] and studied extensively in the literature (see [181] for a survey). Note that every graph G satisfies $\omega(G) \leq \chi(G) \leq \mathrm{tw}(G)+1$, where the first inequality holds with equality for all induced subgraphs of G if and only if G is perfect, and both inequalities hold with equality for all induced subgraphs of G if and only if G is chordal (see Theorem 8.2.3). Thus, in a similar way as χ-boundedness generalizes perfection, (tw, ω)-boundedness generalizes chordality, and every (tw, ω)-bounded graph class is also χ-bounded.

Corollary 12.3.1 (Corollary 2.4 in [76]). *Every* (tw, ω)*-bounded graph class is χ-bounded.*

By Lemma 10.1.11, we immediately get the following corollary of Corollary 12.3.1.

Corollary 12.3.2. *Every graph class with bounded tree-independence number is χ-bounded.*

In fact, Lemma 10.1.11 implies polynomial χ-boundedness.

As we noted in [76], our results on (tw, ω)-bounded graph classes forbidding a single induced minor (see Theorem 9.3.10) also provide new χ-bounded graph classes. To explain this further, let us give the following remark.

Remark 12.3.3 (Remark 4.15 in [76]). If H is an induced subgraph of $K_{1,q}$ for some $q \geq 3$, then χ-boundedness of the class of H-induced-minor-free graphs follows, e.g., from an application of Ramsey's theorem to the class of $K_{1,q}$-free graphs. The cases when H is an induced subgraph of C_4 and K_4^- correspond, respectively, to the classes of chordal and block-cactus graphs (by Observation 8.0.1 and Lemma 9.2.3). In the former case, χ-boundedness follows from the fact that chordal graphs are perfect. In the latter case, we can use the fact that block-cactus graphs have bounded clique-width [144], which is a sufficient condition for χ-boundedness [32, 93]. The case when H is an induced subgraph of

K_4 corresponds to the class of K_4-topological-minor-free graphs, and all such graphs are 3-colorable [83]. In the case when H is isomorphic to any of W_4, K_5^-, $K_{2,3}$, or $K_{2,3}^+$, the above arguments do not apply since none of the resulting classes is contained in the class of perfect graphs or in any graph class of bounded chromatic number or bounded clique-width. (This can be seen using the results of [21], the fact that complete graphs are H-induced-minor-free, and that odd cycles of length at least 5 are H-induced-minor-free but not perfect.)

Another notion related to the notion of tree-independence number is the *tree-chromatic number* of a graph G, denoted by $\mathsf{tree}\text{-}\chi(G)$. Seymour [182] introduced the tree-chromatic number of a graph G as the smallest non-negative integer k such that G admits a tree decomposition, each bag of which induces a k-colorable subgraph. Note that for every graph G, we have $\omega(G) \leq \mathsf{tree}\text{-}\chi(G) \leq \chi(G) \leq \mathrm{tw}(G) + 1$, where the first inequality is shown in [182]. It is a known fact that for every graph G, we have that $|V(G)| \leq \alpha(G) \cdot \chi(G)$. Applying this inequality to every bag of a tree decomposition $\mathcal{T}$ of G with $\alpha(\mathcal{T}) = \mathsf{tree}\text{-}\alpha(G)$, we obtain that $\mathrm{tw}(G) + 1 \leq \mathsf{tree}\text{-}\alpha(G) \cdot \chi(G)$. With a similar reasoning as above, we also obtain that $\mathrm{tw}(G) + 1 \leq \alpha(G) \cdot \mathsf{tree}\text{-}\chi(G)$. A question is whether there exists a common strengthening of these two inequalities.

Question 12.3.4 (Question 8.4 in [77])**.** *Does every graph G satisfy*

$$\mathrm{tw}(G) + 1 \leq \mathsf{tree}\text{-}\alpha(G) \cdot \mathsf{tree}\text{-}\chi(G)\ ?$$

As observed in [77], due to the above mentioned inequalities, Question 12.3.4 holds whenever $\mathsf{tree}\text{-}\chi(G) = \chi(G)$ or whenever $\mathsf{tree}\text{-}\alpha(G) = \alpha(G)$. In particular, this is the case for any class of graphs in which the chromatic number coincides with the clique number, such as the class of perfect graphs.

Remark 12.3.5. Let us remark that one of the difficulties in proving Question 12.3.4 lies in the fact that for a single graph, the three parameters in question might be minimized by distinct tree decompositions. However, it is easy to see that if a graph G admits a tree decomposition with independence number $\mathsf{tree}\text{-}\alpha(G)$ and chromatic number $\mathsf{tree}\text{-}\chi(G)$, then we immediately get the inequality $\mathrm{tw}(G) + 1 \leq \mathsf{tree}\text{-}\alpha(G) \cdot \mathsf{tree}\text{-}\chi(G)$.

Another parameter, called the tree-clique cover number and denoted by $\mathsf{tree}\text{-}\overline{\chi}(G)$, analogous to tree-independence number, was recently introduced by Abrishami et al. [1]. The *clique cover number* of a graph G is the smallest number of cliques needed to cover the graph G. The *clique cover number* of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is defined as the maximum clique cover number over all $t \in V(T)$ of the graph $G[X_t]$. The tree-clique cover number of a graph G is then defined as the minimum clique cover number over all tree decompositions of G. An easy observation shows that for all graphs G, we have $\mathsf{tree}\text{-}\overline{\chi}(G) \leq \mathrm{tw}(G) + 1$. In addition, Abrishami et al. [1] observed that for C_4-free graphs, we obtain the following relation between the tree independence number and the tree-clique cover number.

Lemma 12.3.6 (Abrishami et al. [1]). *Let G be a C_4-free graph. Then*

$$\mathsf{tree}\text{-}\alpha(G) \leq \mathsf{tree}\text{-}\overline{\chi}(G) \leq \binom{\mathsf{tree}\text{-}\alpha(G)+1}{2}.$$

Moreover, they proved that for every non-negative integer C, there exists a graph with $\mathsf{tree}\text{-}\alpha(G) \leq 2$ and $\mathsf{tree}\text{-}\overline{\chi}(G) \geq C$ (see Remark 2.3 in [1] for a construction of such graphs).

Chapter 13

Conclusion

$\mathcal{I}$n Part I, we have studied various vertex and edge colorings of plane graphs and their corresponding graph coloring invariants focusing on the classical vertex coloring and on various colorings with their constraints given on the faces of plane graphs.

In Chapter 4, we considered the classical vertex coloring of plane graphs. In particular, we focused on a result of Grötzsch (Theorem 4.0.2), which states that plane graphs without triangles are 3-colorable. This led us to consider the effect that triangles have on the vertex coloring of plane graphs. Aksenov (Theorem 4.1.1) proved that if a plane graph has at most three triangles, then it is 3-colorable. We mainly focused on certain precolorings of plane graphs which can be extended to a 3-coloring of the whole graph. In this sense, we improved several existing results in the case of triangle-free plane graphs by proving their generalizations in the case of plane graphs containing at most one triangle. In some cases, we managed to generalize the existing results (e.g., Theorem 4.2.1), while in other cases we partially generalized the existing results (e.g., Theorems 4.2.2, 4.2.3, and 4.2.6). In addition, for several of our results, we also provided examples of graphs proving that our results are best possible with respect to the number of triangles. However, this leaves open a possible research direction to consider allowing multiple triangles to only appear in a certain way in a plane graph and still allow for the existence of various precolorings that can be extended to a 3-coloring of the whole graph.

In Chapter 5, we then turn our focus to various facial colorings of plane graphs. In particular, we begin by discussing the cyclic coloring of plane graphs for which Conjecture 5.1.2 remains wide open and seems to be difficult to approach. This led to the introduction of the ℓ-facial vertex-coloring. However, as Conjecture 5.2.1 remains open even in the case when $\ell = 2$, we then turn our focus to the ℓ-facial edge-coloring. In this case, we managed to confirm Conjecture 5.3.1 in the case when $\ell = 3$ (Theorem 5.3.3), thus leaving open the cases when $\ell \geq 4$.

Finally, in Chapter 6, we studied the facial-parity colorings constrained to the faces of plane graphs where each color must appear zero or an odd number of times on each face of a plane graph. Both the vertex and edge versions of the problem have been considered from various points of view. In both variants, we give examples of 2-connected plane graphs that require 12 colors in order to admit a facial-parity vertex(edge)-coloring (Theorems 6.2.4 and 6.1.4), thus slightly tightening the interval on which the correct upper bound lies, which is somewhere between 12 and 16 colors in the case of the edge variant and somewhere between 12 and 97 colors in the case of the vertex variant.

In Part II, we studied the effect of cliques on how close to a tree a graph is by considering the relation between treewidth and clique number.

In Chapter 9, we considered the so-called (tw, ω)-boundedness of graph classes. In particular, we obtained a complete dichotomy in the case of excluding a single forbidden structure with respect to the subgraph, topological minor, and minor relations, as well as their induced variants (Theorems 9.1.4, 9.1.6, 9.1.7, 9.2.1, 9.2.5, and 9.3.10).

In Chapter 10, we studied the tree-independence number of graphs. First we proved several properties of this graph invariant, the most important among them being the implication that every graph class with bounded tree-independence number is, in fact, also (tw, ω)-bounded (Lemma 10.1.11). Then, we continued by proving that in the case of a single forbidden structure, a graph class is (tw, ω)-bounded if and only if it has bounded tree-independence number (Theorems 10.2.1, 10.2.3, 10.2.4, 10.3.5, and Corollaries 10.6.7 and 10.7.6). This also led to Conjecture 12.1.7, which essentially states that for a graph class (tw, ω)-boundedness implies having bounded tree-independence number.

