Some combinatorial aspects of the connections between hypergroups and fuzzy sets

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The theory of hyperstructures was first initiated by F. Marty in 1934, to the 8th Scandinavian Congress of Mathematicians, defining the *hypergroups*, with applications to groups, rational fractions, and algebraic functions.

- In a classical algebraic structure $(H, *)$: $a * b = c \in H, \forall a, b \in H$.
- In an algebraic hyperstructure $(H, \circ)$: $\emptyset \neq a \circ b \subset H, \forall a, b \in H$.

Def. *Hypergroup*

$H \neq \emptyset$, hyperoperation $\circ : H^2 \longrightarrow \mathcal{P}^*(H)$ that satisfies:

- $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in H$ (associativity);
- $x \circ H = H = H \circ x$, for all $x \in H$ (reproducibility).

If $\emptyset \neq A, B \subset H$, then $A \circ B = \bigcup_{\substack{a \in A \\ b \in B}} a \circ b$.

First example

- The first example of hypergroup, which motivated the introduction of these new structures, was given by Marty.

  Let $(G, \cdot)$ be a group and H be a subgroup of G; then the set of left cosets of H, $G/H = \{xH \mid x \in G\}$, with the hyperoperation defined by $xH \circ yH = \{zH \mid z \in xH \cdot yH\}$, is a hypergroup.
- Hypergroups are much varied than groups. For example, if H is of prime cardinality p, there are a large number of non-isomorphic hypergroups on H, while, up to isomorphism, there is only one group $\mathbb{Z}_p$.

Connections with other domains

In the last decades, several examples of hypergroups have been constructed
-using:

- binary /n-ary relations
- graphs
- fuzzy sets
- rough sets

-in connections with:

- geometry
- topology
- codes
- cryptography
- automata
- probabilities, ecc.

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1) P. Corsini. *Prolegomena of hypergroup theory*. Aviani Editore, 1993.
2) P. Corsini, V. Leoreanu. *Applications of Hyperstructures Theory*. Kluwer Academic Publishers, Advances in Mathematics, 2003.

Fuzzy sets were first proposed by Lofti A. Zadeh (University of California) in his 1965 paper entitled none other than: Fuzzy Sets.In the classical set theory, the membership of an element to a set is assessed in binary terms according to a bivalent condition:an element either belongs or not to the set.By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the help of a membership function valued in the real interval $[0, 1]$.

Def.

Let X be a space of points, with a generic element of X denoted by x.
A fuzzy subset A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval $[0, 1]$, with the values of $\mu_A(x)$ at x representing the "grade of membership" of x in A. Thus, the nearer the value of $\mu_A(x)$ to unity, the higher the grade of membership of x in A.

Fuzzy concepts: temperature, weather, age, ecc.

The first connection between algebraic structures and fuzzy sets has been established by Rosenfeld in 1971, when he introduced the concept of fuzzy subgroup, redefined eight years later by Anthony and Sherwood.

Def:fuzzy subgroup

A fuzzy subset μ of a group G is said to be a fuzzy subgroup of G if for any $x, y \in G$, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y), \forall x, y \in G$.

Afterthat, Davvaz extended this definition to the case of hyperstructures, defining fuzzy subhypergroups (1999), fuzzy subhyperrings (1998), or fuzzy subhypermodules (2001).

1) Let S be a projective space and P_S the set of its points. Defining,

- $\forall x, y \in P_S, x \circ y = x + y$ (the set of the points belonging the line through x and y)
- $\forall x \in P_S, x \circ x = x$

then $(P_S, \circ)$ is a hypergroup.

2) Let $(G, +)$ be an abelian group and R the equivalence on G such that $\bar{x} = \{x, -x\}$. Defining the hyperoperation $\circ : G/R \times G/R \longrightarrow \mathcal{P}^*(G/R)$ by $\bar{x} \circ \bar{y} = \{\overline{x+y}, \overline{x-y}\}$, then $(G/R, \circ)$ is a hypergroup.

