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(Magistrsko delo)

A Complexity Study of Distance Variants of Covering and Domination Problems in H-Free Graphs

(Študija zahtevnosti razdaljnih variant problemov pokritja in dominacije v H-prostih grafih)

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Izvleček:
Različne teoretične in praktične motivacije so privedle do posplošitve številnih klasičnih optimizacijskih problemov na grafih na njihove razdaljne variante. Grobo rečeno to pomeni, da se lastnost sosednosti, ki je osnova za definicijo dopustne rešitve problema, nadomesti s splošnejšo lastnostjo, ki temelji na razdaljah v grafih.
V magistrskem delu obravnavamo razdaljne različice naslednjih štirih optimizacijskih problemov na grafih: problem dominantne množice, problem povezavno dominantne množice, problem točkovnega pokritja in problem povezavnega pokritja. Preučimo razmerja med optimalnimi vrednostmi vseh štirih problemov in algoritmično zahtevnost izračuna optimalnih rešitev pod določenimi omejitvami.
Natančneje, v magistrskem delu obravnavamo probleme izračuna namanjše velikosti k-razdaljne dominantne množice, k-razdaljne povezavno dominantne množice, k-razdaljnega točkovnega pokritja in k-razdaljnega povezavnega pokritja. Za vsak $k \geq 1$ in za vsakega od ustreznih štirih problemov popolnoma karakteriziramo družino grafov H, za katere je problem rešljiv v polinomskem času v razredu H-prostih grafov (pod predpostavko, da je $\mathsf{P} \neq \mathsf{NP}$).

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Abstract:
Various theoretical and practical motivations have led to generalizations of many classical graph optimization problems to their distance-based variants. Informally, this means that the adjacency property used to define a feasible solution to the problem is replaced with a relaxed property based on distances in graphs.
In this thesis, we focus on distance-based variants of the following four problems: the dominating set, edge dominating set, vertex cover, and edge cover problems. We consider the relationships between the optimal solution values of the corresponding problems and the algorithmic complexity of their computation under certain restrictions.
More specifically, we study the distance-k dominating set, distance-k edge dominating set, distance-k vertex cover, and distance-k edge cover problems. For every $k \geq 1$ and for each of the four problems, we completely characterize the family of graphs H such that the problem is solvable in polynomial time in the class of H-free graphs (under the assumption that $\mathsf{P} \neq \mathsf{NP}$).

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1 Introduction

Various theoretical and practical motivations have led to generalizations of many classical graph optimization problems to their distance-based variants. Informally, this means that the adjacency property used to defined a feasible solution to a problem is replaced with a relaxed property based on distances in the graph.

In this thesis we consider four optimization problems on graphs and their distance-based variants. The problems are related to the following four types of vertex and edge subsets in a graph. A *dominating set* in a graph G is a set of vertices such that every vertex is either in the set or adjacent to a vertex in the set. An *edge dominating set* in a graph G is a set of edges such that every edge is either in the set or shares a common endpoint with an edge in the set. A *vertex cover* in a graph G is a set of vertices intersecting all edges. An *edge cover* in a graph G is a set of edges such that each vertex is incident with at least one edge in the set. Note that a graph G has an edge cover if and only if it does not have any isolated vertices.

The distance-based variants of the above concepts, along with the corresponding optimization problems—which we present in their decision form—are defined as follows. For a positive integer k, a *distance-k dominating set* in a graph G is a set D of vertices such that every vertex is at distance at most k from a vertex in D. Note that a distance-1 dominating set is the same thing as a dominating set. The DISTANCE-k DOMINATING SET problem is the problem of deciding, given a graph G and an integer ℓ, whether G contains a distance-k dominating set of size at most ℓ.

For a non-negative integer k, a *distance-k edge dominating set* in a graph G is a set F of edges such that every edge has an endpoint that is at distance at most k from an endpoint of an edge in F. Note that a distance-0 edge dominating set is the same thing as an edge dominating set. The DISTANCE-k EDGE DOMINATING SET problem is the problem of deciding, given a graph G and an integer ℓ, whether G contains a distance-k edge dominating set of size at most ℓ.

For a non-negative integer k, a *distance-k edge cover* in a graph G is a set F of edges such that every vertex is at distance at most k from an endpoint of an edge in F. Note that a distance-0 edge cover is the same thing as an edge cover. The DISTANCE-k EDGE COVER problem is the problem of deciding, given a graph G and an integer ℓ, whether G contains a distance-k edge cover of size at most ℓ.

Finally, for a non-negative integer k, a *distance-k vertex cover* in a graph G is a set C of vertices such that every edge has an endpoint which is at distance at most k from a vertex in C. Note that a distance-0 vertex cover is the same thing as a vertex cover. The Distance-k Vertex Cover problem is the problem of deciding, given a graph G and an integer ℓ, whether G contains a distance-k vertex cover of size at most ℓ.

Observe that, informally speaking, for the Distance-k Dominating Set and Distance-k Vertex Cover problems the requirement is to "dominate" (or "cover") at distance all the vertices or edges, respectively, *with vertices*. On the other hand, for the Distance-k Edge Cover and Distance-k Edge Dominating Set problems the vertices or edges, respectively, are dominated at distance *with edges*. Similarly, the Distance-k Dominating Set and Distance-k Edge Cover problems share the property that the objects being dominated are *vertices*, while in the Distance-k Dominating Set and Distance-k Edge Cover problems the objects being dominated are *vertices*.

The smallest interesting choices for k in the definitions of the above problems (that is, $k = 1$ for the Distance-k Dominating Set problem and $k = 0$ for all the other three problems) lead to the well-known Dominating Set, Edge Dominating Set, Edge Cover, and Vertex Cover problems. While the Edge Cover problem is known to be solvable in polynomial time for all graphs by matching techniques (see, e.g., [66]), the Dominating Set, Edge Dominating Set, and Vertex Cover problems are NP-complete for graphs in general (see, e.g., [31]). Moreover, the problems stay NP-complete on some specific graph classes as well (see Chapter 3). Then, a natural question that one can ask is: for which graph classes do the problems remain NP-complete and for which do they become polynomial-time solvable? The same goes for the distance-based variants of the four problems. Those distance-based variants are the focus of this thesis. We look specifically for classes of graphs where we prohibit a specific graph as an induced subgraph.

As an example of a known result of this form, consider the Dominating Set problem. For integers $k \geq 1$, $s \geq 0$, and $t \geq 1$, we denote by $P_k + sP_t$ the disjoint union of a k-vertex path and s copies of the t-vertex path. In 1992, Korobitsin showed that the Dominating Set problem is NP-complete in the class of H-free graphs, unless H is an induced subgraph of some $P_4 + sP_1$, in which case it is solvable in polynomial time (see [47]). That is, he gave a complexity dichotomy in classes of H-free graphs for the Dominating Set problem. We do the same thing for the distance-based variants of the four problems.

We develop the following computational complexity dichotomies. For the distance problems where vertices need to be dominated we show the following dichotomy for any $k \geq 1$ and arbitrary graph H:

- The Distance-k Dominating Set and Distance-k Edge Cover problems are solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and NP-complete otherwise.

For the distance problems where edges need to be dominated we show the following dichotomies for an arbitrary graph H:

- A dichotomy for $k = 1$: the Distance-1 Edge Dominating Set and Distance-1 Vertex Cover problems are solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_4 + sP_2$ for some $s \geq 0$, and NP-complete otherwise.

- A dichotomy for any $k \geq 2$: the Distance-k Edge Dominating Set and Distance-k Vertex Cover problems are solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and NP-complete otherwise.

The thesis is organized as follows. In Chapter 2 we give the necessary definitions and basic graph concepts as well as the problem definitions. In Chapter 3 we survey some of the most important known results about the NP-complete and polynomial-time solvable special cases for the four classical problems and their distance-based variants, discuss what is known regarding their approximability, and see how the problems differ with respect to their parameterized complexity. Chapter 4 presents relations between the optimal solution values of the minimization variants of the four distance-based problems, for each fixed k. In Chapter 5 we consider a number of graph transformation and study their effect on the optimal solution values of these problems, as well as of some other newly defined problems. Then, in Chapter 6 we use the results from Chapter 5 to establish the NP-completeness results for the four main distance-based problems. In Chapter 7, we identify families of graph classes in which the problems are polynomial-time solvable, relying, among others, on the result from Chapter 4. Finally, in Chapter 8 we give a complexity dichotomy for every one of the four distance-based problems. We conclude the thesis in Chapter 9 with a brief summary of the thesis and a statement of some of the key related open problems.

Some of the results presented in this thesis were published in [25].

2 Preliminaries

2.1 Basic concepts about graphs

Before we start with the main topics, let us first overview some basic terminology, notations, and definitions about graphs. We only consider finite, simple, and undirected graphs. A graph will be denoted by $G = (V, E)$ where V is the vertex set and E the edge set of G. Sometimes we will denote the vertex and edge set of a graph G by $V(G)$ and $E(G)$, respectively. The *order* of G is the number of vertices in it. A graph is *nontrivial* if it is of order more than one, that is, a graph with at least two vertices. We denote by $N(v)$ (or $N_G(v)$ if the graph is not clear from the context) the set of neighbors of vertex v in G, that is, the (open) neighborhood of v, and by $N[v] := N(v) \cup \{v\}$ (or $N_G[v]$), the closed neighborhood of v. The *degree* of a vertex v is the cardinality of $N(v)$. An *isolated vertex* in a graph G is a vertex of degree 0. An *isolated edge* in a graph G is an edge whose endpoints have degree one. A graph is said to *cubic* if every if every vertex has degree three, and *subcubic* if every vertex has degree at most three.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* of a graph G is any graph H such that $V(H) \subseteq V(G)$ and $E(H) = \{\{u, v\} \in E(G) : u, v \in V(H)\}$. Given a graph G and a set $S \subseteq V(G)$, we denote by $G[S]$ the *subgraph of* G *induced by* S, that is, the unique induced subgraph of G with vertex set S. Given two graphs G and H, we say that G is H*-free* if no induced subgraph of G is isomorphic to H. More generally, for graphs $H_1, \ldots, H_p$, we say that G is $\{H_1, \ldots, H_p\}$-free if G is H_i-free for all $i \in \{1, \ldots, p\}$. Given two graphs G and H, we denote by $G + H$ their disjoint union. For a non-negative integer s, we denote by sG the disjoint union of s copies of G.

A *path* in a graph G is a sequence $v_1, \{v_1, v_2\}, v_2, \ldots, \{v_{p-1}, v_p\}, v_p$ of distinct vertices v_i $(1 \leq i \leq p)$ of G and edges $\{v_{j-1}, v_j\}$ $(2 \leq j \leq p)$. Such a path is said to be a v_1, v_p*-path*, the vertices v_1 and v_p are the *endpoints* of the path, while the vertices $v_2, \ldots, v_{p-1}$ are its *internal vertices*. We will sometimes identify a path in a graph G with the corresponding subgraph in G. For a positive integer k, we denote by P_k the *path graph* of order k, that is, the graph with vertex set $\{v_1, \ldots, v_k\}$, in which two vertices v_i and v_j with $i < j$ are adjacent if and only if $j = i + 1$. A *cycle* in a graph G is a sequence $v_1, \{v_1, v_2\}, v_2, \ldots, \{v_p, v_1\}, v_1$ such that $p \geq 3$, $v_1, \ldots, v_p$ are pairwise distinct vertices of G, and $\{v_{j-1}, v_j\}$ for all $2 \leq j \leq p$ and $\{v_p, v_1\}$ are edges of G. For an integer $k \geq 3$, we denote by C_k the *cycle graph* of order k, that is, the graph with vertex set $\{v_1, \ldots, v_k\}$, in which two vertices v_i and v_j with $i < j$ are adjacent if and only if $j = i + 1$ or $(i, j) = (1, k)$. The *length* of a path or a cycle is the number of edges in it. A graph G is *complete* if every two distinct vertices of G are adjacent; a complete graph of order n is denoted by K_n. A graph G is *connected* if for every two $u, v \in V(G)$, there is a path in G with endpoints u and v. A graph G' is a *connected component* of a graph G if G' is an induced subgraph of G and for every $u' \in V(G')$ and $u \in V(G) \setminus V(G')$ there is no path in G with endpoints u' and u.

The *girth* of a graph G is the minimum length of a cycle in G (or ∞ is G is acyclic). The *distance* between two vertices u and v in G is defined as the length of a shortest path between u and v (or ∞ if there is no u, v-path in G). Given two sets $A, B \subseteq V(G)$, we denote by $\mathsf{dist}_G(A, B)$ the minimum over all distances in G between a vertex in A and a vertex in B. When clear from context, we may simply write $\mathsf{dist}(A, B)$. For simplicity, if A contains a unique element a, then we may simply write $\mathsf{dist}(a, B)$, and similarly for B. For an edge e and a set of edges F, we denote by $\mathsf{dist}(e, F)$ the minimum over all distances between an endpoint of e and an endpoint of an edge in F. For $A \subseteq V(G)$ and a set of edges F, we denote by $\mathsf{dist}(A, F)$ the minimum over all distances between a vertex in A and an endpoint of an edge in F. For simplicity, if A contains a unique element a, then we may simply write $\mathsf{dist}(a, F)$, and similarly for F.

An *independent set* in a graph G is a set of pairwise non-adjacent vertices; a *clique* is a set of pairwise adjacent vertices. A clique is *maximal* if it is not contained in any larger clique. A *matching* in a graph G is a set M of edges of G such that no two of them share an endpoint. A matching is *maximal* if it is not contained in any larger matching. An *induced matching* in a graph G is a matching M such that G contains no edge whose endpoints belong to different edges of M.

The operation of *subdividing* en edge $\{u, v\}$ in a graph G means deleting the edge and introducing a new vertex w adjacent precisely to u and v.

The *claw* is the graph with four vertices and three edges, all having an endpoint in common. A *fork* is a graph with vertex set $\{v_1, v_2, \ldots, v_5\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_5\}\}$, that is, a graph obtained from the claw by subdividing one of its edges. The *line graph* of a graph G is the graph, denoted by $L(G)$, with vertex set $E(G)$ in which two distinct vertices are adjacent if and only if the corresponding edges of G have an endpoint in common. It is well known, and easily observed, that line graphs are claw-free.

A *chord* in a cycle is an edge that does not belong to the cycle but connects two vertices of the cycle. A graph is *chordal* if it does not contain an induced cycle of length at least four, *strongly chordal* if it is chordal and every cycle of even length greater than six has an odd chord, that is, an edge that connects two vertices that are an odd distance, greater than one, apart from each other in the cycle, *bipartite* if its vertex set can be partitioned into two independent sets, and *planar* if it can be drawn on the plane with no two edges crossing. The *clique graph* of G has the maximal cliques of G as vertices, two of them being adjacent if and only if their intersection is not empty. A graph is said to be *dually chordal* if it is the clique graph of some chordal graph. Given a finite family of sets $S_1, \ldots, S_n$, its *intersection graph* is the graph G obtained by creating one vertex v_i for each set S_i, and connecting two vertices v_i and v_j by an edge whenever the corresponding two sets have a nonempty intersection, that is, $E(G) = \{\{v_i, v_j\} \mid i \neq j, S_i \cap S_j \neq \emptyset\}$. A graph is an *interval graph* if it is the intersection graph of a family of closed intervals on the real line. A *circle graph* is the intersection graph of a family of chords of a circle. The *chromatic number* of a graph is the smallest number of colors needed to color the vertices of a graph so that no two adjacent vertices have the same color. A graph in which the chromatic number of every induced subgraph equals the maximum size of a clique in that subgraph is called *perfect*.

A *cut-vertex* in a graph G is a vertex v such that the graph $G[V(G) \setminus \{v\}]$ has more connected components than G. A *block* of a graph G is a maximal connected subgraph of G that does not contain any cut-vertex of G. A graph G is a *block graph* if every block of G is complete. A *undirected path graph* is the intersection graph of the vertex sets of a family of paths in a tree. The class of undirected path graphs is a superclass of interval graphs, which are exactly the intersection graphs of subpaths of a path, and a subclass of chordal graphs, which are exactly the intersection graphs of subtrees of a tree. If $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ is any permutation of the numbers from 1 to n, then one may define a *permutation graph* from σ, in which there are n vertices $v_1, v_2, \ldots, v_n$, and in which there is an edge $v_i v_j$ for any two indices i and j for which $i < j$ and $\sigma_i > \sigma_j$. A *linear forest* is a disjoint union of path graphs.

Three vertices of a graph form an *asteroidal* triple if every two of them are connected by a path avoiding the closed neighbourhood of the third. A graph is *AT-free* if it does not contain any asteroidal triple.

A *vertex cover* of a graph G is a subset $C \subseteq V(G)$ such that every edge $e \in E(G)$ has an endpoint in C. We denote by $\tau(G)$ the minimum size of a vertex cover of G. A *minimum vertex cover* of a graph G is a vertex cover with size $\tau(G)$. Given a non-negative integer k and a graph G, a *distance-k vertex cover* in G is a set C of vertices such that for all edges $e \in E(G)$, it holds $\mathsf{dist}(e, C) \leq k$. We denote by $\tau_k(G)$ the minimum size of a distance-k vertex cover of G. A *minimum distance-k vertex cover* of a graph G is a distance-k vertex cover with size $\tau_k(G)$. Note that $\tau(G) = \tau_0(G)$.

An *edge cover* of a graph G is a subset $F \subseteq E(G)$ such that every vertex $u \in V(G)$ is incident with an edge in F. Given a graph G without isolated vertices, we denote by $\rho(G)$ the minimum size of an edge cover of G. A *minimum edge cover* of G is an edge cover with size $\rho(G)$. Given a non-negative integer k and a graph G, a *distance-k edge cover* in G is a set F of edges such that for all vertices $u \in V(G)$, it holds $\mathsf{dist}(u, F) \leq k$.

Given a graph G without isolated vertices and an integer $k \geq 0$, we denote by $\rho_k(G)$ the minimum size of a distance-k edge cover of G and refer to a distance-k edge cover of G with size $\rho_k(G)$ as a *minimum distance-k edge cover* of G. Note that $\rho(G) = \rho_0(G)$.

A *dominating set* of a graph G is a subset $D \subseteq V(G)$ such that every vertex $u \in V(G)$ is either in D or adjacent to a vertex in D. We denote by $\gamma(G)$ the minimum size of a dominating set in G. A *minimum dominating set* of a graph G is a dominating set of with size $\gamma(G)$. A *connected dominating set* in a (connected) graph G is a dominating set D such that $G[D]$ is a connected graph. Given a positive integer k and a graph G, a *distance-k dominating set* in G is a set D of vertices such that for all vertices $u \in V(G)$, it holds $\mathsf{dist}(u, D) \leq k$. We denote by $\gamma_k(G)$ the minimum size of a distance-k dominating set of G. A *minimum distance-k dominating set* of a graph G is a distance-k dominating set of G with size $\gamma_k(G)$. Note that $\gamma(G) = \gamma_1(G)$.

An *edge dominating set* of a graph G is a subset $F \subseteq E(G)$ such that every edge $e \in E(G)$ is either in F or shares an endpoint with an edge in F. We denote by $\gamma'(G)$ the minimum size of an edge dominating set in G. A *minimum edge dominating set* of a graph G is an edge dominating set with size $\gamma'(G)$. Given a non-negative integer k and a graph G, a *distance-k edge dominating set* in G is a set F of edges such that for all edge $e \in E(G)$, it holds $\mathsf{dist}(e, F) \leq k$. We denote by $\gamma'_k(G)$ the minimum size of a distance-k edge dominating set of G. A *minimum distance-k edge dominating set* of a graph G is a distance-k edge dominating set with size $\gamma'_k(G)$. Note that $\gamma'(G) = \gamma_0(G)$.

Let G be a graph. A *P_3 factor* of G is a spanning subgraph of G whose components are isomorphic to P_3. A *P_3 cover* of G is a set of 3-vertex paths in G such that each vertex of the graph is a vertex of at least one of these paths. We denote by $\Lambda(G)$ the minimum size of a P_3 cover in G. Note that a P_3 factor is a special case of the P_3 cover. For an integer $k \geq 0$, we say that a *distance-k P_3 dominating set* $\mathcal{P}$ of G is a set of 3-vertex paths in G such that the vertex set of every 3-vertex path of G is at distance at most k from the vertex set of some 3-vertex path in $\mathcal{P}$. For a graph G in which every vertex is contained in some 3-vertex path (that is, G does not have any isolated vertices and isolated edges), we denote by $\Lambda_k(G)$ the minimum size of a distance-k P_3 dominating set in G. A *distance-k P_3 edge dominating set* $\mathcal{P}$ of G is a set of 3-vertex paths in G such that every edge of G is at distance at most k from the vertex set of some 3-vertex path in $\mathcal{P}$. For a graph G in which every edge is contained in some 3-vertex path (that is, G does not have any isolated edges), we denote by $\Lambda_k^e(G)$ the minimum size of a distance-k P_3 edge dominating set in G. A *distance-k edge P_3 dominating set* F of G is a set of edges in G such that the vertex set of every 3-vertex path of G is at distance at most k from an edge in F. We denote by $\varepsilon_k^{\Lambda}(G)$ the minimum size of a distance-k edge P_3 dominating set in G.

An (integer) *graph invariant* is a mapping from the class of all graphs to the set $\mathbb{N}$ of non-negative integers that is constant on any set of pairwise isomorphic graphs.

Some "invariants" corresponding to minimization problems are not defined on the class of all graphs. For example, the value of $\rho_k(G)$, the minimum size of a distance-k edge cover of G, is not defined if G has an isolated vertex. In such cases, it is convenient to extend the co-domain $\mathbb{N}$ of the mapping to the set $\mathbb{N} \cup \{\infty\}$ and assign value ∞ to each graph on which the value of the invariant would otherwise not be defined.

For graph theoretic notions not defined here, we refer to West [66].

2.2 Problem definitions

Before we give formal definitions of our problems let us first recall the definitions of certain classes of problems defined with respect to the computational complexity of a problem. A problem that for a given input asks whether the answer to some question is *yes* or *no* is called a *decision problem*. On the other hand, an *optimization problem* is the problem of finding an optimal solution from all feasible solutions. More precisely, an *instance* of an optimization problem is an ordered triple $(\mathcal{S}, f, opt)$, where $\mathcal{S}$ is an (implicitly given) set of feasible solutions, $f : \mathcal{S} \to \mathbb{R}$ is the objective function and $opt \in \{\min, \max\}$ is the type of the problem (minimization or maximization). We are looking for the value of $OPT := opt\{f(x) \mid x \in \mathcal{S}\}$. For example, for a given graph G, the problem of computing its vertex cover number $\tau(G)$ is an optimization problem in which each instance is an ordered triple $(\mathcal{S}_G, f_G, opt)$, where $\mathcal{S}_G$ is the set of all vertex covers of the graph G, the function f_G assigns to each vertex cover of G its cardinality, and $opt = \min$.

Given a decision or optimization problem Π, we say that Π is *solvable in polynomial time* if there exists an algorithm that solves Π in time that is bounded by a polynomial function of the input size.

The complexity class P consists of decision problems solvable in polynomial time. A decision problem Π is said to be *solvable in non-deterministic polynomial time* if for any input I such that $\Pi(I)$ gives answer *yes* there exists a certificate C such that using C, the fact that $\Pi(I)$ gives answer *yes* can be verified in time polynomial in the size of input I. We denote by NP the complexity class of all such problems. Note that for a problem Π in NP it may not be known whether there exists a polynomial-time algorithm that solves it.

A problem Π is said to be NP-*hard* if the existence of a polynomial time algorithm that solves Π implies the existence of a polynomial time algorithm for any problem in the class NP. A problem that is both in NP and NP-hard is said to be NP-*complete*. Note that not all problems that are NP-hard are decision problems.

A decision problem Π_1 can be *polynomially reduced* to a decision problem Π_2 if there exists a function f that, given any input I_1 for Π_1, constructs an input $I_2 = f(I_1)$ for

Π_2 and has the following properties: $f(I_1)$ can be computed in time that is polynomial in the size of I_1 and problem Π_1 has answer yes for input I_1 if and only if problem Π_2 has answer yes for input $f(I_1)$.