Finally, in Chapter 11, we used the notion of tree-independence number in its refined version, the ℓ-refined tree-independence number, in order to prove that the MWIS problem (and also some of its generalization the Maximum Weight Independent Packing problem) are solvable in polynomial time in all (tw, ω)-bounded classes of graphs presented in Chapter 9. In particular, we used the ℓ-refined tree-independence number for the case of excluding a single induced minor (Theorem 11.3.1). For the other five considered graph containment relations, it turns out that we can prove polynomial-time solvability without the help of the tree-independence number. These are one of the first results proving that the tree-independence number might be useful when considering various algorithmic problems on graphs. To what extent can one use this and similar notions, however, remains to be seen. Defining known graph invariants on the subgraphs induced by the bags of tree decompositions instead of the whole graph leaves a wide range of possible research directions to be explored.

Bibliography

[1] T. Abrishami, B. Alecu, M. Chudnovsky, S. Hajebi, S. Spirkl, and K. Vušković. Tree independence number for (even hole, diamond, pyramid)-free graphs. 2023. arXiv:2305.16258.

[2] T. Abrishami, M. Chudnovsky, S. Hajebi, and S. Spirkl. Induced subgraphs and tree decompositions IV. (Even hole, diamond, pyramid)-free graphs. 2022. arXiv:2203.06775.

[3] B. Ahat, T. Ekim, and Z. C. Taşkın. Integer programming formulations and benders decomposition for the maximum induced matching problem. *INFORMS J. Comput.*, 30(1):43–56, 2018.

[4] V. A. Aksenov. On continuation of 3-colouring of planar graphs (in Russian). *Diskret. Anal. Novosibirsk*, 26:3–19, 1974.

[5] V. A. Aksenov. Chromatic connected vertices in planar graphs (in Russian). *Diskret. Anal.*, 31:5–16, 1977.

[6] V. A. Aksenov, O. V. Borodin, and A. N. Glebov. Continuation of a 3-coloring from a 6-face to a plane graph without 3-cycles. *Diskretn. Anal. Issled. Oper., Ser. 1*, 10(3):3–11, 2003.

[7] V. A. Aksenov, O. V. Borodin, and A. N. Glebov. Continuation of a 3-coloring from a 7-face onto a plane graph without 3-cycles. *Sib. Èlektron. Mat. Izv.*, 1:117–128, 2004.

[8] V. A. Aksenov and L. S. Mel'nikov. Some counterexamples associated with the three-color problem. *J. Combin. Theory Ser. B*, 28(1):1–9, 1980.

[9] J. Alman and V. Vassilevska Williams. A refined laser method and faster matrix multiplication. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 522–539. [Society for Industrial and Applied Mathematics (SIAM)], Philadelphia, PA, 2021.

[10] N. Alon. Combinatorial Nullstellensatz. *Combin. Probab. Comput.*, 8(1–2):7–29, 1999.

[11] O. Amini, L. Esperet, and J. van den Heuvel. A unified approach to distance-two colouring of graphs on surfaces. *Combinatorica*, 33:253–296, 2013.

[12] K. Appel and W. Haken. Every planar map is four colorable. *Bull. Amer. Math. Soc.*, 82:711–712, 1976.

[13] K. Appel and W. Haken. Every planar map is four colorable. part I: Discharging. *Illinois J. Math.*, 21:429–490, 1977.

[14] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. part II: Reducibility. *Illinois J. Math.*, 21:491–567, 1977.

[15] S. Arnborg, J. Lagergren, and D. Seese. Easy problems for tree-decomposable graphs. *J. Algorithms*, 12(2):308–340, 1991.

[16] V. K. Balakrishnan. *Schaum's Outline of Graph Theory: Including Hundreds of Solved Problems*. Schaum's outline series. McGraw-Hill Education, 1997.

[17] T. Bálint and J. Czap. Facial Parity 9-Edge-Coloring of Outerplane Graphs. *Graphs Combin.*, 31(5):1177–1187, 2015.

[18] J. Bang-Jensen, J. Huang, and E. Prisner. In-tournament digraphs. *J. Combin. Theory Ser. B*, 59(2):267–287, 1993.

[19] J. Beisegel, M. Chudnovsky, V. Gurvich, M. Milanič, and M. Servatius. Avoidable vertices and edges in graphs. In *Algorithms and data structures*, volume 11646 of *Lecture Notes in Comput. Sci.*, pages 126–139. Springer, Cham, 2019.

[20] J. Beisegel, M. Chudnovsky, V. Gurvich, M. Milanič, and M. Servatius. Avoidable vertices and edges in graphs: existence, characterization, and applications. *Discrete Appl. Math.*, 309:285–300, 2022.

[21] R. Belmonte, Y. Otachi, and P. Schweitzer. Induced minor free graphs: isomorphism and clique-width. *Algorithmica*, 80(1):29–47, 2018.

[22] W. Ben-Ameur, M.-A. Mohamed-Sidi, and J. Neto. The k-separator problem: polyhedra, complexity and approximation results. *J. Comb. Optim.*, 29(1):276–307, 2015.

[23] A. Berry, J. R. S. Blair, J.-P. Bordat, and G. Simonet. Graph extremities defined by search algorithms. *Algorithms (Basel)*, 3(2):100–124, 2010.

[24] A. Berry and J.-P. Bordat. Separability generalizes Dirac's theorem. *Discrete Appl. Math.*, 84(1-3):43–53, 1998.

[25] A. Berry and G. Simonet. Computing a clique tree with the algorithm maximal label search. *Algorithms (Basel)*, 10(1):Paper No. 20, 23, 2017.

[26] J. R. S. Blair and B. Peyton. An introduction to chordal graphs and clique trees. In *Graph theory and sparse matrix computation*, volume 56 of *IMA Vol. Math. Appl.*, pages 1–29. Springer, New York, 1993.

[27] H. L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. *SIAM J. Comput.*, 25(6):1305–1317, 1996.

[28] H. L. Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoret. Comput. Sci.*, 209(1-2):1–45, 1998.

[29] H. L. Bodlaender and A. M. C. A. Koster. Treewidth computations II. Lower bounds. *Inform. and Comput.*, 209(7):1103–1119, 2011.

[30] H. L. Bodlaender and R. H. Möhring. The pathwidth and treewidth of cographs. *SIAM J. Discrete Math.*, 6(2):181–188, 1993.

[31] H. L. Bodlaender, H. Ono, and Y. Otachi. Degree-constrained orientation of maximum satisfaction: graph classes and parameterized complexity. *Algorithmica*, 80(7):2160–2180, 2018.

[32] M. Bonamy and M. Pilipczuk. Graphs of bounded cliquewidth are polynomially χ-bounded. *Adv. Comb.*, pages Paper No. 8, 21, 2020.

[33] O. Borodin. A new proof of Grünbaum's 3 color theorem. *Discrete Math.*, 169(1-3):177–183, 1997.

[34] O. V. Borodin. Criterion of chromaticity of a degree prescription. In *Abstracts of IV All-Union Conf. on Theoretical Cybernetics*, pages 127–128, 1977.

[35] O. V. Borodin. Solution of Ringel's problems on vertex-face coloring of plane graphs and coloring of 1-planar graphs. *Met. Diskret. Anal.*, 41:12–26, 1984.

[36] O. V. Borodin. Cyclic coloring of plane graphs. *Discrete Math.*, 100:281–289, 1992.

[37] O. V. Borodin. A new proof of the 6 color theorem. *J. Graph Theory*, 19:507–521, 1995.

[38] O. V. Borodin. Colorings of plane graphs: A survey. *Discrete Math.*, 313(4):517–539, 2013.

[39] O. V. Borodin, Z. Dvořák, A. V. Kostochka, B. Lidický, and M. Yancey. Planar 4-critical graphs with four triangles. *Europ. J. Combin.*, 41:138–151, 2014.

[40] O. V. Borodin, A. N. Glebov, and A. Raspaud. Planar graphs without triangles adjacent to cycles of length from 4 to 7 are 3-colorable. *Discrete Math.*, 310(20):2584–2594, 2010.

[41] O. V. Borodin, A. V. Kostochka, B. Lidický, and M. Yancey. Short proofs of coloring theorems on planar graphs. *Europ. J. Combin.*, 36:314–321, 2014.

[42] O. V. Borodin, D. P. Sanders, and Y. Zhao. On cyclic colorings and their generalizations. *Discrete Math.*, 203(1):23–40, 1999.

[43] V. Bouchitté and I. Todinca. Treewidth and minimum fill-in of weakly triangulated graphs. In *STACS 99 (Trier)*, volume 1563 of *Lecture Notes in Comput. Sci.*, pages 197–206. Springer, Berlin, 1999.

[44] V. Bouchitté and I. Todinca. Listing all potential maximal cliques of a graph. *Theoret. Comput. Sci.*, 276(1-2):17–32, 2002.

[45] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[46] B. Brešar, T. R. Hartinger, T. Kos, and M. Milanič. 1-perfectly orientable K_4-minor-free and outerplanar graphs. *Discrete Appl. Math.*, 248:33–45, 2018.

[47] R. L. Brooks. On colouring the nodes of a network. *Math. Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[48] D. P. Bunde, K. Milans, D. B. West, and H. Wu. Parity and strong parity edge-colouring of graphs. *Congr. Numer.*, 187:193–213, 2007.

