Pythagorean-hodograph curves

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Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ be a planar polynomial parametric curve

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix},$$

where

$$x(t) = \sum_{k=0}^{n} a_k \, t^k \quad \text{and} \quad y(t) = \sum_{k=0}^{n} b_k \, t^k$$

are polynomials of degree $\leq n$.

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Curve **r** is a Pythagorean-hodograph curve, if there exists a polynomial σ, such that

$$x'(t)^2 + y'(t)^2 = \sigma(t)^2.$$

Such a curve will be (shortly) called a PH curve.

PH curves have several important properties:

- If a curve is a regular one (the derivative never vanishes), then

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- Their arc length

$$\int_a^b \|r'(t)\| \, dt = \int_a^b \sigma(t) \, dt$$

can be computed exactly (without quadrature rules).

- Unit tangent **t** is a rational curve

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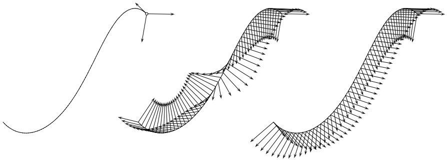
Application:

- Any orthonormal adapted frame (first vector is a unit tangent) of a spatial curve may define the motion of a rigid body.
- It is important to have a rational rotation minimizing frame.
- To have a rational unit tangent, the curve must be a PH curve.



- Their offset curves

$$\mathbf{r}_d(t) := \mathbf{r}(t) + d\,\mathbf{n}(t),$$

where $d > 0$ and $\mathbf{n}(t)$ is a unit normal to $\mathbf{r}$ at parameter t,

$$\mathbf{n}(t) := \frac{(-y'(t), x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}},$$

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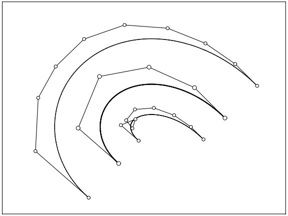
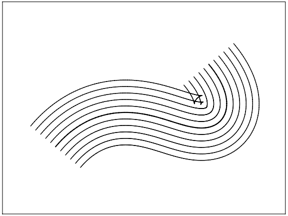
  - If the degree of $\mathbf{r}$ is n, then the degree of $\mathbf{r}_d$ is $2n - 1$.

Theorem (Kubota)

The Pythagorean condition

$$a^2(t) + b^2(t) = c^2(t)$$

is satisfied by polynomials a, b and c iff there exist polynomials u, v and w, such that

$$\begin{aligned} a(t) &= w(t)\ \left(u^2(t) - v^2(t)\right), \\ b(t) &= 2\,w(t)\,u(t)\,v(t), \\ c(t) &= w(t)\ \left(u^2(t) + v^2(t)\right), \end{aligned}$$

where u and v are relatively prime.

From the theorem it follows that the hodograph of a PH curve has to be of the form

$$x'(t) = w(t)\ \left(u^2(t) - v^2(t)\right),$$
$$y'(t) = 2\,w(t)\,u(t)\,v(t).$$

(the order is not important, since PH curves are rotational invariant).

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- $w \neq 0$ and $u = \pm v$ or at least one of u, v equals zero; PH curve is nonuniformly parameterized straight line.

Therefore, we will only consider cases, where $u, v, w \neq 0$, and u, v not both constants.

Such PH curves are of degree at least 3.

PH curves have clearly less degrees of freedom than general polynomial curves

Lema

PH curve of degree n has (at most) $n+3$ degrees of freedom ($n-1$ less than general polynomial curve of degree n).

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Conjecture:

PH curve can interpolate $n+1$ points in the geometric sense.

For prescribed parameters PH curve can interpolate at most $\lfloor (n+3)/2 \rfloor$ points.

- First interesting case are cubic PH curves, where $w = 1$, $\max\{\deg(u), \deg(v)\} = 1$.
- Non-constant $w(t)$ may produce irregular curves, if w has real roots within the curve parameter domain. Therefore, we take $w(t) = 1$. Then the PH curve is always of odd degree.

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- Let us write u and v in the Bernstein-Bézier form

$$u(t) = u_0\, B_0^1(t) + u_1\, B_1^1(t),$$
$$v(t) = v_0\, B_0^1(t) + v_1\, B_1^1(t),$$

where

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k}\, t^k$$

and assume $u_0 : u_1 \neq v_0 : v_1$.

We obtain the cubic PH curve

$$\mathbf{r}(t) = (x(t), y(t))^T = \sum_{k=0}^{3} \mathbf{p}_k \, B_k^n(t)$$

as

$$x(t) = \int_0^t \left(u^2(t) - v^2(t)\right) \, dt,$$
$$y(t) = \int_0^t 2\, u(t)\, v(t)\, dt.$$

Theorem

Control points of a cubic Bézier PH curve satisfy

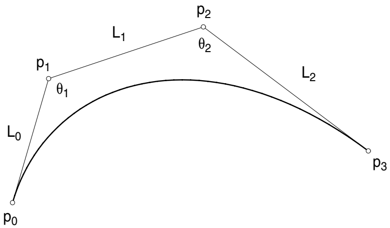
$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{3}\left(u_0^2 - v_0^2, 2\,u_0\,v_0\right)^T,$$
$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{3}\left(u_0\,u_1 - v_0\,v_1, u_0\,v_1 + u_1\,v_0\right)^T,$$
$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{1}{3}\left(u_1^2 - v_1^2, 2\,u_1\,v_1\right)^T,$$

where $\mathbf{p}_0$ *is an arbitrary control point, satisfying constants from integration.*

