Obtaining planarity by contracting few edges

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joint work with

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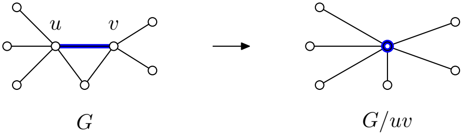
What's the problem?

PLANAR CONTRACTION

Input:	Graph G, integer k.
Question:	Can G be made planar by contracting $\leq k$ edges?



Graph modification problems

problem	vd	ed	ea	ec	target
VERTEX COVER	✓				edgeless
FEEDBACK VERTEX SET	✓				acyclic
ODD CYCLE TRANSVERSAL	✓				bipartite
EDGE BIPARTIZATION		✓			bipartite
INTERVAL COMPLETION			✓		interval
MINIMUM FILL-IN			✓		chordal
CLUSTER EDITING		✓	✓		P_3-free

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Vertex Cover	✓				edgeless
Feedback Vertex Set	✓				acyclic
Odd Cycle Transversal	✓				bipartite
Edge Bipartization		✓			bipartite
Interval Completion			✓		interval
Minimum Fill-In			✓		chordal
Cluster Editing		✓	✓		P_3-free
Path Contraction				✓	path
Tree Contraction				✓	tree
Bipartite Contraction				✓	bipartite

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CLUSTER EDITING		✓	✓		P_3-free
PATH CONTRACTION				✓	path
TREE CONTRACTION				✓	tree
BIPARTITE CONTRACTION				✓	bipartite
PLANAR CONTRACTION				✓	planar

How hard is our problem?

PLANAR CONTRACTION

Input: Graph G, integer k.
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Can we solve PLANAR CONTRACTION in $n^{O(1)}$ time?

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Can we solve PLANAR CONTRACTION in $n^{O(1)}$ time? Bad news:

Theorem (Asano & Hirata; 1983)

Let $\mathcal{H}$ be a class of graphs. Then $\mathcal{H}$-CONTRACTION is NP-complete if $\mathcal{H}$ satisfies the following three properties:

- *$\mathcal{H}$ is non-trivial on connected graphs;*
- *$\mathcal{H}$ is closed under edge contractions; and*
- *for every graph H, we have that $H \in \mathcal{H}$ if and only if all biconnected components of H are in $\mathcal{H}$.*

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Corollary

Planar Contraction *is NP-complete.*

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So that's it? No!

NP-hard problems, beware of...

Parameterized Complexity!

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How hard is our problem, really?

Planar Contraction

Input: Graph G, integer k.
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Can we solve Planar Contraction in $f(k) \cdot n^{O(1)}$ time?

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Theorem

For every fixed integer k *and every constant* $\epsilon > 0$, k-Planar Contraction *can be solved in* $O(n^{2+\epsilon})$ *time.*

How hard is our problem, really?

k-PLANAR CONTRACTION

Input: Graph G.
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Theorem

For every fixed integer k *and every constant* $\epsilon > 0$, k-PLANAR CONTRACTION *can be solved in* $O(n^{2+\epsilon})$ *time.*

Related work on modification into planar graphs

Planar Vertex Deletion

Input:	Graph G, integer k.
Question:	Can G be made planar by deleting $\leq k$ vertices?

Planar Edge Deletion

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Related work on modification into planar graphs

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Observation (easy)

Planar Vertex Deletion *and* Planar Edge Deletion *can be solved in polynomial time for every fixed integer* k.

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Planar Vertex Deletion *and* Planar Edge Deletion *can be solved in* $n^{O(k)}$ *time.*

Observation (easy, but using "heavy machinery")

Planar Vertex Deletion *can be solved in* $f_1(k) \cdot n^3$ *time.*

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Theorem (Marx & Schlotter; WG 2007)

Planar Vertex Deletion *can be solved in* $f_2(k) \cdot n^2$ *time.*

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Theorem (Kawarabayashi; FOCS 2009)

Planar Vertex Deletion *can be solved in* $f_3(k) \cdot n$ *time.*

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Theorem (Kawarabayashi & Reed; STOC 2007)

Planar Edge Deletion *can be solved in* $f_4(k) \cdot n$ *time.*

Related work on $\mathcal{H}$-CONTRACTION

PATH CONTRACTION

Input: Graph G, integer k.
Question: Can G be made into a path by contracting $\leq k$ edges?

