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(Magistrsko delo)

Classes of graphs with tree-independence number at most two

(Razredi grafov z drevesnim neodvisnostnim številom največ 2)

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Izvleček:

V teoriji grafov imamo pogosto opravka s problemi, ki so algoritmično zelo zahtevni. Natančneje, tudi najboljši znani algoritmi ne morejo rešiti takšnih problemov v polinomskem času. V takšnih primerih se lahko odločimo, da vhodne podatke omejimo na način, ki nam omogoča, da problem rešimo v polinomskem času na omejeni vhodni množici. V zadnjem času lahko pogosto vidimo, da se kot način za učinkovito omejevanje vhoda uporabljajo različne mere strukturne zapletenosti grafa, imenovane širinski parametri grafov. V magistrskem delu se ukvarjamo s širinskim parametrom, imenovanim drevesno neodvisnostno število, in obravnavamo razrede grafov, za katere je vrednost tega parametra navzgor omejena z 2. Spoznamo tudi tri grafe s drevesnim neodvisnostnim številom vsaj tri, katerih ima vsak pravi induciran minor drevesno neodvisnostno število navzgor omejeno z 2. V celoti opišemo razred grafov, za katere sta tako drevesna širina kot drevesno neodvisnostno število omejena z 2. Natančneje, pokažemo, da je drevesno neodvisnostno število grafa G z drevesno širino največ 2 omejeno z 2 natanko takrat, ko graf G ne vsebuje induciranega minorja izomorfnega grafu, ki ga poimenujemo C_6^*, sicer pa je enako 3.

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Abstract:

In graph theory, we often need to solve the problems that are algorithmically very difficult. In particular, even the best known algorithms cannot solve such problems in polynomial time. In such cases, we may decide to restrict the input in a way that allows us to solve the problem on this input set in polynomial time. In recent times, we can often see various measures of the structural complexity of a graph, called graph width parameters being used as a way to restrict the input efficiently. In this thesis we consider one such width parameter called tree-independence number. In particular, we identify several classes of graphs for which this parameter is bounded by two. We also discover three graphs with tree-independence number at least three, whose every proper induced minor has tree-independence number bounded by two. We completely characterize the class of graphs with treewidth and tree-independence number both bounded by two. In particular, we show that within graphs with treewidth at most two, the tree-independence number is at most 2 if one graph in this class, which we name C_6^*, is excluded as an induced minor, and otherwise it equals 3.

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List of Abbreviations

i.e.	that is
e.g.	for example
w.l.o.g.	without loss of generality

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1 INTRODUCTION

In graph theory, one often needs to tackle problems that are algorithmically very difficult. In particular, even the best known algorithms cannot solve such problems in polynomial time. When faced with such a problem, one often needs to find a way to compromise either generality, or the accuracy of the solution, in order to successfully obtain a polynomial time algorithm. Based on what is more important for a particular application, one can either find some polynomial-time *ρ-approximation algorithm* (for some positive $\rho > 1$) that produces a solution that is not necessarily correct, however the output is at most by factor ρ away from the actual solution of the problem, or if an exact solution is required, one can sacrifice the generality of the solution, by restricting the input in a way that allows us to solve the problem on this input set in polynomial time.

If one decides to take the second approach, there are many ways to restrict the input that may be applied. Obviously, the main goal is to have a restriction on the input that is as weak as possible, while still allowing for polynomial-time solutions.

In recent times, we can often see various measures of the structural complexity of a graph, called *graph width parameters* being used as a way to restrict the input efficiently.

When a width parameter is bounded by a fixed constant in some family of graphs $\mathcal{F}$, this often leads to a development of an efficient dynamic programming algorithm on a graph from $\mathcal{F}$ that exploits the properties that the graphs with bounded width parameter possess, usually by applying a divide-and-conquer approach on some particular decomposition. An example of such a decomposition is *tree decomposition*, where the graph is decomposed into a tree-like structure, where each node of the tree corresponds to a subset of vertex set of the original graph, satisfying certain conditions.

In this thesis, we consider the classes of graphs that have the width parameter called *tree-independence number* bounded by 2. For the relevant definitions, see the next chapter.

The *maximum weight independent set* problem asks, given a graph G and a non-negative weight function on vertices of G, what is the maximum weight of an independent set in G. This problem is known to be NP-hard in general (see, e.g., [18]).

However, as Dallard, Milanič and Štorgel observed in [10], in the families of graphs with the tree-independence number bounded by a fixed constant k, if a tree decomposition that corresponds to this independence number is known, then the maximum weight independent set problem is solvable in polynomial time. In particular, they provided an algorithm that uses such a tree decomposition to obtain the solution to the maximum weight independent set problem in time $\mathcal{O}(n^{k+1})$, where n is the number of vertices of the input graph.

We notice that the time complexity of their algorithm depends on the independence number of a tree decomposition, so if we can efficiently decide if a graph has small tree-independence number, while producing a tree decomposition that corresponds to this tree-independence number, we can also efficiently solve the maximum weight independent set problem for graphs this particular family.

It is known that we can recognize graphs that have tree-independence number at most one in polynomial time (see, e.g., [10]). Also, for each fixed $k \geq 4$, recognition of graphs with tree-independence number at most k is known to be NP-hard (see [9]).

Thus the next natural step is to consider the graphs with tree-independence number at most two and ask if we can recognize them efficiently in general. A bit easier question, that we will explore in this thesis is, what are some particular classes of graphs in which we can efficiently recognize the graphs that have tree-independence number at most 2.

In this thesis, we explore this question and find some classes of graphs where this is true. We start off by giving some definitions and graph concepts that are required to develop the results of the thesis. We proceed by giving a few examples of graph classes that are completely contained in the class of graphs with tree-independence number bounded by 2, by using some previously known bounds and developing algorithms that yield a tree decomposition of independence number at most 2, given any graph from these classes. We then develop polynomial-time algorithms for deciding if H-free graphs have tree-independence number at most 2, for various choice of small graphs H. Then in the fifth chapter, we find three minimal examples of graphs with tree-independence number greater than 2. In particular, for each of those graphs, every proper induced minor has tree-independence number at most 2, while the graph has tree-independence number strictly larger than 2. We proceed by developing some necessary and sufficient conditions for having tree-independence number bounded by 2, within the class of graphs whose treewidth is bounded by 2. We show that within this class of graphs, it is sufficient to forbid a single graph, as an induced minor in order to have tree-independence number bounded by 2. Then, in the seventh chapter, we look into some of the sufficient conditions for graphs to have tree-independence number bounded by 2 in class of chordal bipartite graphs. We conclude the thesis with a brief summary of the obtained results and by stating some of the relevant open problems.

2 PRELIMINARIES

Before proceeding to the main part, we first overview some basic definitions and notations used in the thesis.

2.1 GRAPH THEORY PRELIMINARIES

Unless stated otherwise, we are only considering the finite, simple, connected, undirected graphs and given a graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$ respectively. Sometimes, if it is clear from the context which graph we are referring to, we only use V and E to denote the vertex set, and the edge set respectively.

Two vertices u and v are *adjacent* in a graph G if there is an edge between them. We denote this by $u \sim v$. If no such edge exists, then they are said to be *nonadjacent*. The *order* of a graph G represents the cardinality of $V(G)$. A graph G is *trivial* if its order is 1. Otherwise, it is *nontrivial*.

The *neighbourhood* of a vertex v, denoted $N(v)$, is the set of all vertices that are adjacent to v. The *degree* of a vertex v, denoted $\deg(v)$, is equal to the cardinality of its neighbourhood. The *closed neighbourhood* of a vertex v, denoted $N[v]$, is the neighbourhood of v together with v itself.

Let $\{u, v\}$ be an edge in a graph G. The *subdivision* of $\{u, v\}$ is the operation of deleting the edge $\{u, v\}$, adding a new vertex w, and adding two edges, one between u and w and another between v and w.

Given a positive integer k, we say a graph is *k-regular* if every vertex of the graph has degree equal to k. If a graph is 3-regular, we say that it is *cubic*.

A *path* on $n \geq 3$ vertices is a graph P whose vertices can be labeled as $v_1, \ldots, v_n$ so that $E(P) = \{\{v_i, v_{i+1}\} \mid i \in \{1, \ldots, n-1\}\}$. We say that $n-1$ is the *length* of such a path and a path is called *trivial* if it has length 0. We denote the path where $v_i = i$ for all i, as in the definition above, as P_n.

A *cycle* on $n \geq 3$ vertices is a graph C whose vertices can be labeled as $v_1, \ldots, v_n$ so that $E(C) = \{\{v_i, v_{i+1}\} \mid i \in \{1, \ldots, n-1\}\} \cup \{v_1, v_n\}$. We denote the cycle where $v_i = i$, for all i, as in the definition above, as C_n.

A *complete graph* is a graph in which every two distinct vertices are adjacent. We denote the complete graph on n vertices as K_n.

A graph is *bipartite* if its vertex set can be partitioned into two sets A and B such that every edge in G has one endpoint in A and the other in B.

A graph is *complete bipartite* if there exists a partition of vertex set into sets A and B such that every vertex in A is adjacent to every vertex in B and there are no other edges. If $|A| = m$ and $|B| = n$, we denote such a graph as $K_{m,n}$. A *claw* is the complete bipartite graph $K_{1,3}$. Adding an edge between two nonadjacent vertices in a claw results in a graph called the *paw*. We refer to the complement of the paw graph as *co-paw*.

A graph G is *complete multipartite* if its vertex set can be partitioned into k independent sets $X_1, \ldots, X_k$ such that whenever $u \in X_i$ and $v \in X_j$, for some $i \neq j$, then u and v are adjacent.

Two graphs G and H are *isomorphic*, if there exists a bijection

$$\phi : V(G) \to V(H)$$

such that $u \sim v$ if and only if $\phi(u) \sim \phi(v)$. Such function is called an *isomorphism* between G and H. If G and H are isomorphic, we write $G \cong H$.

If G is a graph, we say that $S \subseteq V(G)$ is a *cutset* if $G - S$ is disconnected. For a positive integer k, a graph G is *k-connected* if G has more than k vertices and no cutset with fewer than k vertices. If S is a cutset such that no proper subset of S' is a cutset, we say that S is a *minimal cutset*. A *clique cutset* is a cutset that is a clique. If $S = \{v\}$ is a cutset, we say that the vertex v is a *cut-vertex*. A *cut partition* of a graph G is an ordered partition (A, B, C) of the vertex set of G such that A and B are nonempty and no edge in G connects a vertex in A with a vertex in B.

Let G and H be graphs. Then H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Example 1. A path on n vertices is a subgraph of a cycle on n vertices.

Let G be a graph and let $S \subseteq V(G)$. An *induced subgraph* $G[S]$ is the graph whose vertex set is S and for any pair $u, v \in S$, it holds that u and v are adjacent in $G[S]$ if and only if u and v are adjacent in G. Notice that if H is an induced subgraph of G, it is also a subgraph of G. The converse, however does not hold, as can be seen in the following example.

Example 2. P_{n-1} is an induced subgraph of C_n. On the other hand, P_n is a subgraph of C_n, but it is not an induced subgraph.

Given a graph H, G is *H-free* if it contains no induced subgraph isomorphic to H. For a family of graph $\mathcal{F}$, G is $\mathcal{F}$-free if it is H-free for every H in $\mathcal{F}$.

Let G be a graph. We say that H is an *induced minor* of G if H can be obtained from G by deleting vertices and contracting edges. Notice that an induced minor of G

is not necessarily a subgraph of G. For example, K_3 is an induced minor of C_4 (can be obtained from C_4 by contracting any edge), but it is clearly not a subgraph.

Example 3. Let G be the Petersen graph. Then $K_{3,3}$ is an induced minor of G.

Proof. Consider the Petersen graph, labeled as in Figure 1. First contract the edge $\{5, 8\}$ to vertex 5, to obtain the graph as in Figure 2a. Then contract the edge $\{0, 1\}$, to vertex 0, as in Figure 2b. Then contract the edge $\{7, 9\}$, to vertex 7, to obtain the graph as in Figure 2c. And finally delete vertex 6 to obtain $K_{3,3}$ as can be seen in Figure 2d, where the two parts of a bipartition are labeled by red and blue color respectively. □

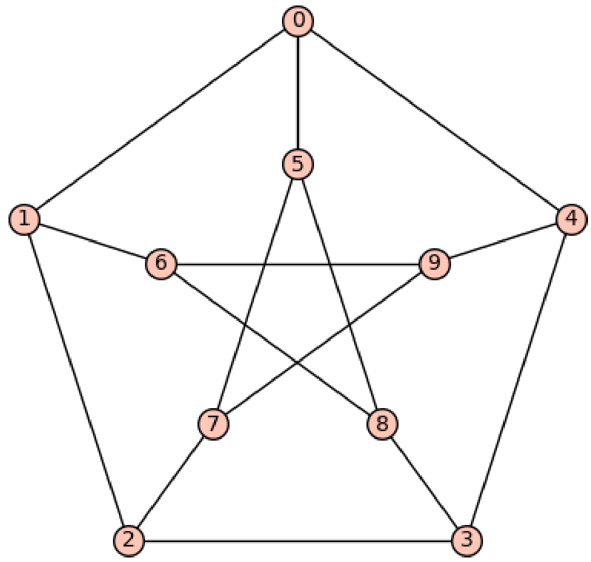
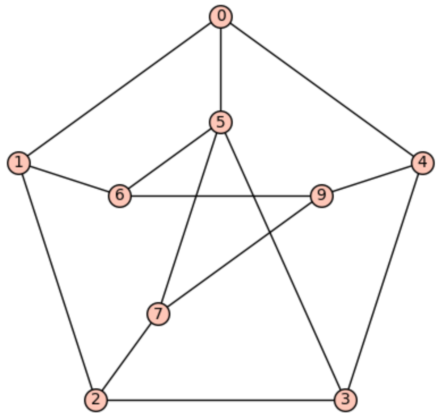


Figure 1: The Petersen graph with vertices labeled.

