When are two graphs really the same?

Ted Dobson

University of Primorska

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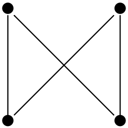

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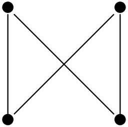

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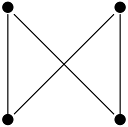

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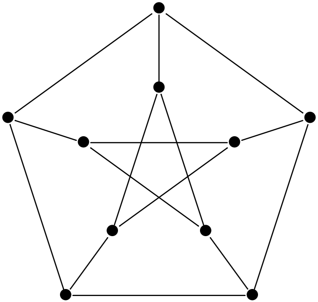

Figure: The Petersen graph

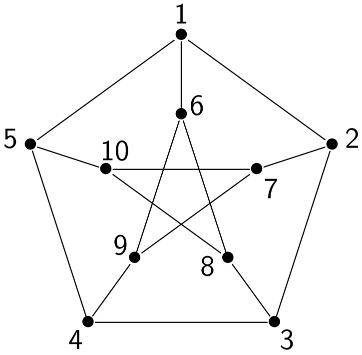

Figure: The Petersen graph with vertices labeled

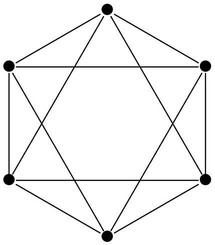

Figure: Octahedron

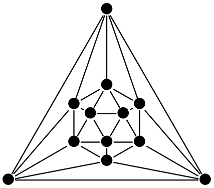

Figure: Icosahedron

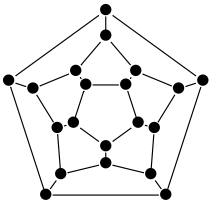

Figure: Dodecahedron

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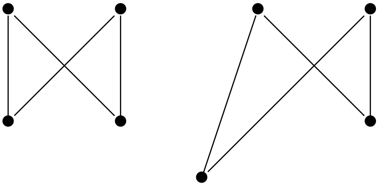

Figure: Two drawings of the same graph

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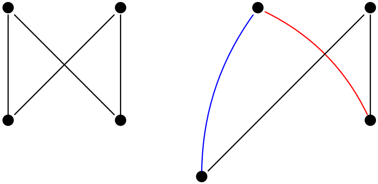

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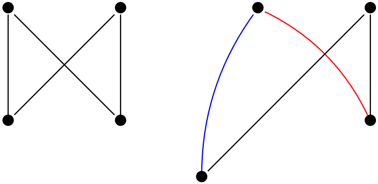

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So we can draw an edge in any shape, length or color, and that does not change the graph.

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Formally, we say that two graphs that are really the same are **isomorphic**. The roots of this word are Greek, where iso means "same", and morph means "form or shape".

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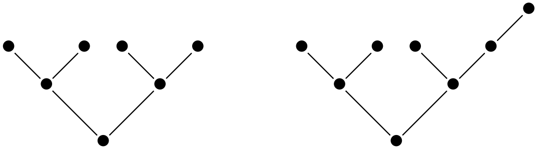

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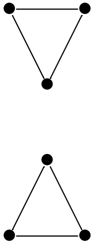
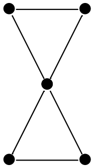


Figure: A graph that is two pieces



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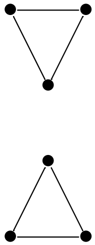

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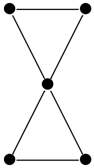

Figure: A graph that is one piece

Notice that the number of vertices of these two graphs are different. These graphs differ in other structural properties, and so there are other reasons why these graphs are not isomorphic.

A definition

A **cycle** in a graph is a sequence of vertices, say, $v_1, v_2, \ldots, v_r, v_1$ which, except for the first and last vertices, are all different, and there is an edge in the graph between successive pairs (i.e. there is an edge between v_1 and v_2, v_2 and v_3, etc., and v_r and v_1). Such a cycle is usually called an r-**cycle**. Here are some examples:

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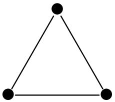


Figure: A 3-cycle or triangle

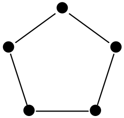
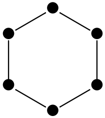


Figure: A 5-cycle or pentagon



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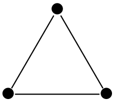

