Counterexamples to Orlin's conjecture on equistable graphs

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Based on joint work with

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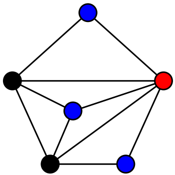
Graphs and stable sets

- $G = (V, E)$ - a finite simple undirected graph
- stable (independent) set: a subset $S \subseteq V$ of pairwise non-adjacent vertices
- a stable set is maximal if it is not contained in any other stable set



Equistable graphs

Definition

A graph $G = (V, E)$ is **equistable** if there exists a function $w : V \to \mathbb{N}$ and a positive integer t such that

$\forall S \subseteq V$:

$$S \text{ is a maximal stable set in } G \quad \Leftrightarrow \quad w(S) = \sum_{v \in S} w(v) = t\,.$$

Equivalently:

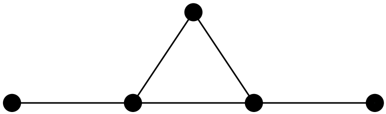
There exists a function $\varphi : V \to \mathbb{R}_+$ such that

$\forall S \subseteq V$:

$$S \text{ is a maximal stable set in } G \quad \Leftrightarrow \quad \varphi(S) = \sum_{v \in S} \varphi(v) = 1\,.$$

Equistable graphs: example

The following graph is equistable:



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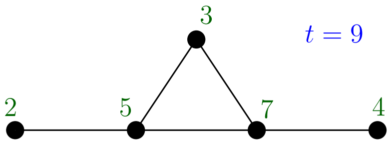

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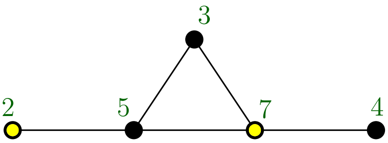

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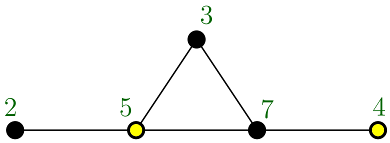

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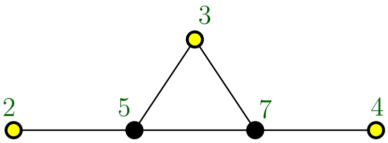
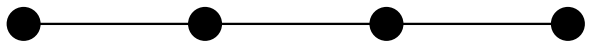


Equistable graphs: example

The following graph is not equistable:



Equistable graphs: example

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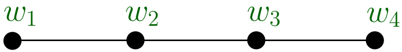


If

$$
\begin{aligned}
w_1 + w_3 &= t \\
w_2 + w_4 &= t \\
w_1 + w_4 &= t
\end{aligned}
$$

then

$$w_2 + w_3 = t\,.$$

Equistable graphs: motivation

- **threshold graphs** (Chvátal-Hammer 1977):
  $\exists w, t$ s.t. $S \subseteq V$ stable $\Leftrightarrow w(S) \leq t$
- **equistable graphs** (Payan 1980):
  $\exists w, t$ s.t. $S \subseteq V$ maximal stable $\Leftrightarrow w(S) = t$

Equistable graphs generalize:

- threshold graphs (Payan, 1980);
- cographs (graphs without an induced 3-edge path) (Mahadev-Peled-Sun, 1994).

Equistable graphs: state of the art

No combinatorial characterization of equistable graphs is known.
Combinatorial characterizations of equistable graphs are known for several graph classes:

- chordal graphs (Peled-Rotics 2003),
- distance-hereditary graphs (Korach-Peled-Rotics 2008),
- outerplanar graphs (Mahadev-Peled-Sun 1994),
- series-parallel graphs (Korach-Peled 2003),
- line graphs (Levit-M 2014),
- very well-covered graphs (Levit-M 2014),
- simplicial graphs (Levit-M 2014),
- various product graphs (Miklavič-M 2011),
- AT-free graphs (Kloks et al. 2003),
- EPT graphs (Alcón-Gutierrez-Kovács-M-Rizzi 2014+).