3) Let (H, E) be a connected graph. On the set of points H define the hyperoperation:

- $x \circ x = \{x\}$
- $x \circ y = [x, y]$ (the set of points z belonging to some path connecting x and y)

Then $(H, \circ)$ is a commutative hypergroup which is a join space iff (H, E) is a tree.

4) Let ρ be a binary relation on H. Define, $\forall x \in H$, $L_x = \{z \in H \mid (x, z) \in \rho\}$ and $\forall x, y \in H$,

$$x \circ y = L_x \cup L_y, \quad x \circ x = L_x.$$

then $(H, \circ)$ is a hypergroupoid which becomes a hypergroup iff:

- **i)** ρ has full domain and full range;
- **ii)** $\rho \subset \rho^2$
- **iii)** if $(a, x) \in \rho^2$, then $(a, x) \in \rho$, whenever x is an outer element of ρ ($\exists h \in H : (h, x) \notin \rho^2$).

$(H, \circ)$ is called Rosenberg hypergroup.

5) Let $H = V_K$ be a vector space over the field K. Set

$$\forall x, y \in H, x \circ y = \{a(x + y) \mid a \in K\}.$$

Then $(H, \circ)$ is a hypergroup.

6) Consider the set $H = \{u, a\}$ and set

$$u \circ u = \{u\}, u \circ a = a \circ u = \{a\}, a \circ a = \{u, a\}.$$

Then $(H, \circ)$ is a hypergroup having u as a scalar unit. This is the simplest hypergroup that is not a group.

Def. (W. Prenowitz, J. Jantosciak, 1972)

A commutative hypergroup $(H, \circ)$ is called a *join space* if, for any $(a, b, c, d) \in H^4$, the following implication holds:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset \ \text{("transposition axiom")},$$

where

$$a/b = \{x \mid a \in x \circ b\} \text{ is the extension of } a \text{ from } b.$$

W. Prenowitz and J. Jantosciak reconstructed from an algebraic point of view the projective, the descriptive and the spherical geometries, “which are join theoretic in character, in the sense that a central role is played by a «join» hyperoperation, that assigns with two distinct points an appropriate connective:

- in descriptive geometry-a segment;
- in spherical geometry-the minor arc of a great circle;
- in projective geometry- a line.

Def.

An application $f : (H_1, \circ_1) \longrightarrow (H_2, \circ_2)$ between two hypergroups is called

- homomorphism if $f(x \circ_1 y) \subset f(x) \circ_2 f(y), \forall x, y \in H_1$
- good homomorphism if $f(x \circ_1 y) = f(x) \circ_2 f(y), \forall x, y \in H_1$
- isomorphism if it is a good bijective homomorphism.

The heart of a hypergroup (Dresher-Ore, 1940)

On a hypergroupoid $(H, \circ)$ define the relation β as follows:

$$a\beta b \Longleftrightarrow \exists n \in \mathbb{N}^*, \exists (x_1, \ldots, x_n) \in H^n, a \in \prod_{i=1}^{n} x_i \ni b$$

(there exists a finite product of elements in H containing both a and b)
β is reflexive and symmetric relation on H, but not transitive.
Denote by β^* the transitive closure of β.
If $(H, \circ)$ is a hypergroup, then $\beta = \beta^*$ and $(H/\beta, \otimes)$ is a group with identity 1:

$$\forall \bar{x}, \bar{y} \in H/\beta, \bar{x} \otimes \bar{y} = \{\bar{z} \mid z \in x \circ y\}$$

Denoting by $\varphi_H : H \longrightarrow H/\beta$ the canonical projection, then the heart of the hypergroup H is the set

$$\omega_H = \{x \in H \mid \varphi_H(x) = 1\}$$

Till now one can distinguish three principal approaches:

- the study of new crisp hypergroups by means of fuzzy sets;
- the study of fuzzy subhypergroups (fuzzy sets having the level sets as crisp subhypergroups);
- the fuzzy of fuzzy hypergroups (hyperstructures endowed with fuzzy hyperoperations).