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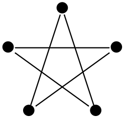

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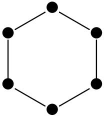

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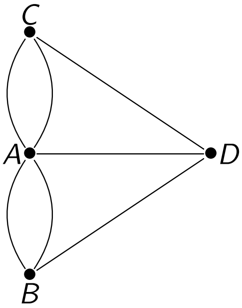

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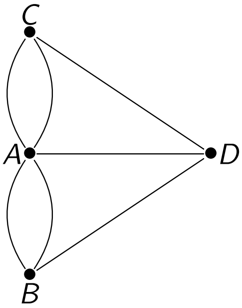

The valency of A is 5, and the valencies of B, C, and D are 3.

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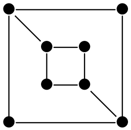
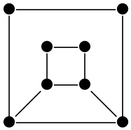


Figure: A graph with degree sequence $(2, 2, 2, 2, 3, 3, 3, 3)$



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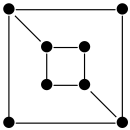

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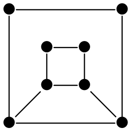

Figure: Another graph with degree sequence $(2, 2, 2, 2, 3, 3, 3, 3)$

These graphs are not isomorphic as on the right hand graph every vertex of degree 3 forms a cycle while on the left hand side they do not.

Focus on cycles

If none of the previous techniques we have discussed give that a pair of graphs are not isomorphic, it is probably time to focus on some small graph that is part of both the graphs and see if this can help. It is usually a good idea to choose something that is not too small (there could be too many of them to do much good) and not too large (they can be hard to find).

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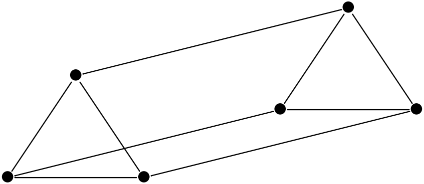

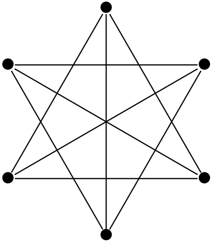

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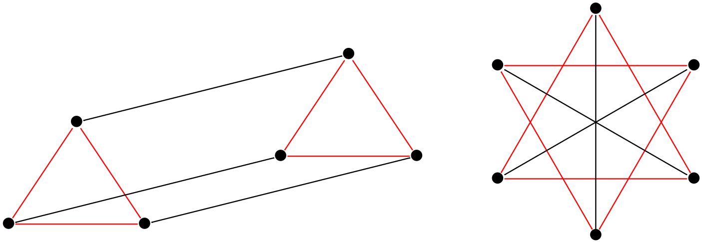

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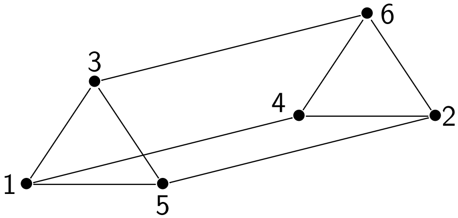

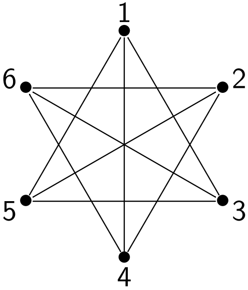

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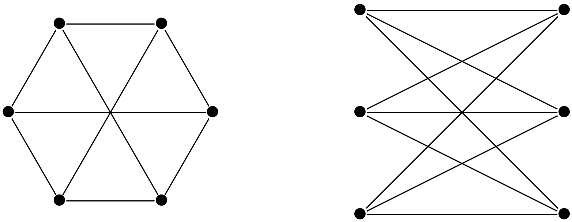

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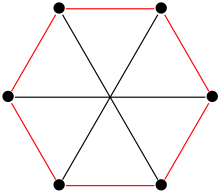

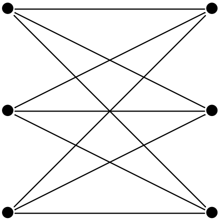

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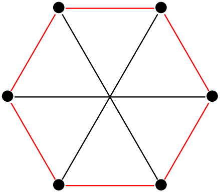