The computational complexity status of recognizing equistable graphs is open.

General partition graphs

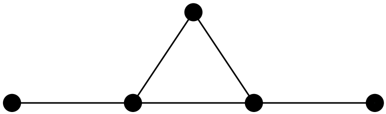
Definition

A graph $G = (V, E)$ is a **general partition graph** if there exists a finite set U and an assignment of nonempty subsets $U_x \subseteq U$ to vertices of V such that

- $xy \in E$ if and only if $U_x \cap U_y \neq \emptyset$, and
- for every maximal stable set S in G, the set $\{U_x : x \in S\}$ forms a partition of U.

General partition graphs: example

The following graph is a general partition graph:



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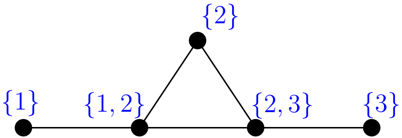


$U = \{1, 2, 3\}$

General partition graphs: a characterization

Theorem (McAvaney-Robertson-DeTemple, 1993)

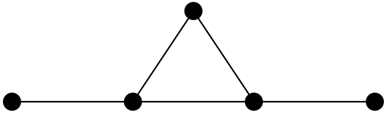
For every graph G, the following are equivalent:

- *G is a general partition graph,*
- *every edge of G is contained in a strong clique.*

strong clique = a clique intersecting all maximal stable sets

General partition graphs: example

The following graph is a general partition graph (every edge is contained in a strong clique):



General partition graphs: example

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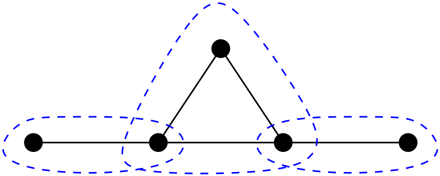
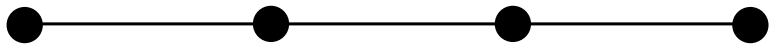


General partition graphs: example

The following graph is not a general partition graph
(there exists an edge not contained in any strong clique):



General partition graphs: example

The following graph is not a general partition graph
(there exists an edge not contained in any strong clique):

Inclusion relations among these classes

The following inclusion relations hold:

$$\begin{array}{c}\text{equistable graphs} \\ \cup\!\mid \\ \text{strongly equistable graphs} \\ \cup\!\mid \\ \text{general partition graphs}\end{array}$$

Strongly equistable graphs

Let $\mathcal{S}(G)$ denotes the set of all maximal stable set of a graph G, and $\mathcal{T}(G)$ the set of all other non-empty subsets of $V(G)$.

Definition

A graph $G = (V, E)$ is **strongly equistable** if
for every $T \in \mathcal{T}(G)$ and every $\gamma \leq 1$
there exists a function $\varphi : V \to \mathbb{R}_+$ such that

$\varphi(S) = 1$ for all $S \in \mathcal{S}(G)$

and
$\varphi(T) \neq \gamma$.

General partition graphs are strongly equistable

Theorem (Jim Orlin, 2009)

Every general partition graph is equistable.

Theorem (Mahadev-Peled-Sun, 1994)

Every equistable graph with a strong clique is strongly equistable.

Theorem (McAvaney-Robertson-DeTemple, 1993)

G is a general partition graph if and only if every edge of G is contained in a strong clique.

Corollary

Every general partition graph is strongly equistable.

Inclusion relations among these classes

$$\begin{array}{c}\text{equistable graphs}\\ \cup\!\text{I}\\ \text{strongly equistable graphs}\\ \cup\!\text{I}\\ \text{general partition graphs}\end{array}$$

Conjecture (Mahadev-Peled-Sun, 1994)

Every equistable graph is strongly equistable.

Conjecture (Jim Orlin, 2009)

Every equistable graph is a general partition graph.

Conjecture (Miklavič-M, 2011)

Every equistable graph contains a strong clique.