We can show that a problem Π is NP-complete by showing that it is in NP and that there exists an NP-complete problem Π' that can be polynomially reduced to Π.

Let Π be an optimization problem such that for every instance $(\mathcal{S}, f, opt)$ of the problem and every feasible solution $x \in \mathcal{S}$, the objective function value is strictly positive (that is, $f(x) > 0$). A *ρ-approximation algorithm* for Π is an algorithm A that runs in polynomial time and for every instance of Π, outputs a feasible solution with objective function value within a factor of ρ of true optimum for the instance. More specifically, if Π is a minimization problem, then for every instance I of Π we have $f_A(I) \leq \rho OPT(I)$, where $f_A(I)$ is the value of the solution returned by the algorithm and $OPT(I)$ is the optimal solution value. Similarly, for maximization problems, $f_A(I) \geq \frac{OPT(I)}{\rho}$.

Let us now formally define our four decision problems.

DISTANCE-k DOMINATING SET
Instance: A graph G and an integer ℓ.
Question: Is there a distance-k dominating set in G with size at most ℓ?

DISTANCE-k EDGE DOMINATING SET
Instance: A graph G and an integer ℓ.
Question: Is there a distance-k edge dominating set in G with size at most ℓ?

DISTANCE-k VERTEX COVER
Instance: A graph G and an integer ℓ.
Question: Is there a distance-k vertex cover in G with size at most ℓ?

DISTANCE-k EDGE COVER
Instance: A graph G and an integer ℓ.
Question: Is there a distance-k edge cover in G with size at most ℓ?

The DISTANCE-k DOMINATING SET problem was introduced by Slater [63] and Henning et al. [35]. For surveys on distance k-domination, we refer to [36, 37].

We will also need the following problem, which was shown to be NP-complete by Kirkpatrick and Hell [45].

P_3 FACTOR
Instance: A graph G.
Question: Is there a P_3 factor in G?

3 A survey of known algorithmic results on covering and domination problems and their distance variants

As already mentioned in the introduction, the basic variants of the four problems were studied before and even though Edge Cover is known to be polynomial-time solvable, the other three problems are known to be NP-complete. Moreover, the problems are known to remain NP-complete on some specific graph classes as well. A natural question that one can ask is: for which graph classes do the problems remain NP-complete and for which do they become polynomial-time solvable? The same goes for the distance variants of the four problems. This chapter gives a brief overview of some known hardness and polynomial results for all the four mentioned problems and their distance variants. We mainly present the most general available polynomial-time results and NP-completeness results for as restricted graph classes as possible.

We also summarize the known approximation results for the problems and have a look at what is known about the parameterized complexity of the classical variants of the domination and covering problems we considered.

3.1 NP-complete and polynomial-time solvable special cases

We start of with the two domination problems and their distance-based variants and summarize the main graph classes for which the problems are known to be NP-complete or polynomial-time solvable.

The Distance-k Dominating Set problem is NP-complete for graphs in general, and remains NP-complete for bipartite graphs and for chordal graphs (see [16]). On the other hand, it is polynomial-time solvable for permutation graphs (see [60]), strongly chordal graphs (see [15]), dually chordal graphs (which generalize strongly chordal graphs, see [10]), AT-free graphs (see [17]) and graphs of bounded mim-width if a decomposition tree is given (see [40]).

In the case when $k = 1$, that is, for the Dominating Set problem, it was additionally shown that the problem is NP-complete on split graphs (see [6] and [22]), undirected path graphs (see [9]), as well as for planar graphs of maximum degree 3 (see [31]) and unit disk graphs (see [20]). Moreover, there is a dichotomy by Korobitsyn [47] showing that the Dominating Set problem is NP-complete if the class of H-free graphs unless H is an induced subgraph of some $P_4 + sP_1$, in which case it is solvable in polynomial time.

Contrary to the Dominating Set problem, in the case of the Edge Dominating Set problem much more is known about the basic variant of the problem than about its distance-based generalizations. In particular, it is known that the Edge Dominating Set problem is NP-complete on subcubic bipartite graphs and subcubic planar graphs (see [67]) as well as on planar bipartite graphs, planar cubic graphs, and line graphs of planar bipartite graphs (see [38]).

The problem is polynomial-time solvable for bipartite permutation graphs (see [64] and [50]), graphs of bounded clique-width (see [29], [46], and [55]), complements of chordal graphs (which generalize the class of split graphs, see [64]), and, more generally, sP_2-free graphs, for any positive integer s. This latter result follows from the fact that each sP_2-free graph has at most polynomially many maximal independent sets (see [2] or [5]) and the fact that the Edge Dominating Set problem is solvable in polynomial time in any class of graphs with at most polynomially many independent sets, which in turn follows from a result from [64] and the fact that in a graph with polynomially many maximal independent sets, all maximal independent sets can be computed in polynomial time (see [21, 51, 65]).

On the other hand, to the best of our knowledge, it is not known for which graph classes the Distance-k Edge Dominating Set problem is NP-hard for any $k \geq 1$. However, for all $k \geq 1$ the problem is polynomial-time solvable for graphs of bounded treewidth. This follows from a meta-theorem of [4] and a result showing that given a graph with treewidth k, a tree decomposition of width at most k can be computed in linear time [7].

Now we will have a look at the covering problems and their distance-based variants and give an overview of the graph classes for which the problems are known to be NP-complete or polynomial-time solvable.

First, the Vertex Cover problem is NP-complete on cubic planar graphs (see [54]) and dually chordal graphs (see [13]), as well as on the class of H-free graphs whenever H has a component that is not a path or a subdivision of the claw (see [1]). For the polynomial-time solvable cases, we first observe that the Vertex Cover problem is equivalent to the Independent Set problem, which takes as input a graph G and an integer ℓ and asks if G contains an independent set with size at least ℓ. In any graph class in which the Independent Set problem is solvable in polynomial time, so is the Vertex Cover. Hence, we obtain the following list of graph classes for which Vertex Cover is polynomial-time solvable: perfect graphs (see [33]) and their subclasses (see [32]), AT-free graphs (see [12]), and graphs of bounded clique-width, which follows from a meta-theorem of Courcelle et al. [23] and a result of Oum [55] showing that an $f(k)$-expression of a graph with clique-width at most k can be computed in polynomial time.

For graph classes defined by forbidding a fixed graph as an induced subgraph the Vertex Cover problem is polynomial-time solvable on claw-free graphs (see two independent works [53] and [62]) and more generally fork-free graphs (see [3]), for H-free graphs where each component of H is a claw (see [11]), as well as for P_6-free graphs (see [34]).

The Distance-k Vertex Cover problem is NP-complete on graphs in general for $k = 1$ (see [39]), as well as for all $k \geq 2$ (see [61]). In the case when $k = 1$, that is, for the Distance-1 Vertex Cover problem, we also have that the problem is NP-complete for bipartite and chordal graphs (see [52]), planar and circle graphs (see [48]), undirected path graphs (see [58]), subcubic bipartite planar and cubic planar graphs (see [68]) and unit disk graphs (see [41]).

Regarding polynomial-time solvability, it is known that, in contrast to the case $k = 0$, for $k \geq 1$ the Distance-k Vertex Cover problem is polynomial-time solvable on dually chordal graphs (which include strongly chordal graphs and interval graphs (see [13]). Let us also note that unless P = NP, the polynomial-time solvability of Distance-k Vertex Cover for strongly chordal graphs for $k \geq 1$ cannot be generalized to the class of perfect graphs, in fact, not even to the class of chordal graphs, for which NP-completeness of the problem is established in this thesis, see Theorem 6.20 on p. 43. Also, the problem is polynomial-time solvable on graphs of bounded clique-width (see [23, 55]). In the case when $k = 1$, the Distance-1 Vertex Cover is polynomial-time solvable for for bipartite permutation graphs (see [57]).

The Edge Cover problem, on the other hand, is known to be polynomial-time solvable for graphs in general. However, for the Distance-k Edge Cover problem, to the best of our knowledge, the only known result is in the case when $k = 1$, where Lewis [48] showed that the problem is NP-complete for bipartite graphs. It follows from [4] and [7] that for all $k \geq 1$ the Distance-k Edge Cover is polynomial-time solvable on graphs of bounded treewidth.

3.2 Approximation algorithms

To discuss the approximability aspects of the considered problems, we need to restrict ourselves to the optimization variants of the problems. For example, the Minimum Dominating Set problem is defined as the optimization problem that takes as input a graph G and the task is to compute $\gamma(G)$, the minimum size of a dominating set in G. The other optimization problems relevant for the discussion in this section are derived analogously from the corresponding decision problems.

Again, we first start with the known results for the domination problems. Given a graph $G = (V, E)$, a natural greedy algorithm provides a factor $1 + \ln |V|$ approximation of a minimum dominating set. This follows from the fact that the MINIMUM DOMINATING SET problem is a special case of the SET COVER problem: given a collection of subsets of a finite set V, find a smallest subcollection whose union equals V. The greedy algorithm provides a factor $1 + \ln |V|$ approximation of the SET COVER problem [19].

Using the analogous result for the SET COVER, proved by Dinur and Steurer [26], one can show that for any $\epsilon > 0$ there is no $((1 - \epsilon) \ln |V|)$-approximation algorithm for the MINIMUM DOMINATING SET problem, unless $\mathsf{P} = \mathsf{NP}$, see [8, Theorem 6.5].

On the other hand, to the best of our knowledge, the distance version of the dominating set problem, that is, the MINIMUM DISTANCE-k DOMINATING SET problem for $k \geq 2$, has not been studied in the literature from the approximation point of view. However, it can be observed that the MINIMUM DISTANCE-k DOMINATING SET problem is also a special case of the SET COVER problem (where the ground set is the vertex of the input graph), and hence the greedy algorithm provides a factor $(1 + \ln |V|)$ approximation of the MINIMUM DISTANCE-k DOMINATING SET problem.

For the MINIMUM EDGE DOMINATING SET problem, let us first remark that this problem is known to have the same optimal value as the MINIMUM MAXIMAL MATCHING problem, the problem of computing the minimum cardinality of a maximal matching [67]. As a consequence, any maximal matching M gives a 2-approximation to the MINIMUM EDGE DOMINATING SET problem. Indeed, M is an edge dominating set; furthermore, M can be at most twice as large as a smallest maximal matching, and, as mentioned above, a smallest maximal matching has the same size as a smallest edge dominating set.

For any $\epsilon > 0$ there is no $(\frac{7}{6} - \epsilon)$-approximation algorithm for the MINIMUM EDGE DOMINATING SET problem, unless $\mathsf{P} = \mathsf{NP}$ ([18]). A stronger bound is also known, as follows. Escoffier et. al showed that for any $\rho \geq 1$, if there is a ρ-approximation algorithm for MINIMUM EDGE DOMINATING SET, then there exists a $(2\rho - 1)$-approximation algorithm for MINIMUM VERTEX COVER ([28]). This result, combined with a $(\sqrt{2} - \epsilon)$-inapproximability result on MINIMUM VERTEX COVER due to Khot, Minzer, and Safra [44] (see below) implies that for any $\epsilon > 0$ there is no $(\frac{\sqrt{2}+1}{2} - \epsilon)$-approximation algorithm for the EDGE DOMINATING SET problem, unless $\mathsf{P} = \mathsf{NP}$.

Again, to the best of our knowledge, the DISTANCE-k EDGE DOMINATING SET problem and its optimization version have not yet been studied in the literature for any $k \geq 1$. Since the MINIMUM DISTANCE-k EDGE DOMINATING SET problem is a special case of the SET COVER problem (where the ground set is the edge set of the input graph), the greedy algorithm provides a factor $1 + \ln |E|$ approximation of the

Minimum Distance-k Edge Dominating Set problem.

As for the covering problems we have the following.

It is known that computing any inclusion maximal matching and returning the set of vertices saturated by the matching gives a 2-approximation algorithm for the Minimum Vertex Cover problem (see [56]). On the other hand, for any $\epsilon > 0$ there is no $(\sqrt{2} - \epsilon)$-approximation algorithm for the Minimum Vertex Cover problem, unless $\mathsf{P} = \mathsf{NP}$ (see [44]).

Regarding the Minimum Distance-k Vertex Cover problem for $k \geq 1$, we are only aware of a result by Lewis [48] showing that there is a constant $c > 0$ such that there is no $(c \ln |V|)$-approximation algorithm for the Minimum Distance-1 Vertex Cover problem on bipartite graphs, unless $\mathsf{P} = \mathsf{NP}$. Since the problem is a special case of the Set Cover problem (where the ground set is the edge set of the input graph), the greedy algorithm provides a factor $(1 + \ln |E|)$ approximation of the Minimum Distance-k Vertex Cover problem.

The Minimum Edge Cover problem can be optimally solved in polynomial time, that is, a 1-approximation algorithm exists for all graphs. Regarding the Minimum Distance-k Edge Cover for $k \geq 1$, we are again only aware of a result of Lewis [48] showing that there is a constant $c > 0$ such that there is no $(c \ln |V|)$-approximation algorithm for the Minimum Distance-1 Edge Cover problem on bipartite graphs, unless $\mathsf{P} = \mathsf{NP}$. Since the Minimum Distance-k Edge Cover problem is a special case of Set Cover (where the ground set is the vertex set of the input graph), the greedy algorithm provides a factor $1 + \ln |V|$ approximation of the Minimum Distance-k Edge Cover problem.

3.3 Parameterized complexity

NP-complete problems can differ also with respect to their parameterized complexity [24]. A parameterized problem is a decision problem in which each input instance comes equipped with a so-called *parameter*, which is (typically) a positive integer associated to the instance. Given a parameterized problem Π, an algorithm that correctly solves Π is said to be *fixed-parameter tractable* (FPT) if it runs in time $\mathcal{O}(f(\ell) \cdot p(n))$, where ℓ is the parameter of the input instance, n is the input size, f is any function, and p is any polynomial function. Thus, a running time of $\mathcal{O}(2^\ell n^3)$ would be acceptable in this definition, but a running time of $\mathcal{O}(n^\ell)$ would not be. A parameterized problem is said to be *fixed-parameter tractable* (FPT) if it admits an FPT algorithm.

Most of the decision problems considered in this thesis are obtained from optimization problems and have the following form: given a graph G and an integer ℓ, is the value of a certain invariant on G at most ℓ? All these problems can be easily turned into parameterized problems, using the so-called natural parameterization, which simply chooses ℓ as the parameter. Let us summarize what is known about the parameterized complexity of the classical variants of the domination and covering problems considered in the thesis. It follows immediately from the definition that for every decision problem that is solvable in polynomial time, its natural parameterization is FPT. In particular, the natural parameterization of the Edge Cover problem is FPT. The natural parameterizations of Vertex Cover and Edge Dominating Set are also fixed-parameter tractable (see [27] and [30], respectively). On the other hand, the existence of an FPT algorithm for the natural parameterization of the Dominating Set problem is considered unlikely (see [24]), and this is also the case for the natural parameterization of Distance-k Dominating Set for all $k \geq 1$ [42, 49]. To the best of our knowledge, the natural parameterizations of Distance-k Edge Cover, Distance-k Vertex Cover and Distance-k Edge Dominating Set have not yet been studied in the literature for any $k \geq 1$.

4 Bounds

Considering the distance-based variants of all the problems, we now establish several inequalities relating the sizes of the optimal solutions of all four problems.

Theorem 4.1. *Let G be a graph without isolated vertices and $k \geq 1$ an integer. Then, the following inequalities hold:*

$$\gamma'_k(G) \leq \tau_k(G) \leq \gamma_k(G) \leq 2\rho_k(G) \leq 2\gamma_k(G)\,.$$

Proof. First we show that $\gamma'_k(G) \leq \tau_k(G)$. Let C be a distance-k vertex cover in G with size $\tau_k(G)$. Note that every vertex u in C is incident with an edge of G as G has no isolated vertices. We construct a set $F \subseteq E(G)$ in the following way: for every vertex $u \in C$ we add one edge $\{u, v\} \in E(G)$ to F.

We claim that F is a distance-k edge dominating set in G with size $\gamma'_k(G)$. Let $\{u, v\} \in E(G)$ be arbitrary. Since C is a distance-k vertex cover in G, we have that $\mathsf{dist}(\{u, v\}, C) \leq k$. However, that implies that $\mathsf{dist}(\{u, v\}, F) \leq \mathsf{dist}(\{u, v\}, C) \leq k$. Hence, F is a distance-k edge dominating set in G with size $|F| \leq |C| = \tau_k(G)$, implying that $\gamma'_k(G) \leq \tau_k(G)$.

Next, we show that $\tau_k(G) \leq \gamma_k(G)$. Let D be a distance-k dominating set in G with size $\gamma_k(G)$. We claim that D is also a distance-k vertex cover in G. Let $\{u, v\} \in E(G)$ be arbitrary. Since D is a distance-k dominating set in G, we have that $\mathsf{dist}(u, D) \leq k$, hence, $\mathsf{dist}(\{u, v\}, D) \leq k$, implying that D is a distance-k vertex cover in G. Thus, $\tau_k(G) \leq |D| = \gamma_k(G)$.

Next, we show that $\gamma_k(G) \leq 2\rho_k(G)$. Let F be a distance-k edge cover in G with size $\rho_k(G)$. Define D to be the set of vertices $u \in V(G)$ such that u is incident with an edge in F. Note that $|D| \leq 2|F|$. We claim that D is a distance-k dominating set in G. Let $v \in V(G)$ be arbitrary. Since F is a distance-k edge cover in G, we have that $\mathsf{dist}(v, F) \leq k$, that is, $\mathsf{dist}(v, u) \leq k$ for some vertex $u \in V(G)$ such that u is incident with an edge in F. Observe that $u \in D$. Therefore, $\mathsf{dist}(v, D) \leq \mathsf{dist}(v, u) \leq k$, implying that D is a distance-k dominating set in G. Thus, $\gamma_k(G) \leq |D| \leq 2|F| = 2\rho_k(G)$.

Finally, we show that $2\rho_k(G) \leq 2\gamma_k(G)$, or equivalently, that $\rho_k(G) \leq \gamma_k(G)$. Let D be a distance-k dominating set in G with size $\gamma_k(G)$. We construct a set $F \subseteq E(G)$ in the following way: for every vertex $u \in D$ we add one edge $\{u, v\} \in E(G)$ to F. We claim that F is a distance-k edge cover in G. Let $u \in V(G)$ be arbitrary. Since D is a distance-k dominating set in G, we have $\mathsf{dist}(u, D) \leq k$. However, that implies that $\mathsf{dist}(u, F) \leq \mathsf{dist}(u, D) \leq k$. Therefore, F is a distance-k edge cover in G. Hence, $\rho_k(G) \leq |F| = |D| = \gamma_k(G)$ or equivalently $2\rho_k(G) \leq 2\gamma_k(G)$. □

As the theorem shows, in any graph G without isolated vertices, the existence of a distance-k dominating set of size at most ℓ guarantees the existence of a distance-k edge dominating set, distance-k vertex cover, as well as distance-k edge cover of size at most ℓ. This result will be used in Chapter 7 to develop polynomial-time algorithms for Distance-k Edge Dominating Set, Distance-k Vertex Cover, as well as Distance-k Edge Cover problems in particular graph classes, by combining the theorem with a result establishing the existence of a distance-k dominating set of bounded size.

5 Graph transformations

In this chapter we consider a number of graph transformations and see how the optimal solution value to one of the problems on the transformed graph can be expressed in terms of an invariant of the the original graph, typically in terms of the optimal solution value of the same or a different problem. The choice of transformations and invariants considered is motivated by the applications of the obtained results in Chapter 6 on NP-completeness; nevertheless, we believe that some of these purely graph theoretic results may be of interest on their own.

We first look at the relations when the transformed instance is a line graph, then when we append paths to vertices, then when we subdivide edges, and finally, when the resulting graph is chordal.

5.1 The line graph transformation

The following lemma is used in the proofs of all theorems in this section.

Lemma 5.1. *Let G be a graph and H its line graph, that is, $H = L(G)$. Then, for any two distinct edges $e, f \in E(G)$, we have $\mathsf{dist}_H(e, f) = \mathsf{dist}_G(e, f) + 1$.*

Proof. Let P_G be a shortest path in G from an endpoint of e to an endpoint of f. In particular, P_G contains exactly one vertex in e and one vertex in f. Let P_H be the path in H with endpoints e and f, and whose internal vertices correspond to the edges of P_G. By the definition of P_G, $\mathsf{dist}_G(e, f) = |E(P_G)|$. On the other hand, we have that $\mathsf{dist}_H(e, f) \leq |E(P_H)| = |V(P_H)| - 1 = (|E(P_G)| + 2) - 1 = \mathsf{dist}_G(e, f) + 1$.

For the converse direction, let P_H be a shortest path in H between e and f. Note that P_H does not contain any vertices corresponding to edges in $E(G') \setminus E(G)$. Let P_G the path in G whose edges correspond to the internal vertices of P_H. Then, P_G contains exactly one vertex incident with e and exactly one vertex incident with f. Thus, $\mathsf{dist}_G(e, f) \leq |E(P_G)| = |V(P_H)| - 2 = (|E(P_H)| + 1) - 2 = \mathsf{dist}_H(e, f) - 1$. □

Given a graph G, the next four theorems express the values of γ_k, γ'_k, τ_k, and ρ_k of the line graph of G in terms of invariants of G, respectively. We start with distance-k domination.

Theorem 5.2. *Let G be a graph, $k \geq 1$ an integer and $H = L(G)$. Then $\gamma_k(H) = \gamma'_{k-1}(G)$.*

Proof. Let F be a distance-$(k-1)$ edge dominating set in G with size $\gamma'_{k-1}(G)$. We claim that F is a distance-k dominating set in H. Let $e \in V(H)$ be arbitrary. Since F is a distance-$(k-1)$ edge dominating set in G, we have $\mathsf{dist}_G(e, f) \leq k-1$ for some $f \in F$. By Lemma 5.1, we get $\mathsf{dist}_H(e, f) = \mathsf{dist}_G(e, f) + 1 \leq k-1+1 = k$, therefore, $\mathsf{dist}_H(e, F) \leq \mathsf{dist}_H(e, f) \leq k$. Hence, F is a distance-k dominating set in H with size $|F| = \gamma'_{k-1}(G)$, implying that $\gamma_k(H) \leq \gamma'_{k-1}(H)$.

Now, let F be a distance-k dominating set in H with size $\gamma_k(H)$. We claim that F is a distance-$(k-1)$ edge dominating set in G. Let $e \in E(G)$ be arbitrary. Since F is a distance-k dominating set in H, we have that $\mathsf{dist}_H(e, f) \leq k$ for some $f \in F$. By Lemma 5.1, we get that $\mathsf{dist}_G(e, f) = \mathsf{dist}_H(e, f) - 1 \leq k-1$, therefore, $\mathsf{dist}_G(e, F) \leq \mathsf{dist}_G(e, f) \leq k-1$. Hence, F is in fact a distance-$(k-1)$ edge dominating set in G with size $|F| = \gamma_k(H)$, implying that $\gamma'_{k-1}(G) \leq \gamma_k(H)$. □

Next, we establish a relation between the minimum size of a distance-$(k-1)$ P_3 dominating set in a graph G (that is, $\Lambda_{k-1}(G)$) and the minimum size of a distance-k edge dominating set in the line graph of G (that is, $\gamma'_k(L(G))$), for every $k \geq 1$.