[49] D. P. Bunde, K. Milans, D. B. West, and H. Wu. Optimal strong parity edge-colouring of complete graphs. *Combinatorica*, 28(6):625–632, 2008.

[50] P. Buneman. A characterisation of rigid circuit graphs. *Discrete Math.*, 9:205–212, 1974.

[51] K. Cameron. Induced matchings. *Discrete Appl. Math.*, 24(1-3):97–102, 1989.

[52] K. Cameron and P. Hell. Independent packings in structured graphs. *Math. Program.*, 105(2-3, Ser. B):201–213, 2006.

[53] I. Choi, J. Ekstein, P. Holub, and B. Lidický. 3-coloring triangle-free planar graphs with a precolored 9-cycle. *Europ. J. Combin.*, 68:38–65, 2018.

[54] M. Chudnovsky. Induced subgraphs and tree decompositions. Talk at the Stony Brook Mathematics Colloquium, October 22, 2020 (online).

[55] M. Chudnovsky. Induced subgraphs and tree decompositions. Talk at the Berlin Mathematical School, MATH+ Friday Colloquium, April 16, 2021 (online).

[56] M. Chudnovsky. Induced subgraphs and tree decompositions. Talk at the Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Noon seminar, May 14, 2021 (online).

[57] M. Chudnovsky. Induced subgraphs and tree decompositions. Invited talk at IWOCA 2021: 32nd International Workshop on Combinatorial Algorithms, 5-7 July 2021, Ottawa, Canada (online).

[58] M. Chudnovsky, I. Penev, A. Scott, and N. Trotignon. Substitution and
χ-boundedness. *J. Combin. Theory Ser. B*, 103(5):567–586, 2013.

[59] M. Chudnovsky, T. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Annals of Mathematics*, 164(1):51–229, 2006.

[60] J. Chuzhoy. Improved bounds for the flat wall theorem. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 256–275. SIAM, Philadelphia, PA, 2015.

[61] F. Cicalese, M. Milanič, and R. Rizzi. On the complexity of the vector connectivity problem. *Theoret. Comput. Sci.*, 591:60–71, 2015.

[62] V. Cohen-Addad, M. Hebdige, D. Kráľ, Z. Li, and E. Salgado. Steinberg's Conjecture is false. *J. Combin. Theory Ser. B*, 122:452–456, 2017.

[63] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33(2):125–150, 2000.

[64] D. W. Cranston and D. B. West. An introduction to the discharging method via graph coloring. *Discrete Math.*, 340(4):766–793, 2017.

[65] M. Cygan, F. V. Fomin, . Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized algorithms*. Springer, Cham, 2015.

[66] J. Czap. Parity vertex coloring of outerplane graphs. *Discrete Math.*, 311(21):2570–2573, 2011.

[67] J. Czap. Facial parity edge coloring of outerplane graphs. *Ars Math. Contemp.*, 5(2):289–293, 2012.

[68] J. Czap, M. Horňák, and S. Jendroľ. A Survey on the Cyclic Coloring and its Relaxations. *Discuss. Math. Graph Theory*, 41(1):5–38, 2021.

[69] J. Czap, S. Jendrol', F. Kardoš, and R. Soták. Facial parity edge colouring of plane pseudographs. *Discrete Math.*, 312:2735–2740, 2012.

[70] J. Czap and S. Jendroľ. Colouring vertices of plane graphs under restrictions given by faces. *Discuss. Math. Graph Theory*, 29(3):521–543, 2009.

[71] J. Czap and S. Jendroľ. Facially-constrained colorings of plane graphs: A survey. *Discrete Math.*, 340(11):2691–2703, 2017.

[72] J. Czap, S. Jendroľ, and F. Kardoš. Facial parity edge colouring. *Ars Math. Contemp.*, 4(2):255–269, 2011.

[73] J. Czap, S. Jendroľ, and M. Voigt. Parity vertex colouring of plane graphs. *Discrete Math.*, 311(6):512–520, 2011.

[74] D. P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete. *Discrete Math.*, 30(3):289–293, 1980.

[75] C. Dallard, F. V. Fomin, P. A. Golovach, T. Korhonen, and M. Milanič. Computing tree decompositions with small independence number. 2022. arXiv:2207.09993.

[76] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus Clique Number. I. Graph Classes with a Forbidden Structure. *SIAM J. Discrete Math.*, 35(4):2618–2646, 2021.

[77] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus clique number. II. Tree-independence number. *arXiv preprint arXiv:2111.04543*, 2021.

[78] C. Dallard, M. Milanič, and K. Štorgel. Treewidth versus clique number. III. Tree-independence number of graphs with a forbidden structure. *arXiv preprint arXiv:2206.15092*, 2022.

[79] G. Di Battista and R. Tamassia. On-line graph algorithms with SPQR-trees. In M. S. Paterson, editor, *Automata, Languages and Programming*, pages 598–611, Berlin, Heidelberg, 1990. Springer Berlin Heidelberg.

[80] G. Di Battista and R. Tamassia. On-line maintenance of triconnected components with SPQR-trees. *Algorithmica*, 15(4):302–318, 1996.

[81] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2017.

[82] K. Diks, L. Kowalik, and M. Kurowski. A New 3-Color Criterion for Planar Graphs. In G. Goos, J. Hartmanis, J. van Leeuwen, and L. Kučera, editors, *Graph-Theoretic Concepts in Computer Science*, pages 138–149, Berlin, Heidelberg, 2002. Springer.

[83] G. A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.

[84] G. A. Dirac. The structure of k-chromatic graphs. *Fund. Math.*, 40:42–55, 1953.

[85] G. A. Dirac. On rigid circuit graphs. *Abh. Math. Sem. Univ. Hamburg*, 25:71–76, 1961.

[86] G. A. Dirac. On the structure of 5- and 6-chromatic abstract graphs. *J. Reine Angew. Math.*, 214–215:43–52, 1964.

[87] F. Dross, B. Lužar, M. Maceková, and R. Soták. Note on 3-choosability of planar graphs with maximum degree 4. *Discrete Math.*, 342(11):3123–3129, 2019.

[88] P. Duchet. Classical perfect graphs: an introduction with emphasis on triangulated and interval graphs. In *Topics on perfect graphs*, vol-

ume 88 of *North-Holland Math. Stud.*, pages 67–96. North-Holland, Amsterdam, 1984.

[89] V. Dujmović, D. Eppstein, G. Joret, P. Morin, and D. R. Wood. Minor-closed graph classes with bounded layered pathwidth. *SIAM J. Discrete Math.*, 34(3):1693–1709, 2020.

[90] Z. Dvořák and B. Lidickỳ. 3-Coloring Triangle-Free Planar Graphs with a Precolored 8-Cycle. *J. Graph Theory*, 80(2):98–111, 2015.

[91] Z. Dvořák. 3-choosability of planar graphs with ($\leq$4)-cycles far apart. *J. Combin. Theory Ser. B*, 104:28–59, 2014.

[92] Z. Dvořák, M. Hebdige, F. Hlásek, D. Král', and J. A. Noel. Cyclic coloring of plane graphs with maximum face size 16 and 17. *European J. Combin.*, 94:103287, 2021.

[93] Z. Dvořák and D. Kráľ. Classes of graphs with small rank decompositions are χ-bounded. *European J. Combin.*, 33(4):679–683, 2012.

[94] Z. Dvořák, D. Kráľ, and R. Thomas. Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies. *J. Combin. Theory Ser. B*, 150:244–269, 2021.

[95] Z. Dvořák, B. Lidický, and R. Škrekovski. Planar graphs without 3-, 7-, and 8-cycles are 3-choosable. *Discrete Math.*, 309(20):5899–5904, 2009.

[96] Z. Dvořák, B. Lidický, and R. Škrekovski. 3-Choosability of Triangle-Free Planar Graphs with Constraints on 4-Cycles. *SIAM J. Discrete Math.*, 24:934–945, 2010.

[97] Z. Dvořák, R. Škrekovski, and M. Tancer. List-Coloring Squares of Sparse Subcubic Graphs. *SIAM J. Discrete Math.*, 22:139–159, 2008.

[98] M. N. Ellingham, H. Fleischner, M. Kochol, and E. Wenger. Colorability of Planar Graphs with Isolated Nontriangular Faces. *Graphs Combin.*, 20(4):443–446, 2004.

[99] H. Enomoto and M. Horňák. A general upper bound for the cyclic chromatic number of 3-connected plane graphs. *J. Graph Theory*, 62:1–25, 2009.

[100] P. Erdős. Graph theory and probability. *Canadian J. Math.*, 11:34–38, 1959.

[101] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *West Coast Conference on Combinatorics, Graph Theory and Computing*, pages 125–157, 1980.

[102] P. Erdös and G. Szekeres. A combinatorial problem in geometry. *Compos. Math.*, 2:463–470, 1935.

[103] L. Euler. Solutio problematis ad geometriam situs pertinentis. *Com-*

mentarii Academiae Scientiarum Imperialis Petropolitanae, 8:128–140, 1736, in Latin.

[104] A. Frank. Some polynomial algorithms for certain graphs and hypergraphs. In *Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975)*, pages 211–226. Congressus Numerantium, No. XV, 1976.

[105] M. R. Garey and D. S. Johnson. *Computers and Intractability: A guide to the theory of NP-completeness.* W. H. Freeman and Co., San Francisco, Calif., 1979.