Known results

Orlin's conjecture holds within the following graph classes:

- chordal graphs,
- graphs obtained from triangle-free graphs by gluing chordal graphs along edges,
- outerplanar graphs,
- series-parallel graphs,
- line graphs,
- EPT graphs,
- very well-covered graphs,
- simplicial graphs,
- AT-free graphs,
- certain product graphs.

Counterexamples

We will show that Orlin's conjecture fails for complements of line graphs.

Recall:

- The **complement** of a graph G is the graph $\overline{G}$ with vertex set $V(G)$ in which two distinct vertices are adjacent if and only if they are non-adjacent in G.
- A graph G is the **line graph** of a graph H, written $G = L(H)$, if $V(G) = E(H)$ and two distinct edges of H are adjacent as vertices of G if they have an endpoint in common.
- Given two graphs G and H, we say that a graph G is H-**free** if no induced subgraph of G is isomorphic to H.

Building counterexamples

We will construct complements of line graphs of triangle-free graphs that are equistable but are not general partition.

We must address two questions:

When is $\overline{L(H)}$ (not) general partition?
When is $\overline{L(H)}$ equistable?

Recall: general partition $\iff$ every edge is contained in a strong clique

Observation

Given a graph $G = (V, E)$, a subset $S \subseteq V$ is a strong clique in $\overline{G}$ if and only if S is a strong stable set in G (that is, a stable set intersecting all maximal cliques).

Building counterexamples

Observation

Let $G = L(H)$. Then:

- *A set $C \subseteq E(H)$ is a maximal clique in G if and only if C is a maximal set in the family of triangles and stars of H.*
  A **triangle** *is (the edge set of) a subgraph of G isomorphic to K_3.*
  A **star** *is the set of all edges incident with a vertex.*
- *maximal stable sets in G = maximal matchings in H*

Suppose in addition that H is triangle-free and of minimum degree at least 2. Then:

- *maximal cliques in G = stars of H.*
- *strong stable sets in G*
  = matchings in H intersecting all stars
  = perfect matchings in H

Building counterexamples

From now on, let H be a triangle-free graph of minimum degree at least 2.

Observation

Every edge of $\overline{L(H)}$ is contained in a strong clique

$\iff$

every non-edge of $L(H)$ is contained in a strong stable set

$\iff$

every pair of disjoint edges in H is contained in a perfect matching

$\iff$

H is 2-extendable

In a paper published in 1980, Plummer defined a graph to be k**-extendable** if it contains a matching of size k and every matching of size k is contained in a perfect matching.

Building counterexamples

Therefore, $\overline{L(H)}$ is not general partition if and only if H is not 2-extendable.

We will construct examples without a perfect matching (clearly such graphs are not 2-extendable).

Now, let us turn to the second question:

When is $\overline{L(H)}$ equistable?

Equistarable graphs

Let H be a triangle-free graph of minimum degree at least 2, and let $F \subseteq E(H)$.

Then:

F is a maximal stable set in $\overline{L(H)}$

$\iff$

F is a maximal clique in $L(H)$

$\iff$

F is a maximal star in H

$\iff$

F is a star in H

Proposition

The graph $\overline{L(H)}$ is equistable if and only if H is equistarable.

Equistarable graphs

Definition

A graph $G = (V, E)$ with at least one edge is **equistarable** if there exists a mapping $\varphi : E \to \mathbb{R}_+$ such that

$\forall F \subseteq E$:

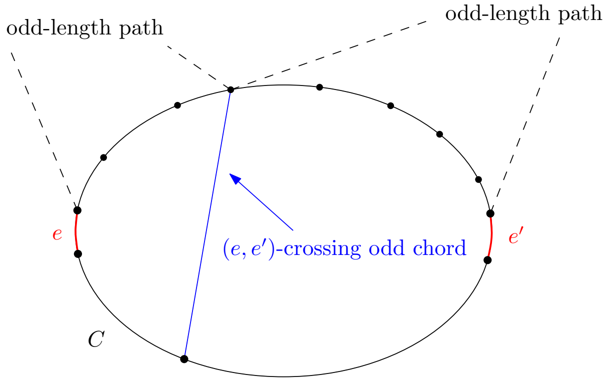
$$F \text{ is a maximal star in } G \quad \Leftrightarrow \quad \varphi(F) = 1\,.$$

To describe a family of equistarable graphs, we need two more definitions.

