A matrix problem and a geometry problem

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Theorem (L. Molnár and W. Timmermann, 2011)

Let $\mathcal{H}$ be a complex separable Hilbert space with $\dim \mathcal{H} \geq 3$. Assume $\phi \colon \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ is a bijection such that

$$\|[\phi(A), \phi(B)]\| = \|[A, B]\| \quad (A, B \in \mathcal{B}_s(\mathcal{H})).$$

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Let $\mathcal{H}$ be a complex separable Hilbert space with $\dim \mathcal{H} \geq 3$. Assume $\phi \colon \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ is a bijection such that

$$\left\|[\phi(A), \phi(B)]\right\| = \left\|[A, B]\right\| \quad (A, B \in \mathcal{B}_s(\mathcal{H})).$$

Then there exist either a unitary or an antiunitary operator U on $\mathcal{H}$ and a function $f \colon \mathcal{B}_s(\mathcal{H}) \to \mathbb{R}$ such that

$$\phi(A) = \pm UAU^* + f(A)I \quad (A \in \mathcal{B}_s(\mathcal{H})).$$

The proof uses the following theorem:

Theorem (L. Molnár and P. Šemrl, 2005)

Let $\mathcal{H}$ be a complex separable Hilbert space with $\dim \mathcal{H} \geq 3$ *and let* $\phi\colon \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ *be a bijective transformation which preserves commutativity in both directions.*

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$$\phi(A) = U f_A(A) U^* \quad (A \in \mathcal{B}_s(\mathcal{H})).$$

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Question

What happens in two dimensions?

Commutativity preservers on $\mathcal{B}_s(\mathbb{C}^2)$

Two linearly independent operators $A, B \in \mathcal{B}_s(\mathbb{C}^2)$ commute $\iff$ $\exists \alpha, \beta \in \mathbb{R}$ s.t. $\alpha A + \beta B = I$.

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E.g. if $\phi\colon \mathcal{B}_s(\mathbb{C}^2) \to \mathcal{B}_s(\mathbb{C}^2)$ is non-singular and linear, then ϕ preserves commutativity in both directions $\iff$ $\phi(I) \in (\mathbb{R} \setminus \{0\}) \cdot I$.

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So the preservation of commutativity provides too few information.

The first step: linear commutativity preservers

Theorem (with G. Nagy, 2014)

Suppose that $d \in \mathbb{N}$, $|||\cdot|||$ *is an arbitrary unitarily invariant norm and* $\phi\colon \mathcal{B}_s(\mathbb{C}^d) \to \mathcal{B}_s(\mathbb{C}^d)$ *is a (real-)linear transformation such that*

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Then there exist either a unitary or an antiunitary operator U *on* $\mathbb{C}^d$ *and a linear functional* $f\colon \mathcal{B}_s(\mathbb{C}^d) \to \mathbb{R}$ *such that*

$$\phi(A) = UAU^* + f(A)I \quad (A \in \mathcal{B}_s(\mathbb{C}^d))$$

or

$$\phi(A) = -UAU^* + f(A)I \quad (A \in \mathcal{B}_s(\mathbb{C}^d)).$$

What can we do in general?

Problem (Reformulation)

Characterize those maps $\phi\colon \mathcal{B}_s(\mathbb{C}^2) \to \mathcal{B}_s(\mathbb{C}^2)$ *s.t.*

$$\det[A,B] = \det[\phi(A),\phi(B)] \qquad (A,B \in \mathcal{B}_s(\mathbb{C}^2)).$$

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Let $Z_2 := \{A \in \mathcal{B}_s(\mathbb{C}^2) \colon \operatorname{Tr} A = 0\}$ and

$$\tilde{\phi}\colon \mathcal{B}_s(\mathbb{C}^2) \to Z_2 \subseteq \mathcal{B}_s(\mathbb{C}^2), \quad \tilde{\phi}(A) = \phi(A) - \frac{\operatorname{Tr} \phi(A)}{2} \cdot I.$$

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We can prove the following:

$$\tilde{\phi}(A) = \pm\psi\left(A - \frac{\operatorname{Tr} A}{2} I\right) \qquad (A \in \mathcal{B}_s(\mathbb{C}^2)).$$