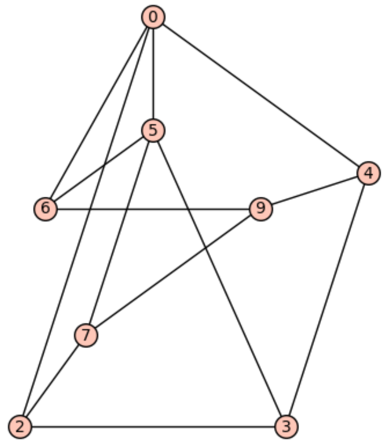
Given a graph H, a graph G is said to be *H-induced-minor-free* if it admits no induced minor isomorphic to H. For a family of graph $\mathcal{F}$, a graph G is $\mathcal{F}$-induced-minor-free if it is H-induced-minor-free for every H in $\mathcal{F}$.

Let G be a graph. Then we say H is a *minor* of G if H can be obtained from G by deleting vertices and edges and contracting edges. Notice that if H is an induced minor of G, it is also a minor of G, but not the other way around. For example, P_3 is a minor of K_3, but not an induced minor. Given a graph H, a graph G is said to be *H-minor-free* if it admits no minor isomorphic to H. For a family of graph $\mathcal{F}$, a graph G is $\mathcal{F}$-minor-free if it is H-minor-free for every H in $\mathcal{F}$.

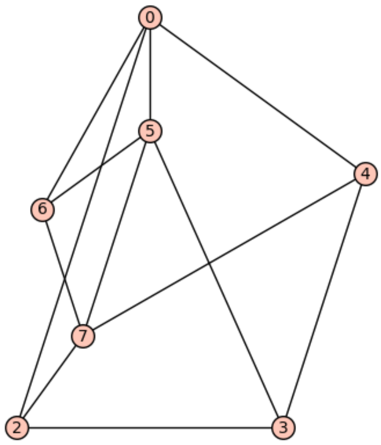
Let G_1 and G_2 be graphs. The *disjoint union* of G_1 and G_2, denoted by $G_1 + G_2$ (sometimes we will also denote it $G_1 \cup G_2$), is the graph obtained from G_1 and G_2 by taking the disjoint unions of their respective vertex sets and edge sets. If G is a graph and k a positive integer, we denote by kG the graph obtained by taking k disjoint copies of G.



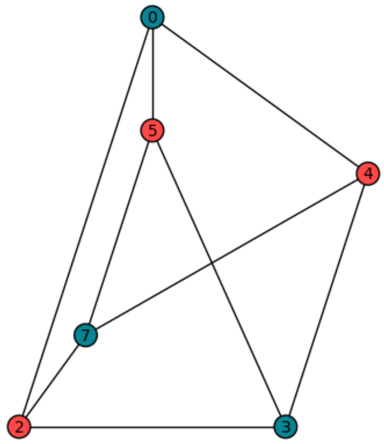
(a) The Petersen graph with edge $\{5, 8\}$ contracted.



(b) The graph from Figure 2a with edge $\{0, 1\}$ contracted.



(c) The graph from Figure 2b with edge $\{7, 9\}$ contracted.



(d) The graph from Figure 2c with vertex 6 deleted, isomorphic to $K_{3,3}$ with parts labeled red and blue.

Figure 2: A sequence of edge contractions and vertex deletions to obtain $K_{3,3}$ from the Petersen graph.

Let G_1 and G_2 be graphs. The *join* of G_1 and G_2, denoted by $G_1 * G_2$, is the graph obtained from the disjoint union of G_1 and G_2 by adding all the edges $\{u, v\}$ where $u \in V(G_1)$ and $v \in V(G_2)$.

Example 4. For every positive integer n, the complete bipartite graph $K_{n,n}$ is isomorphic to the join $nK_1 * nK_1$.

A *Hamiltonian cycle* in a graph G is a subgraph of G that is isomorphic to a cycle of order $|V(G)|$. If G admits a Hamiltonian cycle, then G is *Hamiltonian graph*.

2.2 SOME BASIC GRAPH INVARIANTS

Let $G = (V, E)$ be an undirected simple graph.

We say a set $S \subseteq V$ is an *independent set* if for all $u, v \in S$, u and v are nonadjacent. Similarly, a set $S \subseteq V$ is a *clique* if any two distinct vertices in S are adjacent.

The *independence number* of G is the size of a largest independent set in G and is denoted $\alpha(G)$. The *clique number* of G is the size of a largest clique in G and is denoted $\omega(G)$. A *graph coloring* is a function $c : V \to \mathbb{N}$ such that for any two vertices u, v, whenever $uv \in E$, it holds that $c(u) \neq c(v)$.[1] We say that a graph can be colored with k colors (or is k*-colorable*) if it admits a coloring c with $|\{c(v) \mid v \in V\}| \leq k$. If $c(v) = i$, we say that v is colored with color i.

The *chromatic number* of G is the smallest integer k such that G is k-colorable. It is denoted $\chi(G)$.

Notice that a graph $G = (V, E)$ is bipartite if and only if its chromatic number is at most 2. In that case, the sets A, B, where A is the set of all vertices colored with the first color and $B = V \setminus A$, are the parts of a bipartition.

A *feedback vertex set* in a graph G is a set of vertices intersecting all cycles. The *feedback vertex number* of G is the smallest cardinality of a feedback vertex set in G.

2.3 TREE DECOMPOSITION, TREEWIDTH

In this section, we will introduce a very important and widely used width parameter, called *treewidth* (see, e.g., [1–3]). This parameter was first introduced by Bertelè and Brioschi in 1972 in [3]. It was then rediscovered independently by Halin in 1976 in [17], Robertson and Seymour in 1984 in [20], and Arnborg and Proskurowski in [1].

Given a graph G, we say that a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is called a *tree decomposition* of G if T is a tree, for every node $t \in V(T)$, X_t is a subset of $V(G)$ and the following properties hold:[2]

[1] For simplicity, we denote an edge $\{u, v\}$ as uv.

[2] To make a distinction between the vertices of G and T, we will refer to vertices of T as *nodes*.

- Vertex Coverage:

$$\bigcup_{t \in V(T)} X_t = V(G)$$

- Edge Coverage:

$$\forall uv \in E(G)\ \exists t \in V(T) : u, v \in X_t$$

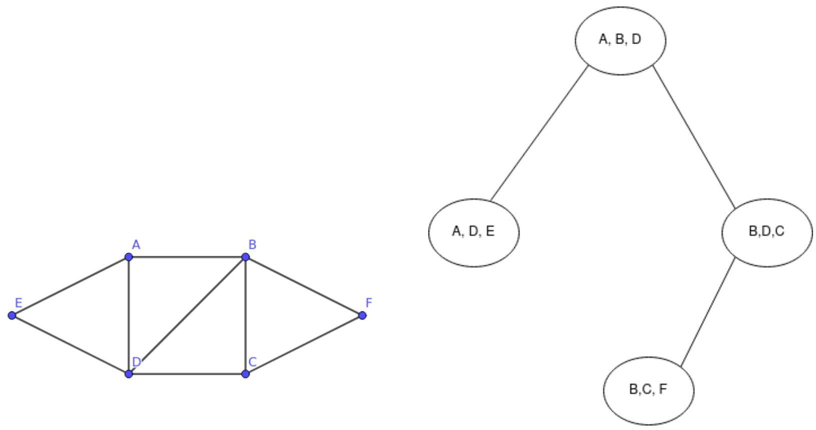
- Consistency:
  For all $u \in V(G)$, the subgraph of T induced by the set $\{t \in V(T) \mid u \in X_t\}$ is connected (in particular forms a subtree of T)

Colloquially, we call any node $t \in V(T)$ a *bag* and we say a vertex $v \in V(G)$ is contained in bag t if $v \in X_t$. In this thesis we will mostly use this terminology.

We denote by T_v the subtree of T induced by all the bags that contain v.

The *width* of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is equal to $\max_{t \in V(T)} |X_t| - 1$. The *treewidth* of a graph G, denoted $\text{tw}(G)$, is equal to the minimum width of a tree decomposition of G.

We can see an example of a graph G in Figure 3a and one possible tree decomposition of G in Figure 3b. From this example we can conclude that tree width of G is at most 2. One can prove that it is indeed equal to two, but more on that a bit later.



(a) (b)

Figure 3: A graph G and a tree decomposition of it.

2.3.1 Treewidth and chordal graphs

Let C be a cycle in a graph G. A *chord* of C is an edge in G that connects two nonadjacent vertices in C.

A graph G is *chordal* if every cycle in G of length at least 4 has a chord. An example of chordal graph can be seen in Figure 3a.

A graph is *perfect* if each of its induced subgraphs has the property that its chromatic number is equal to its clique number. The strong perfect graph theorem says that G is perfect if and only if it does not contain an odd cycle or an odd anticycle (complement of a cycle) of length at least 5 as an induced subgraph (see [7]). From this it follows easily that chordal graphs are perfect.

Now we will see an equivalent definition of treewidth that considers the size of the largest clique in the chordal extension of G (see, e.g., [4]). Namely:

$$\text{tw}(G) = \min\{\omega(H) - 1 \mid G \subseteq H \text{ and } H \text{ is chordal}\}$$

Now from this definition we obtain a few direct consequences.

- If $\text{tw}(G) = k$ then $\chi(G) \leq k + 1$.
- If G is chordal, then $\text{tw}(G) = \omega(G) - 1$.
- If G is a forest, then $\text{tw}(G) = 1$.
- For all positive integers n, $\text{tw}(K_n) = n - 1$.

From the second consequence, we can directly see that the graph G from Figure 3a has treewidth equal to 2.

2.3.2 (tw, ω)-bounded graph classes

An (integer) graph invariant is a mapping from the class of all graphs to the set of nonnegative integers $\mathbb{N}$ that does not distinguish between isomorphic graphs. Following [11], given two graph invariants ρ and σ and a graph class $\mathcal{G}$, we say that $\mathcal{G}$ is *(ρ, σ)-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ (called (ρ, σ)-binding function) such that for every graph $G \in \mathcal{G}$ and every induced subgraph G' of G it holds that $\rho(G') \leq f(\sigma(G'))$.

The most known pair of invariants (ρ, σ), for which the concept of (ρ, σ)-bounded graph classes was studied in the literature corresponds to $(\rho, \sigma) = (\chi, \omega)$. The corresponding graph classes are called *χ-bounded* (see [16]).

We say that a graph class $\mathcal{G}$ is (tw, ω)-*bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ and every induced subgraph G' of G, we have $\text{tw}(G') \leq f(\omega(G'))$. Such a function f is called a *(tw, ω)-binding function* for the class $\mathcal{G}$.

Note that every graph G satisfies $\omega(G) \leq \chi(G) \leq \text{tw}(G) + 1$. Both inequalities hold with equality for all induced subgraphs of G if and only if G is chordal. Thus (tw, ω)-boundedness generalizes chordality in graphs and every (tw, ω)-bounded graph class is also χ-bounded.

2.4 TREE-INDEPENDENCE NUMBER

In this section we give the definition of the most important width parameter for this thesis, namely the *tree-independence number*. This parameter was first introduced by Dallard, Milanič and Štorgel in [10]. Below we state their definition of the parameter.

The *independence number* of a tree decomposition $\mathcal{T}$ of a graph G is defined as the maximum independence number over all subgraphs of G induced by some bag of $\mathcal{T}$. The *tree-independence number* of a graph G is then defined as the minimum independence number over all tree decompositions of G.

A graph class $\mathcal{G}$ has bounded tree-independence number if there exists an integer k such that every graph in $\mathcal{G}$ has tree-independence number at most k. Graph classes of bounded tree-independence number include various families of graph classes: classes of bounded treewidth, classes of bounded independence number, chordal graphs (which are precisely graphs with tree-independence number at most one), etc. All of these bounds and results will be discussed shortly.

Theorem 2.1 (Ramsey's theorem)**.** *For every two positive integers k and ℓ, there exists a least positive integer $R(k,\ell)$ such that every graph with at least $R(k,\ell)$ vertices contains either a clique of size k or an independent set of size ℓ.*

It follows from Ramsey's theorem that every graph class with bounded tree-independence number is polynomially (tw,ω)-bounded. In this sense, boundedness of the tree-independence number is a refinement of (tw,ω)-boundedness.

However, an open question by Dallard, Milanič and Štorgel (see [10]) is whether every (tw,ω)-bounded graph class is polynomially (tw,ω)-bounded, or whether every polynomially (tw,ω)-bounded graph class has bounded tree-independence number.

It is known that the tree-independence number of a graph cannot be increased under the induced minor relation, as stated in the lemma below.

Lemma 2.2 (see [10])**.** *Let G be a graph. If H is an induced minor of G, then* $\text{tree-}\alpha(H) \leq \text{tree-}\alpha(G)$.

Motivated by the previous lemma, we say that G is a *minimal graph of tree-independence number greater than k* if $\text{tree-}\alpha(G) > k$ and $\text{tree-}\alpha(H) \leq k$ for every proper induced minor H of G.

We next state a lemma that we will often use in this thesis that was proved by Dallard, Milanič and Štorgel in [10].

Lemma 2.3. *Let C be a clique cutset in a graph G and let (A,B,C) be a cut partition of G. Let $G_A = G[A\cup C]$ and $G_B = G[B\cup C]$, and let $\mathcal{T}_A, \mathcal{T}_B$ be tree decompositions of G_A and G_B respectively. Then we can compute in polynomial time a tree decomposition $\mathcal{T}$ of G, such that $\alpha(\mathcal{T}) = \max\{\alpha(\mathcal{T}_A), \alpha(\mathcal{T}_B)\}$.*

The following lemma was also proved in [10].

Lemma 2.4. *Let G be a graph and let G' be the join of two copies of G. Then* tree-$\alpha(G') = \alpha(G)$.

Corollary 2.5. *For every positive integer n, we have* tree-$\alpha(K_{n,n}) = n$.

Proof. $K_{n,n}$ is obtained as the join $nK_1 * nK_1$. □

3 EXAMPLES OF GRAPHS WITH TREE-INDEPENDENCE NUMBER AT MOST TWO

In this chapter, we consider some simple examples of graph classes that have tree-independence number at most two.

The following basic observation (see [10, Observation 3.5]) relates the tree-independence number with the independence number.

Lemma 3.1. *Let G be a graph. Then* tree-$\alpha(G) \le \alpha(G)$.