During my presentation I will concentrate on the first line of research, in particular on the sequences of fuzzy sets and join spaces associated with a hypergroup.

Construction of the sequence

For any hypergroup $(H, \circ)$, we may define a fuzzy subset $\widetilde{\mu}$ of H in the following way: for any $u \in H$, we consider

$$\widetilde{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{q(u)}, \qquad (\omega)$$

where $Q(u) = \{(x, y) \in H^2 \mid u \in x \circ y\}, \quad q(u) = |Q(u)|$.

If $Q(u) = \emptyset$, we set $\widetilde{\mu}(u) = 0$.
Besides, with any hypergroupoid H endowed with a fuzzy set, we may associate a join space $({}^1H, \circ_1)$ as follows:

$$\forall (x, y) \in H^2, \;\; x \circ_1 y = \{z \in H \mid \widetilde{\mu}(x) \wedge \widetilde{\mu}(y) \leq \widetilde{\mu}(z) \leq \widetilde{\mu}(x) \vee \widetilde{\mu}(y)\}.$$

Construction of the sequence

Using the same procedure as in (ω), from ${}^1H$ one obtains a membership function $\widetilde{\mu}_1$ and the associated join space ${}^2H$ and so on.

A sequence of fuzzy sets and join spaces $({}^iH = (H, \circ_i), \widetilde{\mu}_{i-1})_{i \geq 1}$ is determined in this way.

Together with Corsini, I started to investigate the properties of the sequence in the general case and also for particular classes of hypergroups: i.p.s. hyergroups, complete hypergroups, 1-hypergroups.

The sequence stops when there exist two consecutive isomorphic join spaces. The length of this sequence, there is the number of non-isomorphic join spaces in the sequence, is called the fuzzy grade of H.

Def. (Corsini-Cristea, 2002)

A hypergroup H has the *fuzzy grade* $m \in \mathbb{N}^*$, and we write $f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces ${}^iH$ and ${}^{i+1}H$ associated with H are not isomorphic (where ${}^0H = H$) and for any $s > m$, ${}^sH$ is isomorphic with ${}^mH$. We say that the hypergroup H has the *strong fuzzy grade* m, and we write $s.f.g.(H) = m$, if $f.g.(H) = m$ and for all $s, s > m$, ${}^sH = {}^mH$.

Let $({}^iH = (H, \circ_i), \widetilde{\mu}_{i-1})_{i\geq 1}$ be the above sequence associated with H.
With any join space ${}^iH$ one may associate an ordered chain $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ and an ordered r-tuple $({}^ik_1, {}^ik_2, ..., {}^ik_r)$, where

- $\forall j \geq 1 : x, y \in {}^iC_j \Longleftrightarrow \widetilde{\mu}_{i-1}(x) = \widetilde{\mu}_{i-1}(y)$;
- for $x \in {}^iC_j$ and $z \in {}^iC_k$, if $j < k$ then $\widetilde{\mu}_{i-1}(x) < \widetilde{\mu}_{i-1}(z)$;
- ${}^ik_j = |{}^iC_j|, \forall j$.

Investigated the properties of the above ordered r-tuple, we answered to the following questions:

1. When two consecutive join spaces in the sequence are isomorphic?
2. Does there exist a hypergroup H such that $f.g.(H) = n$, whenever n is a natural number?
3. Are the join spaces in the sequence reduced hypergroups?
4. What can we say about the fuzzy grade of the i.p.s. hypergroups and complete hypergroups?