Now, we identify elements of Z_2 with vectors of $\mathbb{R}^3$ using the vector space isomorphism

$$\iota\colon \mathbb{R}^3 \to Z_2, \qquad (a,b,c) \mapsto \begin{pmatrix} a & b+ic \\ b-ic & -a \end{pmatrix},$$

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A rather simple calculation shows the following two equations:

$$\det[\iota(a_1,b_1,c_1), \iota(a_2,b_2,c_2)] = 4|(a_1,b_1,c_1) \times (a_2,b_2,c_2)|^2$$

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$$\big|\xi(a_1,b_1,c_1) \times \xi(a_2,b_2,c_2)\big| = |(a_1,b_1,c_1) \times (a_2,b_2,c_2)|.$$

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Characterize those maps $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$ *s.t.*

$$\blacklozenge(\vec{a}, \vec{b}) = \blacklozenge(\phi(\vec{a}), \phi(\vec{b})) \qquad (\forall\ \vec{a}, \vec{b} \in \mathbb{R}^3)$$

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(Beckmann-Quarles; Lester-Martin, linear R-W)

Theorem (G.)

Let E be a real (not necessarily separable) Hilbert space and $\phi\colon E \to E$ be an arbitrary transformation such that

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(i) *If* $\dim E = 2$*, then there exists a linear operator* $A\colon E \to E$ *with* $|\det A| = 1$ *such that the following holds:*

$$\phi(\vec{a}) = \pm A\vec{a} \qquad (\vec{a} \in E). \tag{2}$$

Theorem (G. continued)

(ii) *If* $2 < \dim E < \infty$, *then there exists an orthogonal linear operator* $R\colon E \to E$ *such that*

$$\phi(\vec{a}) = \pm R\vec{a} \qquad (\vec{a} \in E)$$

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Theorem (G. continued)

(ii) *If* $2 < \dim E < \infty$*, then there exists an orthogonal linear operator* $R\colon E \to E$ *such that*

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(iii) *If* $\dim E = \infty$ *and in addition* ϕ *is assumed to be bijective, then there exists a linear, surjective isometry* $R\colon E \to E$ *such that we have*

$$\phi(\vec{a}) = \pm R\vec{a} \qquad (\vec{a} \in E).$$

Outline of the proof for $n = 3$

Let us consider the projectivised $\mathbb{R}^3$ which will be denoted by $\mathcal{P}(\mathbb{R}^3)$. The subspace generated by $\vec{v}$ will be denoted by $[\vec{v}]$. Let

$$g_\phi \colon \mathcal{P}(\mathbb{R}^3) \to \mathcal{P}(\mathbb{R}^3), \quad g_\phi([\vec{v}]) = [\phi(\vec{v})] \ (\vec{v} \neq \vec{0}).$$

Note that $\phi(\vec{v}) = \vec{0} \iff \vec{v} = \vec{0}$.

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STEP 1: We prove that g_ϕ is a homeomorphism.

STEP 2: We consider two linearly independent vectors $\vec{a}, \vec{b} \in E$ and let

$$C_{\vec{a},\vec{b}} := \left\{ \vec{v} \in E \setminus \{0\} \colon \blacklozenge(\vec{v}, \vec{a}) = \blacklozenge(\vec{v}, \vec{b}) \right\} \subseteq E$$

(note that $C_{\vec{a},\vec{b}}$ is a plane iff $|\vec{a}| = |\vec{b}|$)

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$$PC_{\vec{a},\vec{b}} := \left\{ [\vec{v}] \in \mathcal{P}(\mathbb{R}^3) \colon \vec{v} \in C_{\vec{a},\vec{b}} \right\}.$$

We can show that $PC_{\vec{a},\vec{b}}$ contains a loop $\gamma \colon [0,1] \to PC_{\vec{a},\vec{b}}$ not homotopic to the trivial loop $\delta \colon [0,1] \to PC_{\vec{a},\vec{b}}$, $\delta \equiv \gamma(0)$ if and only if $|\vec{a}| = |\vec{b}|$ holds.

STEP 3: Using that g_ϕ is a homeomorphism, we obtain that

$$|\phi(\vec{a})| = \lambda_\phi |\vec{a}| \qquad (\vec{a} \in \mathbb{R}^3)$$

holds with some $\lambda_\phi > 0$.