Theorem 5.3. *Let G be a graph, $k \geq 1$ an integer and $H = L(G)$. Then, $\gamma'_k(H) = \Lambda_{k-1}(G)$.*

Proof. Let $\mathcal{P}$ be a distance-$(k-1)$ P_3 dominating set in G with size $\Lambda_{k-1}(G)$. Let F be the set of edges in H that correspond to the 3-vertex paths in $\mathcal{P}$. We claim that F is a distance-k edge dominating set in H. Let $\hat{e} \in E(H)$ be arbitrary. Let P be the 3-vertex path in G whose edges correspond to the vertices of $\hat{e}$. Since $\mathcal{P}$ is a distance-$(k-1)$ P_3 dominating set in G, there is a $Q \in \mathcal{P}$ such that $\mathsf{dist}_G(V(P), V(Q)) \leq k-1$. In particular, for some $e \in E(P)$ and $f \in E(Q)$ we have $\mathsf{dist}_G(e, f) \leq k-1$. By Lemma 5.1, we have that $\mathsf{dist}_H(e, f) = \mathsf{dist}_G(e, f) + 1 \leq k-1+1 = k$. Let $\hat{f}$ be the edge in H that corresponds to Q in G. Then, $\mathsf{dist}_H(\hat{e}, F) \leq \mathsf{dist}_H(\hat{e}, \hat{f}) \leq \mathsf{dist}_H(e, f) \leq k$. Hence, F is a distance-k edge dominating set of H with size $|F| = |\mathcal{P}| = \Lambda_{k-1}(G)$, implying that $\gamma'_k(H) \leq \Lambda_{k-1}(G)$.

Suppose now that F is a distance-k edge dominating set in H. Let $\mathcal{P}$ be the set of 3-vertex paths in G that correspond to edges in F. We claim that $\mathcal{P}$ is a distance-$(k-1)$ P_3 dominating set in G. Let P be an arbitrary 3-vertex path in G. Let $\hat{e}$ be the edge of H that corresponds to P. Since F is a distance-k edge dominating set in H we have that $\mathsf{dist}_H(\hat{e}, F) \leq k$, which means that $\mathsf{dist}_H(\hat{e}, \hat{f}) \leq k$ for some $\hat{f} \in F$. Then, $\mathsf{dist}_H(e, f) \leq k$ for some endpoint e of $\hat{e}$ and f of $\hat{f}$ in H. By Lemma 5.1 we have that $\mathsf{dist}_G(e, f) = \mathsf{dist}_H(e, f) - 1 \leq k-1$. But e is an edge of P, while f is an edge of Q where Q is the 3-vertex path in G that corresponds to the edge $\hat{f} \in E(H)$. Notice that $Q \in \mathcal{P}$. Since $\mathsf{dist}_G(V(P), V(Q)) \leq \mathsf{dist}_G(e, f)$ we have that $\mathsf{dist}_G(V(P), V(Q)) \leq k-1$. Hence, $\mathcal{P}$ is a distance-$(k-1)$ P_3 dominating set of G with size $|\mathcal{P}| = |F| = \gamma'_k(H)$, implying that $\Lambda_{k-1}(G) \leq \gamma'_k(H)$. □

Our next goal is to show a relation between the minimum size of a distance-$(k-1)$ edge P_3 dominating set in a graph G (that is, $\epsilon^{\Lambda}_{k-1}(G)$) and the minimum size of a distance-k vertex cover in the line graph of G (that is, $\tau_k(L(G))$), for every $k \geq 1$.

Theorem 5.4. *Let G be a graph, $k \geq 1$ an integer and $H = L(G)$. Then, $\tau_k(H) = \varepsilon^{\Lambda}_{k-1}(G)$.*

Proof. Let F be a distance-$(k-1)$ edge P_3 dominating set in G with size $\varepsilon^{\Lambda}_{k-1}(G)$. By definition, F is a set of vertices in H. We claim that F is a distance-k vertex cover in H. Let $\hat{e} \in E(H)$ be arbitrary. Let P be the 3-vertex path in G that corresponds to the edge $\hat{e}$ of H. Since F is a distance-$(k-1)$ edge P_3 dominating set in G, there exists an edge $f \in F$ such that $\mathsf{dist}_G(V(P), f) \leq k-1$, that is, $\mathsf{dist}_G(e, f) \leq k-1$ for some $e \in E(P)$. By Lemma 5.1, we have that $\mathsf{dist}_H(e, f) = \mathsf{dist}_G(e, f) + 1 \leq (k-1) + 1 = k$. However, $\mathsf{dist}_H(\hat{e}, F) \leq \mathsf{dist}_H(e, F) \leq \mathsf{dist}_H(e, f) \leq k$. Hence, F is a distance-k vertex cover of H with size $|F| = \varepsilon^{\Lambda}_{k-1}(G)$, implying that $\tau_k(H) \leq \varepsilon^{\Lambda}_{k-1}(G)$.

Assume now that F is a distance-k vertex cover of H with size $\tau_k(H)$. Note that F is a set of edges in G. We claim that F is a distance-$(k-1)$ edge P_3 dominating set in G. Let P be a 3-vertex path in G such that $\mathsf{dist}_G(V(P), F) > k-1$. Let $\hat{e}$ be the edge in H that corresponds to P. Since F is a distance-k vertex cover in H, we have that $\mathsf{dist}_H(\hat{e}, F) \leq k$ which means that $\mathsf{dist}_H(e, f) \leq k$ for some $e \in \hat{e}$ and $f \in F$. By Lemma 5.1, we have that $\mathsf{dist}_G(e, f) = \mathsf{dist}_H(e, f) - 1 \leq k-1$. Observe that e is an edge of P. Since $\mathsf{dist}_G(V(P), f) \leq \mathsf{dist}_G(e, f) \leq k-1$, we get that $\mathsf{dist}_G(V(P), F) \leq k-1$. Hence, F is a distance-$(k-1)$ P_3 edge dominating set of G with size $|F| = \tau_k(G)$, implying that $\varepsilon^{\Lambda}_{k-1}(G) \leq \tau_k(H)$. □

Finally, we establish a relation between the minimum size of a distance-$(k-1)$ P_3 edge dominating set in a graph G (that is, $\Lambda^{e}_{k-1}(G)$) and the minimum size of a distance-k edge cover in the line graph of G (that is, $\rho_k(L(G))$) for every integer $k \geq 1$.

Theorem 5.5. *Let G be a graph without isolated edges and $H = L(G)$. Then, $\rho_k(H) = \Lambda^{e}_{k-1}(G)$ for all $k \geq 1$.*

Proof. Let $\mathcal{P}$ be a distance-$(k-1)$ P_3 edge dominating set in G with size $\Lambda^{e}_{k-1}(G)$. Let F be the set of edges in H that correspond to the 3-vertex paths in $\mathcal{P}$. We claim that F is a distance-k edge cover in H. Let $e \in V(H)$ be arbitrary. Since e is an edge in G and $\mathcal{P}$ is a distance-$(k-1)$ P_3 edge dominating set in G, there is a $P \in \mathcal{P}$ such that $\mathsf{dist}_G(e, V(P)) \leq k-1$. That is, for some $f \in E(P)$ we have $\mathsf{dist}_G(e, f) \leq k-1$. By Lemma 5.1, we have that $\mathsf{dist}_H(e, f) = \mathsf{dist}_G(e, f) + 1 \leq k - 1 + 1 = k$. However, $\mathsf{dist}_H(e, F) \leq \mathsf{dist}_H(e, E(P)) \leq \mathsf{dist}_H(e, f) \leq k$. Hence, F is a distance-k edge cover of H with size $|F| = |\mathcal{P}| = \Lambda^{e}_{k-1}(G)$, implying that $\rho_k(H) \leq \Lambda^{e}_{k-1}(G)$.

Suppose now that F is a distance-k edge cover of H. Let $\mathcal{P}$ be the set of 3-vertex paths in G that correspond to edges in F. We claim that $\mathcal{P}$ is a distance-$(k-1)$ P_3

edge dominating set in H. Let $e \in E(G)$ be arbitrary. Since F is a distance-k edge cover in H we have that $\mathsf{dist}_H(e, F) \leq k$, which means that $\mathsf{dist}_H(e, \hat{f}) \leq k$ for some $\hat{f} \in F$. Then, $\mathsf{dist}_H(e, f) \leq k$ for some endpoint f of $\hat{f}$ in H. By Lemma 5.1 we have that $\mathsf{dist}_G(e, f) = \mathsf{dist}_H(e, f) - 1 \leq k - 1$. Observe that f is an edge of P where P is a 3-vertex path in G that corresponds to the edge $\hat{f}$. Note that $P \in \mathcal{P}$, therefore, $\mathsf{dist}_G(e, V(P)) \leq \mathsf{dist}_G(e, f) \leq k-1$. Hence, $\mathcal{P}$ is a distance-$(k-1)$ P_3 edge dominating set of G with size $|\mathcal{P}| = |F| = \rho_k(H)$, implying that $\Lambda^e_{k-1}(G) \leq \rho_k(H)$. □

5.2 Path growing transformations

Construction 1. Let G be a graph and t a positive integer. We define G^{+t} to be the graph obtained from G by appending to every vertex $v \in V(G)$ a path of length t. See Figure 1 for an example.

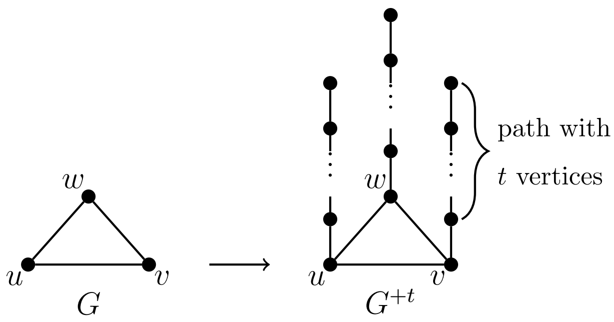

Figure 1: A graph G (left) and the graph G' obtained from it (right), for some positive integer t.

Theorem 5.6. *Let G be a graph and $k \geq 0$ an integer. Let G' be the graph obtained from Construction 1 given G and $t = 1$, that is $G' = G^{+1}$. Then, $\varepsilon^{\Lambda}_k(G') = \gamma'_k(G)$.*

Proof. Let F be a distance-k edge dominating set in G with size $\gamma'_k(G)$. We claim that F is a distance-k edge P_3 dominating set in G'. Note that $F \subseteq E(G')$. Let P be an arbitrary 3-vertex path in G'. Observe that P contains at least one edge, call it $e \in E(G')$, that is also an edge of G. Since F is a distance-k edge dominating set in G, we have that $\mathsf{dist}_G(e, F) \leq k$. However, $\mathsf{dist}_{G'}(V(P), F) \leq \mathsf{dist}_{G'}(e, F)$, which readily implies that $\mathsf{dist}_{G'}(V(P), F) \leq k$, hence F is a distance-k edge P_3 dominating set in G' with size $|F| = \gamma'_k(G)$. This gives $\varepsilon^{\Lambda}_k(G') \leq \gamma'_k(G)$.

Now, let F be a distance-k edge P_3 dominating set in G' with size $\varepsilon^{\Lambda}_k(G')$. Let us first show that G' contains a distance-k edge P_3 dominating set F' of size at most $|F|$ that contains only the edges that are also in G. If $e' \in F$ is an edge that is in G' but not in G then $e' = \{u, u'\}$ for some $u \in V(G)$. Notice that $N_G(u) \neq \emptyset$ since otherwise $F \setminus \{e'\}$ would be a distance-k edge P_3 dominating set in G' with size $\varepsilon^{\Lambda}_k(G') - 1$, which is impossible. Hence, we can define a new set $F' = (F \setminus \{e'\}) \cup \{\{u, v\}\}$ for some arbitrary $v \in N_G(u)$, and this set does not contain e'. So we may assume, without loss of generality, that F is a distance-k edge P_3 dominating set in G' with size $\epsilon_k^{\Lambda}(G')$ that consists only of edges of G. We claim that F is a distance-k edge dominating set in G. Let $e = \{u, v\}$ be an arbitrary edge in G. Let $u' \in V(G') \setminus V(G)$ be such that $\{u, u'\} \in E(G')$ and denote by P the path on the 3 vertices v, u and u'. Since F is a distance-k edge P_3 dominating set in G', we have that $\mathsf{dist}_{G'}(V(P), F) \leq k$, that is, $\mathsf{dist}_{G'}(V(P), f) \leq k$ for some $f \in F$. Observe that $\mathsf{dist}_{G'}(e, f) \leq \mathsf{dist}_{G'}(u, f) = \mathsf{dist}_{G'}(\{u, u'\}, f)$ and $\mathsf{dist}_G(e, f) = \mathsf{dist}_{G'}(e, f)$. Hence, $\mathsf{dist}_G(e, f) = \mathsf{dist}_{G'}(e, f) = \mathsf{dist}_{G'}(V(P), f) \leq k$ and since $f \in F \subseteq E(G)$ we also have $\mathsf{dist}_G(e, F) \leq \mathsf{dist}_G(e, f) \leq k$. Therefore, F is a distance-k edge dominating set in G with size $|F| = \varepsilon_k^{\Lambda}(G')$, implying that $\gamma_k'(G) \leq \varepsilon_k^{\Lambda}(G')$. □

Theorem 5.7. *Let G be a graph without isolated vertices and edges and $k \geq 0$ an integer. Let G' be the graph obtained from Construction 1 given G and $t = k + 2$, that is $G' = G^{+(k+2)}$. Then, $\Lambda_k(G') = \Lambda(G)$.*

Proof. Let $\mathcal{P}$ be a P_3 cover of G with size $\Lambda(G)$. Note that every path in $\mathcal{P}$ is also a path in G'. We claim that $\mathcal{P}$ is a distance-k P_3 dominating set in G'. Let P be an arbitrary 3-vertex path in G'. Then P contains a vertex $u \in V(G)$ or is an induced subgraph of the path appended to some $u \in V(G)$. If P contains a vertex $u \in V(G)$, then $\mathsf{dist}_{G'}(V(P), V(P')) = 0$ for some $P' \in \mathcal{P}$. If P is an induced subgraph of the path appended to some $u \in V(G)$, then $\mathsf{dist}_{G'}(V(P), u) \leq k$. Since $u \in V(G)$, $\mathsf{dist}_{G'}(u, V(P')) = 0$ for some $P' \in \mathcal{P}$, therefore, $\mathsf{dist}_{G'}(V(P), V(P')) \leq \mathsf{dist}_{G'}(V(P), u) + \mathsf{dist}_{G'}(u, V(P')) \leq k$ for some $P' \in \mathcal{P}$. Hence, $\mathcal{P}$ is a distance-k P_3 dominating set in G' with size $|\mathcal{P}| = \Lambda(G)$, implying that $\Lambda_k(G') \leq \Lambda(G)$.

Now, let $\mathcal{P}$ be a distance-k P_3 dominating set in G' with size $\Lambda_k(G)$. We assume, without loss of generality, that $\mathcal{P}$ consists only of 3-vertex paths that are also paths in G. Otherwise, if $P \in \mathcal{P}$ is a 3-vertex path that is a path in G' but not in G, then P contains a vertex w' that belongs to a path appended to some $u \in V(G)$ and then we could replace P with any path P' that contains vertex u and satisfies $V(P') \subseteq V(G)$. We claim that $\mathcal{P}$ is a P_3 cover of G. Consider a vertex $u \in V(G)$. The path with $k+2$ vertices that is appended to it, in G', contains a 3-vertex path, that is at distance exactly k from u. Hence, u has to be contained in some $P \in \mathcal{P}$ and therefore $\mathcal{P}$ is a P_3 cover of G with size $|\mathcal{P}| = \Lambda_k(G)$. Hence, $\Lambda(G) \leq \Lambda_k(G)$. □

Theorem 5.8. *Let G be a graph without isolated vertices and edges and $k \geq 0$ an integer. Let G' be the graph obtained from Construction 1 given G and $t = k + 1$, that is, $G' = G^{+(k+1)}$. Then, $\Lambda_k^e(G') = \Lambda(G)$.*

Proof. Let $\mathcal{P}$ be a P_3 cover of G with size $\Lambda(G)$. We claim that $\mathcal{P}$ is also a distance-k P_3 edge dominating set of G'. Let e be an arbitrary edge of G'. Then e has an endpoint

$u \in V(G)$, or is an edge of the path appended to some $u \in V(G)$. If e has an endpoint $u \in V(G)$, then $\mathsf{dist}_{G'}(e, V(P)) = 0$ for some $P \in \mathcal{P}$. If e is an edge of the path appended to some $u \in V(G)$, then $\mathsf{dist}_{G'}(e, u) \leq k$. Since $u \in V(G)$, we have that $\mathsf{dist}_{G'}(u, V(P)) = 0$ for some $P \in \mathcal{P}$, therefore, $\mathsf{dist}_{G'}(e, V(P)) \leq \mathsf{dist}_{G'}(e, u) + \mathsf{dist}_{G'}(u, V(P)) \leq k$ for some $P \in \mathcal{P}$. Hence, $\mathcal{P}$ is a distance-k P_3 edge dominating set of G' with size $|\mathcal{P}| = \Lambda(G)$, implying that $\Lambda_k^e(G) \leq \Lambda(G)$.

Now, let $\mathcal{P}$ be a distance-k P_3 edge dominating set in G' with size $\Lambda_k^e(G)$. We assume, without loss of generality, that $\mathcal{P}$ consists only of the 3-vertex paths that are fully contained in G. Otherwise, if $P \in \mathcal{P}$ is a 3-vertex path that is a path in G' but not in G, then P contains a vertex w' that belongs to a path appended to some $u \in V(G)$ and then we could replace P with any path P' that contains the vertex u and satisfies $V(P') \subseteq V(G)$. We claim that $\mathcal{P}$ is a P_3 cover in G. Consider a vertex in $u \in V(G)$. The path with $k+1$ vertices that is appended to it, in G', contains an edge that is at distance exactly k from u. Hence, u has to be contained in some $P \in \mathcal{P}$. Therefore, $\mathcal{P}$ is a P_3 cover of G with size $|\mathcal{P}| = \Lambda_k^e(G')$, implying that $\Lambda(G) \leq \Lambda_k^e(G')$. □

5.3 Poljak-type transformations

In this section we generalize the well-known fact, observed first by Poljak in [59], that a double subdivision of an edge increases the minimum size of a vertex cover by exactly one.

We generalize this result to all four distance-based problems. More precisely, we show that subdividing an edge exactly $2k+1$ times results in a unit increase of the minimum size of a distance-k dominating set (see Lemma 5.9), subdividing an edge $2k+3$ times results in a unit increase of the minimum size of a distance-k edge dominating set (see Lemma 5.11), while subdividing an edge $2k+2$ times results in a unit increase of the minimum sizes of a distance-k vertex cover and distance-k edge cover (see Lemmas 5.13 and 5.15).

Distance-k domination

Lemma 5.9. *Let G be a graph, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing edge e exactly $2k+1$ times, for some integer $k \geq 1$. Then $\gamma_k(G') = \gamma_k(G)+1$.*

Proof. Let us denote the endpoints of e by u and v and let X be the set of internal vertices of the path between u and v in G' obtained from the subdivision of the edge $e = \{u, v\}$ in G. Label the elements of X as $X = \{x_1, \ldots, x_{2k+1}\}$ so that u is adjacent to x_1, vertex x_i is adjacent to x_{i+1} for every $i \in \{1, ..., 2k\}$, and x_{2k+1} is adjacent to v.

Let D be a distance-k dominating set in G with size $\gamma_k(G)$. We assume that $\mathsf{dist}_G(u, D) \geq \mathsf{dist}_G(v, D)$, or else we can relabel u, v, and the vertices x_i, for $i \in$

$\{1, \ldots, 2k+1\}$, accordingly. Let $d = \mathsf{dist}_G(v, D) \leq k$ and $D' = D \cup \{x_{d+1}\}$. We claim that D' is a distance-k dominating set in G'. Suppose that there exists a vertex $w \in V(G')$ such that $\mathsf{dist}'_G(w, D') > k$. We consider two cases depending on whether $w \in V(G)$ or not. Consider first the case where $w \in V(G)$. Let P be a shortest path in G from w to D. Then P has length at most k and the assumption that $w \in V(G)$ implies that P contains the edge $\{u, v\}$, for otherwise we would have $\mathsf{dist}_{G'}(w, D') \leq \mathsf{dist}_G(w, D) \leq k$. Thus, $\mathsf{dist}_G(u, D) = \mathsf{dist}_G(v, D) + 1$ (recall that we assumed that $\mathsf{dist}_G(u, D) \geq \mathsf{dist}_G(v, D)$), which in turn implies that $\mathsf{dist}_G(w, v) = \mathsf{dist}_G(w, u) + 1$. Thus, $\mathsf{dist}_G(w, v) \leq k - d$ and $\mathsf{dist}_G(w, u) \leq k - d - 1$. However, $\mathsf{dist}_{G'}(u, x_{d+1}) = d + 1$, which implies that

$$\begin{aligned} \mathsf{dist}_{G'}(w, D') &\leq \mathsf{dist}_{G'}(w, x_{d+1}) \leq \mathsf{dist}_{G'}(w, u) + \mathsf{dist}_{G'}(u, x_{d+1}) \leq \\ &\leq (k - d - 1) + (d + 1) = k, \end{aligned}$$

a contradiction. Now, suppose that $w \notin V(G)$. Then, w is a vertex $x_i \in X$. Observe that for j such that $k + d + 2 \leq j \leq 2k + 1$, we have $\mathsf{dist}_{G'}(x_j, v) \leq k - d$, and hence $\mathsf{dist}_{G'}(x_j, D') \leq \mathsf{dist}_{G'}(x_j, D) \leq \mathsf{dist}_{G'}(x_j, v) + \mathsf{dist}_{G'}(v, D) \leq (k - d) + d = k$. Since $d \leq k$, every vertex x_j such that $j \in \{1, \ldots, k + d + 1\}$ is such that $\mathsf{dist}_{G'}(x_j, x_{d+1}) \leq k$. Thus, every vertex in X is at distance at most k from D', and hence $\mathsf{dist}_{G'}(w, D') \leq k$, a contradiction. We conclude that D' is a distance-k dominating set in G', and therefore $\gamma_k(G') \leq |D'| = |D| + 1 = \gamma_k(G) + 1$.