[106] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoret. Comput. Sci.*, 1(3):237–267, 1976.

[107] P. Gartland, D. Lokshtanov, M. Pilipczuk, M. Pilipczuk, and P. Rzażewski. Finding large induced sparse subgraphs in $c_{>t}$-free graphs in quasipolynomial time. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2021, page 330–341, New York, NY, USA, 2021. Association for Computing Machinery.

[108] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *J. Combinatorial Theory Ser. B*, 16:47–56, 1974.

[109] J. Gimbel and C. Thomassen. Coloring graphs with fixed genus and girth. *Trans. Amer. Math. Soc.*, 349(11):4555–4564, 1997.

[110] P. A. Golovach, D. Paulusma, and B. Ries. Coloring graphs characterized by a forbidden subgraph. *Discrete Appl. Math.*, 180:101–110, 2015.

[111] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*, volume 57. Elsevier Science B.V., Amsterdam, second edition, 2004.

[112] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*, volume 57 of *Annals of Discrete Mathematics*. Elsevier Science B.V., Amsterdam, second edition, 2004.

[113] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1(2):169–197, 1981.

[114] H. Grötzsch. Ein dreifarbensatz für dreikreisfreie netze auf der kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur.*, 8:109–120, 1959.

[115] B. Grünbaum. Grötzsch's theorem on 3-coloring. *Michigan Math. J.*, 10:303–310, 1963.

[116] C. Gutwenger and P. Mutzel. A linear time implementation of SPQR-trees. In J. Marks, editor, *Graph Drawing, 8th International Symposium, GD 2000, Colonial Williamsburg, VA, USA, September 20-23,*

2000, Proceedings, volume 1984 of *Lecture Notes in Computer Science*, pages 77–90. Springer, 2000.

[117] A. Gyárfás. Problems from the world surrounding perfect graphs. *Applicationes Mathematicae*, 19:413–441, 1987.

[118] A. Gyárfás. Problems from the world surrounding perfect graphs. *Zastos. Mat.*, 19(3-4):413–441, 1987.

[119] A. Gyárfás and M. Zaker. On (δ, χ)-bounded families of graphs. *Electron. J. Combin.*, 18(1):Paper 108, 8, 2011.

[120] P. Hall. On Representatives of Subsets. *J. Lond. Math. Soc.*, s1–10(1):26–30, 1935.

[121] T. R. Hartinger. *New Characterizations in Structural Graph Theory: 1-Perfectly Orientable Graphs, Graph Products, and the Price of Connectivity.* PhD thesis, University of Primorska, 2017.

[122] T. R. Hartinger and M. Milanič. Partial characterizations of 1-perfectly orientable graphs. *J. Graph Theory*, 85(2):378–394, 2017.

[123] D. J. Harvey and D. R. Wood. The treewidth of line graphs. *J. Combin. Theory Ser. B*, 132:157–179, 2018.

[124] I. Havel. On a Conjecture of B. Grünbaum. *J. Combin. Theory Ser. B*, 7:184–186, 1969.

[125] I. Havel. O zbarvitelnosti rovinných graføu třemi barvami. *Mathematics (Geometry and Graph Theory)*, pages 89–91, 1970.

[126] F. Havet, D. Král', S. J.-S., and R. Škrekovski. Facial colorings using Hall's Theorem. *European J. Combin.*, 31(3):1001–1019, 2010.

[127] F. Havet, J.-S. Sereni, and R. Škrekovski. 3-facial coloring of plane graphs. *SIAM J. Discrete Math.*, 22(1):231–247, 2008.

[128] P. J. Heawood. On the four-colour map theorem. *Quart. J. Pure Appl. Math.*, 29:270–285, 1898.

[129] M. Hebdige and D. Kráľ. Third Case of the Cyclic Coloring Conjecture. *SIAM Journal on Discrete Mathematics*, 30(1):525–548, 2016.

[130] P. Heggernes, J. A. Telle, and Y. Villanger. Computing minimal triangulations in time $O(n^\alpha \log n) = o(n^{2.376})$. *SIAM J. Discrete Math.*, 19(4):900–913, 2005.

[131] D. Hermelin, J. Mestre, and D. Rawitz. Optimization problems in dotted interval graphs. *Discrete Appl. Math.*, 174:66–72, 2014.

[132] P. v. t. Hof, M. Kamiński, D. Paulusma, S. Szeider, and D. M. Thilikos. On graph contractions and induced minors. *Discrete Appl. Math.*, 160(6):799–809, 2012.

[133] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Comput.*, 10:718–720, 1981.

[134] J. E. Hopcroft and R. E. Tarjan. Algorithm 447: efficient algorithms for graph manipulation. *Communications of the ACM*, 16(6):372–378, 1973.

[135] J. E. Hopcroft and R. E. Tarjan. Dividing a graph into triconnected components. *SIAM J. Comput.*, 2:135–158, 1973.

[136] M. Horňák and S. Jendroľ. On a conjecture by Plummer and Toft. *J. Graph Theory*, 30:177–189, 1999.

[137] M. Horňák and S. Jendroľ. On vertex types and cyclic colourings of 3-connected plane graphs. *Discrete Math.*, 212:101–109, 2000.

[138] M. Horňák, B. Lužar, and K. Štorgel. 3-facial edge-coloring of plane graphs. *Discrete Math.*, 346(5):113312, 2023.

[139] M. Horňák and J. Zlámalová. Another step towards proving a conjecture by Plummer and Toft. *Discrete Math.*, 310(3):442–452, 2010.

[140] M. V. I. Fabrici, S. Jendroľ. Unique-maximum edge-colouring of plane graphs with respect to faces. *Discrete Appl. Math.*, 185:239–234, 2015.

[141] T. R. Jensen and C. Thomassen. The color space of a graph. *J. Graph Theory*, 34(3):234–245, 2000.

[142] T. R. Jensen and B. Toft. *Graph coloring problems.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1995.

[143] T. Kaiser, O. Rucký, M. Stehlík, and R. Škrekovski. Strong parity vertex coloring of plane graphs. *Discrete Math. Theoret. Comput. Sci.*, 16(1):143–158, 2014.

[144] M. Kamiński, V. Lozin, and M. Milanič. Recent developments on graphs of bounded clique-width. *Discrete Appl. Math.*, 157(12):2747–2761, 2009.

[145] Y. Kang, L. Jin, and Y. Wang. The 3-colorability of planar graphs without cycles of length 4, 6 and 9. *Discrete Math. Algorithms Appl.*, 339(1):299–307, 2016.

[146] R. Karp. Reducibility Among Combinatorial Problems. volume 40, pages 85–103, 1972.

[147] R. M. Karp. Reducibility among combinatorial problems. In *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*, pages 85–103, 1972.

[148] D. C. Kay and G. Chartrand. A characterization of certain ptolemaic graphs. *Canadian J. Math.*, 17:342–346, 1965.

[149] M. Kochol. Three colorability characterized by shrinking of locally

connected subgraphs into triangles. *Inform. Process. Lett.*, 135:33–35, 2018.

[150] A. Kostochka and M. Yancey. Ore's conjecture for $k = 4$ and Grötzsch's Theorem. *Combinatorica*, 34(3):323–329, 2014.

[151] A. Kostochka and M. Yancey. A Brooks-Type Result for Sparse Critical Graphs. *Combinatorica*, 38(4):887–934, 2018.

[152] A. V. Kostochka. Lower bound of the Hadwiger number of graphs by their average degree. *Combinatorica*, 4(4):307–316, 1984.

[153] D. Král', T. Madaras, and R. Škrekovski. Cyclic, diagonal and facial colorings–a missing case. *European J. Combin.*, 28(6):1637–1639, 2007.

[154] D. Kráľ, T. Madaras, and R. Škrekovski. Cyclic, diagonal and facial colorings. *European J. Combin.*, 26(3–4):473–490, 2005.

[155] M. Kriesell. Contractions, cycle double covers, and cyclic colorings in locally connected graphs. *J. Combin. Theory Ser. B*, 96:881–900, 2006.

[156] K. Kuratowski. Sur le probléme des courbes gauches en topologie. *Fund. Math.*, 15:271–283, 1930.

[157] H. La, B. Lužar, and K. Štorgel. Further extensions of the Grötzsch Theorem. *Discrete Math.*, 345(6):112849, 2022.

[158] P. C. B. Lam, W. C. Shiu, and Z. M. Song. The 3-choosability of plane graphs of girth 4. *Discrete Math.*, 294(3):297–301, 2005.

[159] E. Lee. Partitioning a graph into small pieces with applications to path transversal. *Math. Program.*, 177(1-2, Ser. A):1–19, 2019.

[160] V. Lozin and I. Razgon. Tree-width dichotomy. *European J. Combin.*, 103:Paper No. 103517, 8, 2022.

[161] H. Lu, Y. Wang, W. Wang, Y. Bu, M. Montassier, and A. Raspaud. On the 3-colorability of planar graphs without 4-, 7- and 9-cycles. *Discrete Math.*, 309(13):4596–4607, 2009.

[162] B. Lužar, M. Mockovčiaková, R. Soták, R. Škrekovski, and P. Šugerek. ℓ-facial edge colorings of graphs. *Discrete Appl. Math.*, 181:193–200, 2015.

[163] B. Lužar, M. Petruševski, and R. Škrekovski. Odd edge coloring of graphs. *Ars Math. Contemp.*, 9(2):277–287, 2015.