Even and odd chords

C: odd cycle in a graph
e, e': disjoint edges in C

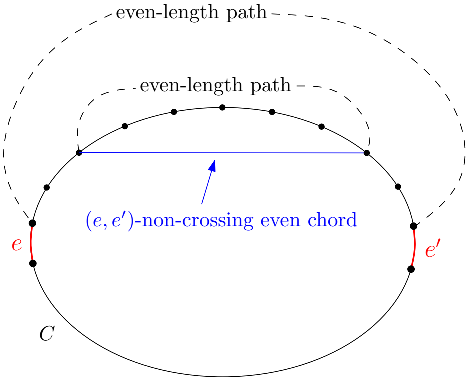
(e, e')-crossing odd chord of C:



Even and odd chords

C: odd cycle in a graph
e, e': disjoint edges in C

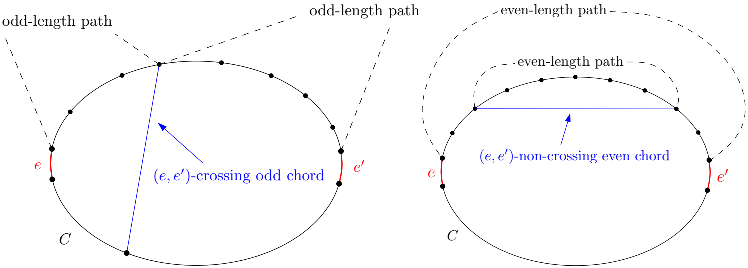
(e, e')-non-crossing even chord of C:



Bad graphs

Definition

A triangle-free graph G with $|V(G)|$ odd is said to be **bad** if
G has a Hamiltonian cycle C such that
for every two disjoint edges $e, e' \in E(C)$,
G contains either an (e, e')-crossing odd chord of C, or
a (e, e')-non-crossing even chord of C.



Bad graphs

Definition

A triangle-free graph G with $|V(G)|$ odd is said to be **bad** if G has a Hamiltonian cycle C such that for every two disjoint edges $e, e' \in E(C)$, G contains either an (e, e')-crossing odd chord of C, or a (e, e')-non-crossing even chord of C.

Theorem

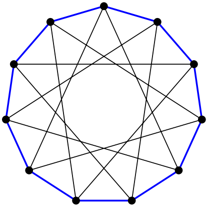
Every bad graph is equistarable.

Corollary

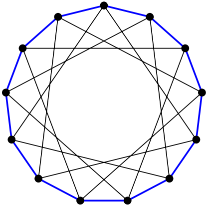
If G is a bad graph, then $\overline{L(G)}$ is an equistable graph without a strong clique. In particular, $\overline{L(G)}$ is a counterexample to the conjecture of Orlin and even to the (weaker) conjecture of Miklavič-M.

Examples of bad graphs: circulants

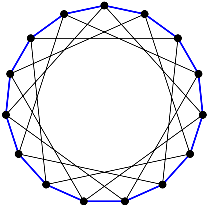
For every odd $n \geq 11$, the circulant $C_n(\{\pm 1, \pm 4\})$ is bad.



$n = 11$



$n = 13$



$n = 15$

Proof sketch of the main theorem

Theorem

Every bad graph is equistarable.

We construct an equistarable weight function φ of G in two steps.

Step 1:

- $F = E(G) \setminus C = \{f_1, \ldots, f_r\}$
- let $\epsilon \in (0, 1/(3r))$
- let $\alpha_1, \ldots, \alpha_r \in (0, \epsilon)$ be positive real numbers, algebraically independent over $\mathbb{Q}$, that is, if $\sum_{i=1}^{r} q_i \alpha_i = 0$ where $q_i \in \mathbb{Q}$ for all $i \in \{1, \ldots, r\}$, then $q_1 = \ldots = q_r = 0$.