For the converse inequality, let D' be a distance-k dominating set in G' with size $\gamma_k(G')$ minimizing $|D' \cap X|$. First, we claim that D' contains exactly one vertex in X. It is clear that D' must contain at least one vertex from X, since $\mathsf{dist}_{G'}(x_{k+1}, V(G') \backslash X) > k$. Suppose that $|D' \cap X| \geq 2$ and let $D^* = (D' \setminus X) \cup \{v, x_1\}$. Observe that $D^* \subseteq V(G')$, $|D^*| \leq |D|$, and every vertex in X is at distance at most k from v or x_1. Furthermore, for every vertex w of G' which is also a vertex of G, we have that $\mathsf{dist}_{G'}(w, D^*) \leq \mathsf{dist}_{G'}(w, D') \leq k$, and thus D^* is a distance-k dominating set in G' with size at most $|D'|$. However, $|D^* \cap X| = 1 < 2 \leq |D' \cap X|$, a contradiction with the definition of D'. So we have that $|D' \cap X| = 1$. Thus, there exists a unique $i \in \{1, \ldots, 2k+1\}$ such that $D' \cap X = \{x_i\}$. We assume without loss of generality that $i \leq k + 1$ (the other case is symmetric). Let $D = D' \setminus \{x_i\}$ and note that $D \subseteq V(G)$. We claim that D is a distance-k dominating set in G. Suppose for a contradiction that this is not the case. Then there exists a vertex $w \in V(G)$ such that $\mathsf{dist}_G(w, D) > k$. This implies that $\mathsf{dist}_{G'}(w, D' \setminus \{x_i\}) > k$, and since D' is a distance-k dominating set in G', we must have $\mathsf{dist}_{G'}(w, x_i) \leq k$. Let P be a shortest path between w and x_i, and notice that P contains u or v. Since we assumed that $i \leq k + 1$, we have that $\mathsf{dist}_{G'}(v, x_i) \geq k + 1$, and hence path P must contain u. This implies that $\mathsf{dist}_{G'}(w, x_i) = \mathsf{dist}_{G'}(w, u) + i \leq k$. We claim that $\mathsf{dist}_G(v, D) \leq i - 1$. Note that $\mathsf{dist}_{G'}(x_{i+k+1}, x_i) > k$ (where $x_{2k+2} = v$). So there exists a vertex $\hat{w} \in D' \setminus \{x_i\} = D$ such that $\mathsf{dist}_{G'}(x_{i+k+1}, w) \leq k$. As we have $\mathsf{dist}_{G'}(x_{i+k+1}, v) = 2k+2-(i+k+1) = k+1-i$ and v belongs to every shortest $x_{i+k+1}, \hat{w}$-path in G', we obtain that $\mathsf{dist}_{G'}(v, \hat{w}) \leq k-(k+1-i) = i-1$. Since $\hat{w} \in D$, we get that $\mathsf{dist}_G(v, D) \leq \mathsf{dist}_{G'}(v, \hat{w}) \leq i-1$, as claimed. Note that $\mathsf{dist}_G(w, u) \leq \mathsf{dist}_{G'}(w, u)$, and since $\mathsf{dist}_{G'}(w, u) \leq k-i$, we get that $\mathsf{dist}_G(w, u) \leq k-i$. Hence, $\mathsf{dist}_G(w, D) \leq \mathsf{dist}_G(w, u) + \mathsf{dist}_G(u, v) + \mathsf{dist}_G(v, D) \leq (k-i)+1+(i-1) = k$, a contradiction with the assumption that $\mathsf{dist}_G(w, D) > k$. Thus, D is a distance-k dominating set in G, and we obtain that $\gamma_k(G) \leq |D| = |D'|-1 = \gamma_k(G')-1$. □

Corollary 5.10. *Let G be a graph, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $p(2k+1)$ times, for some two integers $k \geq 1$ and $p \geq 0$. Then $\gamma_k(G') = \gamma_k(G) + p$.*

Proof. Fix $k \geq 1$. We use induction on p. For $p = 0$ the statement is trivial, and for $p = 1$ this is just Lemma 5.9. Now let $p > 1$, let G' be as in the claim, and let G'' be the graph obtained from G by subdividing the edge $\{u, v\}$ exactly $(p-1)(2k+1)$ times. Denoting by P the path replacing $\{u, v\}$ in G'', observe that G' can be obtained from G'' by subdividing one of the edges of P exactly $2k+1$ times. By the induction hypothesis, we have $\gamma_k(G'') = \gamma_k(G) + p - 1$. Since we also have $\gamma_k(G') = \gamma_k(G'') + 1$ by 5.9, we infer that $\gamma_k(G') = \gamma_k(G) + p$, as claimed. □

Distance-k edge domination

Lemma 5.11. *Let G be a graph, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $2k+3$ times, for some integer $k \geq 0$. Then $\gamma'_k(G') = \gamma'_k(G) + 1$.*

Proof. Let us denote the endpoints of e by u and v and let X be the set of internal vertices of the path between u and v in G' obtained from the subdivision of the edge $\{u, v\}$ in G. We label the elements of X as $X = \{x_1, \ldots, x_{2k+3}\}$ so that u is adjacent to x_1, vertex x_i is adjacent to x_{i+1} for every $i \in \{1, \ldots, 2k+2\}$, and x_{2k+3} is adjacent to v.

Let F be a distance-k edge dominating set in G with size $\gamma'_k(G)$. We assume that $\mathsf{dist}_G(u, F) \geq \mathsf{dist}_G(v, F)$, since otherwise we can relabel u, v, and the vertices x_i, for $i \in \{1, \ldots, 2k+3\}$, accordingly. We consider two cases based on whether the edge e is in F or not. Suppose first that $e \in F$. Define $F' = (F \setminus \{e\}) \cup \{\{u, x_1\}, \{v, x_{2k+3}\}\}$. Then $F' \subseteq E(G')$. Observe that for every edge $f \in E(G)$, we have $\mathsf{dist}_G(f, F \setminus \{e\}) \leq k$ or $\mathsf{dist}_G(f, e) \leq k$. We claim that F' is a distance-k edge dominating set in G'. Suppose there exists an edge $e' \in E(G')$ such that $\mathsf{dist}_{G'}(e', F') > k$. Then either $e' \in E(G)$ or $e' \in E(G') \setminus E(G)$. Consider first the case when $e' \in E(G)$. Let P be a shortest path in G from e' to F. Then P has length at most k and the assumption that $e' \in E(G)$ implies that P contains the edge $\{u, v\}$, for otherwise we would have $\mathsf{dist}_{G'}(e', F') \leq \mathsf{dist}_G(e', F) \leq k$. In particular, $\mathsf{dist}_G(e', e) \leq k$. Note that $\mathsf{dist}_G(e', e) = \min\{\mathsf{dist}_{G'}(e', \{u, x_1\}), \mathsf{dist}_{G'}(e', \{v, x_{2k+3}\})\}$ since $e \in E(P)$. We thus get $\mathsf{dist}_{G'}(e', F) \leq \min\{\mathsf{dist}_{G'}(e', \{u, x_1\}), \mathsf{dist}_{G'}(e', \{v, x_{2k+3}\})\} = \mathsf{dist}_G(e', e) \leq k$, contradicting the assumption that $\mathsf{dist}_{G'}(e', F) > k$. Now consider the case when $e' \in E(G') \setminus E(G)$. Since $\mathsf{dist}_{G'}(\{u, x_1\}, \{x_j, x_{j+1}\}) \leq k$ for all j such that $1 \leq j \leq k+1$ and $\mathsf{dist}_{G'}(\{v, x_{2k+3}\}, \{x_j, x_{j+1}\}) \leq k$ for all j such that $k+2 \leq j \leq 2k+2$, we have that e' is at distance at most k from $\{u, x_1\}, \{x_{2k+3}, v\}$ thus from F' as well, contradicting the assumption. So, whenever $e \in F$ we have that F' is a distance-k edge dominating set of G' with size $|F'| = |F| + 1 = \gamma'_k(G) + 1$. Thus, $\gamma'_k(G') \leq \gamma'_k(G) + 1$ in this case.

Now suppose that $e \notin F$. Let $d = \mathsf{dist}_G(v, F)$. Then $d \leq k$, which in turn implies that $\mathsf{dist}_{G'}(v, F) = \mathsf{dist}_G(v, F)$. Let also $F' = F \cup \{\{x_{d+1}, x_{d+2}\}\}$. Note that $F' \subseteq E(G')$. We claim that F' is a distance-k edge dominating set in G'. Suppose that there exists an edge $e' \in E(G')$ such that $\mathsf{dist}_{G'}(e', F') > k$. We consider two cases depending on whether $e' \in E(G)$ or not. Consider first the case when $e' \in E(G)$. Let P be a shortest path in G from e' to F. Then P has length at most k and the assumption that $e' \in E(G)$ implies that P contains the edge $\{u, v\}$, for otherwise we would have $\mathsf{dist}_{G'}(e', F') \leq |E(P)| \leq k$. Thus, $\mathsf{dist}_G(u, F) = \mathsf{dist}_G(v, F) + 1$ (recall that we assumed that $\mathsf{dist}_G(u, F) \geq \mathsf{dist}_G(v, F)$), which in turn implies that $\mathsf{dist}_G(e', v) = \mathsf{dist}_G(e', u) + 1$. Thus, $\mathsf{dist}_G(e', v) \leq k - d$ and $\mathsf{dist}_G(e', u) \leq k - d - 1$. However, $\mathsf{dist}_{G'}(u, x_{d+1}) = d + 1$, which implies that

$$\begin{aligned} \mathsf{dist}_{G'}(e', F') &\leq \mathsf{dist}_{G'}(e', \{x_{d+1}, x_{d+2}\}) = \mathsf{dist}_{G'}(e', u) + \mathsf{dist}_{G'}(u, \{x_{d+1}, x_{d+2}\}) \\ &\leq (k - d - 1) + (d + 1) = k\,, \end{aligned}$$

a contradiction.

Now, suppose that $e' \notin E(G)$. Then, e' contains a vertex $x_i \in X$. Observe that for all j such that $k + d + 4 \leq j \leq 2k + 3$, we have $\mathsf{dist}_{G'}(x_j, v) \leq k - d$, and hence $\mathsf{dist}_{G'}(x_j, F') \leq \mathsf{dist}_{G'}(x_j, F) \leq \mathsf{dist}_{G'}(x_j, v) + \mathsf{dist}_{G'}(v, F) = \mathsf{dist}_{G'}(x_j, v) + \mathsf{dist}_G(v, F) \leq (k - d) + d = k$. If $j \leq d+1$, then $\mathsf{dist}_{G'}(x_j, x_{d+1}) \leq d \leq k$. If $d+2 \leq j \leq k+d+2$, then $\mathsf{dist}_{G'}(x_j, x_{d+2}) \leq k$. Thus, x_{k+d+3} is the unique vertex in X that is at distance more than k from F'. But then every edge containing an endpoint in X is at distance at most k from F', implying that $\mathsf{dist}_{G'}(e', F') \leq k$, a contradiction. We conclude that F' is a distance-k edge dominating set in G', and therefore $\gamma'_k(G') \leq |F'| = |F| + 1 = \gamma'_k(G) + 1$.

For the converse inequality, let F' be a distance-k edge dominating set in G' with size $\gamma'_k(G')$. Note that F' must contain at least one edge from $E(G') \setminus E(G)$, since $\mathsf{dist}_{G'}(\{x_{k+2}, x_{k+3}\}, V(G') \setminus X) > k$. We consider separately the cases when $|F' \cap (E(G') \setminus E(G))| = 1$ and when $|F' \cap (E(G') \setminus E(G))| \geq 2$. Assume first that $|F' \cap (E(G') \setminus E(G))| = 1$. Then $F' \cap (E(G') \setminus E(G)) = \{\{x_i, x_{i+1}\}\}$ for some $x_i \in X$. Without loss of generality, we assume that $i \leq k + 1$ (the other case is symmetric). Let $F = F' \setminus \{\{x_i, x_{i+1}\}\}$. Then $F \subseteq E(G)$. We claim that F is a distance-k edge dominating set in G. Suppose for a contradiction that there exists an edge $e' \in E(G)$ such that $\mathsf{dist}_G(e', F) > k$. Observe that $e' \neq e$ as $\mathsf{dist}_G(e, F) \leq \mathsf{dist}_{G'}(\{v, x_{2k+3}\}, F' \setminus \{\{x_i, x_{i+1}\}\}) = \mathsf{dist}_{G'}(\{v, x_{2k+3}\}, F') \leq k$. Since $e' \neq e$, we have $e' \in E(G')$. Therefore, since F' is a distance-k edge dominating set in G', there exists an edge $f' \in F'$ such that $\mathsf{dist}_{G'}(e', f') \leq k$. If $f' \neq \{x_i, x_{i+1}\}$, then $f' \in E(G)$, hence, $\mathsf{dist}_G(e', F) \leq \mathsf{dist}_G(e', f') \leq \mathsf{dist}_{G'}(e', f') \leq k$, which contradicts the assumption that $\mathsf{dist}_G(e', F) > k$. Therefore, $f' = \{x_i, x_{i+1}\}$. Notice that $\mathsf{dist}_{G'}(\{x_{i+k+2}, x_{i+k+3}\}, f') > k$ (where $x_{2k+4} = v$) so there exists an edge $f'' \in F' \setminus \{x_i, x_{i+1}\} = F$ such that $\mathsf{dist}_{G'}(\{x_{i+k+2}, x_{i+k+3}\}, f'') \leq k$. As $\mathsf{dist}_{G'}(\{x_{i+k+2}, x_{i+k+3}\}, v) = k - (i-1)$, we have that $\mathsf{dist}_{G'}(f'', v) \leq i - 1$. Moreover, since $f'' \in E(G)$ we have that $\mathsf{dist}_G(f'', v) \leq i - 1$. Then,

$$\begin{aligned}
\mathsf{dist}_G(e', F) &\leq \mathsf{dist}_G(e', u) + \mathsf{dist}_G(u, v) + \mathsf{dist}_G(v, f'') \\
&\leq (\mathsf{dist}_{G'}(e', f') - \mathsf{dist}_{G'}(f', u)) + 1 + (i-1) \\
&\leq (k-i) + 1 + (i-1) = k\,,
\end{aligned}$$

which contradicts the assumption that $\mathsf{dist}_G(e', F) > k$. Hence, F is a distance-k edge dominating set of G with size $|F| = |F'| - 1 = \gamma'_k(G') - 1$ when $|F' \cap (E(G') \setminus E(G)) = 1$, implying that $\gamma'_k(G) \leq \gamma'_k(G') - 1$.

Suppose now that $|F' \cap (E(G') \setminus E(G))| \geq 2$. In this case we define $F = (F' \setminus E(G')) \cup \{e\}$. Note that $F \subseteq E(G)$ and $|F| \leq |F'| - 1$. We claim that F is a distance-k edge dominating set in G. Suppose for a contradiction that there exists an edge $e' \in E(G)$ such that $\mathsf{dist}_G(e', F) > k$. As $\mathsf{dist}_G(e', F) > 0$ and $e \in F$, we have that $e' \neq e$, hence, $e' \in E(G')$ and because F' is a distance-k edge dominating set in G', we have that $\mathsf{dist}_{G'}(e', f') \leq k$ for some $f' \in F'$. If $f' \in E(G)$, then $\mathsf{dist}_G(e', f') \leq \mathsf{dist}_{G'}(e', f') \leq k$, which is not possible since $\mathsf{dist}_G(e', F) \leq \mathsf{dist}_G(e', f')$ and we assumed that $\mathsf{dist}_G(e', F) > k$. If $f' \in E(G') \setminus E(G)$, then

$$\min\{\mathsf{dist}_G(e', u), \mathsf{dist}_G(e', v)\} = \min\{\mathsf{dist}_{G'}(e', u), \mathsf{dist}_{G'}(e', v)\} \leq \mathsf{dist}_{G'}(e', f') \leq k\,.$$

However, we obtain that $\mathsf{dist}_G(e', e) \leq k$, which contradicts the assumption that $\mathsf{dist}_G(e', F) > k$ since $\mathsf{dist}_G(e', F) \leq \mathsf{dist}_G(e', e)$. Therefore, F is a distance-k edge dominating set in G with size $|F| = |F'| - 1 = \gamma'_k(G') - 1$, implying that $\gamma'_k(G) \leq \gamma'_k(G') - 1$. $\square$

An iterative application of Lemma 5.11 leads to the following result.

Corollary 5.12. *Let G be a graph, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $p(2k+3)$ times, for some two integers $k \geq 0$ and $p \geq 0$. Then $\gamma'_k(G') = \gamma'_k(G) + p$.*

Proof. Fix $k \geq 0$. We use induction on p. For $p = 0$ the statement is trivial, and for $p = 1$ this is just Lemma 5.11. Now let $p > 1$, let G' be as in the claim, and let G''

be the graph obtained from G by subdividing the edge $\{u, v\}$ exactly $(p-1)(2k+3)$ times. Denoting by P the path replacing $\{u, v\}$ in G'', observe that G' can be obtained from G'' by subdividing one of the edges of P exactly $2k+3$ times. By the induction hypothesis, we have $\gamma_k'(G'') = \gamma_k'(G) + p - 1$. Since we also have $\gamma_k'(G') = \gamma_k'(G'') + 1$ by Lemma 5.11, we infer that $\gamma_k'(G') = \gamma_k'(G) + p$, as claimed. □

Distance-k vertex cover

Lemma 5.13. *Let G be a graph, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $2k+2$ times, for some integer $k \geq 0$. Then $\tau_k(G') = \tau_k(G) + 1$.*

Proof. Let us denote the endpoints of e by u and v and let X be the set of internal vertices of the path between u and v in G' obtained from the subdivision of the edge $\{u, v\}$ in G. We label the elements of X as $X = \{x_1, \ldots, x_{2k+2}\}$ so that u is adjacent to x_1, vertex x_i is adjacent to x_{i+1} for every $i \in \{1, \ldots, 2k+1\}$, and x_{2k+2} is adjacent to v.

Let C be a distance-k vertex cover in G with size $\tau_k(G)$. We assume that $\mathsf{dist}_G(u, C) \geq \mathsf{dist}_G(v, C)$, or else we can relabel u, v, and the vertices x_i, for $i \in \{1, \ldots, 2k+2\}$, accordingly. Let $d = \mathsf{dist}_G(v, C) \leq k$ and $C' = C \cup \{x_{d+1}\}$. Suppose that there exists an edge $e' \in E(G')$ such that $\mathsf{dist}_{G'}(e', C') > k$. We consider two cases depending on whether $e' \in E(G)$ or not. Consider first the case where $e' \in E(G)$. Let P be a shortest path in G from e' to C. Then P has length at most k and the assumption that $e' \in E(G)$ implies that P contains the edge $\{u, v\}$, for otherwise we would have $\mathsf{dist}_{G'}(e', C') \leq \mathsf{dist}_G(e', C) \leq k$. Thus, $\mathsf{dist}_G(u, C) = \mathsf{dist}_G(v, C) + 1$ (recall that we assumed that $\mathsf{dist}_G(u, C) \geq \mathsf{dist}_G(v, C)$), which in turn implies that $\mathsf{dist}_G(e', v) = \mathsf{dist}_G(e', u) + 1$. Thus, $\mathsf{dist}_G(e', v) \leq k - d$ and $\mathsf{dist}_G(e', u) \leq k - d - 1$. However, $\mathsf{dist}_{G'}(u, x_{d+1}) = d + 1$, which implies that $\mathsf{dist}_{G'}(e', C') \leq \mathsf{dist}_{G'}(e', x_{d+1}) = \mathsf{dist}_{G'}(e', u) + \mathsf{dist}_{G'}(u, x_{d+1}) \leq k - d - 1 + d + 1 = k$, a contradiction. Now, suppose that $e' \notin E(G)$. Then, e' contains a vertex $x_i \in X$. Observe that for $j \in \{k+d+3, \ldots, 2k+2\}$, we have $\mathsf{dist}_{G'}(x_j, v) \leq k-d$, and hence $\mathsf{dist}_{G'}(x_j, C') \leq \mathsf{dist}_{G'}(x_j, C) \leq \mathsf{dist}_{G'}(x_j, v) + \mathsf{dist}_{G'}(v, C) \leq (k-d)+d = k$. Since $d \leq k$, every vertex x_j such that $j \in \{1, \ldots, k+d+1\}$ is such that $\mathsf{dist}_{G'}(x_j, x_{d+1}) \leq k$. This implies that x_{k+d+2} is the unique vertex in X at distance more than k from C' in G'. Thus, every edge with an endpoint in X is at distance at most k from C', and hence $\mathsf{dist}_{G'}(e', C') \leq k$, a contradiction. We conclude that C' is a distance-k vertex cover in G', and therefore $\tau_k(G') \leq |C'| = |C| + 1 = \tau_k(G) + 1$.

Let C' be a distance-k vertex cover in G' with size $\tau_k(G')$ minimizing $|C' \cap X|$. First, we claim that C' contains exactly one vertex in X. It is clear that C' must contain at least one vertex from X, since $\mathsf{dist}_{G'}(\{x_{k+1}, x_{k+2}\}, V(G') \setminus X) > k$. Suppose that $|C' \cap X| \geq 2$ and let $C^* = (C' \setminus X) \cup \{v, x_1\}$. Observe that $|C^*| \leq |C'|$ and that every edge with an endpoint in X is at distance at most k from v or x_1. Furthermore, for every edge f of G' which is also an edge of G, we have that $\mathsf{dist}_{G'}(f, C^*) \leq \mathsf{dist}_{G'}(f, C') \leq k$, and thus C^* is a distance-k vertex cover in G' with size at most $|C'|$. However, $|C^* \cap X| = 1 < 2 \leq |C' \cap X|$, a contradiction with the definition of C'. So we can assume that $|C' \cap X| = 1$. Thus, there exists a unique $i \in \{1, \ldots, 2k+2\}$ such that $C' \cap X = \{x_i\}$. We assume without loss of generality that $i \leq k+1$ (the other case is symmetric). Let $C = C' \setminus \{x_i\}$ and note that $C \subseteq V(G)$. We claim that C is a distance-k vertex cover in G. Suppose for a contradiction that this is not the case. Then there exists an edge $e \in E(G)$ such that $\mathsf{dist}_G(e, C) > k$. This implies that $\mathsf{dist}_{G'}(e, C' \setminus \{x_i\}) > k$, and since C' is a distance-k vertex cover in G', we must have $\mathsf{dist}_{G'}(e, x_i) \leq k$. Let P be a shortest path between e and x_i, and notice that P contains u or v. Since we assumed that $i \leq k+1$, we have $\mathsf{dist}_{G'}(v, x_i) \geq k+1$, and hence path P must contain u. This implies that $\mathsf{dist}_{G'}(e, x_i) = \mathsf{dist}_{G'}(e, u) + i \leq k$. We claim that $\mathsf{dist}_G(v, C) \leq i-1$. Since $i \leq k+1$, we get that $\mathsf{dist}_{G'}(\{x_{i+k+1}, x_{i+k+2}\}, x_i) > k$, where $x_{2k+3} = v$. So there exists a vertex $w \in C' \setminus \{x_i\} = C$ such that $\mathsf{dist}_{G'}(\{x_{i+k+1}, x_{i+k+2}\}, w) \leq k$. As we have $\mathsf{dist}_{G'}(\{x_{i+k+1}, x_{i+k+2}\}, v) = 2k+3-(i+k+2) = k+1-i$, we obtain that $\mathsf{dist}_{G'}(v, w) \leq k-(k+1-i) = i-1$. Since $w \in C$, we get that $\mathsf{dist}_G(v, C) \leq i-1$, as claimed. Note that $\mathsf{dist}_G(e, u) \leq \mathsf{dist}_{G'}(e, u)$, and since $\mathsf{dist}_{G'}(e, u) \leq k-i$, we get that $\mathsf{dist}_G(e, u) \leq k-i$. Hence, $\mathsf{dist}_G(e, C) \leq \mathsf{dist}_G(e, u) + \mathsf{dist}_G(u, v) + \mathsf{dist}_G(v, C) \leq (k-i) + 1 + (i-1) = k$, a contradiction with the assumption that $\mathsf{dist}_G(e, C) > k$. Thus, C is a distance-k vertex cover in G, and we obtain that $\tau_k(G) \leq |C| = |C'| - 1 = \tau_k(G') - 1$. □

An iterative application of Lemma 5.13 leads to the following result.

Corollary 5.14. *Let G be a graph, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $p(2k+2)$ times, for some two integers $k \geq 0$ and $p \geq 0$. Then $\tau_k(G') = \tau_k(G) + p$.*

Proof. Fix $k \geq 0$. We use induction on p. For $p = 0$ the statement is trivial, and for $p = 1$ this is just Lemma 5.13. Now let $p > 1$, let G' be as in the claim, and let G'' be the graph obtained from G by subdividing the edge $\{u, v\}$ exactly $(p-1)(2k+2)$ times. Denoting by P the path replacing $\{u, v\}$ in G'', observe that G' can be obtained from G'' by subdividing one of the edges of P exactly $2k+2$ times. By the induction hypothesis, we have $\tau_k(G'') = \tau_k(G) + p - 1$. Since we also have $\tau_k(G') = \tau_k(G'') + 1$ by Lemma 5.13, we infer that $\tau_k(G') = \tau_k(G) + p$, as claimed. □

Distance-k edge cover

Lemma 5.15. *Let G be a graph without isolated vertices, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $2k+2$ times, for some integer $k \geq 0$. Then $\rho_k(G') = \rho_k(G) + 1$.*

Proof. Let us denote the endpoints of e by u and v and let X be the set of internal vertices of the path between u and v in G' obtained from the subdivision of the edge $\{u,v\}$ in G. We label the elements of X as $X = \{x_1,\ldots,x_{2k+2}\}$ so that u is adjacent to x_1, vertex x_i is adjacent to x_{i+1} for every $i \in \{1,\ldots,2k+1\}$, and x_{2k+2} is adjacent to v.

