Invariant subspace problem and hyperreflexivity

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It is well-known that any matrix $A \in \mathbb{C}^{n\times n}$ is similar to its Jordan canonical form, or equivalently there exists an ordered basis of $C^{n\times 1}$ consisting of (chains of) generalized eigenvectors of A. This basis generates a system of subspaces invariant for the operator (matrix) A. If, moreover, $A^*A = AA^*$, i.e. A is a normal matrix, then A is unitarily equivalent to a diagonal matrix. The latter result has a generalization to infinite dimensional complex Hilbert spaces, the well-known spectral theorem.

The set of all closed subspaces invariant for a given operator T forms a lattice $\operatorname{Lat} T$. Generally, it is not known whether there exists a bounded linear operator T on a complex separable Hilbert H space for which

$$\operatorname{Lat} T = \big\{\{0\}, H\big\}.$$

This question, open since 1930th, is known as the *Invariant subspace problem*.

The closure in weak operator topology of the set of all polynomials in an operator T is denoted by $\operatorname{Alg} T$. Clearly $A \in \operatorname{Alg} T \implies \operatorname{Lat} T \subset \operatorname{Lat} A$. If there exist many invariant subspaces for T in the sense

$$\{A\colon \operatorname{Lat} T \subset \operatorname{Lat} A\} = \operatorname{Alg} T$$

then T is said to be reflexive (Sarason, 1966). The notion of reflexive operator was generalized to linear subspaces of operators Shulman, 1972), and a stronger quantitative notion, hyperreflexivity, was defined and studied first by Arveson (1975).

In the lecture, first, some known examples of operators for which the existence of non-trivial invariant spaces is known will be considered. Then reflexivity and hyperreflexivity in finite dimensional spaces will be considered in more detail.