Proof. Consider the following tree decomposition

$$\mathcal{T}_0 = ((\{t\}, \emptyset), \{X_t\})$$

where $X_t = V(G)$. Then clearly X_t induces the entire graph G and we thus obtain that

$$\text{tree-}\alpha(G) = \min_{\mathcal{T}} \alpha(\mathcal{T}) \le \alpha(\mathcal{T}_0) = \alpha(G[X_t]) = \alpha(G),$$

where the minimum iterates over all tree decompositions $\mathcal{T}$ of G. □

Corollary 3.2. *Every graph with independence number at most 2 also has tree-independence number at most 2.*

Lemma 3.3. *For any integer $n \ge 2$, the complete bipartite graph $K_{2,n}$ has tree-independence number equal to 2.*

Proof. Fix a bipartition $\{R, B\}$ of $K_{2,n}$ with $|R| = 2$ and $|B| = n$, say $R = \{r_1, r_2\}$ and $B = \{b_i \mid i = 1, \ldots, n\}$. We refer to the vertices in R and B as *red* and *blue* respectively. Consider the tree decomposition

$$\mathcal{T} = \left(P_{\lceil \frac{n}{2} \rceil}, \{X_t\}_{t=1,\ldots,\lceil \frac{n}{2} \rceil}\right)$$

of $K_{2,n}$, where $P_{\lceil \frac{n}{2} \rceil}$ denotes the path graph with vertices $1, \ldots, \lceil \frac{n}{2} \rceil$ in order along the path and the bags are defined as follows:

$$X_t = \{r_1, r_2, b_{2t-1}, b_{2t}\}$$

for all $t = 1, \ldots, \lceil \frac{n}{2} \rceil$, with the exception that the last bag equals $\{r_1, r_2, b_n\}$ if n is odd. We can see an example for $n = 4$ in Fig. 4. It is easy to see that this construction

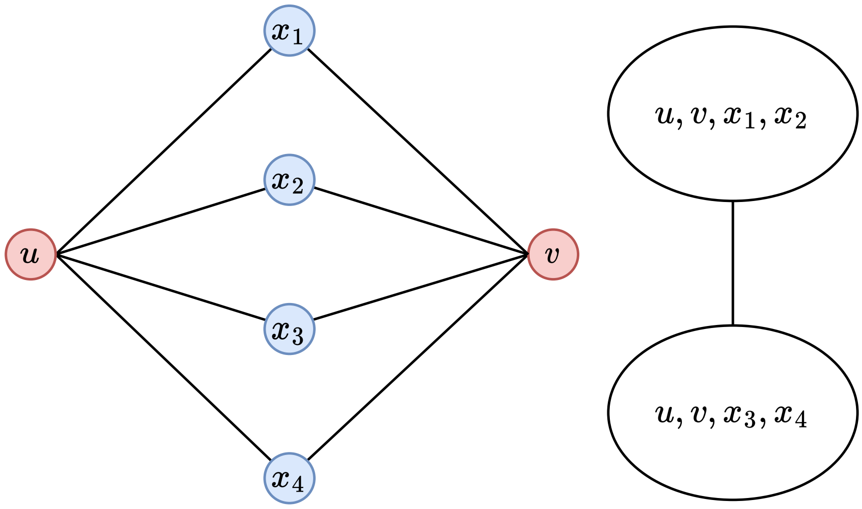


Figure 4: The graph $K_{2,4}$ and a tree decomposition of it.

is indeed a tree decomposition, as each blue vertex is contained in exactly one bag, each red vertices is contained in every bag, and these bags form the path, which is a connected graph. The first two properties of tree decomposition are trivially satisfied.

Now we need to prove that the independence number of this decomposition is at most 2. Each of the bags induces a subgraph isomorphic to $K_{2,2}$, which clearly has independence number 2. Thus we can conclude that tree-$\alpha(K_{2,n}) \leq 2$.

To conclude that the equality holds, we can argue that, since $n \geq 2$, $K_{2,n}$ clearly has $K_{2,2} \cong C_4$ as induced subgraph, namely a cycle of order 4 without a chord. However Dallard, Milanič, and Štorgel proved in [10] that a graph has tree-independence number at most 1 if and only if it is chordal. Thus tree-$\alpha(K_{2,n}) = 2$ as desired. □

We will state the result cited at the end of the last proof as a lemma, since we will use it a few more times.

Lemma 3.4. *A graph G has tree-independence number at most* 1 *if and only if it is chordal.*

Let H be a graph and $\mathcal{F}$ be a family of connected subgraphs of H. The *intersection graph* $\mathcal{F}$ is the graph G whose vertices correspond to the members of $\mathcal{F}$ and there is an edge between two vertices that correspond to subgraphs H_1 and H_2 if and only if $V(H_1) \cap V(H_2)$ is nonempty.

A *feedback vertex set* in a graph G is a set of vertices intersecting all cycles. The *feedback vertex number* of G is the smallest cardinality of a feedback vertex set in G. The *clique cover number* of a graph G is the minimum integer k such that $V(G)$ is a union of k cliques.

Proposition 3.5. *Let H be a graph with feedback vertex number at most k and let G be the intersection graph of a family $\mathcal{F}$ of connected subgraphs of H. Then* tree-$\alpha(G) \leq k+1$.

Proof. Let X be a feedback vertex set in H such that $|X| \leq k$. Let S be the set of those connected subgraphs from $\mathcal{F}$ that are fully contained in $H - X$.

By definition, $H - X$ is a forest, and thus, by [14] the graph $G[S]$ is a chordal graph. From this, by Lemma 3.4 we can compute a tree decomposition $\mathcal{T}$ of $G[S]$ with independence number at most 1.

Now we consider the subgraphs from $\mathcal{F}$ that are not in S. Clearly, if we fix any vertex v from $V(G) \setminus S$, the set of all subgraphs from $\mathcal{F}$ that contain v forms a clique. Hence, the clique cover number of $G - S$ is at most $|X|$. Therefore, if we add $G - S$ to every bag of $\mathcal{T}$, we obtain a tree decomposition of G with independence number at most $|X| + 1$, which is at most $k + 1$. The result follows. □

Observation 3.6. *The bound from the previous proposition is sharp. More precisely, for each nonnegative integer k there exists a graph H with with feedback vertex number k and a family $\mathcal{F}$ of connected subgraphs of H such that the intersection graph G of $\mathcal{F}$ satisfies* tree-$\alpha(G) = k + 1$.

Proof. Let $n = k + 1$, let B be the complete bipartite graph $K_{n,n}$, and let H be the graph obtained from B by subdividing each edge exactly once. For each $v \in V(B)$, let us denote by H_v the subgraph of H induced by the closed neighborhood of v, and let $\mathcal{F} = \{H_v \mid v \in V(B)\}$. Finally, let G be the intersection graph of $\mathcal{F}$.

By Proposition 3.5, it suffices to show that the feedback vertex number of H is at most $n - 1$, while tree-$\alpha(G) \geq n$. By construction, the graph G is isomorphic to $B \cong K_{n,n}$, and hence tree-$\alpha(G) = n$ by Corollary 2.5. It remains to show that the feedback vertex number of H is at most $n - 1$. To this end, observe that each cycle of H corresponds to a unique cycle in B and consequently each feedback vertex set of B is also a feedback vertex set of H. Furthermore, any subset of size $n - 1$ of one of the two parts in a bipartition of B is a feedback vertex set of B. □

Corollary 3.7. *Let H be a graph that contains a vertex v such that $H - v$ is a forest. Let G be the intersection graph of a family $\mathcal{F}$ of connected subgraphs of H. Then* tree-$\alpha(G) \leq 2$.

Proof. Follows directly from the Proposition 3.5, by taking $k = 1$ and considering the corresponding feedback vertex set $X = \{v\}$. □

A *block* of a graph G is a maximal connected subgraph of G without cut-vertices. The *block-cutpoint graph* H of a connected graph G is a bipartite graph that has a vertex corresponding to every block in G and a vertex corresponding to every cut-vertex in G. There is an edge between a vertex of H corresponding to a block B and a vertex corresponding to a cut-vertex v in if the vertex v is contained in the block B in the graph G. It follows directly from the definition that the block-cutpoint graph is connected and acyclic, hence a tree.

A graph G is a *cactus graph* if it is connected and every block of G is either K_2 or a cycle, or if $G \cong K_1$.

Lemma 3.8. *Let H be a cactus graph and let G be the intersection graph of a family $\mathcal{F}$ of connected subgraphs of H. Then* tree-$\alpha(G) \leq 2$.

Proof. If H is a K_1, then G is a complete graph and the result is trivial. So suppose H has at least 2 vertices.

Let H' be the block-cutpoint graph of H. We proceed by induction on the number of vertices of H' (equivalently, the number of blocks in H). If H' is isomorphic to K_1, or K_2, then we are done by Corollary 3.7. Assume that for some $n > 1$ and for all block-cactus graphs with at most $n-1$ blocks, any intersection graph G of a family of connected subgraphs satisfies tree-$\alpha(G) \leq 2$.

Suppose H has n blocks. Since H' is a tree, it has a vertex v of degree 1 (a leaf) and this vertex cannot correspond to a cut-vertex in H.

Consider the block B of H that corresponds to a leaf of H'. This block contains a vertex v that is a cut-vertex in H. Thus the vertices in G that correspond to those subgraphs from $\mathcal{F}$ that contain v form a clique cutset C in G.

Moreover C cuts G in two (not necessarily connected) parts. The first part, G_1, corresponds to those subgraphs of H that are completely contained in $B - v$ and this part obviously induces a chordal graph. The second part, G_2, together with C, corresponds to the intersection graph G' of a family of connected subgraphs of the graph $H - (B \setminus \{v\})$, which has $n-1$ blocks.

By Lemma 2.3, the tree-independence number of G is at most

$$\max\{\text{tree-}\alpha(G[V(G_1) \cup C]), \text{tree-}\alpha(G[V(G_2) \cup C])\}\,.$$

Thus, since the graph G_1 is a chordal graph, by Lemma 3.4, we can obtain a tree decomposition $\mathcal{T}$ of independence number at most 1. We can then add a clique C to every bag of $\mathcal{T}$ to obtain a tree decomposition of $G[V(G_1) \cup C]$ with independence number at most 2.

For the graph $G' = G[V(G_2) \cup C]$, we apply the induction hypothesis to conclude that it has tree-independence number at most 2.

Thus tree-$\alpha(G) \leq 2$ as desired. □

A graph G is *outerplanar* if it has a planar drawing such that each vertex is incident with the outer face. It is known that a graph G is outerplanar if and only if it is

$\{K_{2,3}, K_4\}$-minor-free (see [21]). Before considering the tree-independence number of outerplanar graphs, we first prove the following structural lemma.

Lemma 3.9. *If G is a connected outerplanar graph without clique cutsets, then G is either K_1, K_2, or a cycle.*

Proof. Let G be a connected outerplanar graph with no clique cutset. If G has less than three vertices, it is clearly either a K_1 or a K_2.

So assume that G has at least 3 vertices. Since G has no clique cutsets, it has no cut vertices. Thus G is 2-connected. Then by [21], G has a unique Hamiltonian cycle in a planar embedding such that each vertex in incident with the outer face.

To conclude the proof, it is sufficient to observe that if an outerplanar graph has a cycle of length at least 4 that contains a chord e, then the endpoints of this chord belong to the outer face, by definition, and thus they form a clique cutset of size 2. □

Proposition 3.10. *Let G be an outerplanar graph. Then* tree-$\alpha(G) \leq 2$.

Proof. If G has a clique cutset, then we can apply Lemma 2.3. Thus we can assume without loss of generality that G contains no clique cutsets.

Then by the previous lemma, it is either K_1, K_2, or a cycle. In any case, we are done, as K_1, K_2 are trivial, and if we have a cycle, we may apply Corollary 3.7. □

A graph G is a *split graph* if its vertex set can be partitioned into two sets C, I such that C is a clique and I is an independent set. Clearly, split graphs are chordal, hence by Lemma 3.4, they have tree-independence number at most 1.

A slightly more general class are the so-called pseudo-split graphs. A graph G is *pseudo-split* if its vertex set can be partitioned into three sets C, I, S, such that C is a clique, I is an independent set and S is either an empty set, or induces a C_5 in G, furthermore, every vertex in C is adjacent to every vertex in S and every vertex in I is nonadjacent to every vertex in S.

Lemma 3.11. *If G is a pseudo-split graph, then* tree-$\alpha(G) \leq 2$.

Proof. Let G be a pseudo-split graph. Then $V(G)$ can be partitioned into a clique C, an independent set I and a set S that is either empty, or induces a C_5, as in the definition.

If S is empty, we are done, as we observed above, so we assume that S is nonempty. In particular, S induces a C_5.

Note that either C, or I may also be empty, but if C is empty, then G is either disconnected, in which case we are done by strong induction and Lemma 2.3, or G is isomorphic to C_5, in which case we are done by Lemma 3.1. On the other hand, if I is empty, by construction, we obtain a graph that has independence number at most 2 (the join of a complete graph and a C_5), and by Lemma 3.1, we are also done.

Hence, we may assume without loss of generality that all the three sets C, S and I are nonempty. Construct a path decomposition of G as follows. Let $k = |I| + 1$ be the number of bags. Put C in each bag and add a distinct vertex from I to each of the first $k - 1$ bags arbitrarily. Finally add S to the last bag.

Clearly, every vertex will be in at least one bag.

Given an edge e in $E(G)$, observe that it either has at least one endpoint in C or connects two vertices in S. But by construction of our path decomposition, for every vertex u in $V(G)$ and $v \in C$, there exists at least one bag that contains both u and v. Moreover, S is contained in the last bag along the path. Thus for any edge in G, there exists a bag that contains both of its endpoints.

Finally, notice that given a vertex $u \in C$, the graph T_u is isomorphic to a path, namely, it is connected. Furthermore, given $v \in I$, the graph T_v is isomorphic to K_1. Similarly for $w \in S$, T_w is isomorphic to K_1. Thus T_v is connected for any $v \in V(G)$, and the construction provided is indeed a tree decomposition.

Given an arbitrary bag, it is either a union of a clique with a single vertex, or induces a join of a clique and a C_5. In any case, it is clear that the independence number of the graph induced by any bag is at most 2. This concludes our proof. □

4 CLASSES OF H-FREE GRAPHS FOR VARIOUS SMALL GRAPHS H

In this chapter, we will develop necessary and sufficient conditions for H-free graphs having tree-independence number at most two, for various choice of small graphs H.

Observation 4.1. *Let G be $3K_1$-free graph. Then* tree-$\alpha(G) \leq 2$.