TREE CONTRACTION

Input: Graph G, integer k.
Question: Can G be made into a tree by contracting $\leq k$ edges?

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Input: Graph G, integer k.
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Observation (easy, but again using "heavy machinery")

Both PATH CONTRACTION *and* TREE CONTRACTION *are FPT when parameterized by* k.

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Observation (easy, but again using "heavy machinery")

Both PATH CONTRACTION *and* TREE CONTRACTION *can be solved in* $f_5(k) \cdot n$ *time.*

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Both Path Contraction *and* Tree Contraction *can be solved in* $f_5(k) \cdot n$ *time.*

Theorem (Heggernes et al.; IPEC 2011)

Path Contraction *can be solved in* $2^{k+o(k)} + n^{O(1)}$ *time.*

Related work on $\mathcal{H}$-Contraction

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Observation (easy, but again using "heavy machinery")

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Theorem (Heggernes et al.; IPEC 2011)

Tree Contraction *can be solved in* $4.98^k \cdot n^{O(1)}$ *time.*

Related work on $\mathcal{H}$-Contraction

Bipartite Contraction

Input:	Graph G, integer k.
Question:	Can G be made bipartite by contracting $\leq k$ edges?

Theorem (Heggernes et al.; FSTTCS 2011)

Bipartite Contraction *is FPT when parameterized by* k*.*

Back to our problem

Planar Contraction

Input: Graph G, integer k.
Question: Can G be made planar by contracting $\leq k$ edges?

k-Planar Contraction

Input: Graph G.
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Back to our problem

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Input: Graph G.
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Theorem

For every fixed integer k and every constant $\epsilon > 0$, k-PLANAR CONTRACTION can be solved in $O(n^{2+\epsilon})$ time.

Corollary

PLANAR CONTRACTION *is FPT when parameterized by k.*

Before we start...

A useful observation:

Observation

If G can be made planar by contracting $\leq k$ edges, then there is a set $S \subseteq V(G)$ with $|S| \leq k$ such that $H := G - S$ is planar.

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If G can be made planar by contracting $\leq k$ edges, then there is a set $S \subseteq V(G)$ with $|S| \leq k$ such that $H := G - S$ is planar.

And a famous theorem:

Theorem (Wagner; 1937)

A graph is planar if and only if it contains neither K_5 nor $K_{3,3}$ as a minor.

Main idea of the algorithm

Let G be an instance of k-Planar Contraction.

One of the following two cases applies:

1. treewidth of G is bounded by $f(k)$
   $\Rightarrow$ solve instance using Courcelle's Theorem
2. G has a large wall W as a subgraph
   $\Rightarrow$ find and contract an irrelevant edge in W

Repeat until instance is solved.

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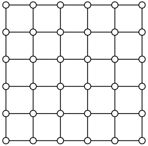
Distinguishing between cases 1 and 2

A 6×6 grid:



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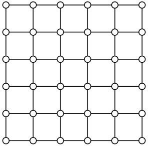

Theorem (Robertson & Seymour; 1994)

Any planar graph with treewidth more than $6r - 5$ *has an* $r \times r$*-grid as a minor.*

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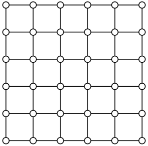
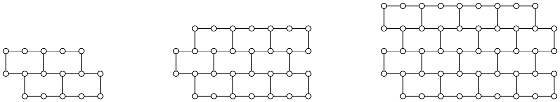


Theorem (Robertson & Seymour; 1994)

Any planar graph with "large" treewidth has a "large" grid as a minor.