Let F be a distance-k edge cover in G with size $\rho_k(G)$. We assume that $\mathsf{dist}_G(u,F) \geq \mathsf{dist}_G(v,F)$, since otherwise we can relabel u, v, and the vertices x_i, for $i \in \{1,\ldots,2k+2\}$, accordingly. We consider two cases depending on whether the edge e is in F or not. Suppose first that $e \in F$. Then we define $F' = (F \setminus \{e\}) \cup \{\{u,x_1\},\{v,x_{2k+2}\}\}$. Note that $F' \subseteq E(G')$. Observe that for every vertex $w \in V(G)$, we have $\mathsf{dist}_G(w,F\setminus\{e\}) \leq k$ or $\mathsf{dist}_G(w,e) \leq k$. We claim that F' is a distance-k edge cover in G'. Suppose there exists a vertex $w \in V(G')$ such that $\mathsf{dist}_{G'}(w,F') > k$. Then either $w \in V(G)$ or $w \in X$. Consider the case when $w \in V(G)$. Let P be a shortest path in G from w to F. Then P has length at most k and the assumption that $w \in V(G)$ implies that P contains the edge $\{u,v\}$, for otherwise we would have $\mathsf{dist}_{G'}(w,F') \leq \mathsf{dist}_G(w,F) \leq k$. However, $\mathsf{dist}_{G'}(w,F') \leq \min\{\mathsf{dist}_{G'}(w,\{u,x_1\}), \mathsf{dist}_{G'}(w,\{u,x_1\})\} = \mathsf{dist}_G(w,e) \leq \mathsf{dist}_G(w,F) \leq k$ which contradicts the assumption that $\mathsf{dist}_{G'}(w,F') > k$.

Now consider the case when $w \in X$. Since $\mathsf{dist}_{G'}(\{u,x_1\},x_j) \leq k$ for all j such that $1 \leq j \leq k+1$ and $\mathsf{dist}_{G'}(\{v,x_{2k+2}\},x_j) \leq k$ for all j such that $k+2 \leq j \leq 2k+2$, we have that every vertex in X is at distance at most k from $\{u,x_1\},\{x_{2k+3},v\}$, hence, from F' as well. In particular, w is at distance at most k from F' contradicting the assumption. So, whenever $e \in F$ we have that F' is a distance-k edge cover of G' with size $|F'| = |F|+1 = \rho_k(G)+1$, and thus $\rho_k(G') \leq \rho_k(G)+1$.

Now suppose that $e \notin F$. Let $d = \mathsf{dist}_G(v,F)$. Then $d \leq k$, which in turn implies that $\mathsf{dist}_G(v,F) = \mathsf{dist}_{G'}(v,F)$. Let also $F' = F \cup \{\{x_{d+1},x_{d+2}\}\}$. Note that $F \subseteq E(G')$. We claim that F' is a distance-k edge cover in G'. Suppose that there exists a vertex $w \in V(G')$ such that $\mathsf{dist}_{G'}(w,F') > k$. We consider two cases depending on whether $w \in V(G)$ or not. Consider first the case when $w \in V(G)$. Let P be a shortest path in G from w to F. Then P has length at most k and the assumption that $w \in E(G)$ implies that P contains the edge $\{u,v\}$, for otherwise we would have $\mathsf{dist}_{G'}(w,F') \leq |E(P)| \leq k$. Thus, $\mathsf{dist}_G(u,F) = \mathsf{dist}_G(v,F)+1$ (recall that we assumed that $\mathsf{dist}_G(u,F) \geq \mathsf{dist}_G(v,F)$), which in turn implies that $\mathsf{dist}_G(w,v) = \mathsf{dist}_G(w,u)+1$. Thus, $\mathsf{dist}_G(w,v) \leq k-d$ and $\mathsf{dist}_G(w,u) \leq k-d-1$. However, $\mathsf{dist}_{G'}(u,x_{d+1}) = d+1$, which implies that

$$\begin{aligned}\mathsf{dist}_{G'}(w,F') &\leq \mathsf{dist}_{G'}(w,\{x_{d+1},x_{d+2}\}) = \mathsf{dist}_{G'}(w,u) + \mathsf{dist}_{G'}(u,\{x_{d+1},x_{d+2}\}) \\ &\leq (k-d-1)+(d+1) = k,\end{aligned}$$

a contradiction.

Now, suppose that $w \notin V(G)$. Then, w is a vertex $x_i \in X$. Observe that for all j

such that $k+d+3 \leq j \leq 2k+2$, we have $\mathsf{dist}_{G'}(x_j, v) \leq k-d$, and hence

$$\begin{aligned}\mathsf{dist}_{G'}(x_j, F') &\leq \mathsf{dist}_{G'}(x_j, F) \leq \mathsf{dist}_{G'}(x_j, v) + \mathsf{dist}_{G'}(v, F) \\ &= \mathsf{dist}_{G'}(x_j, v) + \mathsf{dist}_{G}(v, F) \\ &\leq (k-d)+d = k.\end{aligned}$$

If $j \leq d+1$, then $\mathsf{dist}_{G'}(x_j, x_{d+1}) \leq d \leq k$. If $d+2 \leq j \leq k+d+2$ then $\mathsf{dist}_{G'}(x_j, x_{d+2}) \leq k$. Thus, every vertex in X is at distance at most k from F', implying that $\mathsf{dist}_{G'}(w, F') \leq k$, a contradiction. We conclude that F' is a distance-k edge cover in G', and therefore $\rho_k(G') \leq |F'| = |F|+1 = \rho_k(G)+1$.

For the converse inequality, let F' be a distance-k edge cover in G' with size $\rho_k(G')$. Note that F' must contain at least one edge from $E(G') \setminus E(G)$, since $\mathsf{dist}_{G'}(\{x_{k+1}, x_{k+2}\}, V(G') \setminus X) > k$. We look separately at the cases when $|F' \cap (E(G') \setminus E(G))| = 1$ and when $|F' \cap (E(G') \setminus E(G))| \geq 2$.

Suppose that $|F' \cap (E(G') \setminus E(G))| = 1$. Then $F' \cap (E(G') \setminus E(G)) = \{\{x_i, x_{i+1}\}\}$ for some $x_i \in X$. Without loss of generality, we assume that $i \leq k+1$ (the other case is symmetric). Let $F = F' \setminus \{\{x_i, x_{i+1}\}\}$. Then $F \subseteq E(G)$. We claim that F is a distance-k edge cover in G. Suppose for a contradiction that there exists a vertex $w \in V(G)$ such that $\mathsf{dist}_G(w, F) > k$. Observe that $w \notin e$ as

$$\mathsf{dist}_G(e, F) \leq \mathsf{dist}_{G'}(\{v, x_{2k+2}\}, F' \setminus \{\{x_i, x_{i+1}\}\}) = \mathsf{dist}_{G'}(\{v, x_{2k+2}\}, F') \leq k\,.$$

Since $w \in V(G')$ and F' is a distance-k edge cover in G', there exists an edge $f' \in F'$ such that $\mathsf{dist}_{G'}(w, f') \leq k$. If $f' \neq \{x_i, x_{i+1}\}$, then $f' \in E(G)$, hence, $\mathsf{dist}_G(w, F) \leq \mathsf{dist}_G(w, f') \leq \mathsf{dist}_{G'}(w, f') \leq k$, which contradicts the assumption that $\mathsf{dist}_G(w, F) > k$. Therefore, $f' = \{x_i, x_{i+1}\}$. Notice that $\mathsf{dist}_{G'}(x_{i+k+2}, f') > k$ (where $x_{2k+3} = v$) so there exists an edge $f'' \in F' \setminus \{x_i, x_{i+1}\} = F$ such that $\mathsf{dist}_{G'}(x_{i+k+2}, f'') \leq k$. As $\mathsf{dist}_{G'}(x_{i+k+2}, v) = k-(i-1)$, we have that $\mathsf{dist}_{G'}(f'', v) \leq i-1$. Moreover, since $f'' \in E(G)$ we have that $\mathsf{dist}_G(f'', v) \leq i-1$. Then,

$$\begin{aligned}\mathsf{dist}_G(w, F) &\leq \mathsf{dist}_G(w, u) + \mathsf{dist}_G(u, v) + \mathsf{dist}_G(v, f'') \\ &\leq (\mathsf{dist}_{G'}(w, f') - \mathsf{dist}_{G'}(f', u)) + 1 + (i-1) \\ &\leq (k-i)+1+(i-1) = k\,,\end{aligned}$$

which contradicts the assumption that $\mathsf{dist}_G(w, F) > k$. Hence, F is a distance-k edge cover of G with size $|F| = |F'|-1 = \rho_k(G')-1$ when $|F' \cap (E(G') \setminus E(G))| = 1$, implying that $\rho_k(G) \leq \rho_k(G')-1$.

Suppose now that $|F' \cap (E(G') \setminus E(G))| \geq 2$. Then, we define $F = (F' \setminus E(G')) \cup \{e\}$. Notice that $F \subseteq E(G)$ and $|F| \leq |F'|-1$. We claim that F is a distance-k edge cover in G. Suppose for a contradiction that there exists a vertex $w \in V(G)$ such that $\mathsf{dist}_G(w, F) > k$. Since $w \in V(G')$ and F' is a distance-k edge cover in G', we have that

$\mathsf{dist}_{G'}(w, f') \le k$ for some $f' \in F'$. If $f' \in E(G)$ then $\mathsf{dist}_G(w, f') \le \mathsf{dist}_{G'}(w, f') \le k$, which is not possible since $\mathsf{dist}_G(w, F) \le \mathsf{dist}_G(w, f')$ and we assumed that $\mathsf{dist}_G(w, F) > k$. If $f' \in E(G') \setminus E(G)$ then

$$\min\{\mathsf{dist}_G(w, u), \mathsf{dist}_{G'}(w, v)\} = \min\{\mathsf{dist}_{G'}(w, u), \mathsf{dist}_{G'}(w, v)\} \le \mathsf{dist}_{G'}(w, f') \le k.$$

But then $\mathsf{dist}_G(w, e) \le k$ contradicts the assumption that $\mathsf{dist}_G(w, F) > k$, since $\mathsf{dist}_G(w, F) \le \mathsf{dist}_G(w, e)$. Therefore, F is a distance-k edge cover in G with size $|F| = |F'| - 1 = \rho_k(G') - 1$, implying that $\rho_k(G) \le \rho_k(G') - 1$. □

An iterative application of Lemma 5.15 leads to the following result.

Corollary 5.16. *Let G be a graph without isolated vertices, let $e \in E(G)$, and let G' be the graph obtained from G by subdividing the edge e exactly $p(2k+2)$ times, for some two integers $k \ge 0$ and $p \ge 0$. Then $\rho_k(G') = \rho_k(G) + p$.*

Proof. Fix $k \ge 0$. We use induction on p. For $p = 0$ the statement is trivial, and for $p = 1$ this is just Lemma 5.15. Now let $p > 1$, let G' be as in the claim, and let G'' be the graph obtained from G by subdividing the edge $\{u, v\}$ exactly $(p-1)(2k+2)$ times. Denoting by P the path replacing $\{u, v\}$ in G'', observe that G' can be obtained from G'' by subdividing one of the edges of P exactly $2k + 2$ times. By the induction hypothesis, we have $\rho_k(G'') = \rho_k(G) + p - 1$. Since we also have $\rho_k(G') = \rho_k(G'') + 1$ by Lemma 5.15, we infer that $\rho_k(G') = \rho_k(G) + p$, as claimed. □

5.4 Transformations resulting in chordal graphs

Construction 2. Given a graph G containing at least one edge and an integer $k \ge 1$, we construct a graph G' as follows. First, we take a complete graph on a set Q of $|V(G)|$ new vertices such that for every vertex $u \in V(G)$ there exists a unique vertex u' in Q. Then, for each edge $\{u, v\} \in E(G)$, we create a u,v-paw as follows. We create a path of order k and connect one of its endpoints to both u' and v'. Figure 2 shows the the edge $\{u, v\}$ in G and the corresponding u,v-paw in G'.

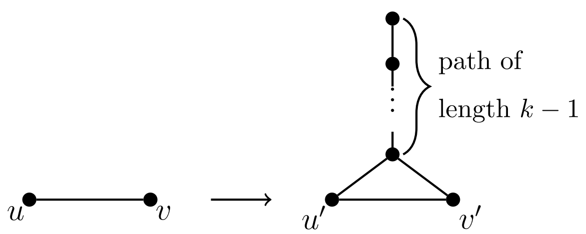

Figure 2: An edge $\{u, v\}$ (left) and its corresponding u,v-paw (right).

Lemma 5.17. *Let G be a graph and k a positive integer. Let G' be a graph obtained from Construction 2 given G and k. Then, G' is chordal and $2P_{k+1}$-free for every $k \geq 1$.*

Proof. Notice that any induced subgraph of G' isomorphic to P_{k+1} contains at least one vertex in the clique Q, which implies that G' cannot contain $2P_{k+1}$ as an induced subgraph, that is, G' is $2P_{k+1}$-free. Moreover, G' is chordal by construction since it does not contain an induced cycle of length more than 3. □

Theorem 5.18. *Let G be a graph containing at least one edge and k a positive integer. Let G' be the graph obtained from Construction 2 given G and k. Then, $\gamma_k(G') = \tau(G)$.*

Proof. Let C be a vertex cover in G with size $\tau(G)$ and $D = \{u' : u \in C\}$. Note that $D \subseteq Q \subseteq V(G')$; we claim that D is a distance-k dominating set in G'. Suppose that this is not the case. Then there exists a vertex $w \in V(G')$ at distance more than k from D. Observe that $D \subseteq Q$, so every vertex in Q is at distance at most 1 from D in G'. Therefore, as w is at distance more than k from D, $w \in V(G') \setminus Q$, and hence belongs to the u,v-paw for some edge $\{u, v\} \in E(G)$. Since C is a vertex cover in G, at least one of u or v belongs to C. We assume without loss of generality that u belongs to C. Since w belongs to the u,v-paw and $u' \in D$, we have $\mathsf{dist}_{G'}(w, D) \leq \mathsf{dist}_{G'}(w, u') \leq k$, a contradiction. Thus, D is a distance-k dominating set in G' with size $|D| = |C| = \tau(G)$, implying that $\gamma_k(G') \leq \tau(G)$.

Let D be a distance-k dominating set in G' with size $\gamma_k(G')$. Observe that if $w \in V(G') \setminus Q$, then w is a vertex in a u,v-paw for some $\{u, v\} \in E(G)$. Furthermore, by construction of G', every vertex x of G' with $\mathsf{dist}_{G'}(x, w) \leq k$ is such that $\mathsf{dist}_{G'}(x, u') \leq k$. Hence, if $w \in D$, then the set $(D \setminus \{w\}) \cup \{u'\}$ is also a distance-k dominating set in G' with size $\gamma_k(G')$. Hence, we may assume that $D \subseteq Q$. Let $C = \{u \in V(G) : u' \in D\}$. We claim that C is a vertex cover in G. Suppose not. Then there is an edge $\{u, v\} \in V(G)$ such that $u, v \notin C$. Therefore, $u', v' \notin D$, but then the vertex w in the u,v-paw that is at maximum distance from $\{u', v'\}$ is such that $\mathsf{dist}_{G'}(w, D) > \mathsf{dist}_{G'}(w, \{u', v'\}) = k$, a contradiction. Hence, C is a vertex cover in G with size $|C| = |D| = \gamma_k(G')$, implying that $\tau(G) \leq \gamma_k(G')$. □

Theorem 5.19. *Let G be a graph containing at least one edge and k a positive integer. Let G' be the graph obtained from Construction 2 given G and k. Then, $\rho_k(G') = \lceil \frac{\tau(G)}{2} \rceil$.*

Proof. Let C be a vertex cover in G with size $\tau(G)$ and $C' = \{u' : u \in C\}$. Let M be a maximal matching in $G'[C']$. We now define a set $F \subseteq E(G')$ as follows. If $|C'|$ is even, then we take $F = M$, and if $|C'|$ is odd, then we take $F = M \cup \{\{u', w'\}\}$ where u' is the unique vertex in C' that is not incident with any edge in M and w' is an arbitrary vertex in $Q \setminus \{u'\}$. Note that $Q \setminus \{u'\} \neq \emptyset$ as G contains at least one edge. Furthermore, every vertex in C' is incident with an edge in F. We claim that

F is a distance-k edge cover in G'. Suppose that this is not the case. Then there exists a vertex $w \in V(G')$ at distance more than k from F. Observe that $F \neq \emptyset$ as G contains at least one edge. Moreover, $F \subseteq E(G'[Q])$, and thus, every vertex of G' that is also in Q is at distance at most 1 from F. Therefore, as w is at distance more than k from F, it must be a vertex in $V(G') \setminus Q$, and hence belongs to the u,v-paw for some edge $\{u, v\} \in E(G)$. Since C is a vertex cover in G, at least one of u or v belongs to C. We assume without loss of generality that u belongs to C and thus $u' \in C'$. Since every vertex of C' is incident with an edge of F and w belongs to the u,v-paw, we have $\mathsf{dist}_{G'}(w, F) \leq \mathsf{dist}_{G'}(w, C') \leq \mathsf{dist}_{G'}(w, u') \leq k$, a contradiction. Thus, F is a distance-k edge cover in G' with size $|F| = \lceil \frac{|C|}{2} \rceil = \lceil \frac{\tau(G)}{2} \rceil$, implying that $\rho_k(G') \leq \lceil \frac{\tau(G)}{2} \rceil$.

Let F be a distance-k edge cover in G' with size $\rho_k(G')$. Observe that if $\hat{f} \in E(G') \setminus E(G'[Q])$, then $\hat{f}$ is an edge in the u,v-paw for some $e = \{u, v\} \in E(G)$. Observe also that, by construction of G', every vertex w of G' with $\mathsf{dist}_{G'}(w, \hat{f}) \leq k$ is such that $\mathsf{dist}_{G'}(w, \{u', v'\}) \leq k$. Hence, if $\hat{f} \in F$, then the set $(F \setminus \{\hat{f}\}) \cup \{\{u', v'\}\}$ is also a distance-k edge cover in G' with size at most $\rho_k(G)$, and therefore with size exactly $\rho_k(G)$. We may thus assume that $F \subseteq E(G'[Q])$. Let $C = \{u \in V(G) : u' \in f$ for some $f \in F\}$. We claim that C is a vertex cover in G. Suppose not. Then there is an edge $\{u, v\} \in E(G)$ such that $u, v \notin C$. Therefore, $\{u', v'\} \notin F$. But then, the vertex w of the u,v-paw that is at maximal distance from $\{u', v'\}$ is such that $\mathsf{dist}_{G'}(w, F) > \mathsf{dist}_{G'}(w, \{u', v'\}) = k$, a contradiction. Hence, C is a vertex cover in G with size $|C| \leq 2|F| = 2\rho_k(G')$, implying that $\tau(G) \leq 2\rho_k(G')$, or equivalently, $\frac{\tau(G)}{2} \leq \rho_k(G')$. Since $\rho_k(G')$ is an integer, we obtain $\lceil \frac{\tau(G)}{2} \rceil \leq \rho_k(G')$. □

Construction 3. Given a graph G containing at least one edge and an integer $k \geq 1$, we construct a graph G' as follows. First, we take a complete graph on a set Q of $|V(G)|$ new vertices such that for every vertex $u \in V(G)$ there exists a unique vertex u' in Q. Then, for each edge $\{u, v\} \in E(G)$, we create a u,v-*ladder* as follows. We create a path $P_{u,v}$ of order k and connect one of its endpoints to both u' and v'; then for each such vertex w of $P_{u,v}$ we add a new vertex w' and make it adjacent exactly to the vertices in $N[w]$ (in particular, this means that $N[w'] = N[w]$ in the resulting graph). We call the unique edge e of the u, v-ladder such that $\mathsf{dist}_{G'}(e, \{u, v\}) = k$ the *opposite edge* of the edge $\{u', v'\}$. Figure 3 shows the the edge $\{u, v\}$ in G and the corresponding u,v-ladder in G'.

Lemma 5.20. *Let G be a graph and k a positive integer. Let G' be the graph obtained from Construction 3 given G and k. If $k = 1$ then G' is $2P_3$-free and if $k \geq 2$ then G' is $2P_{k+1}$-free. Moreover, G' is chordal for every $k \geq 1$.*

Proof. Notice that any induced subgraph of G' isomorphic to $P_{\max\{k+1,3\}}$ contains at least one vertex in the clique Q, which implies that G' cannot contain $2P_{\max\{k+1,3\}}$ as

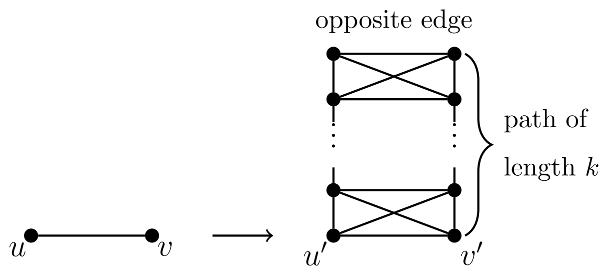

Figure 3: An edge $\{u, v\}$ (left) and its corresponding u,v-ladder (right).

an induced subgraph, that is, if $k = 1$, then G' is $2P_3$-free, and if $k \geq 2$, then G' is $2P_{k+1}$-free. Furthermore, if $k = 1$, then G' is also P_5-free. To see this, consider an induced path P in G' of order 4. Then P has both its endpoints in $V(G) \setminus Q$ and its two internal vertices in Q, as otherwise P would not be induced. This readily implies that P is a maximal path, and thus that G' is P_5-free.

Moreover, G' is chordal by construction, since there is no induced cycle of length more than 3. $\square$

Theorem 5.21. *Let G be a graph containing at least one edge and k a positive integer. Let G' be the graph obtained from Construction 3 given G and k. Then, $\gamma'_k(G') = \lceil \frac{\tau(G)}{2} \rceil$.*

Proof. Let C be a vertex cover in G with size $\tau(G)$ and $C' = \{u' : u \in C\}$. Let M be a maximal matching in $G'[C']$. We define a set $F \subseteq E(G')$ as follows. If $|C'|$ is even, then we take $F = M$, and if $|C'|$ is odd, then we take $F = M \cup \{\{u', w'\}\}$ where u' is the unique vertex in C' that is not incident with any edge in M and $w' \in Q \setminus \{u'\}$. Note that $Q \setminus \{u'\} \neq \emptyset$ as G contains at least one edge. Furthermore, every vertex in C' is incident with an edge in F. We claim that F is a distance-k edge dominating set in G'. Suppose that this is not the case. Then there exists an edge $f \in E(G')$ at distance more than k from F. Observe that $F \neq \emptyset$ as G contains at least one edge. Moreover, $F \subseteq E(G'[Q])$, and thus, every edge of G' having an endpoint in Q is at distance at most 1 from F. Therefore, as f is at distance more than k from F, it must have both endpoints in $V(G') \setminus Q$, and hence belongs to the u,v-ladder for some edge $\{u, v\} \in E(G)$. Since C is a vertex cover in G, at least one of u or v belongs to C. We assume without loss of generality that u belongs to C. Since f belongs to the u,v-ladder, $u' \in C'$, and u' is incident with an edge of F, we have $\mathsf{dist}_{G'}(f, F) \leq \mathsf{dist}_{G'}(f, C') \leq \mathsf{dist}_{G'}(f, u') \leq k$, a contradiction. Thus, F is a distance-k edge dominating set in G' with size $|F| = \lceil \frac{|C|}{2} \rceil = \lceil \frac{\tau(G)}{2} \rceil$, implying that $\gamma'_k(G') \leq \lceil \frac{\tau(G)}{2} \rceil$.