Th.1 (Corsini-Leoreanu, 1995)

Let ${}^{i}H$ and ${}^{i+1}H$ be two consecutive join spaces associated with H and $(k_1, k_2, ..., k_{r_1})$ be the r_1-tuple associated with ${}^{i}H$, $(k'_1, k'_2, ..., k'_{r_2})$ be the r_2-tuple associated with ${}^{i+1}H$.

The join spaces ${}^{i}H$ and ${}^{i+1}H$ are isomorphic if and only if $r_1 = r_2$ and $(k_1, \dots, k_{r_1}) = (k'_1, \dots, k'_{r_1})$ or $(k_1, \dots, k_{r_1}) = (k'_{r_1}, \dots, k'_1)$.

Th.2 (Cristea-Stefanescu, 2008)

Let $(k_1, k_2, ..., k_r)$ be the r-tuple associated with the join space ${}^{i}H$.
If $(k_1, k_2, ..., k_r) = (k_r, k_{r-1}, ..., k_1)$, then the join spaces ${}^{i+1}H$ and ${}^{i}H$ are not isomorphic, so the sequence doesn't stop.

Th. (Cristea-Stefanescu, 2008)

Let $H = \{x_1, x_2, \ldots, x_s\}$, where $s = 2^n$, $n \in \mathbb{N}^* \setminus \{1, 2\}$, be the commutative hypergroupoid defined by the hyperproduct

$$x_i \circ x_i = x_i, 1 \leq i \leq n,$$
$$x_i \circ x_j = \{x_i, x_{i+1}, \ldots, x_j\}, 1 \leq i < j \leq n.$$

Then $s.f.g.(H) = n$. Moreover, H is a join space.

Def. (J. Jantosciak, 1991)

Two elements x, y in a hypergroup $(H, \circ)$ are called:

1. *operationally equivalent*, and write $x \sim_o y$, if $x \circ a = y \circ a$, and $a \circ x = a \circ y$, for any $a \in H$.
2. *inseparable*, and write $x \sim_i y$, if, for all $a, b \in H$, $x \in a \circ b \Longleftrightarrow y \in a \circ b$.
3. *essentially indistinguishable* if they are operationally equivalent and inseparable.

A *reduced hypergroup* has the equivalence class of each element with respect to the essentially indistinguishable relation a singleton.

Th.

The join space ${}^{i}H$ is a reduced hypergroup iff the n-tuple associated with ${}^{i-1}H$ is $(1, 1, \ldots, 1)$, with $n = |H|$. Moreover, the first join space in the sequence is the unique one that could be reduced.

K. Atanassov (1986) introduced the concept of *intuitionistic fuzzy set* in X, as an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\},$$

where, for any $x \in X$, the *degree of membership* of x (namely $\mu_A(x)$) and the *degree of non-membership* of x (namely $\lambda_A(x)$) verify the relation $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$.

We denote an Atanassov intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$, by $A = (\mu, \lambda)$.

Lower and upper intuitionistic fuzzy grade

Let $(H, \circ)$ be a finite hypergroupoid of cardinality $n \in \mathbb{N}^*$. For any $u \in H$, denote $Q(u) = \{(a, b) \in H^2 \mid u \in a \circ b\}$ and $\overline{Q}(u) = \{(a, b) \in H^2 \mid u \notin a \circ b\}$, and consider

$$\overline{\mu}(u) = \frac{\displaystyle\sum_{(x,y)\in Q(u)} \frac{1}{|x \circ y|}}{n^2}, \qquad \overline{\lambda}(u) = \frac{\displaystyle\sum_{(x,y)\in \overline{Q}(u)} \frac{1}{|x \circ y|}}{n^2}. \tag{1}$$

If $Q(u) = \emptyset$, then we put $\overline{\mu}(u) = 0$ and similarly for $\overline{\lambda}$.