Proof. If G is $3K_1$-free, then for any three vertices u, v, w, there is at least one edge between them (as otherwise $G[\{u, v, w\}]$ is a $3K_1$). Equivalently $\alpha(G) \leq 2$, and by Lemma 3.1, we are done. □

Observation 4.2. *Let G be P_3-free graph. Then* tree-$\alpha(G) \leq 1$.

Proof. Suppose G contains an induced cycle C of length at least 4. Then consider any three consecutive vertices in that cycle. Call them u, v, w in order. The induced subgraph $G[\{u, v, w\}]$ is connected by the choice of the vertices, and thus can either be a P_3 or a K_3. Since G is P_3-free, $G[\{u, v, w\}]$ is a K_3. This implies that there is an edge between u and w, which is a chord in C. Thus G is chordal and consequently has tree-independence number at most 1 by Lemma 3.4. □

Proposition 4.3. *Let G be $\overline{P_3}$-free graph. Then G is a complete multipartite graph. Furthermore,* tree-$\alpha(G) \leq 2$ *if and only if there exists at most one part with more than two vertices.*

Proof. Assume first that G has tree-independence number at most two and suppose for a contradiction that there are at least two parts with more than two vertices. We notice that deleting a vertex from G cannot increase the tree-independence number, that is, tree-$\alpha(G[X]) \leq$ tree-$\alpha(G)$ for any $X \subseteq V(G)$. However, $K_{3,3}$ is an induced subgraph of G and since tree-$\alpha(K_{3,3}) = 3$ by Corollary 2.5, we obtain a contradiction.

Conversely, suppose that there is at most one part with more than two vertices. Since deleting a vertex cannot increase the tree-independence number, it suffices to consider the case when there exists a part with more than 2 vertices. Call this part X and denote its vertices by x_i, $i \in \{1, \ldots, t\}$. Now we construct a path decomposition of G, as follows. Make a path with t bags. In each bag put every vertex from $V \setminus X$. Finally in each bag i put one vertex x_i from X. It is easy to see that this construction results in a path decomposition. Furthermore, since the set $V \setminus X$ is a union of at most two cliques in G and every x_i is adjacent to all vertices from $V \setminus X$, the independence number of such a decomposition is 2, as desired. □

The class of *cographs* is defined recursively as follows.

- K_1 is a cograph.
- Whenever G_1, G_2 are cographs, their disjoint union $G_1 \cup G_2$ is also a cograph.
- Whenever G is a cograph, then so is its complement $\overline{G}$.
- There are no other cographs.

It is known that cographs are exactly P_4-free graphs (see, e.g., [8]).

Proposition 4.4. *A cograph G has tree-independence number ≤ 2 if and only if it is $K_{3,3}$-free.*

Proof. Suppose first that a graph G has tree-independence at most 2. To show that it does not contain $K_{3,3}$ as an induced subgraph, we only have to notice that deleting vertices cannot increase the tree-independence number and since by Corollary 2.5, $K_{3,3}$ has tree-independence number 3, we are done.

Conversely, suppose that a cograph G is $K_{3,3}$-free and proceed by strong induction on number of vertices. First, we may notice that tree-$\alpha(K_1) = 1$. Then suppose that for any $K_{3,3}$-free cograph G that has at most n vertices, tree-$\alpha(G) \leq 2$. Consider any disconnected $K_{3,3}$-free cograph G with $n+1$ vertices. This graph is a disjoint union of two $K_{3,3}$-free cographs G_1 and G_2, each having at most n vertices. Then the induction hypothesis applies and both G_1 and G_2 have the tree-independence number at most 2. Therefore, if we take a tree decomposition of G_1 with independence number 2 and a tree decomposition of G_2 with independence number 2, we can take their union and add a single edge between them to obtain a tree decomposition of G with independence number 2. Hence, we may assume that G is connected. But, as any cograph can be obtained from a copies of the one-vertex graph by applying the operations disjoint union and join, we notice that G can be obtained by taking several cographs with at most n vertices each, which we will refer to as components for the rest of this proof, and joining them together, adding all the edges between every pair of components.

Assume that every such component has independence number at most 2. Notice that joining two graphs, both having independence number at most k, yields a graph with independence number at most k. A simple inductive argument can be applied to show that this implies that G has independence number at most 2 and by Lemma 3.1 the tree-independence number of G is at most 2.

So we may assume that there is at least one component in G with independence number greater than 2. Suppose there are two separate components G_1, G_2 with independence number greater than 2. But then take any independent set of size 3 of G_1 and any independent set of G_2 of size 3 and notice that they induce $K_{3,3}$ in G, which is a contradiction.

Finally, consider the case when G has exactly one component C with independence number more than 2. Since the order of each component is strictly smaller than that of G, we can apply the induction hypothesis to infer that tree-$\alpha(C) \leq 2$. Notice that $G - V(C)$ has the independence number at most 2. So to conclude the proof, it is sufficient to show that a join of a graph with independence number at most two and a graph with tree-independence number at most two has tree-independence number at most two. Let G_1 be a graph such that tree-$\alpha(G_1) \leq 2$. Let G_2 be a graph such that $\alpha(G_2) \leq 2$. Take any tree decomposition of G_1 with independence number at most 2. Now add to each bag all the vertices from G_2. Notice that this yields a tree decomposition of $G_1 * G_2$ and furthermore, as each bag induces a graph obtained by joining two graphs with independence numbers at most two, it follows that this decomposition has independence number at most two. Thus $G_1 * G_2$ has tree-independence number at most two and this concludes the proof of our lemma. □

In the last proof, we demonstrated a fact about the tree-independence number that may be useful in the future, so it would be convenient to write it separately as a corollary.

Corollary 4.5. *Let G_1 and G_2 be graphs such that* tree-$\alpha(G_1) \leq 2$ *and* $\alpha(G_2) \leq 2$. *Then* tree-$\alpha(G_1 * G_2) \leq 2$.

We now consider the class of co-paw-free graphs. First we state a theorem due to Olariu.

Theorem 4.6 (Olariu [19]). *Let G be a connected paw-free graph. Then G is either K_3-free or complete multipartite.*

We say a graph G is *co-connected* if its complement $\overline{G}$ is connected. Taking the complementary statement of Olariu's theorem, we obtain the following corollary directly.

Corollary 4.7. *Let G be a co-connected co-paw-free graph. Then G is either $3K_1$-free or P_3-free.*

Proof. If G is a co-connected co-paw-free graph, then $\overline{G}$ is a connected paw-free graph. By Olariu's theorem, $\overline{G}$ is either K_3-free, or complete multipartite. Thus G is either $\overline{K_3} \cong 3K_1$-free, or it is a complement of a complete multipartite graph, which is just a disjoint union of complete graphs, and hence P_3-free. □

Lemma 4.8. *If G is a co-connected co-paw-free graph then* tree-$\alpha(G) \leq 2$.

Proof. Follows directly from the previous corollary and Observations 4.1 and 4.2. □

Lemma 4.9. *If G is a co-paw-free graph that is not co-connected, then* tree-$\alpha(G) \leq 2$ *if and only if G is $K_{3,3}$-free.*

Proof. If G is not co-connected, then its complement is disconnected, thus can be separated into two subgraphs G_1, G_2, such that there are no edges between them. But this means that G can be written as $\overline{G_1} * \overline{G_2}$. Suppose first that tree-$\alpha(G) \leq 2$. Then to show that it is $K_{3,3}$-free, we notice that deleting vertices cannot increase the tree-independence number and we know that $K_{3,3}$ has tree-independence number equal to 3 by Corollary 2.5.

Conversely, assume that G is $K_{3,3}$-free. Then clearly at most one of the subgraphs $\overline{G_1}$ and $\overline{G_2}$ has independence number more than 2. We may assume without loss of generality that $\alpha(\overline{G_1}) \leq 2$. Consequently tree-$\alpha(\overline{G_1}) \leq 2$, by Lemma 3.1.

Now consider $\overline{G_2}$. If $\overline{G_2}$ is co-connected, then tree-$\alpha(\overline{G_2}) \leq 2$ follows from Lemma 4.8. Otherwise, we may apply a simple inductive argument to show that tree-$\alpha(\overline{G_2}) \leq 2$.

Thus, since tree-$\alpha(\overline{G_1})$ and tree-$\alpha(\overline{G_2})$ are both at most 2, using Corollary 4.5, also tree-$\alpha(G) \leq 2$. □

Proposition 4.10. *If G is a co-paw-free graph, then* tree-$\alpha(G) \leq 2$ *if and only if G is $K_{3,3}$-free.*

Proof. Follows directly from the previous two lemmas. □

5 EXAMPLES OF GRAPHS WITH TREE-INDEPENDENCE NUMBER GREATER THAN TWO

In this chapter, we will give a few examples of graphs that have tree-independence number strictly greater than 2.

The following lemma was proved by Dallard et al. in [10].

Lemma 5.1. *Let G be a graph and let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree decomposition of G. Then there exists a vertex $v \in G$ and a node $t \in V(T)$ such that $N[v] \subseteq X_t$.*

As immediate consequence, we get the following result.

Corollary 5.2. *If G is a cubic triangle-free graph, then* tree-$\alpha(G) > 2$.

Proof. By the previous lemma, in any tree decomposition of G, there exists a bag that contains the closed neighbourhood of a vertex. Since G is cubic, the neighbourhood of any vertex contains 3 vertices. We claim that these vertices form an independent set in G. Suppose for a contradiction that the neighbourhood of any fixed vertex u does not form an independent set.

Then there exist vertices $v, w \in N(u)$ such that $v \sim w$. But then u, v, w induce a K_3, which contradicts the assumption that G is triangle free. Thus, any tree decomposition of G has independence number at least 3. □

Another simple example of graphs with tree-independence number at least three are the complete bipartite graphs $K_{m,n}$ for any $m, n \geq 3$.

Lemma 5.3. *If $m, n \geq 3$, then* tree-$\alpha(K_{m,n}) \geq 3$.

Proof. By Corollary 2.5, tree-$\alpha(K_{3,3}) = 3$. Clearly if $m, n \geq 3$, then $K_{3,3}$ is an induced minor of $K_{m,n}$. Then by Lemma 2.2, tree-$\alpha(K_{m,n}) \geq$ tree-$\alpha(K_{3,3}) = 3$, as desired. □

Proposition 5.4. *The graph $K_{3,3}$ is a minimal graph with tree-independence number greater than two.*

Proof. From Corollary 2.5, it follows that tree-$\alpha(K_{3,3}) = 3$, so it is sufficient to argue that if we delete a vertex, or contract an edge, we always obtain a graph with tree-independence number at most two.

We observe that deleting any vertex yields the same graph up to isomorphism, namely $K_{2,3}$. By Lemma 3.3, tree-$\alpha(K_{2,3}) = 2$.

Hence, we consider the operation of contracting an edge. If we contract any edge in $K_{3,3}$, we obtain a chordal graph, which has tree-independence number at most one by Lemma 3.4. □

Let Q_3 denote the graph from Fig. 5a, colloquially known as the cube graph.

Lemma 5.5. *It holds that* tree-$\alpha(Q_3) = 3$.

Proof. By Corollary 5.2, tree-$\alpha(Q_3) \geq 3$, so it is sufficient to find the tree decomposition of Q_3 that has independence number equal to 3. Consider a tree decomposition from Fig. 5b. Clearly, the first and the last bags induce a claw and the inner bag induces a C_6 in Q_3, all of which have independence number equal to 3. □

Lemma 5.6. *If we contract an edge, or remove a vertex from Q_3 we get a graph with tree-independence number equal to 2.*

Proof. First notice that all the vertices and edges of cube are the same up to isomorphism. Formally, this means that for any pair of vertices u, v there exists an automorphism of the cube graph mapping u to v and for every pair of edges e, f, there exists an automorphism mapping e to f.

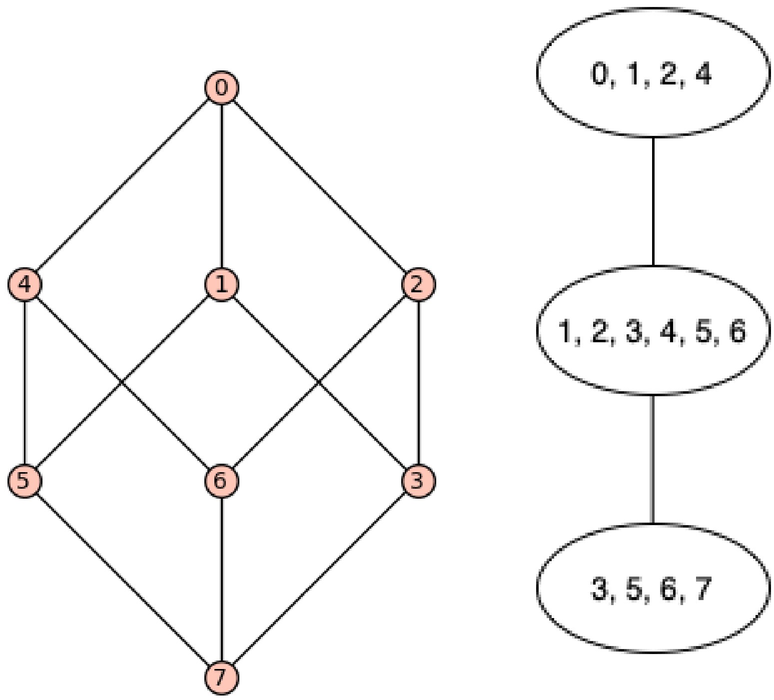
Thus it is enough to consider the operations of deleting the vertex 0 and contracting the edge $\{0, 1\}$.

Consider the cube graph with vertex 0 deleted depicted on Fig. 6a and its tree decomposition depicted on Fig. 6b. Clearly the first bag from top induces a path P_3 in our graph and the remaining 3 bags induce a P_4 in the graph. Thus this tree decomposition has independence number equal to two and furthermore, the graph $Q_3 - 0$ has independence number at most two. However, we notice that the vertices $\{1, 3, 5, 7\}$ induce a C_4, and hence this graph is not chordal, and in particular tree-$\alpha(Q_3 - 0) = 2$.

Consider now the cube with edge $\{0, 1\}$ contracted on Fig. 6c and its tree decomposition on Fig. 6d. Clearly the first and the last bags induce the same graph (up to isomorphism), which has 4 vertices and the clique number equal to 3. It follows that independence number is equal to 2. Finally, the inner two bags induce a P_4. Thus the independence number of this decomposition is at most 2 and again, by noticing that the vertices $\{2, 3, 6, 7\}$ induce a C_4, we can conclude the equality. □

As an immediate consequence of the previous two lemmas, we obtain the following.