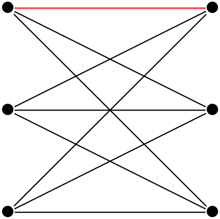

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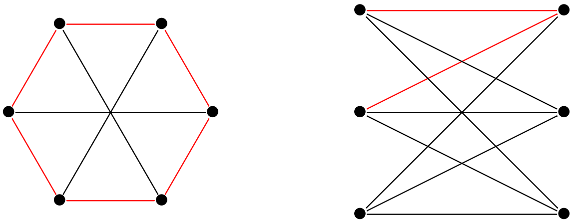

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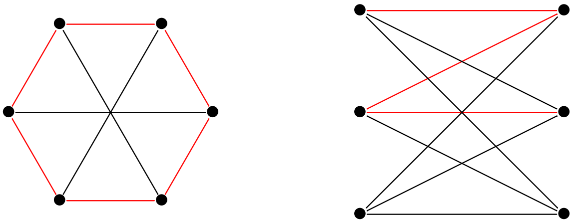

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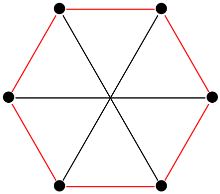

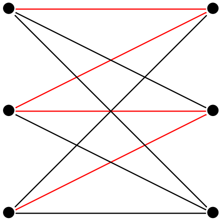

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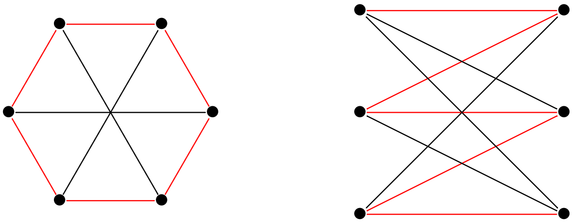

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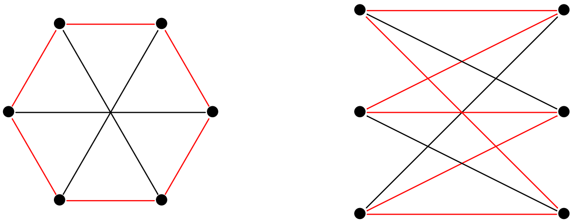

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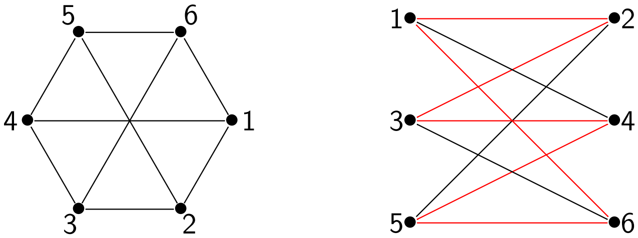

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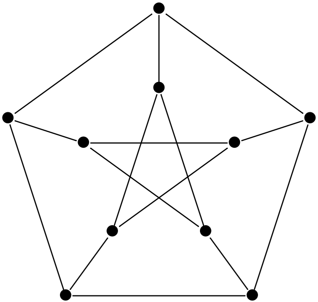

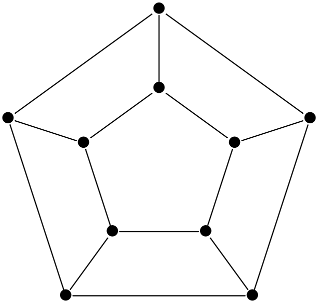

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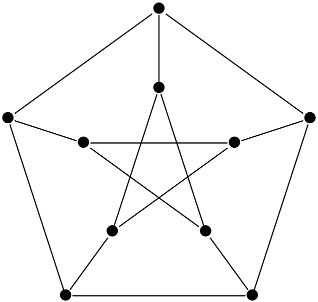