Let $A = (\mu, \lambda)$ be an Atanassov intuitionistic fuzzy set on H. We may associate two join spaces with H:

$$(H, \circ_{\mu \wedge \lambda}), (H, \circ_{\mu \vee \lambda}),$$

where, for any fuzzy set α on H, the hyperproduct "$\circ_\alpha$" is defined by

$$x \circ_\alpha y = \{u \mid \alpha(x) \wedge \alpha(y) \leq \alpha(u) \leq \alpha(x) \vee \alpha(y)\}.$$

In this way, using formula (1) repeatedly, we may construct two sequences of join spaces and Atanassov intuitionistic fuzzy sets associated with a set H:

$$({}_iH = ({}_iH, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i; \overline{\lambda}_i))_{i \geq 0}$$

and

$$({}^iH = ({}^iH, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i}); \overline{A}_i = (\overline{\mu}_i; \overline{\lambda}_i))_{i \geq 0}.$$

Def. (Cristea-Davvaz, 2010)

A set H endowed with an Atanassov intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *lower Atanassov intuitionistic fuzzy grade* m and we write $l.i.f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces $({}_iH, \circ_{\overline{\mu}_i \wedge \overline{\lambda}_i})$ and $({}_{i+1}H, \circ_{\overline{\mu}_{i+1} \wedge \overline{\lambda}_{i+1}})$ associated with H are not isomorphic and, for any $s \geq m$, ${}_sH$ is isomorphic with ${}_{m-1}H$.

Def.

A set H endowed with an Atanassov intuitionistic fuzzy set $A = (\mu, \lambda)$ has the *upper Atanassov intuitionistic fuzzy grade* m and we write $u.i.f.g.(H) = m$ if, for any $0 \leq i < m$, the join spaces $({}^iH, \circ_{\overline{\mu}_i \vee \overline{\lambda}_i})$ and $({}^{i+1}H, \circ_{\overline{\mu}_{i+1} \vee \overline{\lambda}_{i+1}})$ associated with H are not isomorphic and, for any $s \geq m$, ${}^sH$ is isomorphic with ${}^{m-1}H$.

We may start the construction of the same sequences from a hypergroupoid $(H, \circ)$. But, in this case, we obtain only one sequence of join spaces, since the join spaces $({}_0H, \circ_{\overline{\mu}\wedge\overline{\lambda}})$ and $({}^0H, \circ_{\overline{\mu}\vee\overline{\lambda}})$ are always isomorphic.

Def.

We say that a hypergroupoid H has the *Atanassov intuitionistic fuzzy grade* $m \in \mathbb{N}^*$, and we write $i.f.g.(H) = m$, if $l.i.f.g.(H) = m$.

With any join space ${}_iH$ one may associate an *ordered chain* $({}^iC_1, {}^iC_2, ..., {}^iC_r)$ and an *ordered r-tuple* $({}^ik_1, {}^ik_2, ..., {}^ik_r)$, where

1. $\forall j \geq 1$: $x, y \in {}^iC_j \Longleftrightarrow \overline{\mu}_i \wedge \overline{\lambda}_i(x) = \overline{\mu}_i \wedge \overline{\lambda}_i(y)$,
2. for $x \in {}^iC_j$ and $z \in {}^iC_k$, if $j < k$ then $\overline{\mu}_i \wedge \overline{\lambda}_i(x) < \overline{\mu}_i \wedge \overline{\lambda}_i(z)$
3. ${}^ik_j = |{}^iC_j|$, for all j.

Def.-complete hypergroup

A hypergroup $(H, \circ)$ is a *complete hypergroup* if it can be written as the union of its subsets $H = \bigcup_{g \in G} A_g$, where

1. $(G, \cdot)$ is a group;
2. for any $(g_1, g_2) \in G^2$, $g_1 \neq g_2$, we have $A_{g_1} \cap A_{g_2} = \emptyset$;
3. if $(a, b) \in A_{g_1} \times A_{g_2}$, then $a \circ b = A_{g_1 g_2}$.