Proposition 5.7. *The cube graph Q_3 is a minimal graph with tree-independence number greater than two.*



(a) Cube graph Q_3

(b) A tree decomposition of Q_3 with independence number 3

Figure 5: Cube graph Q_3 with a tree decomposition

We now define another graph, for which we show that it has tree-independence number equal to three, and that it is a minimal such graph. Let C_6^* be the graph constructed as follows.

- Start with an edgeless graph on three vertices A, B, C.
- Add two paths of length two between any pair of vertices A, B, C. (See Fig. 7.)

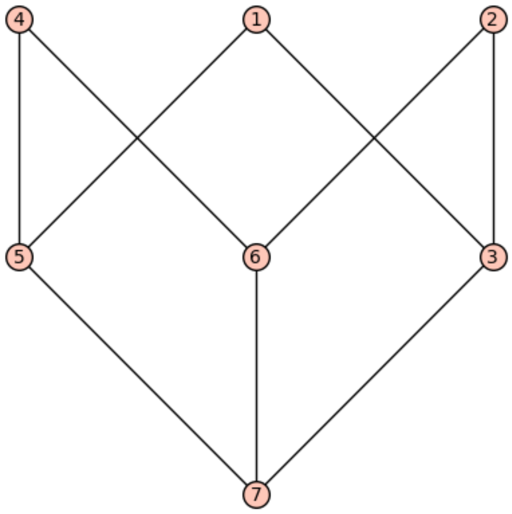
Lemma 5.8. *It holds that* tree-$\alpha(C_6^*) = 3$.

Proof. We first show that tree-$\alpha(C_6^*) \leq 3$, by giving a tree decomposition with independence number 3. We assume that the vertices of C_6^* are labeled as in Fig. 7. Let $\mathcal{T} = (P_6, \{X_t\}_{1 \leq t \leq 6})$ where P_6 denotes the path graph with vertices $1, \ldots, 6$ in order along the path and the bags are defined as follows

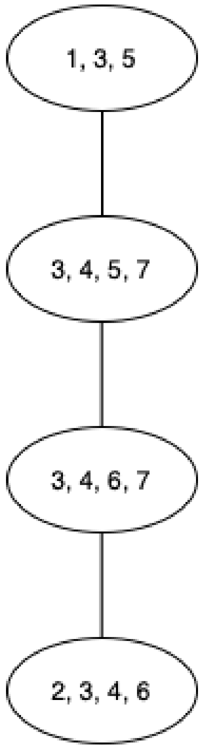
$$X_t = \{A, B, C, x_t\}$$

for all $t \in \{1, \ldots, 6\}$. It is easy to see that $\mathcal{T}$ is indeed a tree decomposition of C_6^* and furthermore that the independence number of the subgraph induced by each bag X_t is 3. Hence tree-$\alpha(C_6^*) \leq 3$, as claimed.

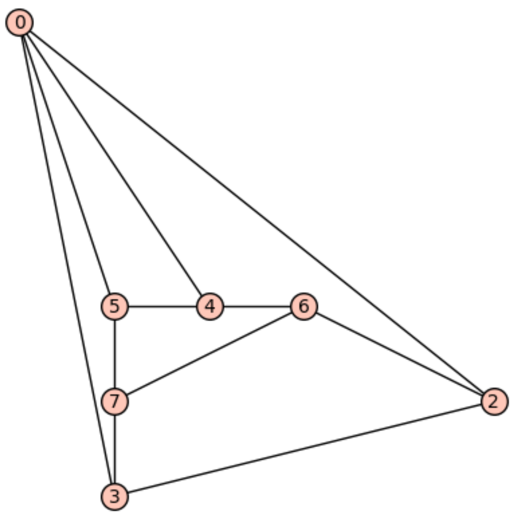
To show the converse inequality tree-$\alpha(C_6^*) \geq 3$, suppose for a contradiction that tree-$\alpha(C_6^*) \leq 2$. Then, there exists a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of C_6^*



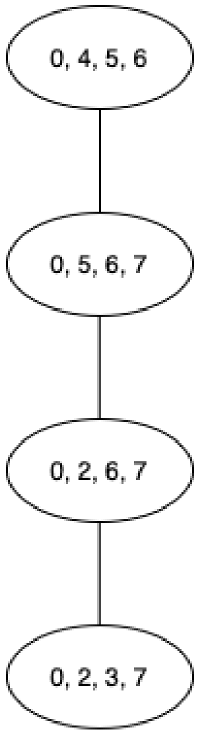
(a) The cube graph Q_3 with vertex 0 removed



(b) A tree decomposition of Q_3 with vertex 0 removed



(c) The cube graph Q_3 with edge $\{0, 1\}$ contracted



(d) A tree decomposition of Q_3 with edge $\{0, 1\}$ contracted

Figure 6: The maximal proper induced minors of the cube graph Q_3, together with their tree decompositions of independence number two.

such that the independence number of $\mathcal{T}$ is at most two. Notice that this means that there is no bag that contains all of the vertices A, B, C, as otherwise we would get a bag that induces a subgraph of C_6^* with independence number ≥ 3. Using [10, Lemma 2.3], we can see that this implies that there exists a pair of vertices from $\{A, B, C\}$ such that no bag contains this pair. We may assume without loss of generality that this pair is A, B (since otherwise we can relabel the vertices accordingly).

But, since x_1, x_2 are the common neighbours of A and B, there is a shortest path P from T_A to T_B (of length at least 1) in our tree decomposition and furthermore any bag in this path contains both x_1 and x_2. Note that the latter is true, since both T_{x_1} and T_{x_2} intersect both T_A and T_B, and since the set of all bags containing x_1 forms a subtree, there must be a (shortest) path from T_A to T_B where every bag contains x_1 and similarly with x_2. And since we are in a tree, this path is unique.

But then, since we assumed that $\alpha(\mathcal{T}) \leq 2$, we know exactly what this path looks like. Namely, it starts with a bag that is exactly equal to $\{A, x_1, x_2\}$ and then has $t \geq 0$ bags that contain only $\{x_1, x_2\}$ and ends with a bag equal to $\{B, x_1, x_2\}$. Notice that if any bag on this path contained any other vertex from C_6^*, we would obtain a bag that induces a subgraph of C_6^* with independence number greater than 2, which is a contradiction.

We order the bags of P linearly so that the first bag is in T_A, the last bag is in T_B, and the bags between them are ordered in the natural order.

Similarly, we know that there exists a (possibly trivial) path Q between T_B and T_C where every bag on the path contains x_3, x_4. Furthermore, each bag on this path contains exactly x_3, x_4 and some subset of $\{B, C\}$. Order the bags along this path similarly as with P, so that the first bag is in T_B and the last bag is in T_C (notice that here the first bag might also be the last).

Finally, using the same arguments as above, there exists a path R from T_A and T_C where each bag contains x_5, x_6. Furthermore, each bag on this path contains exactly x_5, x_6 and some subset of $\{A, C\}$. Similarly, we order the bags so that we begin the path in T_A and finish in T_C.

Fix any bag $X_u \in T_C$ and let X_v be the bag corresponding to the endpoint of P in T_A. There is a unique path in T_A from this bag to the first bag in R.

Then we can reach any bag in T_C (in particular X_u) from this one by just following R. Notice that for any internal vertex of this path, the corresponding bag does not contain any of the vertices x_1, x_2, by the assumption that $\alpha(\mathcal{T}) \leq 2$.

Similarly, from X_v we could use P to reach T_B and then there exists a unique path to the first bag of Q. Then we can follow Q to reach a bag in T_C from where we can reach any bag from T_C (in particular X_u) by following a unique path in T_C. Notice that this path contains at least one bag which contains vertices x_1, x_2.

Notice that we know the exact structures of P, Q and R and in particular, by the

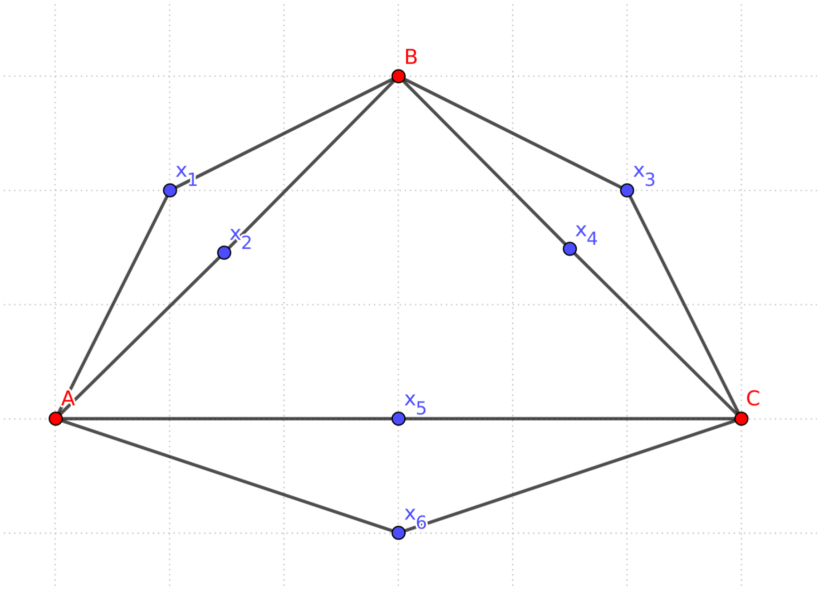


Figure 7: Construction of the C_6^* graph.

assumption that $\alpha(\mathcal{T}) \leq 2$, they have to be disjoint. Thus the gluing of the paths described above indeed yields paths.

We thus obtain a contradiction, since we found two distinct paths between two fixed vertices X_u and X_v in a tree. □

We now proceed to show that this graph is a minimal graph with tree-independence number greater than 2.

Lemma 5.9. *Any proper induced minor G of C_6^* has* tree-$\alpha(G) \leq 2$.

Proof. Assume that C_6^* is labeled as in Fig. 7. We will first consider removing a vertex from C_6^*. There are two choices of a vertex to remove up to isomorphism. Namely, we can remove the vertex A, or the vertex x_6 without loss of generality, otherwise we can relabel the graph accordingly.

If we remove A, then the tree decomposition $\mathcal{T} = (P_6, \{X_t\}_{1 \leq t \leq 6})$, where P_6 denotes the path graph with vertices $1, \ldots, 6$ in order along the path and the bags are defined as follows

$$X_t = \{B, C, x_t\}$$

for all $t \in \{1, \ldots, 6\}$ clearly has the independence number 2.

If we remove x_6, then the tree decomposition depicted on Fig. 8 has independence number 2.

Now we consider contracting an edge. There is only one choice up to isomorphism, namely without loss of generality we contract Ax_6 into A (since otherwise we can relabel vertices accordingly). We only have to notice that the tree decomposition from Fig. 8 works for this graph as well and we are done. □

Proposition 5.10. *The graph C_6^* is a minimal graph with tree-independence number greater than two.*

Proof. Follows directly from the previous two lemmas. □

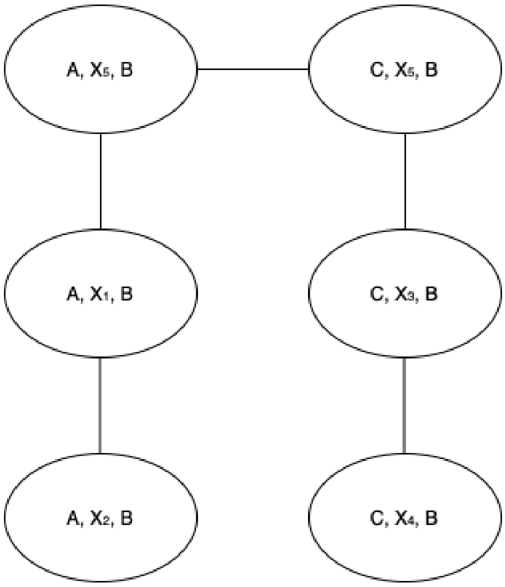


Figure 8: A tree decomposition of the graph $C_6^* - x_6$ with independence number 2

6 SERIES PARALLEL GRAPHS

The following definition originates from [13]. A graph G is *two-terminal series parallel*, with terminals s and t, if it can be produced by a sequence of the following operations:

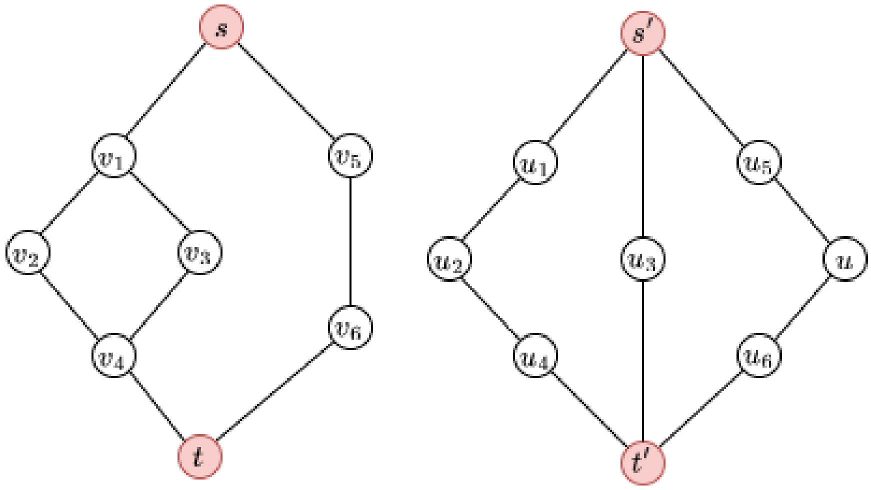
- Create a new graph, consisting of a single edge connecting s and t.
- Given two two-terminal series parallel graphs X and Y, with terminals s_X, t_X and s_Y, t_Y, respectively, form a new graph $G = P(X, Y)$ by identifying $s = s_X = s_Y$ and $t = t_X = t_Y$. This is known as the *parallel composition* of X and Y. An example of parallel composition of two graphs from Fig. 9a can be seen on Fig. 9c.
- Given two two-terminal series parallel graphs X and Y, with terminals s_X, t_X and s_Y, t_Y, respectively, form a new graph $G = S(X, Y)$ by identifying $s = s_X$, $t_X = s_Y$, and $t = t_Y$. This is known as the *series composition* of X and Y. An example of series composition of two graphs from Fig. 9a can be seen on Fig. 9b.