For the converse inequality, let F be a distance-k edge dominating set in G' with size $\gamma'_k(G')$. Observe that if $\hat{f} \in E(G') \setminus E(G'[Q])$, then $\hat{f}$ is an edge in a u,v-ladder for some $e = \{u, v\} \in E(G)$. Observe also that, by construction of G', every edge f of G' with $\mathsf{dist}_{G'}(f, \hat{f}) \leq k$ is such that $\mathsf{dist}_{G'}(f, \{u', v'\}) \leq k$. Hence, if $\hat{f} \in F$, then the set $(F \setminus \{\hat{f}\}) \cup \{\{u', v'\}\}$ is also a distance-k edge dominating set in G' with size at most $\gamma'_k(G)$, therefore, with size $\gamma'_k(G)$. Thus, we may assume that $F \subseteq E(G'[Q])$. Let $C = \{u \in V(G) : u' \in f \text{ for some } f \in F\}$. We claim that C is a vertex cover in G. Suppose, for a contradiction, that C is not a vertex cover in G. Then there is an edge $\{u, v\} \in E(G)$ such that $u, v \notin C$. Therefore, $\{u', v'\} \notin F$, but then the opposite edge $\hat{e}$ of the edge $\{u', v'\}$ is such that $\mathsf{dist}_{G'}(\hat{e}, F) > \mathsf{dist}_{G'}(e, \{u', v'\}) = k$, a contradiction. Hence, C is a vertex cover in G with size $|C| \leq 2|F| = 2\gamma'_k(G')$, implying that $\tau(G) \leq 2\gamma'_k(G')$, or equivalently, $\frac{\tau(G)}{2} \leq \gamma'_k(G')$. Since $\gamma'_k(G')$ is an integer, we obtain $\lceil \frac{\tau(G)}{2} \rceil \leq \gamma'_k(G')$. □

Theorem 5.22. *Let G be a graph containing at least one edge and $k \geq 1$ an integer. Let G' be the graph obtained from Construction 3 given G and k. Then, $\tau_k(G') = \tau(G)$.*

Proof. Let C be a vertex cover in G with size $\tau(G)$ and $C' = \{u' : u \in C\}$. Note that $C' \subseteq Q \subseteq V(G')$; we claim that C' is a distance-k vertex cover in G'. Suppose that this is not the case. Then there exists an edge $f \in E(G')$ at distance more than k from C'. Observe that $C' \subseteq Q$, and thus, every edge of G' having one endpoint in Q is at distance at most 1 from C'. Therefore, as f is at distance more than k from C', it must have both endpoints in $V(G') \setminus Q$, and hence belongs to the u,v-ladder for some edge $\{u, v\} \in E(G)$. Since C is a vertex cover in G, at least one of u or v belongs to C. We assume without loss of generality that u belongs to C. Since f belongs to the u,v-ladder and $u' \in C'$, we have $\mathsf{dist}_{G'}(f, C') \leq \mathsf{dist}_{G'}(f, u') \leq k$, a contradiction. Thus, C' is a distance-k vertex cover in G' with size $|C'| = |C| = \tau(G)$, implying that $\tau_k(G') \leq \tau(G)$.

Let C' be a distance-k vertex cover in G' with size $\tau_k(G')$. Observe that if $w \in V(G') \setminus Q$, then w is a vertex in the u,v-ladder for some $\{u, v\} \in E(G)$. Observe also that, by construction of G', every edge f of G' with $\mathsf{dist}_{G'}(f, w) \leq k$ is such that $\mathsf{dist}_{G'}(f, u') \leq k$. Hence, if $w \in C'$, then the set $(C' \setminus \{w\}) \cup \{u'\}$ is also a distance-k vertex cover in G' with size $\tau_k(G')$. Hence, we may assume that $C' \subseteq Q$. Let $C = \{u \in V(G) : u' \in C'\}$. Suppose that C is not a vertex cover in G. Then there is an edge $\{u, v\} \in E(G)$ such that $u, v \notin C$. Therefore, $u', v' \notin C'$, but then the opposite edge e of the edge $\{u, v\}$ is such that $\mathsf{dist}_{G'}(e, C') > \mathsf{dist}_{G'}(e, \{u', v'\}) = k$, a contradiction. Hence, C is a vertex cover in G with size $|C| = |C'| = \tau_k(G')$, implying that $\tau(G) \leq \tau_k(G')$. □

6 NP-completeness results

In this chapter we build on results from Chapter 5 and develop NP-completeness results for the distance-based variants of the four problems for H-free graphs where H has some specific properties. More precisely, we show that the problems are NP-complete for H-free graphs when H contains an induced claw or a cycle, as well as in some special cases when H is a linear forest.

Observation 1. Let G be a graph and $u, v \in V(G)$. Then determining the distance between u and v can be done in linear time. Moreover, if we denote by n and m the cardinalities of $V(G)$ and $E(G)$ respectively, then calculating the distance for all vertex pairs $u, v \in V(G)$ can be done in time at most $\mathcal{O}(n(n+m))$ (using breadth-first search from each vertex u), which is polynomial with respect to the size of the graph.

Corollary 6.1. *Let $k \in \mathbb{N}$ and let Π_k be one of the* Distance-k Dominating Set, Distance-k Edge Dominating Set, Distance-k Vertex Cover *and* Distance-k Edge Cover *problems. Then Π_k is in* NP.

Proof. Let I be an instance of Π_k. Then I consists of a graph G and an integer ℓ. Suppose that $\Pi_k(I)$ gives answer yes. If we are given a certificate C, that is, a set of vertices or edges satisfying the defining property of sets corresponding to problem Π_k, then we can check in polynomial time whether the set C has cardinality at most ℓ and satisfies the desired property, due to Observation 1. Hence, Π_k is in NP. □

Lemma 6.2. *Let μ and η be two graph invariants and let G and G' be two graphs satisfying $\eta(G') = f(\mu(G))$, where $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijective function. Then, for every integer ℓ, we have $\mu(G) \leq \ell$ if and only if $\eta(G') \leq f(\ell)$.*

Proof. Assume first that $\mu(G) \leq \ell$. Since f is increasing, we get $f(\mu(G)) \leq f(\ell)$, which is is the same as $\eta(G') \leq f(\ell)$. Assume now that $\eta(G') \leq f(\ell)$. Since f is bijective and strictly increasing, f^{-1} exists and is also strictly increasing. Hence we get $f^{-1}(\mu(G')) \leq f^{-1}(f(\ell))$, which is the same as $\mu(G) \leq \ell$. □

6.1 Edge cover at distance

When H contains a claw

Theorem 6.3. *For every fixed integer $k \geq 1$,* Distance-k Edge Cover *is* NP-*complete for line graphs.*

Proof. The problem is in NP by Corollary 6.1. To prove NP-hardness, we reduce from the P_3 FACTOR problem, which is NP-complete (see [45]).

Consider an arbitrary input to the P_3 FACTOR problem, consisting of a graph G. Let $\ell = \frac{|V(G)|}{3}$. Note that G has a P_3 factor if and only if G contains a P_3 cover with size at most ℓ, that is, $\Lambda(G) \leq \ell$. Let G' be the graph obtained from Construction 1 (defined on p. 21) given G and $t = k$, that is, $G' = G^{+k}$, and $H = L(G')$. By Theorems 5.5 and 5.8, we have $\rho_k(H) = \Lambda^e_{k-1}(G') = \Lambda(G)$. By Lemma 6.2, G contains a P_3 cover with size at most ℓ if and only if H contains a distance-k edge cover with size at most ℓ. The claimed NP-hardness follows. □

Since every line graph is claw-free, Theorem 6.3 implies the following result.

Corollary 6.4. *Let H be a graph containing a claw as an induced subgraph. Then for every fixed integer $k \geq 1$,* DISTANCE-k EDGE COVER *is* NP*-complete on H-free graphs.*

When H contains a cycle

Theorem 6.5. *Let H be a graph containing a cycle. Then for every fixed integer $k \geq 1$,* DISTANCE-k EDGE COVER *is* NP*-complete on H-free graphs.*

Proof. Let G be any graph and $k \geq 1$ an integer. Denote by g the girth of H and let G' be the graph obtained from G by subdividing every edge of G exactly $g(2k+2)$ times. Note that G' is obtained in polynomial time and by Corollary 5.16, $\rho_k(G') = \rho_k(G) + g|E(G)|$. Moreover, notice that G' has no cycle of length g, and thus is H-free. By Theorem 6.3, DISTANCE-k EDGE COVER is NP-complete on line graphs, therefore NP-complete on graphs in general. Since for any integer ℓ, graph G contains a distance-k edge cover with size at most ℓ if and only if G' contains a distance-k edge cover with size at most $\ell + g|E(G)|$, we conclude that DISTANCE-k EDGE COVER remains NP-complete on H-free graphs when H contains a cycle. □

Corollary 6.4 and Theorem 6.5 imply NP-completeness for the case when H is not a linear forest. Indeed, in this case H either contains a cycle, in which case Theorem 6.5 applies, or H is acyclic but contains an induced claw, in which case Corollary 6.4 applies.

Corollary 6.6. *Let H be a graph that is not a linear forest. Then for any fixed integer $k \geq 1$,* DISTANCE-k EDGE COVER *is* NP*-complete on H-free graphs.*

When H is a linear forest

Theorem 6.7. Distance-k Edge Cover *is* NP*-complete on* $2P_{k+1}$*-free chordal graphs for every fixed integer* $k \geq 1$*.*

Proof. The problem is in NP. To prove NP-hardness, we reduce from Vertex Cover, which is NP-complete (see [43]).

Consider an input to the Vertex Cover problem consisting of a graph G containing at least one edge and an integer ℓ. Let $2G$ be the graph consisting of two disjoint copies of G and let G' be the graph obtained from Construction 2 (defined on p. 32) given $2G$ and k. Note that $\tau(2G) = 2\tau(G)$ and hence $\rho_k(G') = \lceil \frac{\tau(2G)}{2} \rceil = \tau(G)$. By Theorem 5.19 and Lemma 6.2, G contains a vertex cover with size at most ℓ if and only if G' contains a distance-k edge cover with size at most ℓ. The claimed NP-hardness follows. □

6.2 Edge domination at distance

The Edge Dominating Set problem, which in our context is equivalent to the Distance-0 Edge Dominating Set problem, is known to be NP-complete.

Theorem 6.8 (Yannakakis and Gavril [67]). Edge Dominating Set *is* NP*-complete, even for cubic bipartite graphs and cubic planar graphs.*

Construction 4. Given a graph G and an integer $k \geq 1$, we define a graph G' obtained from G as follows: for each edge $\{u, v\} \in E(G)$, create a path $P_{u,v}$ made of $2k$ new vertices and connect the endpoints of $P_{u,v}$ to u and v, respectively. The path $P_{u,v}$ together with the edge $\{u, v\}$ is called the *u,v-gadget*. Note that the u,v-gadget is an induced cycle in G' of length $2k + 2$. In particular, there exists a unique edge $e' \in E(G')$ of the u,v-gadget such that $\mathsf{dist}_{G'}(e', u) = \mathsf{dist}_{G'}(e', v) = k$. We call the edge e' the *opposite edge* of the edge $\{u, v\}$. See Figure 4 for an example.

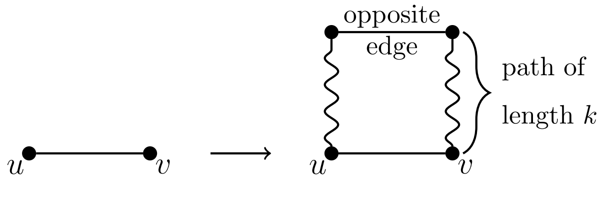

Figure 4: An edge $\{u, v\}$ (left) and its corresponding u,v-gagdet (right).

Lemma 6.9. *Let G be a graph, k a positive integer, and G' the graph obtained from Construction 4 given G and k. Let $F \subseteq E(G)$ and $\{u, v\} \in E(G)$. Then the following conditions are equivalent:*

1. *For every edge f' of the u,v-gadget it holds* $\mathsf{dist}_{G'}(f', F) \leq k$.

2. *The opposite edge e' of $\{u, v\}$ satisfies* $\mathsf{dist}_{G'}(e', F) \leq k$.

3. $\mathsf{dist}_{G'}(\{u, v\}, F) = 0$.

Proof. The implication from the first condition to the second one is trivial. To show that the second condition implies the third one, suppose that $\mathsf{dist}_{G'}(e', F) \leq k$. Observe that $\mathsf{dist}_{G'}(e', \{u, v\}) = k$. Suppose that $\mathsf{dist}_{G'}(\{u, v\}, F) > 0$. Then, no edge in F has u or v as endpoints. However, this implies that $\mathsf{dist}_{G'}(e', F) > k$, a contradiction.

Finally, assume that $\mathsf{dist}_{G'}(\{u, v\}, F) = 0$. This implies that at least one of u or v belongs to some edge in $f \in F$; we assume without loss of generality that $u \in f$. Thus, there exists a unique vertex w in the u,v-gadget such that $\mathsf{dist}_{G'}(w, u) \geq k + 1$ (note that w is an endpoint of the opposite edge of $\{u, v\}$). This implies that every edge of the u,v-gadget has at least one endpoint w' such that $\mathsf{dist}_{G'}(w', u) \leq k$. We conclude that for every edge f' of the u,v-gadget it holds $\mathsf{dist}_{G'}(f', F) \leq k$. □

Theorem 6.10. *For every integer $k \geq 1$,* Distance-k Edge Dominating Set *is* NP*-complete, even for bipartite graphs with maximum degree* 6 *and for planar graphs with maximum degree* 6*.*

Proof. Fix an integer $k \geq 1$. The problem is in NP by Corollary 6.1. To prove NP-hardness, we reduce from the Edge Dominating Set problem to the Distance-k Edge Dominating Set problem. Let G' be the graph obtained from Construction 4 given G and k. Note that G' can be obtained in polynomial time. We claim that G contains an edge dominating set F with size at most ℓ if and only if G' contains a distance-k edge dominating set F' with size at most ℓ.

For the forward implication, let F be an edge dominating set in G with size at most ℓ. Let $\{u, v\} \in E(G)$ and e' be its opposite edge. Following the fact that G is a subgraph of G' and that F is an edge dominating set in G, we have $\mathsf{dist}_{G'}(\{u, v\}, F) \leq \mathsf{dist}_{G}(\{u, v\}, F) = 0$. By Lemma 6.9, we have that every edge f' of the u,v-gadget satisfies $\mathsf{dist}_{G'}(f', F) \leq k$. Since this holds for every edge $\{u, v\} \in E(G)$, we obtain that F is a distance-k edge dominating set with size at most ℓ in G'.

For the converse implication, let F' be a distance-k edge dominating set in G' with size at most ℓ minimizing the number of edges in $E(G') \setminus E(G)$. Suppose that F' contains an edge $f \in E(G') \setminus E(G)$ from the u,v-gadget of some edge $e = \{u, v\} \in E(G)$ and let $F^* = (F' \setminus \{f\}) \cup \{e\}$. Note that the opposite edge e' of e is such that $\mathsf{dist}_{G'}(e', e) = k$, and thus by Lemma 6.9 every edge from the gadget of e is at distance at most k from F^*. Besides, every other edge in G' remains at distance at most k from F^*. Hence, we conclude that F^* is also a distance-k edge dominating set in G' with size at most ℓ. However, F^* contains one less edge in $E(G') \setminus E(G)$ than F', a contradiction with the definition of F'. So we may assume that $F' \subseteq E(G)$. Since for any edge $\{u,v\} \in E(G)$, its opposite edge is at distance at most k from F', we get by Lemma 6.9 that $\mathsf{dist}_{G'}(\{u,v\},F') = 0$. Thus, F' is an edge dominating set with size at most ℓ in G.

Note that the maximum degree of G' is exactly twice the maximum degree of G. Furthermore, it is easily observed that G' is bipartite, resp. planar, if and only G is bipartite, resp. planar. Since Edge Dominating Set is NP-complete on cubic bipartite graphs and cubic planar graphs (see Theorem 6.8), we obtain that Distance-k Edge Dominating Set is NP-complete on bipartite graphs with maximum degree 6 and planar graphs with maximum degree 6, as claimed. □

When H contains a claw

Theorem 6.11. *For every fixed integer $k \geq 1$,* Distance-k Edge Dominating Set *is NP-complete for line graphs.*

Proof. The problem is in NP by Corollary 6.1. To prove NP-hardness, we reduce from the P_3 factor problem, which is NP-complete (see [45]). Consider an arbitrary input to the P_3 factor problem, consisting of a graph G. Let $\ell = \frac{|V(G)|}{3}$. Note that G has a P_3 factor if and only if G contains a P_3 cover with size at most ℓ, that is $\Lambda(G) \leq \ell$. Let G' be the graph obtained from Construction 1 (defined on p. 21) given G and $t = k+1$, that is, $G' = G^{+(k+1)}$, and $H = L(G')$. By Theorems 5.3 and 5.7, we have $\gamma_k'(H) = \Lambda_{k-1}(G) = \Lambda(G)$. By Lemma 6.2, G contains a P_3 cover with size at most ℓ if and only if H contains a distance-k edge dominating set with size at most ℓ. The claimed NP-hardness follows. □

Since every line graph is claw-free, Theorem 6.11 implies the following result.

Corollary 6.12. *Let H be a graph containing a claw as an induced subgraph. Then for every fixed integer $k \geq 1$,* Distance-k Edge Dominating Set *is NP-complete on H-free graphs.*

When H contains a cycle

Theorem 6.13. *Let H be a graph containing a cycle. Then for every fixed integer $k \geq 1$,* Distance-k Edge Dominating Set *is NP-complete on H-free graphs.*

Proof. Let G be any graph and $k \geq 1$ an integer. Denote by g the girth of H and let G' be the graph obtained from G by subdividing every edge of G exactly $g(2k+3)$ times. Note that G' is obtained in polynomial time and by Corollary 5.12, $\gamma_k'(G') = \gamma_k'(G) + g|E(G)|$. Moreover, notice that G' has no cycle of length g, and thus is H-free. By Theorem 6.11, Distance-k Edge Dominating Set is NP-complete on line graphs, therefore NP-complete on graphs in general. Since for any integer ℓ, graph G continas a distance-k

edge dominating set with size at most ℓ if and only if G' contains a distance-k edge dominating set with size at most $\ell + g|E(G)|$, we conclude that Distance-k Edge Dominating Set remains NP-complete on H-free graphs when H contains a cycle. □

Corollary 6.12 and Theorem 6.13 imply the following result.

Corollary 6.14. *Let H be a graph that is not a linear forest. Then for any fixed integer $k \geq 1$,* Distance-k Edge Dominating Set *is* NP*-complete on H-free graphs.*

When H is a linear forest

Theorem 6.15. Distance-1 Edge Dominating Set *is* NP*-complete on $\{P_5, 2P_3\}$-free chordal graphs, and for all $k \geq 2$,* Distance-k Edge Dominating Set *is* NP*-complete on $2P_{k+1}$-free chordal graphs.*

Proof. The problem is in NP. To prove NP-hardness, we reduce from Vertex Cover, which is NP-complete (see [43]). Consider an input to the Vertex Cover problem consisting of a graph G containing at least an edge and an integer ℓ. Let $2G$ be the graphs consisting of two disjoint copies of G and let G' be the graph obtained from Construction 3 (defined on p. 34) given $2G$ and k. Note that $\tau(2G) = 2\tau(G)$ and hence $\gamma'_k(G') = \lceil \frac{\tau(2G)}{2} \rceil = \tau(G)$. By Theorem 5.21 and Lemma 6.2, G contains a vertex cover with size at most ℓ if and only if G' contains a distance-k edge dominating set with size at most ℓ. The claimed NP-hardness follows. □

6.3 Vertex cover at distance

When H contains a claw

Theorem 6.16. *For every fixed integer $k \geq 1$,* Distance-k Vertex Cover *is* NP*-complete for line graphs.*

Proof. The problem is in NP by Corollary 6.1. To prove NP-hardness, we reduce from the Distance-$(k-1)$ Edge Dominating Set problem, which is NP-complete by Theorem 6.10. Consider an arbitrary input to the Distance-$(k-1)$ Edge Dominating Set problem, consisting of a graph G and an integer ℓ. Let G' be the graph obtained from Construction 1 (defined on p. 21) given G and $t = 1$, that is, $G' = G^{+1}$, and $H = L(G')$. By Theorems 5.4 and 5.6, it holds that $\tau_k(H) = \varepsilon^{\Lambda}_{k-1}(G') = \gamma'_{k-1}(G)$. By Lemma 6.2, G contains a distance-$(k-1)$ edge dominating set with size at most ℓ if and only if H contains a distance-k vertex cover edge dominating set with size at most ℓ. The claimed NP-hardness follows. □

Since every line graph is claw-free, Theorem 6.16 implies the following result.

Corollary 6.17. *Let H be a graph containing a claw as an induced subgraph. Then for every fixed integer $k \geq 1$,* Distance-k Vertex Cover *is* NP*-complete on H-free graphs.*

When H contains a cycle

Theorem 6.18. *Let H be a graph containing a cycle. Then for every fixed integer $k \geq 1$,* Distance-k Vertex Cover *is* NP*-complete on H-free graphs.*

Proof. Let G be any graph and k a positive integer. Denote by g the girth of H and let G' be the graph obtained from G by subdividing every edge of G exactly $g(2k+2)$ times. Note that G' is obtained in polynomial time and by Corollary 5.14, $\tau_k(G) = \tau_k(G) + g|E(G)|$. Moreover, notice that G' has no cycle of length g, and thus is H-free. By Theorem 6.16, Distance-k Vertex Cover is NP-complete on line graphs, therefore NP-complete on graphs in general. Since for any integer ℓ, graph G contains a distance-k vertex cover with size at most ℓ if and only if G' contains a distance-k vertex cover with size at most $\ell + g|E(G)|$, we conclude that Distance-k Vertex Cover remains NP-complete on H-free graphs when H contains a cycle. □

Corollary 6.17 and Theorem 6.18 imply the following result.

Corollary 6.19. *Let H be a graph that is not a linear forest. Then for any fixed integer $k \geq 1$,* Distance-k Vertex Cover *is* NP*-complete on H-free graphs.*

When H is a linear forest

Theorem 6.20. Distance-1 Vertex Cover *is* NP*-complete on $\{P_5, 2P_3\}$-free chordal graphs, and for all $k \geq 2$,* Distance-k Vertex Cover *is* NP*-complete on $2P_{k+1}$-free chordal graphs.*

Proof. The problem is in NP. To prove NP-hardness, we reduce from Vertex Cover, which is NP-complete (see [43]). Consider an input to the Vertex Cover problem consisting of a graph G containing at least an edge and an integer ℓ. Let G' be the graph obtained from Construction 3 (defined on p. 34) given G and k. By Theorem 5.22 and Lemma 6.2, G contains a vertex cover with size at most ℓ if and only if G' contains a distance-k vertex cover with size at most ℓ. The claimed NP-hardness follows. □

6.4 Domination at distance

When H contains a claw

Theorem 6.21. *For every fixed integer $k \geq 1$,* Distance-k Dominating Set *is* NP*-complete for line graphs.*

Proof. The problem is in NP by Corollary 6.1. To prove NP-hardness, we reduce from DISTANCE-$(k-1)$ EDGE DOMINATING SET, which is NP-complete by Theorems 6.8 (for $k=1$) and 6.10 (for $k \geq 2$). Consider an arbitrary input to the DISTANCE-$(k-1)$ EDGE DOMINATING SET problem, consisting of a graph G and an integer ℓ. Let $H = L(G)$. By Theorem 5.2 and Lemma 6.2, G contains a distance-$(k-1)$ edge dominating set with size at most ℓ if and only if H contains a distance-k dominating set with size at most ℓ. The claimed NP-hardness follows. □

Since every line graph is claw-free, Theorem 6.21 implies the following result.