[164] B. Lužar and R. Škrekovski. Improved bound on facial parity edge coloring. *Discrete Math.*, 313(20):2218–2222, 2013.

[165] S. E. Markossian, G. S. Gasparian, and B. A. Reed. β-perfect graphs. *J. Combin. Theory Ser. B*, 67(1):1–11, 1996.

[166] T. Mátrai. Covering the edges of a graph by three odd subgraphs. *J. Graph Theory*, 53:75–82, 2006.

[167] K. R. Matthews. On the eulericity of graphs. *J. Graph Theory*, 2:143–148, 1978.

[168] M. Milanič and P. Rzążewski. Tree decompositions with bounded independence number: beyond independent sets. 2022. arXiv:2209.12315.

[169] O. Ore and M. D. Plummer. Cyclic coloration of plane graphs. In *Recent Progress in Combinatorics*, (Proceedings of the Third Waterloo Conference on Combinatorics, 1968). Academic Press, 1969.

[170] Y. Orlovich, A. Dolgui, G. Finke, V. Gordon, and F. Werner. The complexity of dissociation set problems in graphs. *Discrete Appl. Math.*, 159(13):1352–1366, 2011.

[171] B. S. Panda, A. Pandey, J. Chaudhary, P. Dane, and M. Kashyap. Maximum weight induced matching in some subclasses of bipartite graphs. *J. Comb. Optim.*, 40(3):713–732, 2020.

[172] M. D. Plummer and B. Toft. Cyclic coloration of 3-polytopes. *J. Graph Theory*, 11:507–515, 1987.

[173] L. Pyber. Covering the edges of a graph by ... *Colloq. Math. Soc. János Bolyai*, 60:583–610, 1991.

[174] F. P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc. (2)*, 30(4):264–286, 1929.

[175] N. Robertson and P. D. Seymour. Graph minors. III. Planar tree-width. *J. Combin. Theory Ser. B*, 36(1):49–64, 1984.

[176] N. Robertson and P. D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986.

[177] N. Robertson and P. D. Seymour. Graph minors. V. Excluding a planar graph. *J. Combin. Theory Ser. B*, 41(1):92–114, 1986.

[178] D. J. Rose, R. E. Tarjan, and G. S. Lueker. Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.*, 5(2):266–283, 1976.

[179] D. P. Sanders and Y. Zhao. A New Bound on the Cyclic Chromatic Number. *J. Combin. Theory Ser. B*, 83(1):102–111, 2001.

[180] P. Scheffler. What graphs have bounded tree-width? In *Proceedings of the 7th Fischland Colloquium, III (Wustrow, 1988)*, number 41, pages 31–38, 1990.

[181] A. Scott and P. Seymour. A survey of χ-boundedness. *J. Graph Theory*, 95(3):473–504, 2020.

[182] P. Seymour. Tree-chromatic number. *J. Combin. Theory Ser. B*, 116:229–237, 2016.

[183] L. Shen and Y. Wang. A sufficient condition for a planar graph to be 3-choosable. *Inform. Process. Lett.*, 104(4):146–151, 2007.

[184] K. Skodinis. Efficient analysis of graphs with small minimal separators. In *Graph-theoretic concepts in computer science (Ascona, 1999)*, volume 1665 of *Lecture Notes in Comput. Sci.*, pages 155–166. Springer, Berlin, 1999.

[185] D. J. Skrien. A relationship between triangulated graphs, comparability graphs, proper interval graphs, proper circular-arc graphs, and nested interval graphs. *J. Graph Theory*, 6(3):309–316, 1982.

[186] J. P. Spinrad. *Efficient Graph Representations*, volume 19. American Mathematical Society, Providence, RI, 2003.

[187] R. Steinberg. The State of the Three Color Problem. In J. Gimbel, J. W. Kennedy, and L. V. Quintas, editors, *Quo Vadis, Graph Theory?*, volume 55 of *Annals of Discrete Mathematics*, pages 211–248. Elsevier, 1993.

[188] P. G. Tait. On the colouring of maps. *Proc. R. Soc. Edinburgh Sect. A*, 10:501–503, 729, 1880.

[189] R. E. Tarjan and M. Yannakakis. Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM J. Comput.*, 13(3):566–579, 1984.

[190] R. Thomas and B. Walls. Three-coloring klein bottle graphs of girth five. *J. Combin. Theory Ser. B*, 92:115–135, 2004.

[191] C. Thomassen. 3-List-Coloring Planar Graphs of Girth 5. *J. Combin. Theory Ser. B*, 64(1):101–107, 1995.

[192] W. T. Tutte. A theory of 3-connected graphs. *Nederl. Akad. Wetensch. Proc. Ser. A 64 = Indag. Math.*, 23:441–455, 1961.

[193] W. T. Tutte. *Connectivity in graphs.* Mathematical Expositions, No. 15. University of Toronto Press, Toronto, Ont.; Oxford University Press, London, 1966.

[194] W. T. Tutte. *Graph theory*, volume 21 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley Publishing Company, Advanced Book Program, Reading, MA, 1984. With a foreword by C. St. J. A. Nash-Williams.

[195] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Met. Diskret. Anal.*, 3:25–30, 1964.

[196] M. Voigt. A not 3-choosable planar graph without 3-cycles. *Discrete Math.*, 146(1):325–328, 1995.

[197] M. Voigt. A non-3-choosable planar graph without cycles of length 4 and 5. *Discrete Math.*, 307(7-8):1013–1015, 2007.

[198] K. Štorgel. Improved Bounds for Some Facially Constrained Colorings. *Discuss. Math. Graph Theory*, 43(1):151–158, 2023.

[199] M. Šurimová, B. Lužar, and T. Madaras. Adynamic coloring of graphs. *Discrete Appl. Math.*, 284:224–233, 2020.

[200] K. Wagner. Über eine eigenschaft der ebenen komplexe. *Math. Ann.*, 114:570–590, 1937.

[201] J. R. Walter. Representations of chordal graphs as subtrees of a tree. *J. Graph Theory*, 2(3):265–267, 1978.

[202] W. Wang and M. Chen. Planar graphs without 4,6,8-cycles are 3-colorable. *Science in China Series A: Mathematics*, 50:1552–1562, 2007.

[203] W. Wang, S. Finbow, and P. Wang. An improved bound on parity vertex colourings of outerplane graphs. *Discrete Math.*, 312(18):2782–2787, 2012.

[204] Y. Wang, H. Lu, and M. Chen. A note on 3-choosability of planar graphs. *Inform. Process. Lett.*, 105(5):206–211, 2008.

[205] Y. Wang, H. Lu, and M. Chen. Planar graphs without cycles of length 4, 5, 8, or 9 are 3-choosable. *Discrete Math.*, 310(1):147–158, 2010.

[206] Y. Wang, Q. Wu, and L. Shen. Planar graphs without cycles of length 4, 7, 8, or 9 are 3-choosable. *Discrete Appl. Math.*, 159(4):232–239, 2011.

[207] B. Xu. On 3-colorable plane graphs without 5- and 7-cycles. *Discrete Math. Algorithms Appl.*, 1:347–353, 2009.

[208] M. Yannakakis. Node-deletion problems on bipartite graphs. *SIAM J. Comput.*, 10(2):310–327, 1981.

[209] N. Yolov. Minor-matching hypertree width. In A. Czumaj, editor, *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018*, pages 219–233. SIAM, 2018.

[210] J. You, J. Wang, and Y. Cao. Approximate association via dissociation. In *Graph-theoretic concepts in computer science*, volume 9941 of *Lecture Notes in Comput. Sci.*, pages 13–24. Springer, Berlin, 2016.

[211] M. Zaker. On lower bounds for the chromatic number in terms of vertex degree. *Discrete Math.*, 311(14):1365–1370, 2011.

[212] H. Zhang, B. Xu, and Z. Sun. Every plane graph with girth at least 4 without 8- and 9-circuits is 3-choosable. *Ars Combin.*, 80:247–257, 2006.

[213] L. Zhang and B. Wu. Three-coloring planar graphs without certain small cycles. *Graph Theory Notes N. Y.*, 46:27–30, 2004.

[214] L. Zhang and B. Wu. A note on 3-choosability of planar graphs without certain cycles. *Discrete Math.*, 297(1):206–209, 2005.

[215] X. Zhu, M. Lianying, and C. Wang. On 3-choosability of plane graphs without 3-,8-, and 9-cycles. *Australas. J. Comb.*, 36:249–254, 2007.

Povzetek v slovenskem jeziku

V tem delu obravnavamo različne invariante grafov povezane z barvanjem ravninskih grafov ter različne invariante in dekompozicije povezane z drevesnimi dekompozicijami grafov v izbranih razredih grafov. S tem namenom je disertacija razdeljena na dva dela. Glavna motivacija za raziskave različnih invariant grafov je poglobitev razumevanja strukturnih lastnosti grafov ter algoritmičnih posledic, ki sledijo. *Grafovska invarianta* je funkcija, definirana na grafih, ki ima enake vrednosti za izomorfne grafe. Skozi leta je bilo definiranih veliko različnih invariant grafov. Mnoge med njimi so težko razumljive, če jih opazujemo nad množico vseh grafov, zato se pri raziskavah pogosto omejimo zgolj na specifične razrede grafov. *Razred grafov* je množica grafov, ki je zaprta za izomorfizem (glej, npr. [45, 111, 186]).