Let $G = \{g_1, \ldots, g_m\}$ be a finite group. Then with $H = \bigcup_{i=1}^{m} A_{g_i}$ we may associate an m-tuple $[k_1, \ldots, k_m]$, where $k_i = |A_{g_i}|$.The formulas for the membership functions associated with a complete hypergroup are:

$$\widetilde{\mu}(u) = \frac{1}{|A_{g_u}|},$$

$$\bar{\mu}(u) = \frac{|Q(u)|}{|A_{g_u}|} \cdot \frac{1}{n^2}, \quad \bar{\lambda}(u) = \left(\sum_{v \notin \hat{u}} \frac{|Q(v)|}{|A_{g_v}|} \right) \cdot \frac{1}{n^2}.$$

Comparison between the two grades-complete hypergroups

Th. (Cristea, 2002; Cristea-Davvaz-Sadrabadi, 2013))

Let H be a complete hypergroup of order $n \leq 6$.

1. There are two non isomorphic complete hypergroups of order 3 that have $s.f.g.(H) = i.f.g.(H) = 1$.
2. Among the five non isomorphic complete hypergroups of order 4, three of them have $s.f.g.(H) = i.f.g.(H) = 1$ and two of them have $s.f.g.(H) = i.f.g.(H) = 2$.
3. There are 12 non isomorphic complete hypergroups of order 5. All of them have $s.f.g.(H) = 1$. Nine of them have $i.f.g.(H) = 1$ and the other 3 have $i.f.g.(H) = 3$.
4. There are 21 non isomorphic complete hypergroups of order 6:
   - 17 with $s.f.g.(H) = 1$ and 4 with $s.f.g.(H) = 2$.
   - 16 with $i.f.g.(H) = 1$, 3 with $i.f.g.(H) = 2$ and 2 with $i.f.g.(H) = 3$.

Comparison between the two grades-complete hypergroups

Th. (Angheluţă-Cristea (2013))

Let H be a complete hypergroup of type

1. $[\underbrace{p, p, ..., p}_{k \ \ times}, kp]$, where $n = |H| = 2kp$. Then $s.f.g.(H) = 2$.
2. $[\underbrace{p, p, \ldots, p}_{s \ \ times}, \underbrace{k, k, \ldots, k}_{t \ \ times}, ps]$, $2 \leq p < k < ps$, $n = |H| = 2ps + kt$.
   - for $n = 4ps$, $s.f.g.(H) = 3$,
   - for $kt \neq 2ps$, $f.g.(H) = 2$.
3. $[\underbrace{k, k, \ldots, k}_{l \ \ times}, \underbrace{p, p, \ldots, p}_{s \ \ times}, \underbrace{s, s, \ldots, s}_{p \ \ times}, \underbrace{l, l, \ldots, l}_{k \ \ times}]$, $2 \leq k < p < s < l$,
   $n = |H| = 2(ps + kl)$.
   - if $kl = ps$, then $s.f.g.(H) = 3$,
   - otherwise $f.g.(H) = 2$.
4. $[\underbrace{1, \ldots, 1}_{k \ \ times}, \underbrace{2, \ldots, 2}_{\frac{k}{2} \ \ times}, \underbrace{3, \ldots, 3}_{\frac{k}{3} \ \ times}, \ldots, \underbrace{m-1, \ldots, m-1}_{\frac{k}{m-1} \ \ times}, k]$,
   $k = m.c.m.(1, 2, \ldots, m-1)$, $n = |H| = mk = 2^{s-1}k$.
   Then $s.f.g.(H) = s$.

Def.-i.p.s. hypergroup

An *i.p.s. hypergroup* (i.e. a hypergroup with partial scalar identities) is a particular type of *canonical hypergroup*, that is a commutative hypergroup $(H, \circ)$ with a scalar identity, where every element $a \in H$ has a unique inverse a^{-1} such that it satisfies the following properties

1. if $y \in a \circ x$, then $x \in a^{-1} \circ y$;
2. $x \in x \circ a \Longrightarrow x = x \circ a$

The canonical hypergroups were introduced for the first time by Krasner in 1983 as the additive structures of the hyperfields. Later on Corsini determined all i.p.s. hypergroups of order less than 9, proving that they are strongly canonical.

