On the complexity of some packing and covering problems in certain graph classes

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(with various coauthors)
Minimum Vertex Cover

$G = (V,E)$ - a finite undirected graph.

$U \subseteq V$ is a vertex cover of $G$ if for every edge $e$ of $G$, $U$ contains at least one vertex of $e$. 
Minimum Vertex Cover
Minimum Vertex Cover
Problem [GT1] of [Garey, Johnson, 1979]:

**INSTANCE:** Graph $G = (V, E)$, positive integer $K \leq |V|$.

**QUESTION:** Is there a vertex cover of size $K$ or less for $G$?
Vertex cover versus other problems

Observation.

(1) $U$ is a vertex cover of $G \iff V - U$ is an independent vertex set in $G$.

(2) $U$ is an $\subseteq$ - maximal independent set in $G \iff U$ is an independent dominating set in $G$. 
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Graph:

- $x_1$ connected to $y_1$, $y_2$, $y_4$, $y_7$, $y_9$
- $x_3$ connected to $y_2$, $y_4$, $y_7$, $y_9$
- $x_4$ connected to $y_4$, $y_7$, $y_9$
- $x_5$ connected to $y_7$, $y_9$
- $x_7$ connected to $y_9$
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Diagram:

- $x_1$ connected to $y_1$
- $x_3$ connected to $y_2$
- $x_4$ connected to $y_4$
- $x_5$ connected to $y_7$
- $x_7$ connected to $y_9$
Minimum Vertex Cover

**Theorem** [D. König, 1930]

In a bipartite graph $B = (X, Y, E)$, minimum size of a vertex cover = maximum size of a matching.
Maximum Independent Sets

\[ G = (V,E) \] - a finite undirected graph, 
\[ w \] - a \textit{vertex weight} function on \( V \).
\[ U \subseteq V \] is \textit{stable} or \textit{independent} if all vertices of \( U \) are pairwise nonadjacent.
\[ \alpha(G) = \max \text{ size of a stable vertex set in } G \]
\[ \alpha_w(G) = \max \text{ weight of a stable vertex set in } G \]
Problem [GT20] of [Garey, Johnson, 1979]:
INSTANCE: Graph $G=(V,E)$, integer $K$.
QUESTION: Does $G$ contain an independent set of size $K$ or more?
Let $\text{MIS}$ ($\text{MWIS}$, resp.) denote the unweighted (vertex-weighted, resp.) problem.
Problem [SP2] of [Garey, Johnson, 1979]:

**INSTANCE:** A finite set $X$ with $|X| = 3q$ and a collection $C$ of 3-element subsets of $X$.

**QUESTION:** Does $C$ contain an exact cover for $X$, that is, a subcollection $D$ of $C$ such that every element of $X$ occurs in exactly one member of $D$?
Efficient domination

Let $G = (V,E)$ be a finite undirected graph. A vertex $v$ is dominated by itself and its neighbors, i.e., $v$ dominates $N[v]$.

[Bange, Barkauskas, Slater 1988]: $D$ is an efficient dominating (e.d.) set in $G$ if

1. it is dominating in $G$ and
2. every vertex is dominated exactly once.
Efficient domination

Efficient dominating sets in $G$ are also called independent perfect dominating sets.

Let $G^2 = (V, E^2)$ with $xy \in G^2$ if the distance between $x$ and $y$ in $G$ is at most 2.

**Fact.** Let $N(G)$ denote the closed neighborhood hypergraph of $G$. Then:

$$G^2 = L(N(G))$$ holds.
Efficient domination

**Fact.** The following are equivalent for $D \subseteq V$:

1. $D$ is an e.d. set in $G$.
2. $D$ dominating in $G$ and independent in $G^2$.
3. the closed neighborhoods $N[v]$, $v \in D$, are an exact cover of $N(G)$.