Ena izmed bolj znanih invariant grafov je *kromatično število grafa*, ki ga označimo s $\chi(G)$. To je najmanjše število barv, ki jih potrebujemo za barvanje vozlišč grafa G tako, da nobeni dve sosednji vozlišči nista pobarvani z isto barvo. Takemu barvanju pravimo *pravilno barvanje* grafa. Raziskave te invariante so se začele s problemom štirih barv, ki ga je postavil Francis Guthrie v letu 1852. Problem sprašuje po obstoju pravilnega barvanja z največ štirimi barvami za vsak ravninski graf, to je graf, ki ga lahko vložimo v ravnino brez sekajočih se povezav. Da za vsak ravninski graf obstaja pravilno barvanje s štirimi barvami so v letih 1976–1977 dokazali Appel in Haken [12, 13] ter Appel, Haken in Koch [14]. Kromatično število grafa predstavlja najmanjše število množic, na katere lahko razdelimo vozlišča grafa tako, da nobena množica ne vsebuje para sosednjih vozlišč. Množicam posamezne barve pogosto pravimo *barvni razredi*. Razdelitve vozlišč grafa lahko seveda naredimo tudi glede na drugačne pogoje. Posledično je bilo skozi leta definiranih veliko različnih invariant grafov povezanih z barvanjem grafov. V prvem delu disertacije obravnavamo pravilna barvanja ravninskih grafov, ciklično barvanje in njegove posplošitve, kot so ℓ-lična barvanja vložitev ravninskih grafov, in nenazadnje tudi lično-parna barvanja vložitev ravninskih grafov.

Po drugi strani pa lahko, namesto razdelitev vozlišč grafov na barvne razrede, kjer ne dopuščamo ponovitev elementov, raziskujemo različne dekompozicije grafov. *Dekompozicija grafa* je razdelitev množice vozlišč na več množic imenovanih *vreče*, kjer dopuščamo, da se lahko posamezen element pojavi v več različnih vrečah. Ob tem mora veljati, da za vsak par sosednjih vozlišč obstaja vreča, ki vsebuje obe vozlišči. Ena izmed takšnih dekompozicij je *drevesna dekompozicija*, kjer zahtevamo, da mora biti vsako vozlišče vsebovano v neki vreči, da mora za vsako povezavo obstajati vreča, ki vsebuje obe njeni krajišči ter, da za vsako vozlišče velja, da vreče, ki vsebujejo izbrano vozlišče, tvorijo drevo. *Drevo* je povezan graf brez ciklov. Invarianta grafov, ki je povezana z drevesnimi dekompozicijami, je *drevesna širina* grafa G, ki jo označimo s $\mathrm{tw}(G)$ in je enaka najmanjši vrednosti med vsemi drevesnimi dekompozicijami velikosti največje vreče drevesne dekompozicije minus ena. V grobem drevesna širina meri podobnost grafa z drevesom. Ta koncept je vpeljal Halin leta 1976. Zanimanje zanj pa se je razširilo predvsem z njegovim ponovnim odkritjem s strani Robertsona and Seymourja leta 1984 v članku z naslovom Graph Minors III [175], ki je le eden izmed mnogih njunih člankov v tej seriji.

Mnoge invariante grafov so NP-težke (glej, npr. [105]). Kljub temu je veliko invariant grafov mogoče izračunati v polinomskem času v razredu grafov z omejeno drevesno širino. Posledica tega je, da je drevesna širina grafov postala pomembna invarianta grafov v mnogih raziskavah. V drugem delu disertacije se tako usmerimo v obravnavo (tw, ω)-omejenih razredov grafov, kjer $\omega(G)$ predstavlja velikost največje *klike* grafa G, to je največje množice paroma sosednjih vozlišč. Nato se usmerimo v obravnavo invariante imenovane drevesno neodvisnostno število grafa in na koncu tudi na uporabo drevesnega neodvisnostnega števila pri razrešljivosti problema najtežje neodvisne množice.

Prvi del

Prvi del disertacije začnemo z obravnavo 3-obarvljivosti ravninskih grafov. Grötzschev izrek [114] pravi, da za vsak ravninski graf brez trikotnikov obstaja pravilno 3-barvanje. Hkrati pa vemo, da obstajajo ravninski grafi s trikotniki, ki potrebujejo 4 barve v vsakem pravilnem barvanju vozlišč. Eden izmed takih grafov je polni graf K_4. To nas vodi do raziskav o vplivu trikotnikov na barvanje vozlišč ravninskih grafov. Aksenov [4] je pokazal, da za vsak ravninski graf z največ tremi trikotniki vedno obstaja pravilno 3-barvanje vozlišč. Kasneje so se pojavili tudi krajši dokazi (glej [33] in [41]). V [41] so avtorji uporabili rezultat Kostochke and Yanceyja [150], ki nam da najmanjše število povezav k-kritičnega graf za poljuben $k \geq 4$. Graf G je *k-kritičen*, če je $\chi(G) = k$ in za vsak induciran podgraf H grafa G velja, da je $\chi(H) < k$. Ob tem avtorji pokažejo, da se določena predbarvanja vložitev ravninskih grafov brez trikotnikov lahko razširijo v pravilna 3-barvanja celotnega grafa. Glavno vprašanje, povezano z Grötzschevim izrekom, je vprašanje o tem, katera predbarvanja vložitev ravninskih grafov z največ enim trikotnikom se lahko razširijo na pravilna 3-barvanja celotnega grafa, kjer nam je v pomoč prej omenjeni rezultat Kostochke and Yanceyja. Najprej pokažemo, da lahko vsako predbarvanje para vozlišč v vložitvi ravninskega grafa z največ enim trikotnikom razširimo na 3-barvanje celotnega grafa (Izrek 4.2.1). Nato pokažemo, da lahko vsako predbarvanje vozlišč lica dolžine največ 4 v vložitvi ravninskega grafa z največ enim trikotnikom razširimo na 3-barvanje celotnega grafa

(Izrek 4.2.3). Na koncu poglavja pa pokažemo, da lahko za vsako vozlišče stopnje največ 3 v vložitvi ravninskega grafa, predbarvanje, kjer vse njegove sosede pobarvamo z isto barvo, razširimo na 3-barvanje celotnega grafa (Izrek 4.2.6). Nenazadnje podamo tudi primere, ki dokazujejo tesnost naših rezultatov. Ob tem omenimo še en rezultat. *Adinamično barvanje* je pravilno barvanje vozlišč grafa G, za katero obstaja vozlišče v stopnje vsaj 2, ki ima vse sosede pobarvane z isto barvo. Iz definicije sledi, da mora za obstoj adinamičnega barvanja obstajati vozlišče z neodvisno soseščino, to je tako vozlišče, ki ne leži na nobenem trikotniku. Šurimová idr. [199] so postavili domnevo, ki pravi, da za vsak ravninski graf z največ enim trikotnikom, ki omogoča adinamično barvanje, obstaja adinamično barvanje z največ tremi barvami. Kot posledico Izreka 4.2.6 pokažemo, da je domneva resnična (Izrek 7.1.3).

Nadaljujemo z obravnavo *cikličnega barvanja* vložitev ravninskih grafov, to je tako barvanje vozlišč vložitve ravninskega grafa, da nobeno lice ni sosednje z dvema vozliščema enake barve. Ciklično barvanje sta definirala Ore in Plummer [169] leta 1969. Iz definicije direktno sledi, da za tako barvanje potrebujemo vsaj toliko barv, kot je dolžina $\Delta^*(G)$ najdaljšega lica. Domneva cikličnega barvanja pa pravi, da $\lfloor \frac{3}{2}\Delta^*(G) \rfloor$ barv vedno zadošča za poljuben $\Delta^*(G) \geq 3$. Leta 1987 sta Plummer in Toft [172] podala prvo družino vložitev ravninskih grafov, ki dosežejo mejo podano v Domnevi cikličnega barvanja. O zahtevnosti Domneve cikličnega barvanja priča dejstvo, da je znano le to, da domneva velja za majhno število primerov, in sicer za $\Delta^*(G) \in \{3, 4, 6\}$. Več je znanega v primeru vložitev 3-povezanih ravninskih grafov, za katere sta Plummer in Toft [172] postavila domnevo, da zadošča $\Delta^*(G) + 2$ barv za vsak $\Delta^*(G) \geq 3$. Različni rezultati (glej [136, 137, 139, 92]) in dejstvo, da za $\Delta^*(G) \in \{3, 4\}$, domneva sovpada z Domnevo cikličnega barvanja, kažejo na resničnost domneve Plummerja in Tofta za vse $\Delta^*(G) \notin \{5, \ldots, 15\}$. Kot posplošitev cikličnega barvanja so leta 2005 Kráľ, Madaras in Škrekovski [154] definirali *ℓ-lično barvanje vozlišč* vložitev ravninskega grafa, to je tako barvanje vozlišč, kjer vsaki dve vozlišči sosednji z istim licem na razdalji največ ℓ na tem licu pobarvamo z različnima barvama. Kráľ, Madaras in Škrekovski [154] so postavili tudi domnevo, da za vsako takšno barvanje potrebujemo največ $3\ell + 1$ barv. Opazimo lahko, da v primeru, ko je $\Delta^*(G) \leq 2\ell + 1$, ℓ-lično barvanje sovpada s cikličnim barvanjem. V splošnem je znano le, da je domneva resnična za $\ell = 1$, kar dokazuje izrek štirih barv, saj je vsako pravilno barvanje vložitve ravninskega grafa tudi 1-lično barvanje. V primeru vložitev ravninskih grafov brez trikotnikov so Dvořák, Škrekovski in Tancer [97] postavili domnevo, da 3ℓ barv vedno zadošča. Podobno kot Izrek štirih barv dokazuje veljavnost domneve ℓ-ličnega barvanja v primeru, ko je $\ell = 1$, Grötzschev izrek (Izrek 4.0.2) v splošnem dokazuje veljavnost domneve ℓ-ličnega barvanja vložitev ravninskih grafov brez trikotnikov. Tudi ta domneva ostaja odprta za preostale vrednosti $\ell \geq 2$. Poseben primer ℓ-ličnega barvanja vozlišč vložitve ravninskega grafa je *ℓ-lično barvanje povezav*, kjer je edina razlika ta, da namesto vozlišč barvamo povezave tako, da vsak par povezav na razdalji največ ℓ na sprehodu po robu lica velja, da sta pobarvani z različno barvo. Takšno barvanje so definirali leta 2015 Lužar, Mockovčiaková, Soták, Škrekovski in Šugerek [162]. Prav tako so postavili domnevo, da zadostuje $3\ell+1$ barv za vsako ℓ-lično povezavno barvanje poljubne vložitve ravninskega grafa. Da je ℓ-lično barvanje povezav poseben primer ℓ-ličnega barvanja vozlišč, sledi iz dejstva, da lahko vsak graf G vedno spremenimo v tako imenovani *medialni graf* $M(G)$, to je, graf, kjer za vsako povezavo grafa G ustvarimo vozlišče v grafu $M(G)$ in dodamo povezave med dvema vozliščema u in v grafa $M(G)$ natanko tedaj, ko sta ustrezni povezavi v grafu G sosednji na skupnem licu. Iz definicij obeh barvanj in medialnega grafa sledi, da je vsako ℓ-lično barvanje vozlišč grafa $M(G)$ tudi ℓ-lično povezavno barvanje grafa G. O domnevi ℓ-ličnega povezavnega barvanja je znanega nekoliko več. Primer, ko je $\ell = 1$, je posledica izreka štirih barv. Primer, ko je $\ell = 2$, so dokazali Lužar idr. [162]. S pomočjo metode prenosa naboja pokažemo, da je domneva veljavna tudi za $\ell = 3$ (Izrek 5.3.3).

