Bourgin-Yang theorems for the $p$-toral groups

from Wacław Marzantowicz at

to Univerza na Primorskem

Thursday 09.06.2016; under the spirit of
In 1933 S. Ulam posed and K. Borsuk (Charles Badger) showed that if $n > m$ then it is impossible to map $f : S^n \to S^m$ preserving symmetry: $f(-x) = -f(x)$.

Next in 1954-55, C. T. Yang, and D. Bourgin, showed that if $f : S^n \to \mathbb{R}^{m+1}$ preserves this symmetry then

$$\dim f^{-1}(0) \geq n - m - 1.$$ 

We will present versions of the latter for some other groups of symmetries and also discuss the case $n = \infty$. 
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We estimate the dimension of set $Z_f = f^{-1}\{0\}$ in terms of $\dim V$ and $\dim W$, if $G$ is the torus $\mathbb{T}^k$, the $p$-torus $\mathbb{Z}_p^k$, or the cyclic group $\mathbb{Z}_p^k$, $p$-prime.
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Finally, we show that for any $p$-toral group:

$e \hookrightarrow \mathbb{T}^k \twoheadrightarrow G \twoheadrightarrow P \twoheadrightarrow e$, $P$ a finite $p$-group,

and a $G$-map $f : S(V) \rightarrow W$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$. 
Proof of the Borsuk-Ulam

Suppose that $f : S(\mathbb{R}^{n+1}) \xrightarrow{\mathbb{Z}_2} S(\mathbb{R}^{m+1})$ and $n > m$ is such a map. Let $\iota : \mathbb{R}^{m+1} \subset \mathbb{R}^{n+1}$ natural embedding.

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i) and ii) lead to a contradiction.
Applications

Theorem (Consequences of B-U theorem)

a) *No subset of* \( \mathbb{R}^n \) *is homeomorphic to* \( S^n \).
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c) The Ham sandwich theorem: For any compact sets $A_1, \ldots, A_n$ in $\mathbb{R}^n$ we can always find a hyperplane dividing each of them into two subsets of equal measure.
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c) *The Ham sandwich theorem: For any compact sets* $A_1, \ldots, A_n$ *in* $\mathbb{R}^n$ *we can always find a hyperplane dividing each of them into two subsets of equal measure.*

d) *The Brouwer fixed-point theorem.*
Example

d) Common practical applications: The case $n = 2$:

$\exists (x, -x)$ on the Earth's surface s.t. $T(x) = T(-x)$,
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e) The case $n = 1$: there always $\exists$ a pair of opposite points on the earth’s equator with the same temperature.
Necklace splitting problem

Suppose a necklace, open at the clasp, has $k \cdot n$ beads, $k \cdot a_i$ beads of colour $i$, $1 \leq i \leq t$. 
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Imre Bárány (1978) gave a simple proof, using the Borsuk-Ulam t. and a lemma of David Gale.
Theorem (Ham sandwich theorem for measures)

Let \( \mu_1, \mu_2, \ldots, \mu_d \) be finite Borel measures on \( \mathbb{R}^d \) such that every hyperplane has measure 0 for each of the \( \mu_i \) (we refer to such measures as mass distributions).

Then there exists a hyperplane \( h \) such that \( \mu_i(h_+) = \frac{1}{2} \mu_i(\mathbb{R}^d) \) for \( i = 1, 2, \ldots, d \), where \( h_+ \) denotes one of the half-spaces defined by \( h \).
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Theorem (Ham sandwich theorem for point sets)

Let $A_1, A_2, ..., A_d \subset \mathbb{R}^d$ be finite point sets. Then there exists a hyperplane $h$ that simultaneously bisects $A_1, A_2, ..., A_d$.

The idea of proof is very simple: replace the points of $A_i$ by tiny balls and apply the ham sandwich theorem for measures. But there are some subtleties along the way.
Borsuk-Ulam

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The pancakes theorem in $\mathbb{R}^2$ with two ingredients = each of distinct cake.