**Corollary.** $G$ has an e.d. $\iff N(G)$ has an exact cover.
Efficient domination

Not every graph has an efficient dominating set!
Efficient domination

Not every graph has an efficient dominating set!
Efficient domination

Not every graph has an efficient dominating set!
Efficient domination

Not every graph has an efficient dominating set!
Efficient domination
Efficient Domination (ED) Problem

**INSTANCE:** A finite graph $G = (V, E)$.

**QUESTION:** Does $G$ have an e.d. set?

**Theorem** [Yen, Lee 1996]

The ED Problem is NP-complete for bipartite graphs and for chordal graphs.

Proof by simple reduction from X3C:
Theorem [Lu, Tang 2002]
The ED problem is NP-complete for chordal bipartite graphs.
(proof by complicated reduction from 1-in-3 3SAT)
Efficient domination

Recall: $D$ is an efficient dominating set in $G$ $\iff D$ is dominating in $G$ and independent in $G^2$.

Let $w(v):=|N[v]|$. Then:

(i) $D$ dominating in $G$ $\Rightarrow |V| \leq w(D)$.

(ii) $D$ independent set in $G^2$ $\Rightarrow w(D) \leq |V|$.
Efficient domination

Recall: \( w(v) := |N[v]|. \)

**Fact 1** [Leitert, 2012]  
\( D \) is an e.d. set in \( G \) \( \iff \) \( D \) is a minimum weight dominating set in \( G \) with \( w(D) = |V| \).

**Fact 2** [Leitert; Milanič, 2012]  
\( D \) is an e.d. set in \( G \) \( \iff \) \( D \) is a maximum weight independent set in \( G^2 \) with \( w(D) = |V| \).
Corollary. Let $C$ be a graph class. If the MWIS problem is solvable in polynomial time for $G^2$, for all $G \in C$, then the ED problem is solvable in polynomial time on $C$.

Examples:

dually chordal graphs: squares are chordal.

AT-free graphs: squares are co-comparability.
Corollary. The ED problem is solvable in polynomial time for dually chordal graphs and thus also for strongly chordal graphs.
Efficient domination

Open [Lu, Tang 2002]
Complexity of ED for convex bipartite graphs and for strongly chordal graphs.

Recall:

\( G \) strongly chordal \( \Rightarrow \) \( G \) dually chordal
\( G \) convex bipartite \( \Rightarrow \) \( G \) interval bigraph
\( \Rightarrow \) \( G \) chordal bipartite
Efficient domination

**Theorem** [Bui-Xuan, Telle, Vatshelle, 2011]
If for a graph class, boolean width is at most $O(\log n)$ then the Minimum Weight Dominating Set problem can be solved in polynomial time.

**Theorem** [Keil, 2012] Boolean width of interval bigraphs is at most $2 \log n$. 
Efficient domination

Corollary.
For interval bigraphs, the ED problem can be solved in polynomial time.

Recall:
\[ G \text{ convex bipartite } \Rightarrow G \text{ interval bigraph} \]
Efficient edge domination

[Grinstead, Slater, Sherwani, Holmes, 1993]:
$M \subseteq E$ is an efficient edge dominating (e.e.d.) set in $G$ if $M$ is dominating in $L(G)$ and every edge of $E$ is dominated exactly once in $L(G)$ (that is, $M$ is an efficient dominating set in $L(G)$).
Not every graph (not every tree !) has an efficient edge dominating set:
G:

\[ L(G): \]
**Efficient Edge Domination (EED) Problem**

**INSTANCE:** A finite graph $G = (V, E)$.

**QUESTION:** Does $G$ have an e.e.d. set?

**Theorem** [Grinstead, Slater, Sherwani, Holmes, 1993]

The EED Problem is NP-complete.
Efficient edge domination

Efficient edge dominating sets are also called *dominating induced matchings* (*d.i.m.*):

**Fact.** $M$ is an e.e.d. in graph $G = (V,E) \iff$

1. $M$ is an *induced matching* in $G$ (that is, pairwise distance of edges in $M$ at least 2),
2. every edge in $E$ is intersected by exactly one edge from $M$. 
Efficient edge domination


EED is NP-complete for bipartite graphs, and is solvable in linear time for bipartite permutation graphs, generalized series-parallel graphs and for chordal graphs.
Efficient edge domination

EED is NP-complete for bipartite graphs, and is solvable in linear time for bipartite permutation graphs, generalized series-parallel graphs and for chordal graphs.