Zadnja tematika prvega dela te disertacije so lično-parna barvanja vložitev ravninskih grafov. Leta 2009 sta Czap in Jendroľ [70] definirala *lično-parno barvanje vozlišč* vložitve 2-povezanega ravninskega grafa kot po licih pravilno barvanje vozlišč, kjer je vsako lice sosednje z nič ali pa z lihim številom vozlišč posamezne barve. Podobno so leta 2011 Czap, Jendroľ in Kardoš [72] definirali *lično-parno barvanje povezav* vložitve povezanega ravninskega grafa brez mostov kot po licih pravilno barvanje povezav, kjer je vsako lice sosednje z nič ali pa z lihim številom povezav posamezne barve. Razlog, da je lično-parno barvanje vozlišč definirano za vložitve 2-povezanega ravninskega grafa in da je lično-parno barvanje povezav definirano za vložitve povezanega ravninskega grafa je v tem, da presečna vozlišča, oz. mostove, štejemo dvakrat na zunanjem licu. Med glavne motivacije za omenjeni barvanji spadajo parna barvanja, ki so jih definirali Bunde, Milans, West in Wu [48, 49]. Definirana so na sledeč način. Za povezavno barvanje enostavnega grafa je *parni sprehod* tak sprehod, da se na njem vsaka barva pojavi sodokrat. *Parno povezavno barvanje* je takšno povezavno barvanje, ki nima nobenega parnega sprehoda. Z drugimi besedami, za vsako pot P obstaja barva c, ki se na poti P pojavi lihokrat. Velja, da je vsako parno povezavno barvanje tudi pravilno barvanje povezav grafa, saj dve sosednji povezavi predstavljata pot dolžine 2, ki mora posledično biti pobarvana z dvema barvama. Še ena izmed motivacij so problemi pokritij grafov. Pravimo, da grafi $H_1, \ldots, H_k$ tvorijo *pokritje* grafa G, če za vsak $1 \leq i \leq k$, velja, da je H_i podgraf grafa G in, da je $\bigcup_{1\leq i\leq k} E(H_i) = E(G)$. Še posebej so zanimiva *liha pokritja*, to so taka pokritja grafa G, kjer ima vsak graf H_i le vozlišča lihih stopenj. Pyber [173] je leta 1991 pokazal, da ima vsak graf liho pokritje z največ štirimi povezavno disjunktnimi podgrafi. Podobno so Lužar idr. [163] pokazali, da lahko vsak multigraf brez zank pokrijemo z največ šestimi povezavno disjunktnimi podgrafi. Naj omenimo še, da je vsako lično-parno barvanje tudi 1-lično barvanje. Če se vrnemo na lično-parno barvanje vozlišč, so Czap, Jendroľ in Voigt [73] pokazali, da lahko vsako vložitev 2-povezanega ravninskega grafa pobarvamo z največ

118 barvami. Tri leta kasneje so Kaiser idr. [143] mejo izboljšali na 97 barv, kar je trenutno najboljši rezultat. V primeru zunanje-ravninskih grafov, to je grafov, ki jih lahko vložimo v ravnino tako, da so vsa vozlišča incidenčna z zunanjim licem, je znanega nekoliko več. Leta 2011 je Czap [66] pokazal, da največ 12 barv zadošča za lično-parno barvanje povezav vložitev 2-povezanih ravninskih grafov. Ob tem je pokazal tudi obstoj zunanje-ravninskega grafa, ki potrebuje 10 barv. Leto kasneje so Wang, Finbow in Wang [203] uspeli pokazati, da za zunanje-ravninske grafe vedno zadostuje 10 barv, pri čemer obstajata le dva grafa, ki to mejo tudi dosežeta. V primeru lično-parnega barvanja povezav so Czap idr. [72] pokazali, da ima vsaka vložitev povezanega ravninskega grafa brez mostov lično-parno barvanje povezav z 92 barvami. Kasneje so Czap, Jendroľ, Kardoš in Soták [69] mejo izboljšali na 20 barv, ta rezultat pa sta dodatno izboljšala Lužar in Škrekovski [164], in sicer na 16 barv, kar je trenutno najboljši rezultat. Podobno kot pri lično-parnem barvanju vozlišč, je tudi v primeru lično-parnega barvanja povezav nekoliko več znanega v primeru zunanje-ravninskih grafov. Leta 2012 je Czap [67] pokazal, da za vsako vložitev povezanega zunanje-ravninskega grafa brez mostov vedno obstaja lično-parno barvanje povezav z največ 15 barvami. Ta rezultat sta tri leta kasneje izboljšala Bálint in Czap [17], ko sta pokazala, da, z izjemo le enega grafa, ima vsaka vložitev povezanega zunanje-ravninskega grafa brez mostov lično-parno povezavno barvanje z največ 9 barvami, edina izjema pa potrebuje natanko 10 barv. Tako za lično-parno barvanje vozlišč kot tudi za lično-parno barvanje povezav pokažemo, da za vsako celo število t, kjer je $6 \leq t \leq 12$, obstaja neskončna družina grafov za katere obstaja lično-parno barvanje vozlišč, oz. povezav, in ki potrebujejo natanko t barv (Izreka 6.2.4 in 6.1.4). Še posebej to pokaže obstoj grafov, ki potrebujejo 12 barv.