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Each plane cutting $\mathbb{R}^2$ along the line of solution in $\mathbb{R}^2$ gives a solution in $\mathbb{R}^3$. Their unit normal vectors form $Z_f = S^1 \subset S(\mathbb{R}^3)$
Borsuk-Ulam: Hundreds of generalizations & several applications

See the article of Steinlein [25] for generalizations
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Fibrewise setting: \( f : S(E) \subset E \rightarrow E' \) over \( B \).

1981 - Jaworowski [12];
1988 - Dold [10] for \( G = \mathbb{Z}_2 \);
1989 - Nakaoka [23] for \( G = \mathbb{Z}_2, \mathbb{Z}_p \) and \( S^1 \);
2007 - de Mattos and dos Santos [9] for \( G = \mathbb{Z}_p \), and a product of spheres;
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**Applications:**

**Nonlinear Analysis:** See *Topological Methods for Variational Problems with Symmetries* of T. Bartsch [3]

**Combinatorics:** See *Using the Borsuk-Ulam Theorem* of J. Matousek [18]
”Classical” Bourgin-Yang theorem

- Bourgin-Yang theorem for the mappings of spheres
  representation $f : S(V) \stackrel{G}{\to} W$

2012 - W. M., de Mattos and dos Santos [16]
for $G = \mathbb{Z}_{p^k}$, a use of equivariant $K$-theory;

2013-2015 - W. M., de Mattos and dos Santos [17]
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- 2016 - W.M., Błaszczyk, Singh [4] for a general setting: $X, Y$ more general, a combination of methods.
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It studies

$$\dim A(f) = \{x \in X : | f(x) = f(gx), \text{ for all } g \in G\}$$

for a map (not equivariant in general) $f : X \to Y$ of $G$-spaces $X$ and $Y$. 
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There are relations but not direct - we will not discuss the latter.
Idea of proof

Theorem (Yang, Bourgin)

If \( f : S(\mathbb{R}^{n+1}) \xrightarrow{\mathbb{Z}_2} \mathbb{R}^{m+1} \) then \( \dim Z_f \geq n - m - 1 \).

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\[ i_G(S^n) = n + 1; \quad i_G(Y) \leq \text{coh dim } Y + 1 \]

Bourgin-Yang theorems for the \( p \)-toral groups
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By the first two properties of $i_G$:

$$n + 1 = i_G(S(\mathbb{R}^{n+1})) \leq i_G(V) + i_G(U),$$
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because $f : V \to \mathbb{R}^{m+1} \setminus \{0\} \sim S(\mathbb{R}^{m+1})$. 

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$$\text{coh.dim} Z_f \geq n - m - 1. \Box$$
Notation

dim \ X: the covering dimension of \ X and coh.dim \ X the cohomological dimension of a space \ X, i.e.,

$$\text{coh.dim} \ X = \max\{n \mid \check{H}^n(\ X) \neq 0\}$$

where \( \check{H}^n(\cdot) \) denotes the Čech cohomology with coefficients \( F = \mathbb{Z}_p \) or \( F = \mathbb{Q} \), depending on whether \( G = \mathbb{Z}_p^k \) or \( G = \mathbb{T}^k \).
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Also, \( H_*(-) \), \( H^*(-) \) (\( \tilde{H}_*(-) \), \( \tilde{H}^*(-) \)) the (reduced) singular (co)homology with coefficients \( F = \mathbb{Z}_p \) or \( F = \mathbb{Q} \), depending on whether \( G = \mathbb{Z}_p^k \) or \( G = \mathbb{T}^k \).
Recall that for $G = \mathbb{Z}_p^k$, with $p$ prime odd, and $G = \mathbb{T}^k \forall$ nontrivial irreducible orthogonal representation is even dimensional and admits the complex structure, $\implies V$ and $W$ admit it too.
Recall that for $G = \mathbb{Z}_p^k$, with $p$ prime odd, and $G = \mathbb{T}^k$ any nontrivial irreducible orthogonal representation is even dimensional and admits the complex structure, $\implies V$ and $W$ admit it too.