**Open** [Lu, Ko, Tang 2002]
Complexity of EED for weakly chordal graphs and for permutation graphs.
Efficient edge domination

Theorem [Cardoso, Lozin 2008]
EED is NP-complete for (very special) bipartite graphs, and is polynomial time solvable for claw-free graphs.
Efficient edge domination

Open [Cardoso, Korpelainen, Lozin 2011]

Complexity of EED for

- $P_k$–free graphs, $k > 4$
- chordal bipartite graphs
- weakly chordal graphs
Efficient edge domination

**Theorem** [B., Hundt, Nevries 2009, LATIN 2010] The EED problem is solvable in
- linear time for chordal bipartite graphs,
- polynomial time for hole-free graphs, and
- is NP-complete for planar bipartite graphs with maximum degree 3.
Theorem [B., Mosca, ISAAC 2011] EED in linear time for $P_7$-free graphs in a robust way.
Efficient edge domination

**Theorem [B., Mosca, ISAAC 2011]**

EED in linear time for P₇-free graphs in a robust way.

**EED in Monadic Second Order Logic:**

**Fact.** $G = (V, E)$ has an e.e.d. $\iff$

$\exists E' \subseteq E \ \forall e \in E \ \exists! e' \in E' \ (e \cap e' \neq \emptyset)$
**Efficient edge domination**

**Recall:** $D$ is an efficient edge dominating set in $G \iff D$ is dominating in $L(G)$ and independent in $L(G)^2$, i.e., $D$ is an e.d. set in $L(G)$.

Let $w(e) := |N[e]|$ (neighborhood w.r.t. $L(G)$).

**Fact.** $M$ is an efficient edge dominating set in $G \iff M$ is a maximum weight independent set in $L(G)^2$ with $w(M) = |E|$. 
Squares of Line Graphs

- $G$ chordal $\Rightarrow L(G)^2$ chordal [Cameron 1989]
- $G$ circular-arc $\Rightarrow L(G)^2$ circular-arc [Golumbic, Laskar 1993]
- $G$ co-comparability $\Rightarrow L(G)^2$ co-comparability [Golumbic, Lewenstei 2000]
- $G$ weakly chordal $\Rightarrow L(G)^2$ weakly chordal [Cameron, Sritharan, Tang 2003]
- stronger result for AT-free graphs [J.-M. Chang 2004]
Efficient edge domination

**Recall:** If the MWIS problem is solvable in polynomial time for the squares of the line graphs of all graphs in $C$ then the EED problem is solvable in polynomial time on $C$.

**Corollary.**
EED in polynomial time for weakly chordal graphs and for permutation graphs.
ED for hypergraphs

\[ H = (V, E) \] - a finite hypergraph.

\( D \subseteq V \) is an e.d. set in \( H \) if \( D \) is an e.d. set in \( 2\text{sec}(H) \).

**Theorem.** The ED problem is NP-complete for \( \alpha \)-acyclic hypergraphs, and is solvable in polynomial time for hypertrees.
EED for hypergraphs

\[ H = (V, E) \] - a finite hypergraph.

\( M \subseteq E \) is an e.e.d. set in \( H \) if \( M \) is an e.e.d. set in \( L(H) \).

**Theorem.** The EED problem is solvable in polynomial time for \( \alpha \)-acyclic hypergraphs, and is NP-complete for hypertrees.
Maximum induced matchings for hypergraphs

$H = (V,E)$ - a finite hypergraph.

$M \subseteq E$ is an *induced matching* in $H$ if $M$ is an independent node set in $L(H)^2$.