We set $\varphi(f_i) = \alpha_i$ for all $i \in \{1, \ldots, r\}$.

Proof sketch of the main theorem

Step 2:

The function φ should satisfy linear constraints of the form

$$\sum_{e \in E(v)} \varphi(e) = 1$$

for every vertex $v \in V(G)$.

Substituting into this system of equations the already fixed values

$$\varphi(f_i) = \alpha_i$$

we obtain a linear system with n variables and n equations.

Since the cycle is odd, the coefficient matrix is of determinant 2, and the system has a unique solution β (which can be explicitly computed).

We set $\varphi(e) = \beta_e$ for all $e \in E(C)$.

Proof sketch of the main theorem

It remains to show that φ is an equistarable weight function of G, that is, that:

- $\varphi(e) \geq 0$ for all $e \in E(G)$.
- For every star $E' \subseteq E(G)$, we have $\varphi(E') = 1$.
- Every set $E' \subseteq E(G)$ such that $\varphi(E') = 1$ is a star.

Recall: since we are in the case of minimum degree at least 2, every star is maximal

Proof sketch of the main theorem

- By construction, we have $\varphi(f) > 0$ for all $f \in F$.
- As $\epsilon \to 0$, we have $\alpha_i \to 0$ for all i.
  Explicit formulas for the β_e's imply that $\beta_e \to 1/2$ for all $e \in E(C)$.
- Since the β_e's are continuous functions of the α_i's, choosing ϵ small enough guarantees that $1/3 < \beta_e < 2/3$ for all $e \in E(C)$.
- By construction, for every star $E' \subseteq E(G)$, we have $\varphi(E') = 1$.

Proof sketch of the main theorem

It remains to prove that every set $E' \subseteq E(G)$ such that $\varphi(E') = 1$ is a star.

- $|E' \cap E(C)| = 2$, otherwise the weight is either too small or too large.
- If the two edges e, e' in $E' \cap E(C)$ share a common endpoint, say v, then the algebraic independence of the α_i's implies that E' is a star rooted at v.
- If the two edges are disjoint then we derive a contradiction using the fact that G is bad:
  G contains either an (e, e')-crossing odd chord of C, or
  an (e, e')-non-crossing even chord of C.
  In either case, a contradiction can be derived.

Implications

Theorem

Every bad graph is equistarable.

Corollary

If G is a bad graph, then $\overline{L(G)}$ is an equistable graph without a strong clique. In particular, $\overline{L(G)}$ is not a general partition graph.

This disproves two conjectures:

Conjecture (Jim Orlin, 2009)

Every equistable graph is a general partition graph.

Conjecture (Miklavič-M, 2011)

Every equistable graph contains a strong clique.

Complements of line graphs of bipartite graphs

What about complements of line graphs of bipartite graphs?

Is there an equistarable bipartite graph H of minimum degree at least 2 that is not 2-extendable?

- Every equistarable weighting on the edges of a bipartite graph defines a **doubly stochastic matrix** (a matrix with non-negative entries, each column sum 1, and each row sum 1).
- It follows that equistarable bipartite graphs are balanced (thus, they cannot have odd order).
- The Birkhoff-von Neumann theorem states that every doubly shochastic matrix is a convex combination of permutation matrices.
- It follows that every edge of an equistarable bipartite graph belongs to a perfect matching, that is, the graph is 1-extendable.

Bipartite graphs?

Can our approach be modified to work for bipartite graphs?

A suitable replacement for the condition requiring the existence of chords with respect to a Hamiltonian cycle is needed.

Perhaps interpreting the problem in the language of doubly stochastic matrices could give some needed insight.

Open problems

The following questions remain open:

Is there an equistarable bipartite graph H of minimum degree at least 2 such that H is not 2-extendable?

Does Orlin's conjecture hold for complements of line graphs of bipartite graphs? For perfect graphs?

Determine the complexity of recognizing equistable graphs.

Conjecture (Mahadev-Peled-Sun, 1994)

Every equistable graph is strongly equistable.

The end

Thank you!