Denote $d(V) = \dim \mathbb{C} V = \frac{1}{2} \dim \mathbb{R} V$, and correspondingly $d(W) = \dim \mathbb{C} W = \frac{1}{2} \dim \mathbb{R} W$. 
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For a $G$-map $f : S(V) \to W$ we study the set

$$Z_f := f^{-1}(0) \subset S(V)$$
Let $\mathcal{A}$ be a set of $G$-spaces, $h^*$ a multiplicative equivariant cohomology theory, and $I \subseteq h^*(pt)$ an ideal.

**Definition**

The $(\mathcal{A}, h^*, I)$-length of a $G$-space $X$ is defined to be the smallest integer $k \geq 1$ such that there exist $A_1, \ldots, A_k \in \mathcal{A}$ with the property that for any $\alpha_i \in I \cap \ker [h^*(pt) \to h^*(A_i)]$, $1 \leq i \leq k$,

$$p^*_X(\alpha_1) \sim \cdots \sim p^*_X(\alpha_k) = 0,$$

where $p_X : X \to pt$. 

Bourgin-Yang theorems for the $p$-toral groups
Theorem ([3, Theorem 4.7])

The length has the following properties:

1. If there exists an $h^*$-functorial $G$-equivariant map $X \rightarrow Y$, then $\ell(X) \leq \ell(Y)$.

2. Let $A, B \subseteq X$ be $G$-invariant subspaces such that $h^*(X, A) \times h^*(X, B) \sim h^*(X, A \cup B)$ is defined. If $A \cup B = X$, then $\ell(X) \leq \ell(A) + \ell(B)$.

3. If $h^* = H^*_G$, $I$ is noetherian and $X$ is paracompact, then any $GA = A$, $\text{cl} A = A \subseteq X$ has an open $G$-neighborhood $U \subseteq X$ such that $\ell(U) = \ell(A)$. 
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Depending on the group $G$, we set:

(1) if $G = (\mathbb{Z}_2)^k$: $h^* = H^*_G(-; \mathbb{Z}_2), I = H^*_G(\text{pt}; \mathbb{Z}_2)$,
(2) if $G = (\mathbb{Z}_p)^k, p > 2$: $h^* = H^*_G(-; \mathbb{Z}_p), I = (c_1, \ldots, c_k)$,
(3) if $G = (S^1)^k$: $h^* = H^*(pt; \mathbb{R}), I = H^*(pt; \mathbb{R})$.
Not difficult to repeat and get

\[ \ell(Z_f) \geq \ell(S(V)) - \ell(S(W)) = d(V) - d(W) \]
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For $G = (\mathbb{Z}_2)^k$, or correspondingly $G = \mathbb{Z}_p$, $p > 2$, and $G = S^1$ one can show that any $G$-invariant closed set

$$\ell(Z) \leq \text{coh.dim}Z + 1,$$

or respectively

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For \( G = (\mathbb{Z}_2)^k \), or correspondingly \( G = \mathbb{Z}_p, \ p > 2 \), and \( G = S^1 \) one can show that any \( G \)-invariant closed set

\[ \ell(Z) \leq \text{coh.dim} Z + 1, \ \text{or respectively} \ 2\ell(Z) \leq \text{coh.dim} Z + 1. \]

The case \( G = (S^1)^k \) can be reduced to \( G = S^1 \). However it is not possible to compare \( \ell(Z) \) and \( \text{coh.dim} Z \) for the group \( G = (\mathbb{Z}_p)^k, \ p > 2 \).
Theorem

Let $V, W$ be two orth. representations of $G = \mathbb{Z}_p^k$ or $G = \mathbb{T}^k$ such that $V^G = W^G = \{0\}$. If $f : S(V) \xrightarrow{G} W$ is $G$-map, then

$$\text{coh.dim} Z_f \geq \dim_{\mathbb{R}} V - \dim_{\mathbb{R}} W - 1.$$
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Corollary

Let $G = \mathbb{Z}_p^k$ with $p > 2$, or $G = \mathbb{T}^k$ and $V$, $W$ as above. Then for any $f : S(V) \to W \dim \mathbb{R} V > \dim \mathbb{R} W$ implies $\text{coh.dim} \mathbb{Z}_f \geq 1$.