**Theorem.** The MIM problem is solvable in polynomial time for $\alpha$–acyclic hypergraphs, and is NP-complete for hypertrees.
Theorem. The Exact Cover problem is NP-complete for \( \alpha \)-acyclic hypergraphs, and is solvable in polynomial time for hypertrees.
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Thank you for your attention!
Exact Cover

Let $H = (V, E)$ be a hypergraph, $C$ a collection of hyperedges, and let $w(e) := |e|$. If the edges in $C$ are pairwise disjoint then $w(C) \leq |V|$.  

**Fact.** The following are equivalent:
- $C$ is an exact cover of $V$.
- edges in $C$ pairwise disjoint and $w(C) = |V|$.
- $C$ maximum independent in $L(H)$, $w(C) = |V|$. 

Corollary. Exact Cover is solvable in polynomial time for every class $C$ of hypergraphs for which MWIS is solvable in polynomial time on the line graphs of $C$.

Example. Exact Cover is solvable in polynomial time for hypertrees: Their line graphs are chordal.
A hypergraph $H$ is a **hypertree** if there is a tree $T$ such that every hyperedge of $H$ induces a connected subgraph in $T$.

**Theorem** [Duchet, Flament, Slater 1976] $H$ is a hypertree $\iff H$ is Helly and $L(H)$ is chordal.
Dually chordal graphs

Let \( N(G) \) denote the *closed neighborhood hypergraph* of graph \( G \).

A graph \( G \) is *dually chordal* if \( N(G) \) is a hypertree.

**Fact.**

\[ G^2 = L(N(G)). \]
Dually chordal graphs

**Theorem** [B., Dragan, Chepoi, Voloshin 1994]

$G$ is dually chordal $\iff G^2$ is chordal and $N(G)$ has the Helly property $\iff$ its clique hypergraph is a hypertree.

(and various other characterizations in [BDCV 1994], [Szwarcfiter, Bornstein 1994], [Gutierrez, Oubina 1996] …)
Dually chordal graphs

**Theorem** [B., Dragan, Chepoi, Voloshin 1994]

$G$ is strongly chordal $\iff G$ is hereditarily dually chordal.
Thank you for your attention!
Thank you for your attention!
Thank you for your attention!
Strongly chordal graphs

$G$ is *strongly chordal* if $G$ is sun-free (i.e., $S_k$-free for any $k \geq 3$) and chordal.

**Theorem.**

$G$ is strongly chordal $\iff$ it is hereditarily dually chordal.

(and many other characterizations …)
Theorem [Lubiw 1982; Dahlhaus, Duchet 1987; Raychaudhuri 1992] For every $k \geq 2$: $G$ strongly chordal $\Rightarrow G^k$ strongly chordal.
Tree Structure of Hypergraphs

Let \( H = (V, E) \) be a hypergraph.

\( T \) is a join tree of \( H \) if \( E \) is the set of nodes of \( T \) and for all vertices \( v \in V \), the occurrences of \( v \) in nodes of \( T \) form subtrees of \( T \).

\( H \) is \( \alpha \)-acyclic if \( H \) has a join tree.
Chordal graphs

**Theorem** [Walter, Gavril, Buneman 1972]
$G$ is chordal $\iff G$ is the intersection graph of subtrees of a tree.

**Corollary.** $G$ is chordal $\iff$ the hypergraph of its $\subseteq$-maximal cliques has a join tree (which is called *clique tree* for graphs).
\[ \text{Diagram} \]

\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \]
Let $H = (V, E)$ be a hypergraph. 

$H$ is \textit{conformal} if every maximal clique of its 2-section graph is contained in a hyperedge. 

\textbf{Theorem} [Duchet, Flament, Slater 1976] 

$H$ is $\alpha$–acyclic $\iff$ $H$ is conformal and its 2-section graph is chordal.
Fact. $H$ is a hypertree $\iff$ its dual is $\alpha$–acyclic.

Theorem [Duchet, Flament, Slater 1976]

$H$ is a hypertree $\iff H$ is Helly and its line graph is chordal.