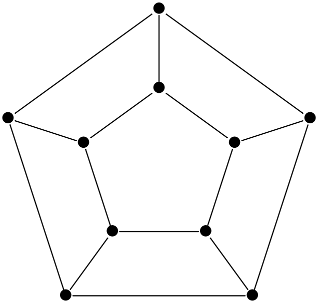

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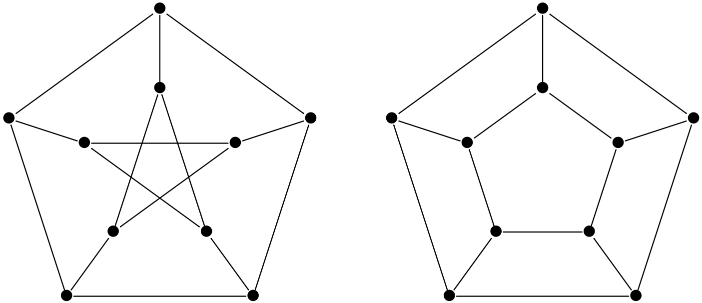

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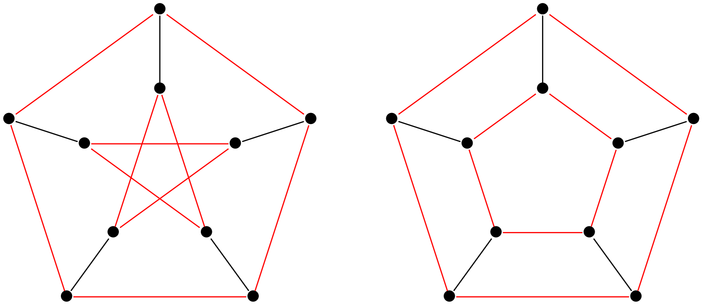

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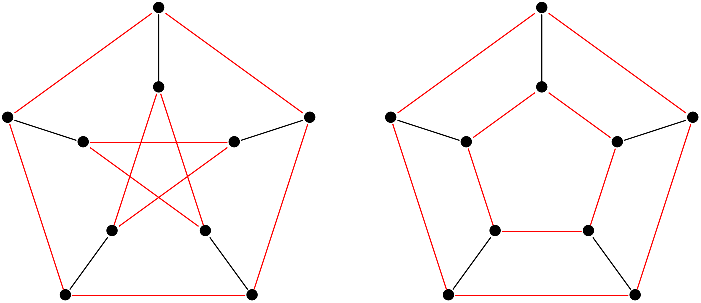

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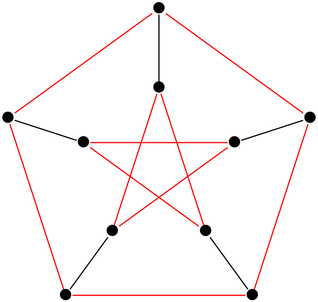
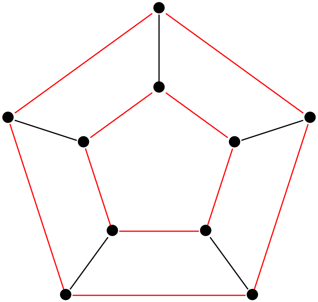
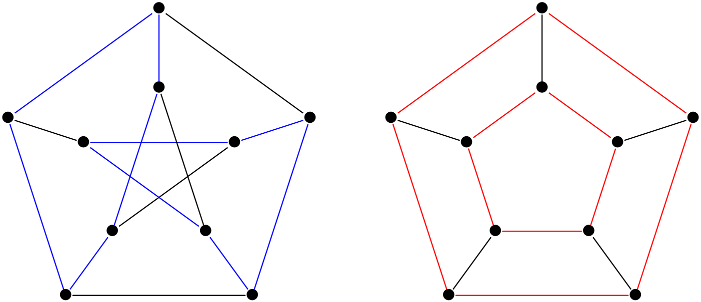




Here, we focus on 5-cycles. Both graphs have an obvious pair of 5-cycles. The graph on the right only has two 5-cycles. The Petersen graph on the left has twelve 5-cycles. We show two more.

Are these two graphs isomorphic?



Here, we focus on 5-cycles. Both graphs have an obvious pair of 5-cycles. The graph on the right only has two 5-cycles. The Petersen graph on the left has twelve 5-cycles. We show two more. So these graphs are not isomorphic.

Where is the hard part?

If you think back to the examples we have seen, to show two graphs are NOT isomorphic, we look for distinguishing features of the graphs, and then compare them.

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What is the state of the art?

In 2015, Lásló Babai used the framework developed by Luks to show that the graph isomorphism problem could be solved not quickly, but not really slowly (technically, he showed the problem could be solved in quasipolynomial time).

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Thanks!