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STEP 4: Since

$$\sqrt{|\vec{a}|^2 \cdot |\vec{b}|^2 - \langle \vec{a}, \vec{b} \rangle^2} = \blacklozenge(\vec{a}, \vec{b})$$

$$= \blacklozenge(\phi(\vec{a}), \phi(\vec{b})) = \sqrt{|\phi(\vec{a})|^2 \cdot |\phi(\vec{b})|^2 - \langle \phi(\vec{a}), \phi(\vec{b}) \rangle^2} \qquad (\vec{a}, \vec{b} \in \mathbb{R}^3),$$

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Finally, we apply Wigner's theorem. ■

Back to the Molnár-Timmermann problem

Theorem (G.)

Fix a unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}^{d\times d}$ where $d \geq 2$. Let $\phi\colon \mathcal{B}_s(\mathbb{C}^d) \to \mathcal{B}_s(\mathbb{C}^d)$ be an arbitrary transformation for which the following holds:

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Then there exist a function $f:\mathcal{B}_s(\mathbb{C}^d)\to\mathbb{R}$ *and a unitary or antiunitary operator* U *such that*

$$\phi(A) = \pm UAU^* + f(A)I \qquad (A\in\mathcal{B}_s(\mathbb{C}^d))$$

is satisfied.

Back to the Molnár-Timmermann problem (cont.)

Theorem (G.)

Let $\mathcal{H}$ be a separable Hilbert space and fix a unitarily invariant norm $|||\cdot|||$ on $\mathcal{B}(\mathcal{H})$. Let $\phi\colon \mathcal{B}_s(\mathcal{H}) \to \mathcal{B}_s(\mathcal{H})$ be a bijection for which the following holds:

$$|||[A,B]||| = |||[\phi(A),\phi(B)]||| \qquad (A, B \in H_d). \tag{4}$$

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k-parallelepipeds

For any k vectors $\vec{a}_1, \dots \vec{a}_k \in E$ let us denote the k-dimensional volume of the parallelepiped spanned by them by the symbol $\blacklozenge_k(\vec{a}_1, \dots \vec{a}_k)$.

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Theorem (G.)

Let E be a real (not necessarily separable) Hilbert space, $2 < k < \infty$, $k \leq \dim E$ and $\phi\colon E \to E$ be a transformation such that

$$\blacklozenge_k(\vec{a}_1, \dots \vec{a}_k) = \blacklozenge_k(\phi(\vec{a}_1), \dots \phi(\vec{a}_k)) \qquad (\forall\, \vec{a}_1, \dots \vec{a}_k \in E). \quad (5)$$

Theorem (G., cont.)

(i) *If* $\dim E = k$, *then there exists a linear operator* $A\colon E \to E$ *with* $|\det A| = 1$ *such that the following holds:*

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(ii) *If* $2 < k < \dim E (\leq \infty)$*, then there exists a linear (not necessarily surjective) isometry* $R\colon E \to E$ *such that*

$$\phi(\vec{a}) = \pm R\vec{a} \qquad (\vec{a} \in E)$$

is satisfied.

The proof is a straightforward consequence of the fundamental theorem of projective geometry.

Back to the parallelogram case

Note that if we knew that $[\vec{c}] \subseteq [\vec{a}, \vec{b}]$, then we could use the fundamental theorem of projective geometry and drop the bijectivity condition.

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Problem

Let E be an arbitrary real Hilbert space. Characterize those transformations $\phi\colon \mathbb{R}^2 \to E$ which preservers the area of parallelograms.

Problem

Describe those transformations $\phi \colon E \to E$ *s. t.*

$$\blacklozenge(\vec{a}, \vec{b}) = 1 \Longrightarrow \blacklozenge(\phi(\vec{a}), \phi(\vec{b})) = 1$$

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Describe those transformations $\phi\colon E \to E$ *s. t.*

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Problem

Characterize those transformations $\phi\colon E \to E$ *s. t.*

$$\blacklozenge(\vec{a}, \vec{b}) = 0 \implies \blacklozenge(\phi(\vec{a}), \phi(\vec{b})) = 0$$

and

$$\blacklozenge(\vec{a}, \vec{b}) = 1 \implies \blacklozenge(\phi(\vec{a}), \phi(\vec{b})) = 1$$

are satisfied.

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This research was also supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP-4.2.4.A/2-11/1-2012-0001 'National Excellence Program'

Thank You for Your Kind Attention