Corollary 6.22. *Let H be a graph containing a claw as an induced subgraph. Then for every fixed integer $k \geq 1$,* DISTANCE-k DOMINATING SET *is* NP*-complete on H-free graphs.*

When H contains a cycle

Theorem 6.23. *Let H be a graph containing a cycle. Then for every fixed integer $k \geq 1$,* DISTANCE-k DOMINATING SET *is* NP*-complete on H-free graphs.*

Proof. Let G be any graph and k a non-negative integer. Denote by g the girth of H and let G' be the graph obtained from G by subdividing every edge of G exactly $g(2k+1)$ times. Note that G' is obtained in polynomial time and by Corollary 5.10, $\gamma_k(G) = \gamma_k(G) + g|E(G)|$. Moreover, notice that G' has no cycle of length g, and thus is H-free. By Theorem 6.21, DISTANCE-k DOMINATING SET is NP-complete on line graphs, therefore NP-complete on graphs in general. Since for any integer ℓ, graph G contains a distance-k dominating set with size at most $\ell + g|E(G)|$, we conclude that DISTANCE-k DOMINATING SET remains NP-complete on H-free graphs when H contains a cycle. □

Corollary 6.22 and Theorem 6.23 imply the following result.

Corollary 6.24. *Let H be a graph that is not a linear forest. Then for any fixed integer $k \geq 1$,* DISTANCE-k DOMINATING SET *is* NP*-complete on H-free graphs.*

When H is a linear forest

Theorem 6.25. DISTANCE-k DOMINATING SET *is* NP*-complete on $2P_{k+1}$-free chordal graphs.*

Proof. The problem is in NP. To prove NP-hardness, we reduce from VERTEX COVER, which is NP-complete (see [43]). Consider an input to the VERTEX COVER problem consisting of a G graph containing at least one edge and an integer ℓ. Let G' be the graph obtained from Construction 2 (defined on p. 32) given G and k. By Theorem

5.22 and Lemma 6.2, G contains a vertex cover with size at most ℓ if and only if G' contains a distance-k dominating set with size at most ℓ. The claimed NP-hardness follows. $\square$

7 Polynomial algorithms

In this section we identify, for each integer $k \geq 1$, an infinite family of graph classes in which Distance-k Dominating Set, Distance-k Edge Dominating Set, Distance-k Vertex Cover and Distance-k Edge Cover can be solved in polynomial time.

7.1 Domination at distance in H-free graphs

Our first result is based on the following structural property of P_t-free graphs.

Theorem 7.1 (Camby and Schaudt [14]). *Let $t \geq 4$ be an integer, let G be a connected P_t-free graph, and let S be any minimum connected dominating set in G. Then the subgraph induced by S in G is either P_{t-2}-free or isomorphic to P_{t-2}.*

Theorem 7.1 has the following consequence for distance-k dominating set in P_{2k+2}-free graphs.

Lemma 7.2. *For every integer $k \geq 0$, every connected P_{2k+2}-free graph G has a distance-k dominating set that induces a path of order at most two.*

Proof. The proof is by induction on k. Suppose first that $k = 0$. In this case, the statement says that every connected P_2-free graph G has a distance-0 dominating set that induces a path of order at most two. This follows directly since G is edgeless and connected, hence, G is a one-vertex graph. Suppose now that $k \geq 1$ and consider a connected P_{2k+2}-free graph G. Let S be a minimum connected dominating set in G and let G' be the subgraph of G induced by S. Following Theorem 7.1, we obtain that G' is either P_{2k}-free or isomorphic to P_{2k}. If G' is P_{2k}-free, then the induction hypothesis implies that G' has a distance-$(k-1)$ dominating set that induces a path of order at most two. If G' is isomorphic to P_{2k}, with vertices $v_1, \ldots, v_{2k}$ in order, then the edge $\{v_k, v_{k+1}\}$ is a distance-$(k-1)$ dominating set in G'. In either case, G' has a distance-$(k-1)$ dominating set S' that induces a path of order at most two. Since every vertex in G is either in S or has a neighbor in S, we infer that S' is a distance-k dominating set in G that induces a path of order at most two. $\square$

In the following theorem, the running time of the algorithm is independent of k, that is, the $\mathcal{O}$ notation does not hide any constants depending on k.

Theorem 7.3. *For every integer $k \geq 1$, there is an algorithm with running time $\mathcal{O}(|V(G)| + |E(G)|^2)$ that takes as input a P_{2k+2}-free graph G and computes a minimum distance-k dominating set of G.*

Proof. Fix a positive integer k and let G be a P_{2k+2}-free graph. To compute a minimum distance-k dominating set of G, we first compute the connected components $G_1, \ldots, G_s$ of G, solve the problem in each connected component G_i, and combine the obtained solutions. By Lemma 7.2, each connected component G_i of G has a distance-k dominating set that induces a path of order at most two. Thus, we immediately obtain a polynomial-time algorithm for computing a minimum distance-k dominating set of a component G_i. We first check if there exists a vertex $u \in V(G_i)$ such that $\{u\}$ is a distance-k dominating set in G_i. If this is the case, then we have an optimal solution; otherwise we check for each edge $\{u, v\} \in E(G_i)$ if $\{u, v\}$ is a distance-k dominating set in G_i. Once we find one, we return it.

It remains to analyze the running time. Let us write, as usual, $n = |V(G)|$, $m = |E(G)|$, and, $n_i = |V(G_i)|$ and $m_i = |E(G_i)|$ for all $i \in \{1, \ldots, s\}$. Computing the connected components of G can be done in time $\mathcal{O}(n + m)$ and, given a component G_i and a set $S \subseteq V(G_i)$, testing if S is a distance-k dominating set of G_i can be done in time $\mathcal{O}(n_i + m_i) = \mathcal{O}(m_i)$[1], by using breadth-first search in G_i up to distance k from S and verifying if all the vertices have been reached. We need to test $\mathcal{O}(n_i + m_i) = \mathcal{O}(m_i)$ sets S, hence the overall time complexity of the algorithm on G_i is $\mathcal{O}(m_i^2)$. Since $\sum_i m_i^2 \leq (\sum_i m_i)^2 = m^2$, summing up the complexities over all components yields the overall running time of $\mathcal{O}(n + m^2)$. □

Lemma 7.4. *For every two integers $k \geq 1$ and $s \geq 0$, every connected $(P_{2k+2}+sP_k)$-free graph G has a distance-k dominating set that induces a linear forest of order at most $f_k(s)$ where*

$$f_k(s) = \begin{cases} 2 & \text{if } s = 0\,, \\ (s+1)k + 2 & \text{if } s \geq 1\,. \end{cases}$$

Proof. Fix an integer $k \geq 1$. We use induction on s. For $s = 0$, the statement follows from Lemma 7.2.

Suppose now that $s \geq 1$ and that every connected $(P_{2k+2} + (s-1)P_k)$-free graph has a distance-k dominating set that induces a linear forest of order at most $f_k(s-1)$. Let G be a connected $(P_{2k+2} + sP_k)$-free graph. If G is $(P_{2k+2} + (s-1)P_k)$-free, then G has a distance-k dominating set that induces a linear forest of order at most $f_k(s-1) \leq f_k(s)$. On the other hand, if G is not $(P_{2k+2} + (s-1)P_k)$-free, then there exists a set $S \subseteq V(G)$ inducing a $P_{2k+2} + (s-1)P_k$. Note that S induces a linear forest

[1]Since G_i is a connected graph, it holds that $m_i \geq n_i - 1$ and consequently $n_i = \mathcal{O}(m_i)$ as soon as G_i has at least one edge – which can assumed, since otherwise the unique vertex of G_i has to be selected into the optimal solution.

of order $(s+1)k+2 = f_k(s)$. It thus suffices to show that S is a distance-k dominating set in G. Let $X = N(S)$ be the set of vertices not in S and with a neighbor in S and $Y = V(G) \setminus (S \cup X)$ be the set of vertices not in S and without a neighbor in S. Let w be a vertex of G. If w belongs to $S \cup X$, then $\mathsf{dist}_G(w, S) \leq 1 \leq k$. So let $w \in Y$. Since G is connected, there exists a shortest path P between w and a vertex in S. Since G is $(P_{2k+2} + sP_k)$-free, the part of P entirely contained in Y has at most $k-1$ vertices. Other than that, P has exactly one vertex in X and exactly one in S. Thus, the length of P is at most k, which implies $\mathsf{dist}_G(w, S) \leq k$. This shows that S is a distance-k dominating set in G and completes the proof. □

Lemma 7.4 implies that for all integers $k \geq 1$ and $s \geq 0$ the minimum size of a distance-k dominating set in a connected $(P_{2k+2} + sP_k)$-free graph G is bounded from above by a function depending only on k and s but independent of G. Thus, we can do a complete enumeration of small subsets of vertices to find a minimum distance-k dominating set in such a graph, and essentially the same approach as the one used in Theorem 7.3 using Lemma 7.2 can be used to prove the following theorem using Lemma 7.4.

Theorem 7.5. *For every two integers $k \geq 1$ and $s \geq 0$, there is a polynomial-time algorithm that takes as input a $(P_{2k+2} + sP_k)$-free graph G and computes a minimum distance-k dominating set in G.*

7.2 Vertex cover at distance in H-free graphs

Lemma 7.6. *For every integer $k \geq 0$ every connected P_{2k+2}-free graph G has a distance-k vertex cover that induces a path of order at most two.*

Proof. Lemma 7.2 implies that G has a distance-k dominating set D that induces a path of order at most two. Theorem 4.1 and its proof imply that D is also a distance-k vertex cover in G proving the statement. □

Theorem 7.7. *For every integer $k \geq 1$, there is an algorithm with running time $\mathcal{O}(|V(G)| + |E(G)|^2)$ that takes as input a P_{2k+2}-free graph G and computes a minimum distance-k vertex cover of G.*

Proof. Fix a positive integer k and let G be a P_{2k+2}-free graph. To compute a minimum distance-k vertex cover of G, we first compute the connected components of G, solve the problem in each component, and then combine the obtained solutions. Given a connected component G_i, we check whether a set $S \subseteq V(G_i)$ is a distance-k vertex cover in G_i in time $\mathcal{O}(|E(G_i)|)$ using breadth-first search up to distance k from S and verifying if all the edges have been reached. The rest of the proof is similar to the proof of Theorem 7.3, except that Lemma 7.6 is used instead of Lemma 7.2. □

Lemma 7.8. *For every integer $s \geq 0$, every connected $(P_4 + sP_2)$-free graph G has a distance-1 vertex cover that induces a linear forest of order at most $2s + 2$.*

Proof. We use induction on s. For $s = 0$, the statement follows from Lemma 7.6.

Suppose now that $s \geq 1$ and that every connected $P_4 + (s-1)P_2$-free graph has a distance-1 vertex cover that induces a linear forest of order at most $2s$. Let G be a connected $(P_4 + sP_2)$-free graph. If G is $P_4 + (s-1)P_2$-free, then G has a distance-1 vertex cover that induces a linear forest of order at most $2s$. On the other hand, if G is not $P_4 + (s-1)P_2$-free, then there exists a set $S \subseteq V(G)$ inducing a $P_4 + (s-1)P_2$. Since G is $(P_4 + sP_2)$-free, every edge e of G either has an endpoint in S or there is an edge connecting an endpoint of e with a vertex of S. In other words, every edge of G is at distance at most 1 from S. Thus, S is a distance-1 vertex cover in G that induces a linear forest of order $2(s-1) + 4 = 2s + 2$. □

Lemma 7.9. *For every two integers $k \geq 2$ and $s \geq 0$, every connected $(P_{2k+2}+sP_k)$-free graph G has a distance-k vertex cover that induces a linear forest of order at most $f_k(s)$ where*

$$f_k(s) = \begin{cases} 2 & \text{if } s = 0\,, \\ (s+1)k + 2 & \text{if } s \geq 1\,. \end{cases}$$

Proof. By Lemma 7.4 we have that G contains a distance-k dominating set D that induces a linear forest of order at most $f_k(s)$. Then Theorem 4.1 and its proof imply that D is also a distance-k vertex cover. Hence G contains a distance-k vertex cover that induces a linear forest of order at most $f_k(s)$. □

Lemmas 7.8 and 7.9 imply that for all integers $k \geq 1$ and $s \geq 0$ the minimum size of a distance-k vertex cover in a connected $(P_{2k+2} + sP_{\max\{k,2\}})$-free graph is bounded from above by a function depending only on k and s but independent of G. Thus, we can do a complete enumeration of small subsets of vertices to find a minimum distance-k vertex cover in such a graph, and essentially the same approach as the one used in Theorem 7.7 using Lemma 7.6 can be used to prove the following theorem using Lemmas 7.8 and 7.9.

Theorem 7.10. *For every two integers $k \geq 1$ and $s \geq 0$, there is a polynomial-time algorithm that takes as input a $(P_{2k+2} + sP_{\max\{k,2\}})$-free graph G and computes a minimum distance-k vertex cover in G.*

7.3 Edge domination at distance in H-free graphs

Lemma 7.11. *For every integer $k \geq 0$ every connected P_{2k+2}-free graph G has a distance-k edge dominating set of size at most one.*

Proof. If G is isomorphic to K_1, then the empty set is the only distance-k edge dominating set of G. Suppose that G is not isomorphic to K_1. Lemma 7.2 shows that G has a distance-k dominating set D that induces a path of order at most 2. If $|D| = 1$, then we take $F = \{\{u, v\}\}$ where $u \in D$ and v is any neighbor of u in G. Note that such a vertex v exists, since G is connected and not isomorphic to K_1. If $|D| = 2$, then we have $D = \{u, v\}$ for a pair u, v of adjacent vertices in G, and we take $F = \{\{u, v\}\}$. In both cases, F consists of a single edge of G. As D is a distance-k dominating set in G we get that for every $w \in V(G)$ we have $\mathsf{dist}_G(w, F) \leq \mathsf{dist}_G(w, D) \leq k$, implying that F is a distance-k edge dominating set in G of size one. □

Note that Lemma 7.11 could be equivalently stated as follows: For every integer $k \geq 0$, every nontrivial connected P_{2k+2}-free graph G satisfies $\gamma'_k(G) = 1$.

Theorem 7.12. *For every integer $k \geq 1$, there is an algorithm with running time $\mathcal{O}(|V(G)| + |E(G)|^2)$ that takes as input a P_{2k+2}-free graph G and computes a minimum distance-k edge dominating set of G.*

Proof. Fix a positive integer k and let G be a P_{2k+2}-free graph. To compute a minimum distance-k edge dominating set of G, we first compute the connected components $G_1, \ldots, G_s$ of G, solve the problem in each connected component G_i, and combine the obtained solutions. If G_i has at least two vertices, Lemma 7.11 guarantees that there exists an edge $e \in E(G_i)$ such that $\{e\}$ is a distance-k edge dominating set in G_i. Thus, we immediately obtain a polynomial-time algorithm for computing a minimum distance-k edge dominating set of a component G_i. Testing if for some edge $e \in E(G_i)$ the set $\{e\}$ is a distance-k dominating set of G_i can be done in time $\mathcal{O}(|E(G_i)|)$, by using breadth-first search in G_i up to distance k from $\{e\}$ and verifying if all the edges have been reached. The rest of the time complexity analysis is done similarly as the proof of Theorem 7.3.

□

Lemma 7.13. *For every integer $s \geq 0$, every connected $(P_4 + sP_2)$-free graph G has a distance-1 edge dominating set of size at most $2s + 2$.*

Proof. By Lemma 7.8, G contains a distance-1 vertex cover of size at most $2s+2$, hence $\tau_1(G) \leq 2s + s$. Then we conclude, by Theorem 4.1, that $\gamma'_1(G) \leq \tau_1(G) \leq 2s + 2$, implying that G contains a distance-1 edge dominating set of size at most $2s + 2$. □

Lemma 7.14. *For every two integers $k \geq 2$ and $s \geq 0$, every connected $(P_{2k+2} + sP_k)$-free graph G has a distance-k edge dominating set of size at most $f_k(s)$ where*

$$f_k(s) = \begin{cases} 2 & \text{if } s = 0\,, \\ (s+1)k + 2 & \text{if } s \geq 1\,. \end{cases}$$

Proof. By Lemma 7.4, G contains a distance-k dominating set of size at most $f_k(s)$, hence, $\gamma_k(G) \leq f_k(s)$. We conclude, by Theorem 4.1, that $\gamma'_k(G) \leq \gamma_k(G) \leq f_k(s)$, implying that G contains a distance-k edge dominating set of size at most $f_k(s)$. $\square$

Similarly as for Theorem 7.10, Lemmas 7.13 and 7.14 imply that for all integers $k \geq 1$ and $s \geq 0$ and for any connected $(P_{2k+2} + sP_{\max\{2,k\}})$-free graph G, the value of $\gamma'_k(G)$ is bounded from above by a function depending only on k and s but independent of G. Thus, we get the following theorem also in a similar way.

Theorem 7.15. *For every two integers $k \geq 1$ and $s \geq 0$, there is a polynomial-time algorithm that takes as input a $(P_{2k+2} + sP_{\max\{k,2\}})$-free graph G and computes a minimum distance-k edge dominating set in G.*

7.4 Edge cover at distance in H-free graphs

Lemma 7.16. *For every integer $k \geq 0$ every nontrivial connected P_{2k+2}-free graph G has a distance-k edge cover of size one.*

Proof. Lemma 7.2 shows that G has a distance-k dominating set D that induces a path of order at most 2. If $|D| = 1$, then we take $F = \{\{u, v\}\}$ where $u \in D$ and v is any neighbor of u in G. Note that such a vertex v exists, since G is connected and nontrivial. If $|D| = 2$, then we have $D = \{u, v\}$ for a pair u, v of adjacent vertices in G, and we take $F = \{\{u, v\}\}$. In both cases, F consists of a single edge of G. As D is a distance-k dominating set in G we get that for every $w \in V(G)$ we have $\mathsf{dist}_G(w, F) \leq \mathsf{dist}_G(w, D) \leq k$, implying that F is a distance-k edge cover of G of size one. $\square$

Note that Lemma 7.16 could be equivalently stated as follows: For every integer $k \geq 0$, every nontrivial connected P_{2k+2}-free graph G satisfies $\rho'_k(G) = 1$.

Theorem 7.17. *For every integer $k \geq 1$, there is an algorithm with running time $\mathcal{O}(|V(G)| + |E(G)|^2)$ that takes as input a P_{2k+2}-free graph G without isolated vertices and computes a minimum distance-k edge cover of G.*

Proof. Fix a positive integer k and let G be a P_{2k+2}-free graph. To compute a minimum distance-k edge cover of G, we first compute the connected components $G_1, \ldots, G_s$ of G. We solve the problem in each connected component G_i, and combine the obtained solutions. By Lemma 7.16, each connected component G_i of G has a distance-k edge cover of size one. Thus, we immediately obtain a polynomial-time algorithm for computing a minimum distance-k edge cover of G_i. We check for every edge $e \in E(G_i)$ if $\{e\}$ is a distance-k edge cover in G_i. This can be done in time $\mathcal{O}(|E(G_i)|)$ using breadth-first search in G_i up to distance k from $\{e\}$ and verifying if all the vertices have been reached. The rest of the time complexity analysis is done similarly as the proof of Theorem 7.3. □

Lemma 7.18. *For every two integers $k \geq 1$ and $s \geq 0$, every nontrivial connected $(P_{2k+2} + sP_k)$-free graph G has a distance-k edge cover of size at most $f_k(s)$ where*

$$f_k(s) = \begin{cases} 2 & \text{if } s = 0\,, \\ (s+1)k + 2 & \text{if } s \geq 1\,. \end{cases}$$

Proof. By Lemma 7.4 G contains a distance-k dominating set of size at most $f_k(s)$, hence, $\gamma_k(G) \leq f_k(s)$. Then we conclude, by Theorem 4.1, that $\rho_k(G) \leq \gamma_k(G) \leq f_k(s)$, implying that G contains a distance-k edge cover of size at most $f_k(s)$. □

Similarly as for Theorem 7.5, Lemma 7.18 implies that for all integers $k \geq 1$ and $s \geq 0$ and any nontrivial connected $(P_{2k+2} + sP_k)$-free graph G, the value of $\rho_k(G)$ is bounded from above by a function depending only on k and s but independent of G. Thus, get the following theorem also in a similar way.

Theorem 7.19. *For every two integers $k \geq 1$ and $s \geq 0$, there is a polynomial-time algorithm that takes as input a $(P_{2k+2} + sP_k)$-free graph G without isolated vertices and computes a minimum distance-k edge cover in G.*

8 Complexity dichotomies

Let us now combine the results obtained in the previous two sections to get the complexity dichotomies for the four distance covering and domination problems in the classes of H-free graphs.

Theorem 8.1. *For every graph H and every integer $k \geq 1$,* Distance-k Dominating Set *is solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and* NP*-complete otherwise.*

Proof. Fix a graph H and let $\mathcal{G}$ be the class of H-free graphs. If H is not a linear forest, then for all $k \geq 1$ Corollary 6.24 implies that Distance-k Dominating Set is NP-complete on $\mathcal{G}$.

Suppose that H is a linear forest and let $k \geq 1$. If H contains $2P_{k+1}$ as an induced subgraph, then $\mathcal{G}$ contains the class of $2P_{k+1}$-free chordal graphs, and hence by Theorem 6.25 Distance-k Dominating Set is NP-complete on $\mathcal{G}$. Otherwise, H is $2P_{k+1}$-free. Let t denote the maximum order of a component of H and let C be a component of H of order t. If $t \leq k$, then every component of H has order at most k. If $t \geq k+1$, then, since H is $2P_{k+1}$-free, every component of H other than C has order at most k. Thus, in either case, every component of H other than C has order at most k, and H is an induced subgraph of $P_t + sP_k$ for some $s \geq 0$, which implies that $t \leq 2k+2$, and thus H is an induced subraph of $P_{2k+2} + sP_k$ for some $s \geq 0$. It follows that every H-free graph is $(P_{2k+2} + sP_k)$-free. Thus, by Theorem 7.5 the problem can be solved in polynomial time for graphs in $\mathcal{G}$. □

Theorem 8.2. *For every graph H, the following holds:*

- Distance-1 Edge Dominating Set *is solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_4 + sP_2$, for some $s \geq 0$, and* NP*-complete otherwise.*

- *For every integer $k \geq 2$,* Distance-k Edge Dominating Set *is solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and* NP*-complete otherwise.*

Proof. Fix a graph H and let $\mathcal{G}$ be the class of H-free graphs. If H is not a linear forest, then for all $k \geq 1$, Corollary 6.14 implies that Distance-k Edge Dominating Set is NP-complete on $\mathcal{G}$. Suppose that H is a linear forest.

Consider first the case when $k = 1$. If H contains P_5 or $2P_3$ as an induced subgraph, then $\mathcal{G}$ contains the class of $\{P_5, 2P_3\}$-free chordal graphs, and hence by Theorem 6.15 Distance-1 Edge Dominating Set is NP-complete on $\mathcal{G}$. Otherwise, we obtain that H is $\{P_5, 2P_3\}$-free. Recall that H is a linear forest. Let us denote by t be the maximum order of a component of H and let C be a component of H of order t. If $t \leq 2$, then every component of H has order at most two. If $t \geq 3$, then, since H is $2P_3$-free, every component of H other than C has order at most two. In either case, every component of H other than C has order at most two, which implies that H is an induced subgraph of $P_t + sP_2$ for some $s \geq 0$. Since H is P_5-free, we have $t \leq 4$, and hence every H-free graph is $(P_4 + sP_2)$-free. Thus, by Theorem 7.15 the problem can be solved in polynomial time for graphs in $\mathcal{G}$.

Suppose now that $k \geq 2$. If H contains $2P_{k+1}$ as an induced subgraph, then $\mathcal{G}$ contains the class of $2P_{k+1}$-free chordal graphs, and hence by Theorem 6.15 Distance-k Edge Dominating Set is NP-complete on $\mathcal{G}$. Otherwise, H is $2P_{k+1}$-free. Again, let t denote the maximum order of a component of H and let C be a component of H of order t. If $t \leq k$, then every component of H has order at most k. If $t \geq k+1$, then, since H is $2P_{k+1}$-free, every component of H other than C has order at most k. In either case, every component of H other than C has order at most k, and H is an induced subgraph of $P_t + sP_k$ for some $s \geq 0$. Since H is $2P_{k+1}$-free, it is also P_{2k+3}-free, which implies that $t \leq 2k+2$, and thus H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$. It follows that every H-free graph is $(P_{2k+2} + sP_k)$-free. Thus, by Theorem 7.15 the problem can be solved in polynomial time for graphs in $\mathcal{G}$. □

Theorem 8.3. *For every graph H, the following holds:*

- Distance-1 Vertex Cover *is solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_4 + sP_2$, for some $s \geq 0$, and* NP*-complete otherwise.*
- *For every integer $k \geq 2$,* Distance-k Vertex Cover *is solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and* NP*-complete otherwise.*

Proof. Fix a graph H and let $\mathcal{G}$ be the class of H-free graphs. If H is not a linear forest, then for all $k \geq 1$, Corollary 6.19 implies that Distance-k Vertex Cover is NP-complete on $\mathcal{G}$. Suppose that H is a linear forest.