Drugi del

V drugem delu disertacije se osredotočimo na raziskave razredov grafov zaprtih za inducirane podgrafe, v katerih je odsotnost velike klike tako potreben kot tudi zadosten pogoj za omejeno drevesno širino. Ta lastnost je imenovana (tw, ω)-omejenost. Za razred grafov $\mathcal{G}$ pravimo, da je (tw, ω)-*omejen*, če obtaja takšna funkcija f, da za vsak graf G iz $\mathcal{G}$ in za vsak induciran podgraf G' grafa G velja, da je drevesna širina grafa G' največ $f(\omega(G'))$. Taki funkciji f pravimo (tw, ω)-*omejitvena funkcija.* Za vsak *tetiven graf*, to je graf brez induciranih ciklov dolžine vsaj 4, velja, da je $\mathrm{tw}(G) = \omega(G) - 1$. Razred tetivnih grafov je eden izmed najosnovnejših razredov grafov, ki je (tw, ω)-omejen. V splošnem velja, da je za vsak graf $\mathrm{tw}(G) \geq \omega(G) - 1$. Vprašamo se lahko torej, za katere razrede grafov je drevesno širino mogoče omejiti z neko funkcijo v odvisnosti od kličnega števila. Ob tetivnih grafih so (tw, ω)-omejeni tudi razredi grafov z omejenim neodvisnostnim številom, kar je posledica Ramseyevega izreka (Izrek 9.0.1). Da bi bolje razumeli lastnost (tw, ω)-omejenosti, se osredotočimo na obravnavo razredov grafov in na šest različnih znanih relacij vsebovanosti v grafih, ki so: podgraf, topološki minor, minor, induciran podgraf, induciran topološki minor in induciran minor. Za vsako od omenjenih relacij v celoti karakteriziramo grafe H, za katere je razred grafov, ki ne vsebujejo grafa H glede na izbrano relacijo, (tw, ω)-omejen (Izreki 9.1.4, 9.1.6, 9.1.7, 9.2.1, 9.2.5 in 9.3.10, katerih rezultati so povzeti v Tabeli 9.1). V primeru prepovedanega podgrafa, topološkega minorja in minorja velja celo več, in sicer, da imajo omejeno drevesno širino. V nekaterih primerih (glej Izreke 9.1.4, 9.1.6, 9.1.7, 9.2.1 in 9.2.5) pokažemo tudi obstoj linearne (tw, ω)-omejitvene funkcije. Eno izmed možnih vprašanj je torej, kakšni so potrebni in/ali zadostni pogoji, da je razred grafov (tw, ω)-omejen z linearno funkcijo. Ob tem naj omenimo, da je odprto tudi vprašanje, ali je vsak (tw, ω)-omejen razred grafov (tw, ω)-omejen s polinomsko funkcijo. Med možnimi nadaljnjimi raziskavami je tudi vprašanje, kateri razredi grafov so (tw, ω)-omejeni, ko prepovemo končno množico grafov glede na eno izmed zgoraj omenjenih šestih relacij vsebovanosti grafov. V primeru prepovedanih induciranih podgrafov nam pri tem pomaga rezultat Lozina in Razgona [160] (glej Izrek 12.1.3). Kot posledico tega rezultata dobimo celotno karakterizacijo (tw, ω)-omejenih razredov grafov, opisanih s končno množico prepovedanih induciranih podgrafov (Posledica 12.1.4). V primeru prepovedanih induciranih topoloških minorjev in induciranih minorjev vprašanje ostaja odprto. Omenimo tudi, da je koncept (tw, ω)-omejenosti del veliko bolj splošne definicije. Za dve invarianti grafov ρ in σ pravimo, da je razred grafov $\mathcal{G}$ *(ρ, σ)-omejen*, če obstaja takšna *(ρ, σ)-omejitvena funkcija* $f : \mathbb{N} \to \mathbb{N}$, da za vsak graf $G \in \mathcal{G}$ in vsak induciran podgraf H grafa G velja, da je $\rho(H) \leq f(\sigma(H))$. Med najbolj znanimi primeri je (χ, ω)-omejenost (glej, npr. [31, 165, 211]), imenovana tudi χ-omejenost, vendar obstajajo tudi drugi primeri (glej, npr. [142, 119, 131]).

Nadaljujemo z obravnavo drevesnega neodvisnostnega števila, ki je definirano na naslednji način. *Neodvisnostno število* drevesne dekompozicije $\mathcal{T}$ grafa G je največje neodvisnostno število med vsemi podgrafi grafa G, ki so inducirani z neko vrečo drevesne dekompozicije $\mathcal{T}$. *Drevesno neodvisnostno število* grafa G je nato definirano kot najmanjše neodvisnostno število med vsemi drevesnimi dekompozicijami grafa G. Ta invarianta, ki jo označimo s $\mathsf{tree}\text{-}\alpha(G)$, je bila definirana s strani Yolova [209] (pod imenom α-drevesna širina) in neodvisno s strani Dallarda idr. [77, 78]. Za razliko od drevesne širine nas ne zanimajo drevesne dekompozicije z najmanjšo širino, ampak drevesne dekompozicije z najmanjšim neodvisnostnim številom. Glavna motivacija za raziskovanje drevesnega neodvisnostnega števila je sledeča. Denimo, da imamo razred grafov, kjer ima vsak graf drevesno dekompozicijo s širino omejeno z neko konstantno vrednostjo. V tem primeru velja, da za izbran problem vsaka vreča drevesne dekompozicije seka optimalno rešitev problema v omejenem številu načinov, ki jih lahko učinkovito (v polinomskem času) naštejemo. Posledično lahko uporabimo princip dinamičnega programiranja, da najdemo optimalno rešitev izbranega problema v polinomskem času. Kot posplošitev te ideje lahko zahtevo po konstantni drevesni širini nadomestimo z zahtevo, da vsaka vreča seka optimalno rešitev le na polinomsko mnogo načinov, ki jih lahko v polinomskem času naštejemo. Ta ideja je bila neodvisno predstavljena s strani Marie Chudnovsky [54, 55, 56, 57]. Kot poseben primer lahko opazimo, da omejeno drevesno neodvisnostno število implicira dejstvo, da je število vozlišč, ki jih lahko optimalna rešitev problema najtežje neodvisne množice vsebuje v posamezni vreči, omejeno. Med rezultati drugega dela te disertacije je med drugim tudi nova karakterizacija tetivnih grafov s pomočjo drevesnega neodvisnostnega števila. Velja namreč, da je drevesno neodvisnostno število grafa G enako 1 natanko tedaj, ko je G tetiven graf (Izrek 10.1.1). Omenimo še, da je problem določitve drevesnega neodvisnostnega števila danega grafa NP-težak (Izrek 10.1.7), kar sledi iz dejstva, da za vsako drevesno dekompozicijo velja, da obstaja takšno vozlišče grafa, da je njegova zaprta soseščina v celoti vsebovana v vsaj eni vreči dekompozicije. Med pomembnimi lastnostmi drevesnega neodvisnostnega števila je tudi ta, da je vsak razred grafov z omejenim drevesnim neodvisnostnim število pravzaprav tudi (tw, ω)-omejen (Lema 10.1.11). Ob tem se pojavi vprašanje kako je z obratno implikacijo, to je ali ima vsak (tw, ω)-omejen razred grafov tudi omejeno drevesno neodvisnostno število. Naša domneva je, da je ta implikacija resnična. Resničnost domneve potrdimo za razrede grafov opisane z enim prepovedanim grafom glede na eno od zgoraj omenjenih relacij vsebovanosti grafov. Bolj natančno, za vsak razred grafov opisan v Tabeli 9.1 pokažemo, da je (tw, ω)-omejen natanko tedaj, ko ima omejeno drevesno neodvisnostno število (Izreki 10.2.1, 10.2.3, 10.2.4, 10.3.5 in Posledici 10.6.7 in 10.7.6).

Na koncu se osredotočimo na vprašanje o tem, kakšne algoritmične implikacije ima (tw, ω)-omejenost na različne probleme, kjer se usmerimo predvsem na problem najtežje neodvisne množice. Izreki 11.0.1, 11.0.2 in 11.0.3 razrešijo vprašanje o polinomski rešljivosti problema v vseh (tw, ω)-omejenih razredih grafov karakteriziranih z enim grafov H glede na eno izmed šestih relacij vsebovanosti grafov z izjemo relacije induciranega minorja. Le-ta je osrednji rezultat tega dela disertacije. Bolj natančno, za vsak $k \geq 1$ pokažemo, da je problem najtežje neodvisne množice rešljiv v polinomskem času v (tw, ω)-omejenih razredih grafov karakteriziranih z enim prepovedanim induciranim minorjem. To je posledica dejstva, da imajo ti razredi grafov omejeno drevesno neodvisnostno število in rezultata s strani Dallarda idr. [75], ki pravi, da obstaja algoritem, ki za dan graf G in pozitivno celo število k v polinomskem času izračuna drevesno dekompozicijo z neodvisnostnim številom največ $8k$ ali pa zaključi, da je drevesno neodvisnostno število grafa G večje od k, s čimer avtorji izboljšajo rezultat, ki ga dobimo z uporabo rezultata Yolova [209]. Dodatno pokažemo, da lahko v polinomskem času rešimo tudi posplošitev problema najtežje neodvisne množice, in sicer problem najtežjega neodvisnega pakiranja. *Problem najtežjega neodvisnega pakiranja* prejme kot vhodni podatek graf G, končno družino $\mathcal{H} = \{H_j\}_{j \in J}$ povezanih nepraznih podgrafov grafa G in utežno funkcijo $w : J \to \mathbb{Q}^+$ za podgrafe v $\mathcal{H}$, kot izhodni podatek pa želimo najtežje neodvisno $\mathcal{H}$-pakiranje v grafu G. Naj bo G graf in naj bo $\mathcal{H} = \{H_j\}_{j \in J}$ družina povezanih podgrafov grafa G. Z $G(\mathcal{H})$ označimo graf z množico vozlišč J, kjer sta dve vozlišči $i, j \in J$ sosednji natanko tedaj, ko imata grafa H_i in H_j skupno vozlišče ali pa obstaja v grafu G povezava med njima. Ta konstrukcija je bila predstavljena s strani Cameron in Hella [52]. Med drugim pokažemo, da za poljuben graf G in za poljubno množico nepraznih povezanih podgrafov grafa G velja, da je neodvisnostno število grafa G vedno vsaj tako veliko, kot je neodvisnostno število grafa $G(\mathcal{H})$ (Izrek 11.2.2). To pomeni, da za vsak razred grafov $\mathcal{G}$ z omejenim neodvisnostnim številom in za vsako množico $\mathcal{F}$ povezanih nepraznih grafov velja, da ima razred $\{G(\mathcal{H}) : G \in \mathcal{G}, \mathcal{H}(G, \mathcal{F})\}$, kjer je $\mathcal{H}(G, \mathcal{F})$ množica vseh podgrafov grafa G izomorfnih nekemu elementu množice iz $\mathcal{F}$, omejeno drevesno neodvisnostno število. To vodi do rezultata, da je problem najtežjega neodvisnega pakiranja mogoče rešiti v polinomskem času v vseh obravnavanih (tw, ω)-omejenih razredih grafov (glej Izrek 11.2.6).

Declaration

I declare that this doctoral dissertation does not contain any materials previously published or written by another person except where due reference is made in the text.

Kenny Štorgel