Distinguishing between cases 1 and 2

Elementary walls of height 2, 3 and 4:



A *wall* of height h is a subdivision of an elementary wall of height h.

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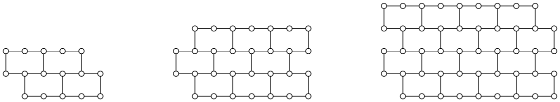

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If a graph G has an $r \times r$-grid as a minor, then G has a wall of height $\lfloor r/2 \rfloor - 1$ as a subgraph.

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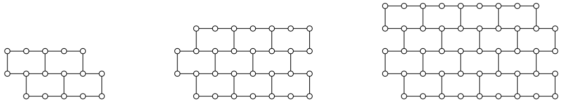

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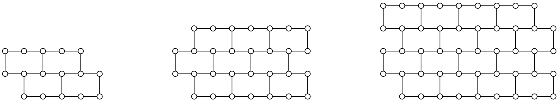


A *wall* of height h is a subdivision of an elementary wall of height h.

We find a large wall in polynomial time using the following result:

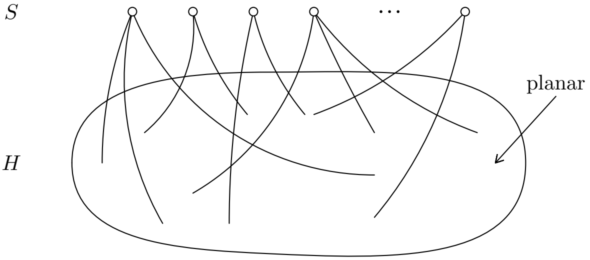
Theorem (Gu & Tamaki; 2011)

Let H be a planar graph, and let h^* be the height of a largest wall that appears as a subgraph in H. For every constant $\epsilon > 0$, there exists a constant $c_\epsilon > 3$ such that a wall in H with height at least h^*/c_ϵ can be constructed in $O(n^{1+\epsilon})$ time.

Outline of our algorithm

Let G be an instance of k-Planar Contraction.

1. Find $S \subseteq V(G)$ with $|S| \leq k$ such that $H := G - S$ is planar.
   - If no such set S exists, output "no".



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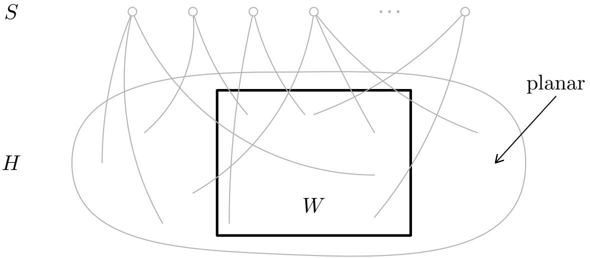

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   - Suppose $h \leq \lceil \sqrt{2k+1} \rceil (12k + 10)$.
     - ▷ $h^* \leq c_\epsilon h \leq c_\epsilon \lceil \sqrt{2k+1} \rceil (12k + 10)$;
     - ▷ H has bounded treewidth;
     - ▷ G has bounded treewidth;
     - ▷ Use Courcelle's Theorem to solve instance in $O(n)$ time.
   - Suppose $h > \lceil \sqrt{2k+1} \rceil (12k + 10)$.
     - ▷ Find and contract an irrelevant edge in W.

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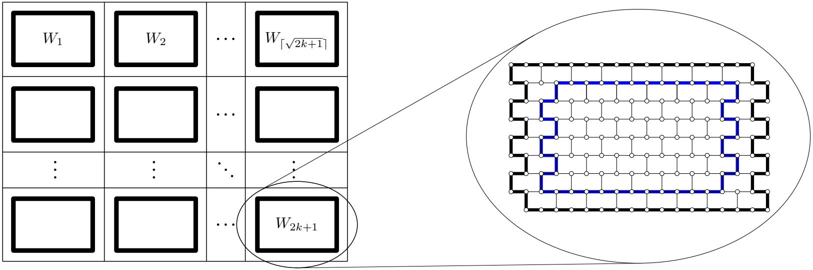
     - ▷ Use Courcelle's Theorem to solve instance in $O(n)$ time.
   - Suppose $h > \lceil\sqrt{2k+1}\rceil(12k+10)$.
     - ▷ Define $2k+1$ disjoint subwalls $W_1, \ldots, W_{2k+1}$ as follows:

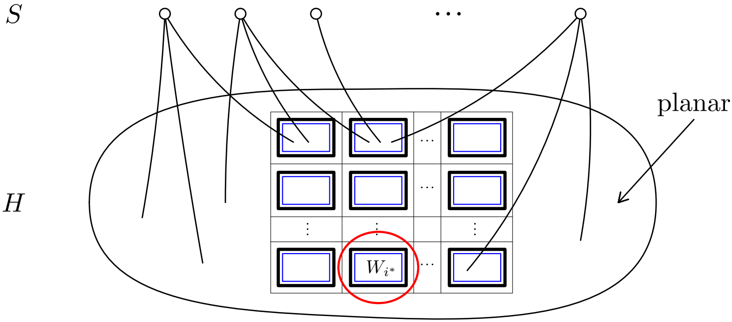
Packing subwalls $W_1, \ldots, W_{2k+1}$ inside W



- Wall W has height $h > \lceil\sqrt{2k+1}\rceil(12k+10)$.
- Each subwall W_i has height $12k+8$.
- Inside each subwall W_i, we choose a subwall W_i' of height $12k+6$.

Defining the sets S_i

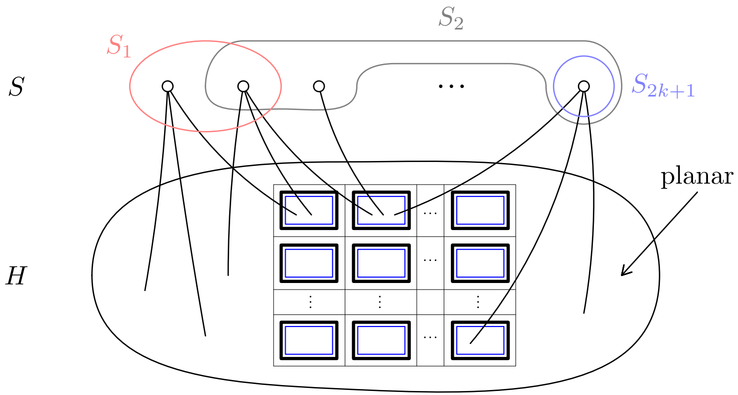
We will show there exists a subwall W_{i^*} such that no vertex of S is adjacent to an interior vertex of W'_{i^*}.



Defining the sets S_i

Definition

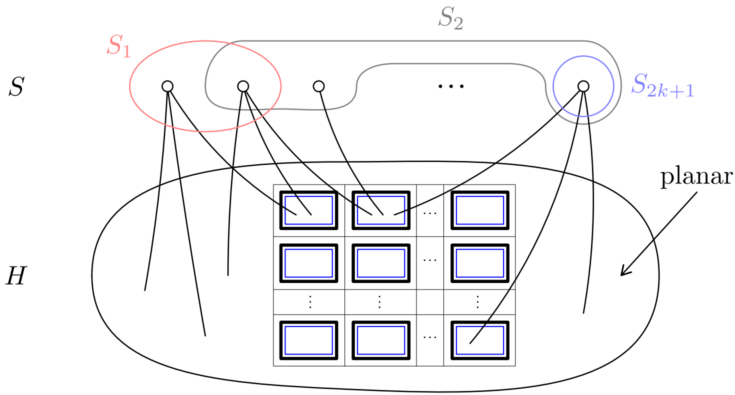
For $i = 1, \ldots, 2k+1$, let $S_i \subseteq S$ be the subset of vertices of S that are adjacent to an interior vertex of W_i'.



Defining the sets S_i

Definition

A set S_i is of *type* 1 if S_i is non-empty and every vertex $y \in S_i$ also belongs to some S_j with $j \neq i$. Otherwise, S_i is of type 2.