Consider first the case when $k = 1$. If H contains P_5 or $2P_3$ as an induced subgraph, then $\mathcal{G}$ contains the class of $\{P_5, 2P_3\}$-free chordal graphs, and hence by Theorem 6.20 Distance-1 Vertex Cover is NP-complete on $\mathcal{G}$. Otherwise, we obtain that H is $\{P_5, 2P_3\}$-free. Recall that H is a linear forest. Using the same arguments as in the proof of Theorem 8.2, we infer that H is an induced subgraph of $P_4 + sP_2$ for some

$s \geq 0$, and hence every H-free graph is $(P_4 + sP_2)$-free. Thus, by Theorem 7.10 the problem can be solved in polynomial time for graphs in $\mathcal{G}$.

Suppose now that $k \geq 2$. If H contains $2P_{k+1}$ as an induced subgraph, then $\mathcal{G}$ contains the class of $2P_{k+1}$-free chordal graphs, and hence by Theorem 6.20 Distance-k Vertex Cover is NP-complete on $\mathcal{G}$. Otherwise, H is $2P_{k+1}$-free. Again, let t denote the maximum order of a component of H and let C be a component of H of order t. If $t \leq k$, then every component of H has order at most k. If $t \geq k+1$, then, since H is $2P_{k+1}$-free, every component of H other than C has order at most k. Thus, in either case, every component of H other than C has order at most k, and H is an induced subgraph of $P_t + sP_k$ for some $s \geq 0$. Since H is $2P_{k+1}$-free, it is also P_{2k+3}-free, which implies that $t \leq 2k+2$, and thus H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$. It follows that every H-free graph is $(P_{2k+2} + sP_k)$-free. Thus, by Theorem 7.10 the problem can be solved in polynomial time for graphs in $\mathcal{G}$. □

Theorem 8.4. *For every graph H and every integer $k \geq 1$,* Distance-k Edge Cover *is solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and* NP*-complete otherwise.*

Proof. Fix a graph H and let $\mathcal{G}$ be the class of H-free graphs. If H is not a linear forest, then for all $k \geq 1$ Corollary 6.6 implies that Distance-k Edge Cover is NP-complete on $\mathcal{G}$. Suppose that H is a linear forest.

Let $k \geq 1$. If H contains $2P_{k+1}$ as an induced subgraph, then $\mathcal{G}$ contains the class of $2P_{k+1}$-free chordal graphs, and hence by Theorem 6.7 Distance-k Edge Cover is NP-complete on $\mathcal{G}$. Otherwise, we obtain that H is $2P_{k+1}$-free. Let t denote the maximum order of a component of H and let C be a component of H of order t. If $t \leq k$, then every component of H has order at most k. If $t \geq k+1$, then, since H is $2P_{k+1}$-free, every component of H other than C has order at most k. In either case, every component of H other than C has order at most k, and H is an induced subgraph of $P_t + sP_k$ for some $s \geq 0$, which implies that $t \leq 2k+2$, and thus H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$. It follows that every H-free graph is $(P_{2k+2} + sP_k)$-free. Thus, by Theorem 7.19 the problem can be solved in polynomial time for graphs in $\mathcal{G}$. □

9 Conclusion

In this thesis, we considered four classical optimization problems on graphs, namely the minimum dominating set, minimum edge dominating set, minimum vertex cover, and minimum edge cover problems, and studied the complexity of their distance-based variants in H-free graphs. After a summary of the known NP-hardness, polynomial-time solvability, approximation algorithms, and parameterized complexity results, we established several inequalities relating the optimal solution values of the four distance problems. Further, for several graph transformations we saw how the optimal solution values to some of these problems on a graph G are related to the optimal solution values to other problems on the transformed graph G'. Those results were used to prove the NP-completeness of the problems on H-free graphs when H contains a cycle as well as when H contains a claw. Combined, these results showed that the problems are NP-complete for H-free graphs whenever H is not a linear forest. Using a similar approach we also developed NP-completeness results for certain cases when a linear forest is excluded. Finally, using a structural property of P_t-free graphs proved by Camby and Schaudt and the inequalities obtained before, we showed that in all other cases, the problems become polynomial-time solvable.

Hence, we obtained complexity dichotomies of the four distance problems with respect to the forbidden induced subgraph H. Namely, we have the following results:

- Dichotomies for any $k \geq 1$ and arbitrary graph H: the Distance-k Dominating Set and Distance-k Edge Cover problems are solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and NP-complete otherwise.

- Dichotomies for $k = 1$ and arbitrary graph H: the Distance-1 Edge Dominating Set and Distance-1 Vertex Cover problems are solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_4 + sP_2$ for some $s \geq 0$, and NP-complete otherwise.

- Dichotomies for any $k \geq 2$ and arbitrary graph H: the Distance-k Edge Dominating Set and Distance-k Vertex Cover problems are solvable in polynomial time in the class of H-free graphs if H is an induced subgraph of $P_{2k+2} + sP_k$ for some $s \geq 0$, and NP-complete otherwise.

Let us note that for every $k \geq 1$, the dichotomies coincide for pairs of problems where the objects *being dominated* are of the same type (vertices, resp. edges). Furthermore, for any graph H, if any of the two problems in which the edges need to be dominated is solvable in polynomial time in the class of H-free graphs, then so are the two problems in which the vertices need to be dominated.

In conclusion, let us remark that a dichotomy for H-free graphs is still an open question for the classical Vertex Cover and Edge Dominating Set problems. In particular, it is an open question whether the Vertex Cover problem (or, equivalently, the Independent Set problem) is polynomial-time solvable in the class of H-free graphs whenever every component of H is either a path or a subdivision of the claw. An affirmative answer would provide a dichotomy (see [1]). For the much less studied Edge Dominating Set problem, the NP-completeness of the problem in the class of line graphs [38] and Corollary 5.12 imply that the problem is NP-complete in the class of H-free graphs whenever H is not a linear forest. However, it is not known whether the problem is polynomial-time solvable in the class of H-free graphs whenever H is a linear forest.

10 Povzetek v slovenskem jeziku

Različne teoretične in praktične motivacije so privedle do posplošitve številnih klasičnih optimizacijskih problemov na grafih na njihove razdaljne variante. Grobo rečeno to pomeni, da se lastnost sosednosti, ki je osnova za definicijo dopustne rešitve problema, nadomesti s splošnejšo lastnostjo, ki temelji na razdaljah v grafih.

V magistrskem delu obravnavamo razdaljne različice naslednjih štirih optimizacijskih problemov na grafih: problem dominantne množice, problem povezavno dominantne množice, problem točkovnega pokritja in problem povezavnega pokritja.

Dominantna množica v grafu G je taka množica $D \subseteq V(G)$, da je vsaka točka grafa G, ki ni v množici D, sosednja z neko točko iz D. Za celo število $k \geq 1$ definiramo k-razdaljno dominantno množico v grafu G kot tako množico $D \subseteq V(G)$, da je vsaka točka v G na razdalji največ k od neke točke v D. Odločitveni problem k-Razdaljna Dominantna Množica sprašuje, za dai graf G in celo število ℓ, ali v grafu G obstaja k-razdaljna dominantna množica velikosti največ ℓ.

Povezavno dominantna množica v grafu G je taka množica $F \subseteq E(G)$, da ima vsaka povezava, ki ni v F, skupno krajišče z neko povezavo v F. Za celo število $k \geq 0$ definiramo k-razdaljno povezavno dominantno množico v grafu G kot tako množico $F \subseteq E(G)$, da ima vsaka povezava v G vsaj eno krajišče na razdalji največ k od krajišča neke povezave v F. Odločitveni problem k-Razdaljna Povezavno Dominantna Množica sprašuje, za dan graf G in celo število ℓ, ali v grafu G obstaja k-razdaljna povezavno dominantna množica velikosti največ ℓ.

Točkovno pokritje v grafu G je taka množica $C \subseteq V(G)$, da ima vsaka povezava vsaj eno krajišče v C. Za celo število $k \geq 0$ definiramo k-razdaljno točkovno pokritje v grafu G kot tako množico $C \subseteq V(G)$, da je vsaka povezava v G na razdalji največ k od neke točke v C. Odločitveni problem k-Razdaljno Točkovno Pokritje sprašuje, za dan graf G in celo število ℓ, ali v grafu G obstaja k-razdaljno točkovno pokritje velikosti največ ℓ.

Povezavno pokritje v grafu G je taka množica $F \subseteq E(G)$, da je vsaka točka grafa krajišče neke povezave v F. Za celo število $k \geq 0$ definiramo k-razdaljno povezavno pokritje v grafu G kot tako množico $F \subseteq E(G)$, da je vsaka točka grafa G na razdalji največ k od krajišča neke povezave v F. Odločitveni problem k-Razdaljno Povezavno Pokritje prašuje, za dan graf G in celo število ℓ, ali v grafu G obstaja k-razdaljno povezavno pokritje velikosti največ ℓ.

V magistrskemu delu so izpeljani naslednji izreki, ki podajajo dihotomije glede na zahtevnost obravnavanih problemov v H-prostih grafih, tj. grafih, ki ne vsebujejo nobenega induciranega podgrafa, izomorfnega nekemu fiksnemu grafu H. Za cela števila $k \geq 1$, $s \geq 0$ in $t \geq 1$ označimo s $P_k + sP_t$ disjunktno unijo poti na k točkah in s kopij poti na t točkah. Za razdaljne probleme, pri katerih je cilj na določen razdalji "dominirati" (ali "pokriti") vse *točke*, smo pokazali naslednje dihotomije za vse $k \geq 1$ in poljuben graf H:

- Problema k-Razdaljna Dominantna Množica in k-Razdaljno Povezavno Pokritje sta v razredu H-prostih grafov rešljiva v polinomskem času, če je H induciran podgraf nekega grafa oblike $P_{2k+2} + sP_k$ za $s \geq 0$, sicer pa sta NP-polna.

Za razdaljne probleme, pri katerih je cilj na določen razdalji dominirati (ali pokriti) vse *povezave*, pa smo pokazali naslednje dihotomije za poljuben graf H:

- Dihotomija za $k = 1$: Problema 1-Razdaljna Povezavna Dominatna Množica in 1-Razdaljno Točkovno Pokritje sta v razredu H-prostih grafov rešljiva v polinomskem času, če je H induciran podgraf nekega grafa oblike $P_4 + sP_2$ za $s \geq 0$, sicer pa sta NP-polna.

- Dihotomija za $k \geq 2$: Problema k-Razdaljna Povezavna Dominatna Množica in k-Razdaljno Točkovno Pokritje sta v razredu H-prostih grafov rešljiva v polinomskem času, če je H induciran podgraf nekega grafa oblike $P_{2k+2} + sP_k$ za $s \geq 0$, sicer pa sta NP-polna.

Bibliography

[1] V. E. Alekseev. "The effect of local constraints on the complexity of determination of the graph independence number". In: *Combinatorial-algebraic methods in applied mathematics* (1982), pp. 3–13.

[2] V. E. Alekseev. "On the number of maximal independent sets in graphs from hereditary classes". In: *Combinatorial-algebraic methods in discrete optimization, University of Nizhny Novgorod* (1991), pp. 5–8.

[3] V. E. Alekseev. "Polynomial algorithm for finding the largest independent sets in graphs without forks". In: *Discrete Applied Mathematics* 135.1-3 (2004), pp. 3–16.

[4] S. Arnborg, J. Lagergren, and D. Seese. "Easy problems for tree-decomposable graphs". In: *Journal of Algorithms* 12.2 (1991), pp. 308–340.

[5] E. Balas and C. S. Yu. "On graphs with polynomially solvable maximum-weight clique problem". In: *Networks* 19.2 (1989), pp. 247–253.

[6] A. A. Bertossi. "Dominating sets for split and bipartite graphs". In: *Information processing letters* 19.1 (1984), pp. 37–40.

[7] H. L. Bodlaender. "A linear-time algorithm for finding tree-decompositions of small treewidth". In: *SIAM Journal on computing* 25.6 (1996), pp. 1305–1317.

[8] F. Bonomo et al. "Domination parameters with number 2: interrelations and algorithmic consequences". In: *Discrete Applied Mathematics* 235 (2018), pp. 23–50.

[9] K. S. Booth and J. H. Johnson. "Dominating sets in chordal graphs". In: *SIAM Journal on Computing* 11.1 (1982), pp. 191–199.

[10] A. Brandstädt, V. D. Chepoi, and F. F. Dragan. "The algorithmic use of hypertree structure and maximum neighbourhood orderings". In: *Discrete Applied Mathematics* 82.1-3 (1998), pp. 43–77.

[11] A. Brandstädt and R. Mosca. "Maximum weight independent set for ℓclaw-free graphs in polynomial time". In: *Discrete Applied Mathematics* 237 (2018), pp. 57–64.

[12] H. Broersma et al. "Independent sets in asteroidal triple-free graphs". In: *SIAM Journal on Discrete Mathematics* 12.2 (1999), pp. 276–287.

[13] A. H. Busch, F. F. Dragan, and R. Sritharan. "New min-max theorems for weakly chordal and dually chordal graphs". In: *Combinatorial optimization and applications. Part II*. Vol. 6509. Lecture Notes in Comput. Sci. Springer, Berlin, 2010, pp. 207–218.

[14] E. Camby and O. Schaudt. "A new characterization of P_k-free graphs". In: *Algorithmica* 75.1 (2016), pp. 205–217.

[15] G. J. Chang. "Labeling algorithms for domination problems in sun-free chordal graphs". In: *Discrete Applied Mathematics* 22.1 (1988), pp. 21–34.

[16] G. J. Chang and G. L. Nemhauser. "The k-domination and k-stability problems on sun-free chordal graphs". In: *SIAM J. Algebraic Discrete Methods* 5.3 (1984), pp. 332–345.

[17] J.-M. Chang, C.-W. Ho, and M.-T. Ko. "Powers of asteroidal triple-free graphs with applications". In: *Ars Combinatoria* 67 (2003), pp. 161–174.

[18] M. Chlebík and J. Chlebíková. "Approximation hardness of edge dominating set problems." In: *J. Comb. Optim.* 11.3 (2006), pp. 279–290.

[19] V. Chvátal. "A greedy heuristic for the set-covering problem". In: *Mathematics of operations research* 4.3 (1979), pp. 233–235.

[20] B. N. Clark, C. J. Colbourn, and D. S. Johnson. "Unit disk graphs". In: *Discrete mathematics* 86.1-3 (1990), pp. 165–177.

[21] A. Conte et al. "Sublinear-space and bounded-delay algorithms for maximal clique enumeration in graphs". In: *Algorithmica* 82.6 (2020), pp. 1547–1573.

[22] D. G. Corneil and Y. Perl. "Clustering and domination in perfect graphs". In: *Discrete Applied Mathematics* 9.1 (1984), pp. 27–39.

[23] B. Courcelle, J. A. Makowsky, and U. Rotics. "Linear time solvable optimization problems on graphs of bounded clique-width". In: *Theory of Computing Systems* 33.2 (2000), pp. 125–150.

[24] M. Cygan et al. *Parameterized algorithms*. Springer, Cham, 2015.

[25] C. Dallard, M. Krbezlija, and M. Milanič. "Vertex Cover at Distance on H-Free Graphs". In: *Combinatorial Algorithms - 32nd International Workshop, IWOCA 2021, Ottawa, ON, Canada, July 5-7, 2021, Proceedings*. Ed. by P. Flocchini and L. Moura. Vol. 12757. Lecture Notes in Computer Science. Springer, 2021, pp. 237–251.

[26] I. Dinur and D. Steurer. "Analytical approach to parallel repetition". In: *Proceedings of the forty-sixth annual ACM symposium on Theory of computing.* 2014, pp. 624–633.

[27] R. G. Downey and M. R. Fellows. *Parameterized complexity.* Springer Science & Business Media, 2012.

[28] B. Escoffier et al. "New results on polynomial inapproximabilityand fixed parameter approximability of edge dominating set". In: *Theory of Computing Systems* 56.2 (2015), pp. 330–346.

[29] W. Espelage, F. Gurski, and E. Wanke. "How to solve NP-hard graph problems on clique-width bounded graphs in polynomial time". In: *International Workshop on Graph-Theoretic Concepts in Computer Science.* Springer. 2001, pp. 117–128.

[30] H. Fernau. "Edge dominating set: Efficient enumeration-based exact algorithms". In: *International Workshop on Parameterized and Exact Computation.* Springer. 2006, pp. 142–153.

[31] M. R. Garey and D. S. Johnson. *Computers and intractability.* A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences. W. H. Freeman and Co., San Francisco, Calif., 1979.

[32] M. C. Golumbic. *Algorithmic graph theory and perfect graphs.* Second. Vol. 57. Annals of Discrete Mathematics. With a foreword by Claude Berge. Elsevier Science B.V., Amsterdam, 2004, pp. xxvi+314.

[33] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric algorithms and combinatorial optimization.* Vol. 2. Springer Science & Business Media, 2012.

[34] A. Grzesik et al. "Polynomial-time algorithm for maximum weight independent set on P_6-free graphs". In: *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms.* SIAM, Philadelphia, PA, 2019, pp. 1257–1271.

[35] M. A. Henning, O. R. Oellermann, and H. C. Swart. "Bounds on distance domination parameters". In: *J. Combin. Inform. System Sci.* 16.1 (1991), pp. 11–18.

[36] M. A. Henning. "Distance domination in graphs". In: *Domination in graphs.* Vol. 209. Monogr. Textbooks Pure Appl. Math. Dekker, New York, 1998, pp. 321–349.

[37] M. A. Henning. "Distance domination in graphs". In: *Topics in domination in graphs.* Vol. 64. Dev. Math. Springer, Cham, [2020] ©2020, pp. 205–250.

[38] J. D. Horton and K. Kilakos. "Minimum edge dominating sets". In: *SIAM Journal on Discrete Mathematics* 6.3 (1993), pp. 375–387.

[39] J. D. Horton and A. López-Ortiz. "On the number of distributed measurement points for network tomography". In: *Proceedings of the 3rd ACM SIGCOMM Internet Measurement Conference, IMC 2003, Miami Beach, FL, USA, October 27-29, 2003*. ACM, 2003, pp. 204–209.

[40] L. Jaffke et al. "Mim-width III. Graph powers and generalized distance domination problems". In: *Theoretical Computer Science* 796 (2019), pp. 216–236.

[41] S. K. Jena and G. K. Das. "Vertex-edge domination in unit disk graphs". In: *Discrete Applied Mathematics* (2021).

[42] M. Jiang and Y. Zhang. "Parameterized complexity in multiple-interval graphs: Domination, partition, separation, irredundancy". In: *Theoretical Computer Science* 461 (2012), pp. 27–44.

[43] R. M. Karp. "Reducibility among combinatorial problems". In: *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*. 1972, pp. 85–103.

[44] S. Khot, D. Minzer, and M. Safra. "On independent sets, 2-to-2 games, and Grassmann graphs". In: *Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing*. 2017, pp. 576–589.

[45] D. G. Kirkpatrick and P. Hell. "On the complexity of general graph factor problems". In: *SIAM J. Comput.* 3 (1983), pp. 601–609.

[46] D. Kobler and U. Rotics. "Edge dominating set and colorings on graphs with fixed clique-width". In: *Discrete Applied Mathematics* 126.2-3 (2003), pp. 197–221.

[47] D. Korobitsin. "On the complexity of domination number determination in monogenic classes of graphs". In: *Discrete Math. Appl.* 2 (1992), pp. 191–199.

[48] J. R. Lewis. "Vertex-edge and edge-vertex parameters in graphs". PhD thesis. Clemson University, 2007.

[49] D. Lokshtanov et al. "Hardness of r-dominating set on graphs of diameter $(r+1)$". In: *International Symposium on Parameterized and Exact Computation*. Springer. 2013, pp. 255–267.

[50] C. L. Lu and C. Y. Tang. "Solving the weighted efficient edge domination problem on bipartite permutation graphs". In: *Discrete Applied Mathematics* 87.1-3 (1998), pp. 203–211.

[51] K. Makino and T. Uno. "New algorithms for enumerating all maximal cliques". In: *Algorithm theory—SWAT 2004*. Vol. 3111. Lecture Notes in Comput. Sci. Springer, Berlin, 2004, pp. 260–272.

[52] A. A. McRae. "Generalizing NP-completeness proofs for bipartite graphs and chordal graphs". PhD thesis. Clemson University, 1994.

[53] G. J. Minty. “On maximal independent sets of vertices in claw-free graphs”. In: *Journal of Combinatorial Theory, Series B* 28.3 (1980), pp. 284–304.

[54] B. Mohar. “Face covers and the genus problem for apex graphs”. In: *Journal of Combinatorial Theory, Series B* 82.1 (2001), pp. 102–117.

[55] S.-I. Oum. “Approximating rank-width and clique-width quickly”. In: *ACM Trans. Algorithms* 5.1 (2009), Art. 10, 20.

[56] C. H. Papadimitriou and K. Steiglitz. *Combinatorial optimization: algorithms and complexity.* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1982, pp. xvi+496.

[57] S. Paul, D. Pradhan, and S. Verma. “Vertex-Edge Domination in Interval and Bipartite Permutation Graphs”. In: *Discussiones Mathematicae Graph Theory* (2021). To appear. DOI: https://doi.org/10.7151/dmgt.2411.

[58] S. Paul and K. Ranjan. “On vertex-edge and independent vertex-edge domination”. In: *International Conference on Combinatorial Optimization and Applications.* Springer. 2019, pp. 437–448.

[59] S. Poljak. “A note on stable sets and colorings of graphs”. In: *Comment. Math. Univ. Carolinae* 15 (1974), pp. 307–309.

[60] A. Rana, A. Pal, and M. Pal. “An efficient algorithm to solve the distance k-domination problem on permutation graphs”. In: *J. Discrete Math. Sci. Cryptogr.* 19.2 (2016), pp. 241–255.

[61] M. Sasaki, L. Zhao, and H. Nagamochi. “Security-aware beacon based network monitoring”. In: *2008 11th IEEE Singapore International Conference on Communication Systems.* IEEE. 2008, pp. 527–531.

[62] N. Sbihi. “Algorithme de recherche d’un stable de cardinalité maximum dans un graphe sans étoile”. In: *Discrete Mathematics* 29.1 (1980), pp. 53–76.

[63] P. J. Slater. “R-domination in graphs”. In: *J. Assoc. Comput. Mach.* 23.3 (1976), pp. 446–450.

[64] A. Srinivasan et al. “Edge domination on bipartite permutation graphs and cotriangulated graphs”. In: *Information Processing Letters* 56.3 (1995), pp. 165–171.

[65] S. Tsukiyama et al. “A new algorithm for generating all the maximal independent sets”. In: *SIAM Journal on Computing* 6.3 (1977), pp. 505–517.

[66] D. B. West et al. *Introduction to graph theory.* Vol. 2. Prentice hall Upper Saddle River, 2001.

[67] M. Yannakakis and F. Gavril. “Edge dominating sets in graphs”. In: *SIAM J. Appl. Math.* 38.3 (1980), pp. 364–372.

[68] R. Ziemann and P. Żyliński. "Vertex-edge domination in cubic graphs". In: *Discrete Mathematics* 343.11 (2020), p. 112075.