Theorem (Characterization)

Let $\mathbf{r}$ *be a cubic Bézier curve with control points* $\mathbf{p}_k$*,* $k = 0, \ldots, 3$*. Moreover let* $L_i := \|\Delta \mathbf{p}_i\|$*,* $i = 0, 1, 2$ *and let* θ_1 *and* θ_2 *be the control polygon angles at the interior vertices* $\mathbf{p}_1$ *and* $\mathbf{p_2}$*. Then* $\mathbf{r}$ *is a PH curve iff*

$$L_1 = \sqrt{L_0 \, L_2} \quad \text{and} \quad \theta_1 = \theta_2.$$



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Let us identify

$$(x, y)^T \in \mathbb{R}^2 \to x + \mathbf{i}\, y.$$

Planar curve **r** therefore becomes

$$\mathbf{r}(t) = (x(t), y(t))^T \to \mathbf{z}(t) := x(t) + \mathbf{i}\, y(t).$$

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Notation:

- $\Pi \ldots$ regular polynomial curves in $\mathbb{R}^2$,
- $\widehat{\Pi} \ldots$ regular PH curves in $\mathbb{R}^2$.

- Clearly

$$\widehat{\Pi} \subset \Pi,$$

since there exist regular curves, which are not PH curves, e.g.,

$$\mathbf{z}(t) = t + \mathrm{i}\, t^2.$$

- Let us define the mapping $\mathcal{P} : \mathbf{z}(t) \to \widehat{\mathbf{z}}(t)$ as:
  - $\mathbf{w}(t) := \mathbf{z}'(t)$,

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  - $\mathbf{w}(t) := \mathbf{z}'(t)$,
  - $\widehat{\mathbf{w}}(t) := \mathbf{w}^2(t)$,
  - $\widehat{\mathbf{z}}(t) := \int \widehat{\mathbf{w}}(\tau)\, d\tau$.

  Here we assume $\mathbf{z}(0) = \widehat{\mathbf{z}}(0) = 0$.

Theorem

Mapping $\mathcal{P}$ is a bijection from Π to $\widehat{\Pi}$.

Remark: Sets Π and $\widehat{\Pi}$ have the same cardinality.

Lemma:

If n is the degree of $\mathbf{z}$ and $\widehat{n}$ is the degree of $\widehat{\mathbf{z}}$, then

$$\widehat{n} = 2\,n - 1\,.$$

Corollary:

There are no regular PH curves of even degree.

Let $\mathbf{r} : [a, b] \to \mathbb{R}^3$ be a spatial polynomial parametric curve

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}.$$

Curve $\mathbf{r}$ is a spatial Pythagorean-hodograph curve, if there exists a polynomial σ, such that

$$x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma(t)^2.$$

Theorem (Kubota)

The Pythagorean condition

$$a^2(t) + b^2(t) + c^2(t) = d^2(t)$$

is satisfied by relatively prime polynomials a, b, c and d iff there exist polynomials u, v, p and q, such that

$$\begin{aligned} a(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t), \\ b(t) &= 2\left[u(t)\,q(t) + v(t)p(t)\right], \\ c(t) &= 2\left[v(t)\,q(t) - u(t)p(t)\right], \\ d(t) &= u^2(t) + v^2(t) + p^2(t) + q^2(t). \end{aligned}$$

From the theorem it follows that the hodograph of a spatial PH curve has to be of the form

$$x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t),$$
$$y'(t) = 2\left[u(t)\,q(t) + v(t)p(t)\right],$$
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If the polynomials u, v, p, q are of degree n, then the PH curve is of degree $2n+1$.

Spatial PH curves are connected with quaternions.

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Quaternions:

- They form a 4-dimensional vector space $\mathbb{H}$ with standard basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$,

$$\mathbf{1} = (1,0,0,0),\ \mathbf{i} = (0,1,0,0),\ \mathbf{j} = (0,0,1,0),\ \mathbf{k} = (0,0,0,1).$$

- First component is a scalar part, other three components form a vector part.
- Let $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$, $\ a, b \in \mathbb{R}$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, then

$$\mathcal{A} + \mathcal{B} = (a + b, \mathbf{a} + \mathbf{b}), \quad \mathcal{AB} = (ab - \mathbf{a} \cdot \mathbf{b}, a\mathbf{b} + b\mathbf{a} - \mathbf{a} \times \mathbf{b}).$$

- $\mathcal{A}^* := (a, -\mathbf{a})$

- Let $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k} \in \mathbb{H}$.
- Then

$$\begin{aligned}\mathbf{r}'(t) &= \mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t)\\ &= \left[u^2(t) + v^2(t) - p^2(t) - q^2(t)\right]\mathbf{i}\\ &+ 2\left[u(t)\,q(t) + v(t)p(t)\right]\mathbf{j}\\ &+ 2\left[v(t)\,q(t) - u(t)p(t)\right]\mathbf{k}.\end{aligned}$$

Procedure:

$$\mathbf{r}(t) := \int_0^t \mathcal{A}(t)\mathbf{i}\mathcal{A}^*(t)\,dt$$

Thank you!