Indeed, $\dim \mathbb{R} V - \dim \mathbb{R} W - 1 = 2d(V) - 2d(W) - 1$

is integral, positive and odd. \qed
We shall use a most general version of the Borsuk-Ulam theorem by Assadi in [2, page 23] (for $p$-torus) and Clapp and Puppe in [8, Theorem 6.4] for the torus and $p$-torus
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**Theorem**

Let $G$ be a $p$-torus or a torus. Let $X$ and $Y$ be $G$-spaces with fixed-points-free actions; moreover, in the case of a torus action assume additionally that $Y$ has finitely many orbit types. Suppose that $\tilde{H}_j(X) = \tilde{H}_j(X) = 0$ for $j < n$, $Y$ is compact or paracompact and finite-dimensional, and $H_j(Y) = H_j(Y) = 0$ for $j \geq n$. Then there exists no $G$-equivariant map of $X$ into $Y$. 
Proof: Denote $m = \dim_{\mathbb{R}} V$ and $n = \dim_{\mathbb{R}} W$ and suppose

$$\text{coh.dim} Z_f < m - n - 1.$$  

Then,

$$\check{H}^i(Z_f) = 0, \text{ for any } i > m - n - 2.$$
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Then,

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By using Poincaré-Alexander-Lefschetz duality and the long exact sequence of the pair $(SV, SV \setminus Z_f)$, we conclude

$$0 = \check{H}^i(Z_f) = H_{m-1-i}(SV, SV \setminus Z_f) = \check{H}_{m-i-2}(SV \setminus Z_f), \text{ for } j = m-i-2.$$ 

$$\check{H}_j(SV \setminus Z_f) = 0, \text{ for } j < n.$$
On the other hand, we have

\[ H_j(W \setminus \{0\}) = H_j(SW) = 0, \text{ for } j \geq n. \]
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On the other hand, we have

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However,

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is a $G$-equivariant map, which contradicts Theorem 4. In particular, if $\dim \mathbb{R} V > \dim \mathbb{R} W$, for a $G$-map $f : S(V) \to W \setminus \{0\} \subset W$ it implies that $\text{coh.dim} Z_f \geq 0$ and, consequently, $Z_f \neq \emptyset$, which gives a contradiction.\[ \square \]
Let $G = (\mathbb{Z}_2)^k, (\mathbb{Z}_p)^k$ or $(\mathbb{S}^1)^k$, with $k \geq 1$.

- Let $X$ be a $G$-space and a $K$-orientable closed topological manifold such that $\tilde{H}^i(X) = 0$ for $i < n - 1$. 

Bourgin-Yang theorems for the $p$-toral groups
Let $G = (\mathbb{Z}_2)^k, (\mathbb{Z}_p)^k$ or $(\mathbb{S}^1)^k$, with $k \geq 1$.

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- Let $Y$ be a $G$-space and $A \subset Y$ a $G$-subspace such that $Y - A$ is compact (or paracompact and finite dimensional) and $H^i(Y - A) = 0$ for $i \geq m$.

Additionally $(Y - A)^G = \emptyset$.

In the case $p = 0$, i.e. the torus, suppose $Y$ has finitely many orbit type.
**Theorem**

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*If $f : X \to Y$ is a $G$-map, then*

$$\dim f^{-1}(A) \geq n - m - 1.$$
Theorem (Characterization of \( p \)-toral groups, WM/DM/ES (12))

a) Let \( G \) be a \( p \)-toral group \( 1 \hookrightarrow \mathbb{T}^k \to G \to P \to 1 \). Then for the sphere \( S(V) \) of a \( G \)-Hilbert space (orthogonal representation) \( V \), \( V^G = \{0\} \), \( \dim V = \infty \), and finite dimensional orthogonal representation \( W \) of \( G \) such that \( W^G = \{0\} \), and a \( G \)-map \( f : S(V) \to W \) we have \( \dim Z_f = l(Z_f) = \infty \).
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b) If $G$ is not $p$-toral then $\exists$ an infinite-dim. fixed point free $G$-Hilbert space $V$, a finite dimensional representation $W$ of $G$ with $W^G = \{0\}$ and a $G$-map $f : S(V) \rightarrow W$ such that $Z_f = \emptyset$, e.g. $\dim Z_f = -1 < \infty$. 