A graph G is *series parallel* if it contains a pair of vertices s and t such that G is a two-terminal series parallel graph with terminals s and t.

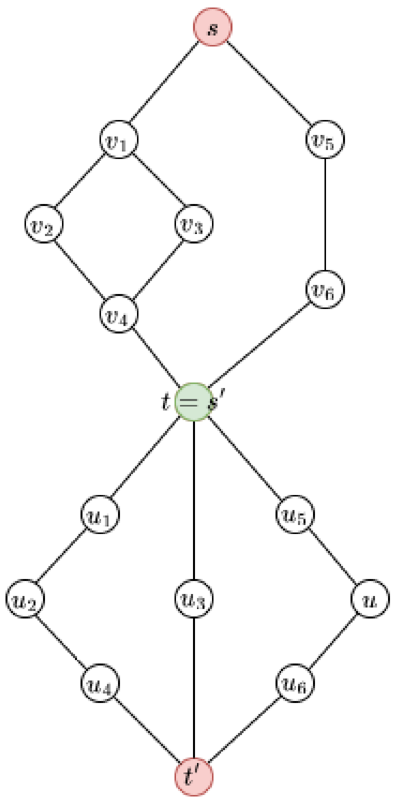
A *maximal parallel decomposition* of a series parallel graph G is a finite sequence $(X_1, \ldots, X_k)$ of induced subgraphs of G defined recursively as follows. If $G \cong K_2$ or G is a series composition of two smaller series parallel graphs, then G has a unique maximal parallel decomposition, namely (X_1) with $X_1 = G$. If G is a parallel composition of two smaller series parallel graphs X and Y, with maximal parallel decompositions $(X_1, \ldots, X_k)$ and $(Y_1, \ldots, Y_\ell)$, then their concatenation $(X_1, \ldots, X_k, Y_1, \ldots, Y_\ell)$ is a maximal parallel decomposition of G. We say that a maximal parallel decomposition $(X_1, \ldots, X_k)$ of a series parallel graph G is *trivial* if $k = 1$ and *nontrivial*, otherwise. If $(X_1, \ldots, X_k)$ is a maximal parallel decomposition of G, we write $G = P(X_1, \ldots, X_k)$.

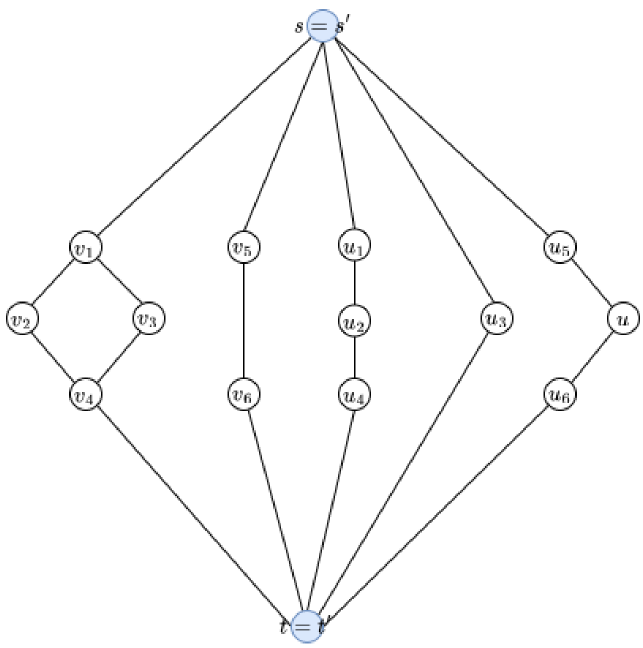
Notice that after fixing the terminal vertices, the number k of parallel components in the maximal parallel decomposition is uniquely determined. Furthermore, the maximal parallel decomposition is invariant under reordering of the X_i's, thus there are exactly $k!$ maximal parallel decompositions that result in the same graph. More formally, $P(X_1, \ldots, X_k) = P(X_{\sigma(1)}, \ldots X_{\sigma(k)})$, for any permutation σ of $\{1, \ldots, k\}$.

A *maximal series decomposition* of a series parallel graph G is a finite sequence $(X_1, \ldots, X_k)$ of induced subgraphs of G defined recursively as follows. If $G \cong K_2$ or G is a parallel composition of two smaller series parallel graphs, then G has a unique maximal series decomposition, namely (X_1) with $X_1 = G$. If G is a series composition



(a) Two series parallel graphs X and Y, with terminal vertices s, t and s', t' respectively, coloured red.





(b) Series composition $S(X, Y)$ of graphs X and Y.

(c) Parallel composition $P(X, Y)$ of graphs X and Y.

Figure 9: Two series parallel graphs and their series and parallel compositions.

of two smaller series parallel graphs X and Y, with maximal series decompositions $(X_1, \dots, X_k)$ and $(Y_1, \dots, Y_\ell)$, then their concatenation $(X_1, \dots, X_k, Y_1, \dots, Y_\ell)$ is a maximal series decomposition of G. We say that a maximal series decomposition $(X_1, \dots, X_k)$ of a series parallel graph G is *trivial* if $k = 1$ and *nontrivial*, otherwise. If $(X_1, \dots, X_k)$ is a maximal series decomposition of G, we write $G = S(X_1, \dots, X_k)$.

Notice that, after fixing the terminal vertices, there are exactly two maximal series decompositions that result in the same graph. In particular $S(X_1, X_2, \dots, X_{k-1}, X_k) \cong S(X_k, X_{k-1}, \dots X_2, X_1)$.

If $G = P(X_1, \dots, X_k)$ (respectively $G = S(X_1, \dots, X_k)$), we say that the X_i's are the *parallel components* of G (respectively *series components*).

Lemma 6.1 (Brandstädt et al. [5])**.** *Every series parallel graph G has treewidth at most two. Furthermore, if G is 2-connected, then it has treewidth at most two if and only if it is series parallel.*

Lemma 6.2 (Dallard et al. [10])**.** *For every graph G,* tree-$\alpha(G) \leq \text{tw}(G) + 1$*, and this bound is sharp.*

Corollary 6.3. *If G is a series parallel graph, then* tree-$\alpha(G) \leq 3$*.*

Proof. Follows directly from Lemmas 6.1 and 6.2. □

By Corollary 6.3, series parallel graphs seem like a good candidate for graphs with tree-independence number at most two. However, we can observe that the graph C_6^* is a series parallel graph with terminal vertices A and C, as labeled in Fig. 7 and has tree-independence number equal to 3 by Lemma 5.8. Thus, a natural question to ask is what are some necessary and sufficient conditions for series parallel graphs to have tree-independence number at most 2. We proceed to show that in the class of series parallel graphs, forbidding C_6^* as an induced minor is not only necessary but also sufficient to bound the tree-independence number by two.

Lemma 6.4. *Let G be a series parallel graph with terminal vertices s and t and $G = P(X_1, \dots, X_r)$, with $r \geq 2$, such that for each i, $X_i = S(Y_1^i, \dots, Y_{s_i}^i)$ with $s_i \geq 2$, and at most one Y_j^i is non-chordal. Furthermore, assume that G has no clique cutsets. Then for each $i \in \{1, \dots, r\}$, there exists a tree decomposition of X_i with independence number at most* 2 *and with a bag that contains both terminal vertices s and t of G.*

Proof. We proceed by strong induction on the number of vertices in the largest parallel component. If all of the parallel components X_i have at most 3 vertices, then each X_i is chordal, and by Lemma 3.4 admits a tree decomposition of independence number 1. Adding the vertex s to every bag of such a tree decomposition yields a desired tree decomposition of X_i.

Suppose that whenever each parallel component X_i of G has at most n vertices, for some $n \geq 3$, the assertion holds. Now assume that in the graph G each parallel component has at most $n+1$ vertices. Sort the parallel components by their number of vertices, so that $|V(X_1)| \geq |V(X_2)| \geq \ldots \geq |V(X_r)|$. Consider the largest parallel component X_1. We first explain how to obtain a desired tree decomposition of X_1. If X_1 is chordal, then by Lemma 3.4 X_1 admits a tree decomposition of independence number 1; adding the vertex s to every bag of such a tree decomposition yields a desired tree decomposition of X_1.

Assume now that X_1 is non-chordal. Denote the series components of X_1 by $Y_1, \ldots, Y_s$. Recall that at most one of these components is non-chordal, and, since X_1 is non-chordal, exactly one of these components is non-chordal. Let Y_j be the non-chordal series component of X_1.

Since $s \geq 2$, each of the parallel components of Y_j has at most n vertices. Thus, the induction hypothesis applies and we can obtain a tree decomposition of Y_j with independence number at most 2 that contains both terminal vertices of Y_j.

Since G has no clique cutsets, it is easy to see that every other (chordal) series component of X_1 has to be isomorphic to K_2.

Denote the terminal vertices of Y_j by x and y so that the length of a shortest path from s to x in X_1 is shorter than the length of a shortest path from s to y. Denote the vertices on the unique path in X_1 from x to s by $x_0 = x, x_1, \ldots, x_p = s$ (possibly $p = 0$) and from y to t by $y_0 = y, y_1, \ldots, y_q = t$ (possibly $q = 0$). Assume w.l.o.g. $p \leq q$.

Construct the tree decomposition as follows.

If $p = 0$, start with the tree decomposition of Y_j as constructed above. Identify a bag B that contains both $x = s$ and y. Add q bags arranged in a path, where the i-th bag along the path contains vertices $y_{i-1}, y_i, x = s$, for all $i = 1, \ldots, q$. Connect the first bag along this path to B. Clearly this construction yields a desired tree decomposition.

Hence, we may assume that $1 \leq p \leq q$. Again, start with a tree decomposition of Y_j as constructed above and identify a bag B that contains both x and y. Add p bags arranged in a path, where the i-th bag along the path contains vertices $x_{i-1}, x_i, y_{i-1}, y_i$ for all $i = 1, \ldots, p$. Connect the first bag along this path to B. Finally, add $q - p$ bags arranged in a path, where i-th bag along the path contains vertices $s = x_p, y_{p+i-1}, y_{p+i}$, for all $i = 1, \ldots, q - p$.

This construction yields the desired tree decomposition for X_1. Repeat for all the parallel components of G with $n+1$ vertices and apply the induction hypothesis for the parallel components with at most n vertices. □

Lemma 6.5. *Let G be a series parallel graph with terminal vertices s and t and no clique cutsets such that $G = P(X_1, \ldots, X_r)$, with $r \geq 2$, where each parallel component has at most one non-chordal series component. Then there exists a tree decomposition* *of G with independence number at most 2 and with a bag that contains both terminal vertices s and t of G.*

Proof. By Lemma 6.4, each of the parallel components $X_1, \dots, X_r$ has a tree decomposition with independence number at most two with a bag that contains both terminal vertices s and t, of G. Take such a tree decomposition of every parallel component of G and identify a bag B_i in parallel component X_i that contains both s and t.

Add an edge between B_i and B_{i+1} for every $i \in \{1, \dots, r-1\}$ to obtain a desired tree decomposition of G. □

Theorem 6.6. *Let G be a series parallel graph. Then* tree-$\alpha(G) \leq 2$ *if and only if G is C_6^*-induced-minor-free.*

Proof. Assume first G has tree-independence number at most 2. By Lemmas 2.2 and 5.8, G must be C_6^*-induced-minor-free.

Conversely, assume that G is C_6^*-induced-minor-free. We assume without loss of generality that G has no clique cutsets, since otherwise we can apply Lemma 2.3 along with an inductive argument on the number of vertices.

If G has a nontrivial maximal series decomposition, it clearly has a cut vertex, so we can assume that G has a trivial maximal series decomposition and a nontrivial maximal parallel decomposition. In particular, let $G = P(X_1, \dots, X_k)$ for some $k \geq 2$.

We consider two cases separately:

- $k = 2$,
- $k \geq 3$.

Assume first that $k \geq 3$. Then, by assumption of no clique cutsets, we can assume that the terminal vertices of G (denoted s and t) are nonadjacent to each other.

If there exists a parallel component of G, X_i, that contains two non-chordal series components Y_j and Y_k, then G contains C_6^* as an induced minor. Thus every X_i contains at most one non-chordal series component and by Lemma 6.5, we are done.

Now consider the case when $k = 2$. We have two subcases:

- both X_1 and X_2 are non-chordal,
- at most one parallel component is non-chordal.

If both X_1 and X_2 are non-chordal, then, since G is C_6^*-induced-minor-free, each of them has at most one non-chordal series component. We are done by Lemma 6.5.

On the other hand, if w.l.o.g. X_2 is chordal, then X_1 may have at most two non-chordal series components. If X_1 has at most one non-chordal series components, we are done similarly as above, so we may assume there are exactly two such non-chordal components.

Let Y_i and Y_j be the non-chordal series components of X_1. Clearly, since G is C_6^*-induced-minor-free and has no clique cutsets, each of the parallel components of Y_i and Y_j has at most one non-chordal series component.

Suppose Y_i has terminal vertices x and y. Then there exists a series parallel graph isomorphic to G, but having x and y as terminal vertices. This graph clearly has parallel components with at most one non-chordal series component, and hence Lemma 6.5 applies again.

This shows that any C_6^*-induced-minor-free series parallel graph with no clique cutsets is isomorphic to a series parallel graph whose every parallel component has at most one non-chordal series component. Applying the result from Lemma 6.5 concludes the proof. □

The following theorem follows directly from the previous one and relates the graphs with treewidth at most two and the graphs with tree-independence number at most two.

Theorem 6.7. *Let G be a graph with* $\text{tw}(G) \leq 2$. *Then* $\text{tree-}\alpha(G) \leq 2$ *if and only if G is C_6^*-induced-minor-free.*

7 CHORDAL BIPARTITE GRAPHS

Let G be a bipartite graph. We say that G is *chordal bipartite* if each cycle of length at least 6 has a chord (see, e.g. [6, 15]).

We first explore a simple class of chordal bipartite with a lot of structure.