Comparison between the two grades- i.p.s. hypergroups

Th. (Corsini-Cristea- 2003,2004, Cristea-Davvaz-Sadrabadi-2013)

1. There exists one i.p.s. hypergroup H of order 3 and $f.g.(H) = i.f.g.(H) = 1$.
2. There exist 3 i.p.s. hypergroups of order 4 with $f.g.(H_1) = i.f.g.(H_1) = 1$, $f.g.(H_2) = 1, i.f.g.(H_2) = 2$, $f.g.(H_3) = i.f.g.(H_3) = 2$.
3. There exist 8 i.p.s. hypergroups of order 5: one has $f.g.(H) = 2$, all the others have $f.g.(H) = 1$; four of them have $i.f.g.(H) = 1$, three have $i.f.g.(H) = 2$ and one is with $i.f.g.(H) = 3$.
4. There exist 19 i.p.s. hypergroups of order 6: 14 of them have $f.g.(H) = 1$, 4 have $f.g.(H) = 2$ and one has $f.g.(H) = 3$;
   ten of them have $i.f.g.(H) = 1$, eight of them have $i.f.g.(H) = 2$ and only for one of them we find $i.f.g.(H) = 3$.
5. There exist 36 i.p.s. hypergroups of order 7: 27 of them with $f.g.(H) = 1$, 8 with $f.g.(H) = 2$, 1 with $f.g.(H) = 3$;
   10 of them with $i.f.g.(H) = 1$, 10 with $i.f.g.(H) = 2$, 9 with $i.f.g.(H) = 3$, 5 with $i.f.g.(H) = 4$, 2 with $i.f.g.(H) = 5$.
   Moreover, for 18 of them we find that the associated sequences of join spaces and intuitionistic fuzzy sets are cyclic.

Concluding remarks

With any hypergroup H we may associate two numerical functions that compute:

- the fuzzy grade of H;
- the intuitionistic fuzzy grade of H.

Grater is the grade (fuzzy or intuitionistic fuzzy), more interesting is the structure of the sequence of join spaces associated with H.

Concluding remarks- i.p.s. hypergroups

- There exist more i.p.s. hypergroups H of order less than 8, having the $i.f.g.(H) > 1$, then those with $f.g.(H) > 1$
- There exist i.p.s. hypergroups of order 7 having a cyclic sequence of j.s.: with completed cycles, or having cycles of length 2 or 3 and also several j.s. out of the cyclic part.
  We can divide any sequence into two parts:
  - the cyclic (periodical) one, that has an infinite number of j.s.
  - the non cyclic (non periodical) part, that has a finite number of terms.

Concluding remarks- complete hypergroups

- I didn't meet the cyclicity property for the compete hypergroups that I studied.
- The $f.g.(H)$ depends just on the m-decomposition of the cardinality n of H,while the $i.f.g.(H)$ depends both on the group G (from which we construct the hypergroup) and on the m-decomposition of n.
  If G_1 and G_2 are non-isomorphic groups of the same order m, and H_1 and H_2 are the correspondent complete hypergroups of order n, than $f.g.(H_1) = f.g.(H_2)$.
  The i.f.g. does not have the same property.
  Let H be a complete hypergroup of order 6 s.t. $[1, 1, 2, 2]$ is the 4-tuple associated with it. There exist two non-isomorphic complete hypergroups of this type:
  - H_1 (obtained from a group $G \sim (\mathbb{Z}_4, +)$)
  - H_2 (obtained from a group $G \sim (K, \cdot)$ Klein group)

  We get $i.f.g.(H_1) = 3$ and $i.f.g.(H_2) = 1$.

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