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An adaptation of proof of a B-U theorem of Bartsch, Clapp & D. Puppe, based on the Borel cohomology of stable cohomotopy theory and used the Segal conjecture (G. Carlson)

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An adaptation of proof of a B-U theorem of Bartsch, Clapp & D. Puppe, based on the Borel cohomology of stable cohomotopy theory and used the Segal conjecture (G. Carlson) \( \widehat{A}(G) = \pi^0_{st}(BG) \).

i) Equivariant cohomology theory \( h^*_G(X) = \omega^0_{st}(X \times_G EG) \) the Borel construction for the stable cohomotopy theory;
An adaptation of proof of a B-U theorem of Bartsch, Clapp & D. Puppe, based on the Borel cohomology of stable cohomotopy theory and used the Segal conjecture (G. Carlson)
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i) Equivariant cohomology theory \( h^*_G(X) = \omega^0_{st}(X \times_G EG) \) the Borel construction for the stable cohomotopy theory;

ii) By the mentioned theorem
\[
 h^*_G(pt) = h^0_G(pt) = \omega^0_{st}(BG) = \pi^0_{st}(BG) = \hat{A}(G),
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where the completion is taken with respect to the ideal $I : = \ker \dim : A(G) \to \mathbb{Z}$.

iii) Next, one should use the fact that the completion map $A(G) \to \hat{A}(G)$ is injective if $P$ is a finite $p$-group (E. Laitinen).
Let $\ell(X)$ be the $G$-length with respect the above $h^*_G(X)$. 

Bourgin-Yang theorems for the $p$-toral groups
Let $\ell(X)$ be the $G$-length with respect the above $h^*_G(X)$

Take $X = S(V)$, $V$ infinite-dimensional Hilbert space. Known [3] that $\ell(X) = \ell(pt)$ if $X$ is contractible $G$-space.
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On the other hand for any finite $G$ there exists an element $\alpha \in A(G)$ such that $\alpha^n \neq 0$ for every $n \in \mathbb{N}$ (T. tom Dieck).

Consequently, its image $\hat{\alpha} \in \widehat{A(G)} = \omega^0_{st}(BG) = h^0_G(pt)$ has the same property for a $p$-group. This shows that $\ell(pt) = \infty$. 
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From it follows that for every $G$-map $f : S(V) \to W$

$$\ell(Z_f) \geq \ell(S(V)) - \ell(S(W)) = \infty$$

as $\ell(S(W)) < \infty$. The remaining task is to adapt it for any toral $p$-group and show that also $\dim Z_f = \infty$.  □
Bourgin-Yang for the cyclic group \( \mathbb{Z}_{p^k}, k \geq 2 \)

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Corollary (already from Bartsch B–U for $G = \mathbb{Z}_{p^k}$)

In particular, if $d(W) < d(V)/p^{k-1}$, then $\dim Z_f \geq 0$, \implies no $G$-equivariant map $f : S(V) \to S(W)$. 
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Example given by T. Bartsch

If $1 \hookrightarrow G_0 \hookrightarrow G \twoheadrightarrow \Gamma \twoheadrightarrow 1$ is a compact Lie group and $\Gamma$ has an element of order $p^2$, then there exist $V, W$ orthogonal repr. $V^G = \{0\} = W^G$, $\dim W < \dim V$ and a $G$-equivariant map $f : S(V) \to S(W)$. 
Corollaries

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In particular if \( G = \Gamma \) a finite group with an element of order \( p^2 \).
Tools

An index of type $i_{\mathbb{Z}_2}$ for a $\mathbb{Z}_{p^k}$ spaces but defined by use of the equivariant $K_G^*$ theory. Its definition, thus the value, depends on the orbits in $X$.

$$l_n(X) = (A_{m,n}, K_G^*, R) - \text{length index of } (X).$$ (2)

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Bourgin-Yang theorems for the $p$-toral groups
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**Bourgin-Yang theorems for the $p$-toral groups**
References


M. Nakaoka, Parametrized Borsuk-Ulam theorems and characteristic polynomials. Topological fixed point theory and Bourgin-Yang theorems for the $p$-torus.