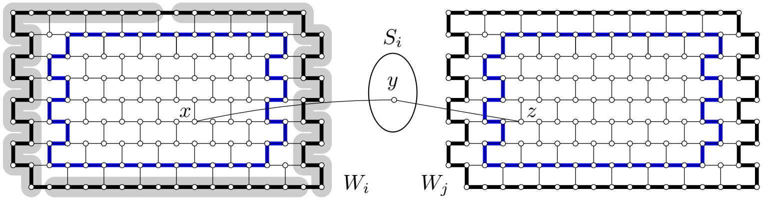
Bounding the number of sets S_i of type 1

Claim

If there are at least $k+1$ sets S_i of type 1, then G is a no-instance.

Proof (sketch). Suppose $S_1, \ldots, S_\ell$ are of type 1, where $\ell \geq k+1$.

For each $i \in \{1, \ldots, \ell\}$, we define a K_5-witness structure $\mathcal{X}_i$ of a subgraph of G as follows:



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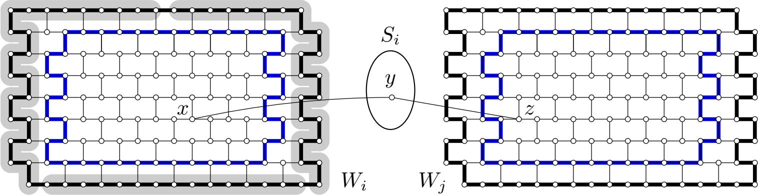

Destroying ℓ witness structures $\mathcal{X}_i$ requires $\ell \geq k+1$ contractions.

Finding an empty set S_i

- Suppose there are at most k sets S_i of type 1.
- There are $2k+1$ sets S_i in total.
- There are at least $k+1$ sets S_i of type 2.

Definition

A set S_i is of *type* 1 if S_i is non-empty and every vertex $y \in S_i$ also belongs to some S_j with $j \neq i$. Otherwise, S_i is of type 2.

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- Each non-empty set S_i of type 2 has a "private" S-vertex.
- Recall that $|S| \leq k$.
- There is at least one set S_i (of type 2) which is empty.

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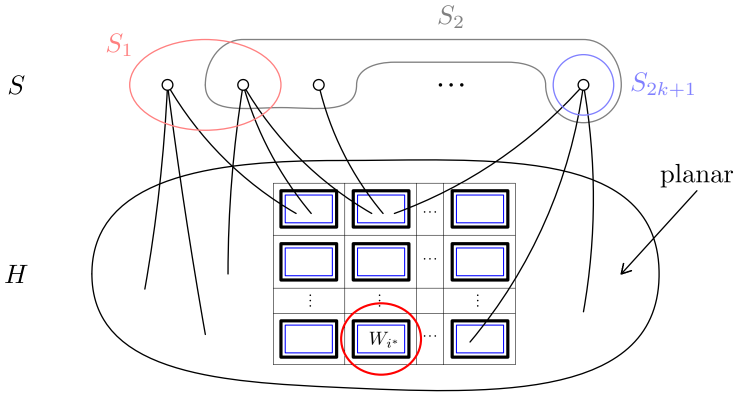
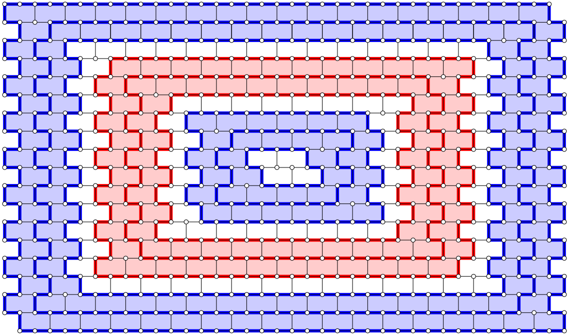
Let us assume that $S_{i^*} = \emptyset$.