Lemma 7.1. *Let T be a tree and let G_T be the graph constructed as follows. Subdivide every edge of T once and add two new vertices u, v. Connect u, v by an edge to every vertex that corresponds to a vertex of T (an example of this construction with $T \cong P_3$ can be seen on Fig. 10). Then G_T is chordal bipartite.*

Proof. It is sufficient to show that such a graph G_T is bipartite and that each cycle of length at least 6 has a chord. Let $P \subseteq V(G_T)$ be the set of the vertices of G_T that correspond to vertices of T and $Q \subseteq V(G_T)$ be the set of vertices of G_T that correspond to edges of T and the universal vertices u and v. It is obvious that every edge of G_T has one endpoint in P and another in Q and that $P \cup Q = V(G_T)$. Thus, G_T is bipartite.

Since G is bilartite, every cycle in G has an even number of vertices. Furthermore, since T is a tree, every cycle contains either u or v. Since u and v are nonadjacent, every cycle must contain at least two vertices corresponding to the vertices of T, x_i and x_j. If these are the only two vertices, then clearly our cycle is of length 4.

So suppose that a cycle C contains more than 2 vertices corresponding to vertices of T. Then observe that C cannot contain both u and v. We may assume without loss of generality that C contains u. But then, let x_i, x_j, x_k be any three vertices in C that correspond to vertices of T, ordered so that the unique path in $C - u$ from x_i to x_k contains x_j. Notice that if such order does not exist, then they cannot be contained in the same cycle in G_T.

But then clearly the edge $\{u, x_j\}$ is a chord in C, showing that we cannot have induced cycle of length greater than 4. □

Lemma 7.2. *Let G_T be a graph as constructed in Lemma 7.1. Then* tree-$\alpha(G_T) \leq 2$.

Proof. First notice that we can partition the vertices of G_T in three parts, where each part forms an independent set.

- The first part consists of two vertices, u and v.
- The second part consists of the vertices that correspond to the vertices of T.

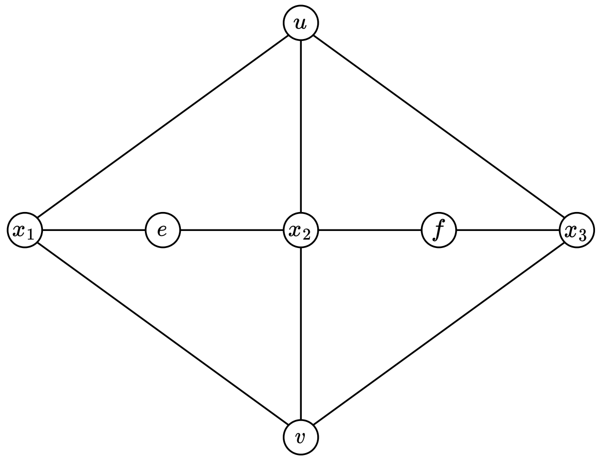


Figure 10: Example of construction from Lemma 7.2 with $T \cong P_3$.

- The third part consists of the vertices that correspond to the edges of T.

Furthermore, each of the vertices in the third part has degree equal to two and is adjacent only to the vertices that correspond to the endpoints of the corresponding edge in T.

We proceed by strong induction on the number of vertices in T. Notice that the claim holds whenever $|V(T)| \leq 2$, as if $T \cong K_1$, then $G \cong P_3$ and if $T \cong K_2$, with vertices x, y and an edge e between them, then G_T is just a P_3 with endpoints x and y together with two vertices, u and v, that are both adjacent to x and y. Thus, we can obtain a tree decomposition with independence number 2 by taking two adjacent bags and putting $\{x, y, e, u\}$ in the first bag and $\{x, y, e, v\}$ in the second. Clearly both bags induce a C_4 and we are done. For later use, observe that the constructed tree decomposition $\mathcal{T}$ has the property that for every vertex z of G_T that corresponds to a vertex of T, there exists a bag in $\mathcal{T}$ that has $\{u, v, z\}$ as a subset.

Before proceeding, we prove the following claim. If T is a tree with $|V(T)| > 2$ and $\mathcal{T} = (T', \{X_t\}_{t \in V(T')})$ is a tree decomposition of graph G_T as described above, with $\alpha(\mathcal{T}) \leq 2$, then for every vertex x of G_T that corresponds to a vertex of T, there exists a bag in $\mathcal{T}$ that has $\{u, v, x\}$ as a subset.

Suppose that this is not true. Then if we look at T_x, there exists a shortest path P in T' that starts with a bag that contains x and u and ends with a bag that contains x, v. Let y and z be two other vertices of T'. Clearly, because of the assumption that $\alpha(\mathcal{T}) \leq 2$, there exists no bag of $\mathcal{T}$ that contains all three vertices x, y, z. Assume that there is a bag that contains x, y and a bag that contains x, z. Then, since y is adjacent to both u and v, T_y intersects with both T'_u and T'_v. But then, since the unique shortest path from T'_u to T'_v in T'_x is P, every bag along this path contains y. Similarly for z, and this contradicts the assumption that no bag contains all x, y, z.

So without loss of generality T'_x and T'_z are disjoint. But then, since T'_z and T'_u

intersect, as well as T'_z and T'_v, the shortest path Q between T'_u and T'_v (possibly trivial) is such that every bag along Q contains z. But then, consider the endpoints of P, call them B_1 and B_2, and any bag B_3 along Q. Clearly, starting from B_1, we can follow (a unique path in) T'_u to reach an endpoint of Q that contains u and then follow Q to reach B_3. Notice that there is exactly one bag along this path that contains x. Alternatively, we can follow P to reach B_2 and then follow (a unique path in) T'_v to reach an endpoint of Q that contains v. Then we follow Q to reach B_3. Notice that this path consists of at least two vertices that contain x. Thus, we have obtained two distinct paths in $\mathcal{T}$ that connect B_1 and B_3, which yields a contradiction. This proves the claim.

Suppose that for some $n \geq 3$, for every tree T with $|V(T)| < n$, there exists a tree decomposition $\mathcal{T}$ of G with $\alpha(\mathcal{T}) \leq 2$.

Let T be any tree with n vertices. Consider a leaf x in T and suppose that it is adjacent to a vertex y and that the edge between them is labeled e.

We construct a tree decomposition of the corresponding graph G_T.

Take two adjacent bags B_1 and B_2 and place x, y, u, v into B_1 and x, y, e into B_2. Consider a tree decomposition $\mathcal{T}'$ of a graph G_{T-x} such that $\alpha(\mathcal{T}') \leq 2$. Note that such a tree decomposition $\mathcal{T}'$ exists by the induction hypothesis. By the above claim and the explicit construction for the case $|V(T-x)| = 2$, we may assume that there exists a bag B' of $\mathcal{T}'$ that contains u, v, y. Connect B_1 to B' to obtain a tree decomposition of G_T with independence number at most two. □

We can notice that the construction from previous lemma almost always yields a graph with an induced $K_{2,3}$. We now consider the family of $K_{2,3}$-free chordal bipartite graph, and show that every such graph has tree-independence number at most two.

Before showing this, we need to prove a lemma regarding the structure of such graphs. An edge $\{x, y\}$ in G is called *bisimplicial* if $N(x) \cup N(y)$ induces a complete bipartite graph.

Lemma 7.3. *Let $\mathcal{G}$ be the class of connected $K_{2,3}$-free chordal bipartite graphs. Then $G \in \mathcal{G}$ if and only if G satisfies one of the following conditions.*

- *G is isomorphic to either K_1, K_2, or C_4.*
- *G contains a cut-vertex v such that there exists a cut partition $(A, B, \{v\})$ of G such that the graphs $G[A \cup \{v\}]$ and $G[B \cup \{v\}]$ both belong to $\mathcal{G}$.*
- *G contains a clique cutset C of size 2 such that there exists a cut partition (A, B, C) of G such that the graphs $G[A \cup C]$ and $G[B \cup C]$ both belong to $\mathcal{G}$.*

Proof. Assume first that G is a graph that satisfies one of the three conditions. If G is isomorphic to either K_1, K_2, or C_4, then G is a $K_{2,3}$-free chordal bipartite graph.

Assume next that G admits a cut partition (A, B, C) such that C is a clique of size one or two and the graphs $G[A \cup C]$ and $G[B \cup C]$ both belong to $\mathcal{G}$. Suppose for a contradiction that $G \notin \mathcal{G}$. Since C is nonempty and the induced subgraphs $G[A \cup C]$ and $G[B \cup C]$ are both connected, the graph G is also connected. We thus infer that G contains an induced subgraph H isomorphic to either $K_{2,3}$ or to a cycle of length other than four. However, no such graph H admits a clique cutset, and therefore H is an induced subgraph of either $G[A \cup C]$ or $G[B \cup C]$. This contradicts the fact that both of these graphs belong to $\mathcal{G}$.

Conversely, assume that G is a connected $K_{2,3}$-free chordal bipartite graph. If G has 1, or 2 vertices, then G is isomorphic to either K_1 or K_2, respectively, as those are the only such connected graphs. Assume now that G has at least three vertices. Since G is connected, it has an edge. Golumbic and Goss showed in [15] that every chordal bipartite graph that has an edge has a bisimplicial edge. Let $\{u, v\}$ be a bisimplicial edge in G. We may assume without loss of generality that $\deg(u) \leq \deg(v)$. Since the edge $\{u, v\}$ is bisimplicial, the subgraph of G induced by $N[u] \cup N[v]$ is isomorphic to the complete bipartite graph $K_{m,n}$ where $m = \deg(u)$ and $n = \deg(v)$.

If $m = 1$, then v is a cut-vertex in G and taking $A = \{u\}$ and $B = V(G) \setminus \{u, v\}$, we obtain a cut partition $(A, B, \{v\})$ of G such that the graphs $G[A \cup \{v\}]$ and $G[B \cup \{v\}]$ both belong to $\mathcal{G}$.

Assume now that both m and n are at least 2. If, on the other hand, $n \geq 3$, then G contains an induced $K_{2,3}$, contradicting the assumption that $G \in \mathcal{G}$. So we conclude that $m = n = 2$, that is, the set $N[u] \cup N[v]$ induces a $K_{2,2} \cong C_4$ in G. Denote the other two vertices in this set by w and z, where $v \sim w$ and $u \sim z$. If $|V(G)| = 4$, then $V(G) = \{u, v, w, z\}$ and G is isomorphic to C_4. Assume now $|V(G)| > 4$. By connectedness, we may assume that there exists a vertex $x \notin \{v, z\}$ such that $x \sim w$. But then notice that in $G - \{w, z\}$ there is no path between x and u. Therefore, taking $A = \{u, v\}$, $B = V(G) \setminus \{u, v, z, w\}$, and $C = \{w, z\}$, we obtain a clique cutset C of size 2 and a cut partition (A, B, C) of G such that the graphs $G[A \cup C]$ and $G[B \cup C]$ both belong to $\mathcal{G}$. □

The following corollary is a direct consequence of the previous lemma.

Corollary 7.4. *If G is a $K_{2,3}$-free chordal bipartite graph, then* tree-$\alpha(G) \leq 2$.

Proof. The statement follows from the previous lemma, as K_1, K_2, and C_4 all have tree-independence number at most 2 and otherwise, G has a clique cutset, so we can proceed recursively, using Lemma 2.3 to construct a tree decomposition with independence number at most 2. □

We consider another class of chordal bipartite graphs and show that in this class it is sufficient to forbid $K_{3,3}$ as an induced subgraph. This is the class of $2K_2$-free bipartite graphs. It is easy to see that this class is indeed a subclass of chordal bipartite class of graphs, since every cycle with at least six vertices contains an induced $2K_2$. The structure of $2K_2$-free bipartite graphs is well known (see, e.g., [22]). We state the two lemmas from [12] that will help us prove our desired result. Given a graph G, a set S of vertices of G, and another set of vertices U, we say that U is *universal to* S if every vertex in S has a neighbour in U. The definition is similar if U is a vertex, an edge, or a subgraph of G.

Lemma 7.5. *Let G be a connected graph and S be any minimal cutset of G. Let $G_1, G_2, \dots, G_l$, $(l \geq 2)$ be the connected components in $G - S$. Then G is $2K_2$-free if and only if it satisfies the following conditions:*

- *$G - S$ contains at most one nontrivial component which is again $2K_2$-free.*
- *Every trivial component of $G - S$ is universal to S.*
- *Every edge in the nontrivial component of $G - S$ is universal to S.*
- *The subgraph induced by S is either connected or has at most one nontrivial component which is again $2K_2$-free.*
- *If S and $G - S$ both have a nontrivial component, say S_1 and G_1, respectively, then every edge $e = \{u, v\}$ in S_1 is universal to M, where $M = \{x \in V(G_1) \mid \{x, y\} \in E(G)$ for all $y \in S - (N(u) \cup N(v))\}$.*

Lemma 7.6. *If G is a connected $2K_2$-free bipartite graph, then for any minimal cutset S of G, the following conditions are satisfied.*

- *S is an independent set.*
- *If $G - S$ has a nontrivial component, say G_1, then for every vertex $x \in S$, $N(x) \cap V(G_1)$ is an independent set.*
- *For every edge $\{u, v\}$ in the nontrivial component G_1, either u or v is universal to S. Moreover, if u (without loss of generality) is universal to S, then $N(v) \cup S = \emptyset$.*

We can now prove the announced result.

Theorem 7.7. *Let G be a $2K_2$-free bipartite graph. Then* tree-$\alpha(G) \leq 2$ *if and only if G is $K_{3,3}$-free.*

Proof. Let G be a $2K_2$-free bipartite graph.

Assume first tree-$\alpha(G) \leq 2$. Suppose for contradiction that G contains $K_{3,3}$ as induced subgraph. But, since tree-$\alpha(K_{3,3}) = 3$ by Corollary 2.5 and deleting a vertex cannot increase tree-independence number, it follows tree-$\alpha(G) \geq 3$, which is a contradiction.

Conversely, assume that G is $K_{3,3}$-free. By Lemma 2.3, we can assume without loss of generality that G contains no clique cutsets. Let S be any minimal cutset of G.

We proceed by considering two separate cases:

- $|S| \geq 3$
- $|S| \leq 2$

Suppose first $|S| \geq 3$. By Lemma 7.6, if $G - S$ has at least 3 components, then G contains $K_{3,3}$ (since vertices of S form an independent set and every component has at least one vertex that is universal to S). Hence, since S is a cutset, we can assume $G - S$ has exactly two components.