Exploiting the fact that $S'_{i^*} = \emptyset$



Finding and contracting an irrelevant edge within W'_{i^*}

- Define $2k+1$ nested triple layers within subwall W'_{i^*}.



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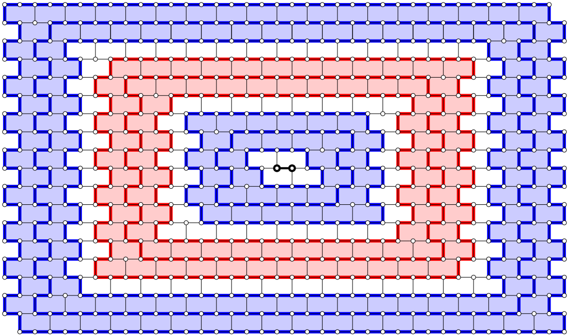

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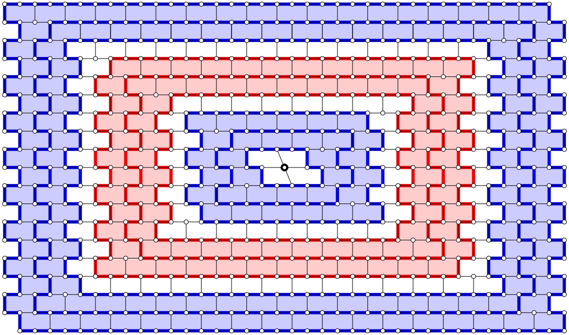

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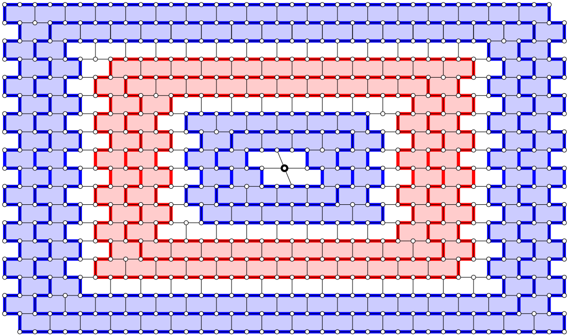

Claim

*G is a yes-instance of k-*PLANAR CONTRACTION* iff G' is.*

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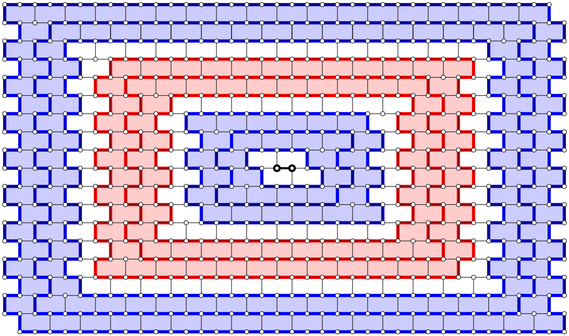

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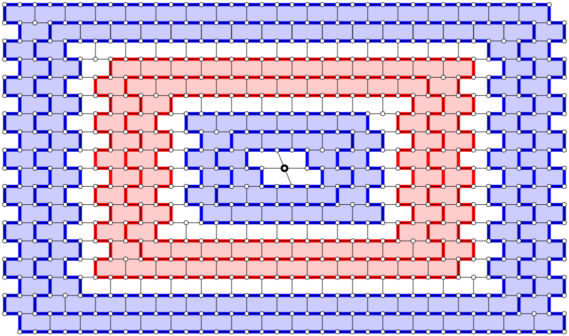

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Concluding remarks

Theorem (Abello, Klavík, Kratochvíl & Vyskočil; IPEC 2012)

Planar Contraction *can be solved in* $f(k) \cdot n^2$ *time.*

Further research:

1. Determine whether or not Planar Contraction admits a polynomial kernel.
2. Determine the parameterized complexity of $\mathcal{H}$-Contraction for classes $\mathcal{H}$ that are H-minor free.

That's it!

Hvala vam!

Dank u wel!

Thank you!

Takk!