Then by lemma Lemma 7.5, at most one of these two components is nontrivial.

Suppose that no component of $G - S$ is nontrivial. Then G is isomorphic to $K_{2,|S|}$ and we are done by Lemma 3.3.

Hence, we assume that there exists a nontrivial component G_1. Now if there exists at most one vertex in G_1 that is universal to S, it is clear that this vertex is a cut-vertex in G which contradicts the assumption of no clique cutsets.

On the other hand, if there are at least two vertices in G_1 that are universal to S, clearly these two vertices together with the trivial component G_2 and any three vertices in S form a $K_{3,3}$, which is a contradiction to the assumption that G is $K_{3,3}$-free.

This concludes the proof for the case when $|S| \geq 3$.

Assume now that $|S| \leq 2$. Notice that if $|S| = 1$, S is a cut-vertex in G and this contradicts our assumption that G has no clique cutsets. Thus, we assume that $|S| = 2$ and label the vertices in S as u and v arbitrarily.

Again by Lemma 7.5, $G - S$ has at most one nontrivial component. First assume that $G - S$ has no nontrivial components. Then, if we denote the components of $G - S$ by $G_1, \ldots, G_r$ for some $r \geq 2$, we can notice that for every $i \in \{1, \ldots, r\}$, the set $V(G_i) \cup S$ induces a P_3. Hence we can construct a path decomposition by taking r bags along the path and putting $V(G_i) \cup S$ in the i-th bag. Clearly this yields a valid tree decomposition of independence number 2.

If on the other hand, there exists a nontrivial component of $G - S$, say G_1, it is unique and we can observe that it is sufficient to prove that there exists a tree decomposition of $G[V(G_1) \cup S]$ that contains a bag B such that $S \subseteq B$. Indeed, if this is true, then we can apply the same argument as above, namely construct a path decomposition where each bag contains one trivial component together with S, and connect the first bag along this path to a bag in a tree decomposition of $G[V(G_1) \cup S]$ that contains S.

So consider the graph $H = G[V(G_1) \cup S]$. If G_1 is a tree, then we claim that the structure of H corresponds the structure of a graph from Lemma 7.2, with the vertices u, v from the lemma corresponding to the two vertices u, v in S.

To argue this, it is sufficient to show that every vertex in G_1 that is not universal to S has degree equal to two, as other properties follow directly from Lemma 7.6.

Indeed, if there exists a vertex in G_1 that is not universal to S and has degree 1, then the vertex adjacent to it is a cut vertex and since G has no cut vertices, this is impossible. On the other hand, if there exists a vertex x that is not universal to S and has degree at least 3, we can notice that, by Lemma 7.6 every vertex in its neighbourhood is universal to S and clearly $N[x] \cup S$ induce a $K_{3,n}$ for some $n \geq 3$, thus $K_{3,3}$ is an induced subgraph, which contradicts the assumption that G does not have an induced $K_{3,3}$.

Thus we conclude that the structure of H corresponds the structure of graphs from Lemma 7.2, and as we already argued above, there exists a tree decomposition of this graph with a bag B that contains both vertices u, v and we are done.

Thus, we assume that G_1 contains a cycle. Notice that if G_1 cannot contain a cycle of odd length, since G has to be bipartite. Similarly, since G is $2K_2$-free, the only allowed cycle is C_4.

We now prove that H has a tree decomposition with independence number 2 that has a bag containing both u and v.

Let G be a $2K_2$-free bipartite graph with no induced $K_{3,3}$. Let S be a minimal cutset of G of order two with vertices u and v and let G_1 denote the largest component of $G - S$. Let $H = G[V(G_1) \cup S]$.

We proceed by strong induction on number of cycles in the largest component of $G - S$. If this number is equal to 0, as argued above, we have the desired property.

Suppose now that for every G and H, defined as above, such that G_1 has at most $n - 1$ cycles, for some $n \geq 1$, H has a tree decomposition $\mathcal{T}$ with $\alpha(\mathcal{T}) \leq 2$ and a bag B such that $S \subseteq B$.

Consider a graph G such that its nontrivial component G_1 contains n cycles. We observe the graph H. We notice that H is still a $2K_2$-free bipartite graph with no induced $K_{3,3}$.

Furthermore, if we consider any 4-cycle *abcd* in G_1. By Lemma 7.6, there exists a pair of nonadjacent vertices of this cycle (w.l.o.g. a, c), that is universal to S.

We claim that $\{a, c\}$ is a minimal cutset in H. It is sufficient to show that if we remove $\{a, c\}$ from H, there exists no path between b and d.

We prove this by showing that the shortest paths of different lengths are not possible.

- If there exists a shortest path of length 1 between b and d, then $\{a, b, c, d\}$ induces either a diamond or a K_4, none of which is bipartite.

- If there exists a path of length 2 between b and d, label the vertices along the path by b, x_1, d, then b, u, v together with x_1, a, c form $K_{3,3}$.

- If there exists a shortest path of length $r \geq 4$ (since G is bipartite, no path of length three between b and d can exist), label the vertices along this path $b, x_1, \ldots, x_{r-1}, d$. Then the set $\{b, x_1, d, x_{r-1}\}$ induces a $2K_2$, which is again a contradiction.

Hence, H is a $2K_2$-free bipartite graph with a minimal cutset $\{a, c\}$.

Now, if $H - \{a, c\}$ has no nontrivial components, as we argued above, we can construct a path decomposition where each bag contains one trivial component and a, c. Witout loss of generality, assume that the bag containing u and the one containing v are adjacent. Then we can put another bag between them that contains u, v, a, c, which clearly induces a C_4 and we are done.

On the other hand, if $H - \{a, c\}$ has a nontrivial component H_1, observe that this component has fewer than n cycles by construction. By the induction hypothesis, there exists a tree decomposition of $H[V(H_1) \cup \{a, c\}]$ with independence number at most 2, such that a bag in this decomposition contains $\{a, c\}$ as a subset.

But then, we add all the trivial components to this composition by attaching the path decomposition as constructed above to it, and again, we may assume w.l.o.g. that the bag that contains u and the one that contains v are adjacent and again, we add a bag that contains a, c, u, v between the two bags.

Thus there exists a tree decomposition of $G[V(G_1) \cup S]$ with independence number at most 2, such that u and v are contained in a same bag and as argued above, this is sufficient to give us the desired result and conclude the proof. □

8 CONCLUSION AND FURTHER WORK

In this thesis, we considered some classes of graphs with tree-independence number at most 2.

We first gave an overview of known results and bounds on tree-independence number and why are we even studying it. Then, using these bounds and explicit tree-decomposition constructions, we gave several examples of classes of graphs that have tree-independence number at most two. We concluded that chapter by looking into intersection graphs of a graph H for various choices of H.

Then, we considered the classes of H-free graphs for various small graphs H. In particular, we considered all connected graphs of order 3 except K_3 and showed that if any one of these graphs is forbidden as an induced subgraph, we obtain an efficiently testable characterization of graphs with tree-independence number at most two. We obtained similar result for some selections of forbidden graphs H of order 4.

After this, we found three minimal obstructions for tree-independence number at most two, that is, graphs that have tree-independence number greater than 2, but whose every proper induced minor has tree-independence number at most 2. Those include the cube graph Q_3, the complete bipartite graph $K_{3,3}$, and a particular 9-vertex series parallel graph C_6^*.

Using the inequality tree-$\alpha(G) \leq \text{tw}(G) + 1$, we then observed that graphs with treewidth at most 2 necessarily have tree-independence number at most 3. We then focused on the class of series parallel graphs, a class that contains all 2-connected graphs with treewidth at most 2. We first observed that the graph C_6^* is series parallel graph with tree-independence number equal to 3, and showed that this graph is indeed the only minimal obstruction in the class of graphs with treewidth at most two. More precisely, we showed that within this class of graphs it is sufficient to forbid the graph C_6^* as an induced minor in order to bound the tree-independence number by 2.

Finally, we explored the class of chordal bipartite graphs, and in particular, we looked into some sufficient conditions for these graphs to have tree-independence number at most two, like being $\{2K_2, K_{3,3}\}$-free, or $K_{2,3}$-free.

However, even though we managed to find many graph classes for which we found constructive proofs that lead to polynomial-time algorithms for computing tree decompositions with independence number at most two, this thesis probably opened more interesting questions than it answered. Below, we list a few of the most interesting research questions related to this thesis.

Clearly, the most general and probably the hardest question to answer would be: Given a graph G, is the problem of deciding if tree-$\alpha(G) \leq 2$ NP-complete? Regardless of what the answer to this question is, it is still interesting to see if we can find some more general classes in which we can recognize graphs with tree-independence nubmer at most two efficiently.

In this thesis, we considered the class of chordal bipartite graphs. We showed that, in this class, it is sufficient to forbid $K_{2,3}$ to have tree independence number at most 2. One trivial necessary condition is that we always have to forbid $K_{3,3}$. Is this also sufficient?

We also found a couple of minimal graphs with tree-independence number at least 3. Can we find more such graphs? Is this list finite or infinite? Furthermore, given a graph G with tree-independence number at least 3, can we determine in polynomial time if G is a minimal graph with tree-independence number greater than 2? That is, can we tell if there exists a proper induced minor H of G such that tree-$\alpha(H) \geq 3$?

9 DALJŠI POVZETEK V SLOVENSKEM JEZIKU

V teoriji grafov moramo pogosto reševati probleme, ki so algoritmično zelo zahtevni. Natančneje, tudi najboljši znani algoritmi ne morejo rešiti takšnih problemov v polinomskem času.

V takšnih primerih se lahko odločimo, da vhodne podatke omejimo na način, ki nam omogoča, da problem rešimo v polinomskem času na omejeni vhodni množici. Obstaja veliko načinov omejevanja vhodnih podatkov, ki jih lahko uporabimo. Jasno je, da je glavni cilj, da je omejitev čim splošnejša, obenem pa še vedno omogoča rešitve v polinomskem času.

V zadnjem času lahko pogosto vidimo, da se kot način za učinkovito omejevanje vhoda uporabljajo različne mere strukturne zapletenosti grafa, imenovane širinski parametri grafov. Kadar je širinski parameter v neki družini grafov $\mathcal{F}$ navzgor omejen s poljubno fiksno konstanto, to pogosto omogoča razvoj učinkovitih algoritmov z uporabo dinamičnega programiranja na grafu iz družine $\mathcal{F}$, ki izkorišča strukturne lastnosti grafa, običajno z uporabo pristopa deli in vladaj na ustrezni razgradnji grafa.

Primer takšne razgradnje je drevesna dekompozicija. Drevesna dekompozicija grafa G je tak par $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$, da je T drevo in za vse $t \in V(T)$ je $X_t \subseteq V(G)$ ter velja:

- pokritje točk:
$$\bigcup_{t\in V(T)} X_t = V(G)$$
- pokritje povezav:
$$\forall uv \in E(G)\ \exists t \in V(T) : u, v \in X_t$$
- konsistentnost:
za vsako točko $u \in V(G)$ je podgraf drevesa T, induciran z množico $\{t \in V(T) \mid u \in X_t\}$ povezan.

Točke drevesa T imenujemo vreče in pravimo, da je točka v vsebovana v vreči t, če je $v \in X_t$.

Številni širinski parametri so definirani z uporabo drevesne dekompozicije. Najpomembnejša takšna parametra za to magistrsko delo sta drevesna širina in drevesno neodvisno število.

Širina drevesne dekompozicije je za ena zmanjšana moč največje vreče v dekompoziciji. Drevesna širina grafa G, označena s $\mathrm{tw}(G)$, je najmanjša možna širina drevesne dekompozicije grafa G.

Neodvisnostno število drevesne dekompozicije $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$, grafa G, označeno z $\alpha(\mathcal{T})$, je definirano kot $\alpha(\mathcal{T}) = \max_{t\in V(T)} \alpha(G[X_t])$. Drevesno neodvisnostno število grafa G je definirano kot najmanjše možno neodvisnostno število drevesne dekompozicije grafa G in označeno s tree-$\alpha(G)$.

Dallard, Milanič in Štorgel so v [10] dokazali, da so grafi z drevesnim neodvisnostnim številom največ 1 natanko tetivni grafi. Prav tako so Dallard, Fomin, Golovach, Korhonen in Milanič v [9] dokazali, da je za vsako konstanto $k \geq 4$ NP-težko odločiti, ali ima dan graf drevesno neodvisnostno število največ k.

Naravni naslednji korak je obravnava grafov z drevesnim neodvisnostnim številom največ 2. V magistrskem delu so obravnavani razredi grafov z drevesnim neodvisnostnim številom največ 2 in so izpeljani naslednji rezultati:

- Našli smo različne primere družin grafov z drevesnim neodvisnostnim številom največ 2.
- V [10] so Dallard, Milanič in Štorgel dokazali, da če je H induciran minor grafa G, tedaj je tree-$\alpha(H) \leq$ tree-$\alpha(G)$. Zato definiramo, da je G minimalen graf z drevesnim neodvisnostnim številom, večjim od k, če je tree-$\alpha(G) > k$ in za vsak pravi induciran minor H grafa G velja tree-$\alpha(H) \leq k$. V magistrskem delu so najdeni naslednji grafi, ki so minimalni z drevesnim neodvisnostnim številom, večjim od 2: kocka (Q_3), poln dvodelni graf $K_{3,3}$ in graf C_6^*, prikazan na sliki 7.
- Za različne izbire majhnih grafov H smo dokazali, da bodisi za vsak H-prost graf G velja, da je tree-$\alpha(G) \leq 2$, ali pa za vsak H-prost graf G velja, da je tree-$\alpha(G) \leq 2$, če in samo če graf G ne vsebuje induciranega podgrafa, izomorfnega grafu $K_{3,3}$.
- Obravnavali smo nekaj podrazredov tetivnih dvodelnih grafov, kot so na primer $2K_2$-prosti dvodelni grafi, ali $K_{2,3}$-prosti tetivni dvodelni grafi.
- Grafi ki imajo drevesno širino omejeno z 2, imajo drevesno neodvisnostno število omejeno s 3. Naj bo C_6^* graf, ki je prikazan na sliki 7. V magistrskem delu smo dokazali, da je drevesno neodvisnostno število grafa G z drevesno širino največ 2 omejeno z 2 natanko takrat ko graf G ne vsebuje induciranega minorja izomorfnega grafu C_6^*, sicer pa je enako 3.

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