Regular polytopes and almost simple groups

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Chapter 1

Basic Theory

1.1 Posets

A *partially ordered set* $(\mathcal{P}, \leq)$ is a set $\mathcal{P}$ equipped with a binary relation $\leq$ which is

- *reflexive*: $a \leq a$ for all $a \in \mathcal{P}$;
- *antisymmetric*: if $a \leq b$ and $b \leq a$ then $a = b$; and
- *transitive*: if $a \leq b$ and $b \leq c$ then $a \leq c$.

Examples include the real numbers under their standard ordering, subsets of a set under inclusion, and the positive integers under divisibility.

We define $a < b$ to mean that $a \leq b$ but $a \neq b$. If $a \leq b$ or $b \leq a$, then a and b are said to be *comparable*, and otherwise *incomparable*. Any subset of $\mathcal{P}$ in which every pair of elements is comparable is called a *chain*, or *linear order*, while any subset in which no two distinct elements are comparable is called an *antichain*.

An element $x \in \mathcal{P}$ is called a *maximum* element of $\mathcal{P}$ if $a \leq x$ for all $a \in \mathcal{P}$, or a *minimum* element of $\mathcal{P}$ if $x \leq a$ for all $a \in \mathcal{P}$. Weaker but still important are these: an element $x \in \mathcal{P}$ is called a *maximal* element of $\mathcal{P}$ if there is no element $a \in \mathcal{P}$ apart from x for which $x \leq a$, or a *minimal* element of $\mathcal{P}$ if there is no element $a \in \mathcal{P}$ apart from x for which $a \leq x$. Every maximum element is maximal (and every minimum element is minimal), but not vice versa.

1.2 Abstract polytopes

We now follow [21]. The propositions/theorems are numbered as in [21] to facilitate the connection with the book.

An *abstract polytope* is a partially ordered set $(\mathcal{P}, \leq)$ with special properties that hold for certain solid geometric objects (known as geometric polytopes).

(P1) The polytope has a maximum element (which is usually called $\mathcal{P}$ as well), and a minimum element, which is often denoted by ϕ (like the empty set).

All other elements of the polytope are called *faces* of the polytope, and they occur in 'layers'. Specifically, for some positive integer n there is a *rank* function $r : \mathcal{P} \to \{-1, 0, 1, 2, \ldots, n\}$, such that $r(\phi) = -1$ and $r(\mathcal{P}) = n$, and $r(x) < r(y)$ whenever $x < y$ in $\mathcal{P}$.

The elements of rank k are called the k-faces of $\mathcal{P}$. In particular, the elements of rank 0 are called the *vertices*, the elements of rank 1 are called the *edges*, and the elements of rank $n-1$ are called the *facets* of the polytope $\mathcal{P}$.

(P2) Every maximal chain in $\mathcal{P}$ is required to have length $n+2$ and therefore be of the form

$$\phi \ < \ x_0 \ < \ x_1 \ < \ \ldots \ < \ x_{n-1} \ < \ \mathcal{P}$$

where each x_k is a k-face. Every such maximal chain is called a *flag* of P.

We denote by $\mathcal{F}(\mathcal{P})$ the set of flags of $\mathcal{P}$.

To complete the definition of an abstract polytope, we require two more properties, which depend on the following concepts. We say that two flags are *adjacent* if one can be obtained from the other by replacing just one of its faces; and if x and z are elements of $\mathcal{P}$, then we call the set $[x, z] = \{y \in \mathcal{P} : x \leq y \leq z\}$ a *section* of $\mathcal{P}$.

(P3) The polytope $\mathcal{P}$ must be *strongly connected*, which means that in any section $\mathcal{Q}$ of $\mathcal{P}$, for any two faces F and G, there exists a finite sequence of proper faces $F = H_0, \ldots, H_k = G$ such that H_{i-1} and H_i are comparable for each $i = 1, \ldots k$.

We say that a polytope $\mathcal{P}$ is *flag-connected* if any flag F' can be obtained from any other flag F by a sequence $(F_0, F_1, F_2, \ldots, F_m)$ of flags where $F_0 = F$ and $F_m = F'$, and the flag F_{i-1} is adjacent to the flag F_i for $1 \leq i \leq m$.

A polytope $\mathcal{P}$ is *strongly flag-connected* if each section of $\mathcal{P}$ is flag-connected.

Proposition 2A1 Let $\mathcal{P}$ be a poset with properties (P1) and (P2). Then $\mathcal{P}$ is strongly connected if and only if it is strongly flag-connected.

$\Rightarrow$ **(P3)'** The polytope $\mathcal{P}$ must be *strongly flag-connected*.

(P4) The polytope $\mathcal{P}$ must satisfy the *diamond condition*, which says that for each rank k, if x and z are elements with $x \leq z$ and $r(x) = k-1$ and $r(z) = k+1$, then there are exactly two elements of rank k in the section $[x,z]$; in other words, the section $[x,z]$ is like a diamond, with top element z, bottom element x, and two intermediate elements y and y'.

If $n \geq 1$, then, for each $0 \leq j \leq n-1$, and each flag Φ of $\mathcal{P}$, we denote by Φ^j the flag adjacent to Φ that differs in the j-face. By extension, for $r \geq 2$, we write

$$\Phi^{j(1)\ldots j(r-1)j(r)} := \left(\Phi^{j(1)\ldots j(r-1)}\right)^{j(r)}$$

The following lemma summarises basic facts about flags.

Lemma 2A2 Let Φ be a flag of an abstract n-polytope $\mathcal{P}$, and let $0 \leq j \leq k \leq n-1$. Then

1. $\Phi^{jj} = \Phi$;

2. $\Phi^{jk} = \Phi^{kj}$ for $k \geq j+2$.

For example, an abstract polytope of rank 2 is a standard polygon. Note that every edges has exactly two vertices, and every vertex lies is exactly two edges. Similarly, an abstract polytope of rank 3 is a non-degenerate map, whose faces are the elements of rank 2. (By non-degenerate, we mean that the underlying graph is simple, and every edges lies in two faces.)

We usually write $\mathcal{P}$ instead of $(\mathcal{P}, \leq)$.

Let $(\mathcal{P}, \leq)$ and $(\mathcal{Q}, \leq')$ be two abstract polytopes.

A map $\varphi : \mathcal{P} \to \mathcal{Q}$ between the element-sets of two polytopes $\mathcal{P}$ and $\mathcal{Q}$ is called a *homomorphism* if it preserves *incidence* (i.e. $F \leq G$ in $\mathcal{P}$ implies $F\varphi \leq' G\varphi$ in $\mathcal{Q}$).

An *isomorphism* $\varphi : (\mathcal{P}, \leq) \to (\mathcal{Q}, \leq')$ is a bijection for which both φ and φ^{-1} are homomorphisms.

$\mathcal{P}$ and $\mathcal{Q}$ are said to be *isomorphic* if there exists an isomorphism from one to the other. In that case we write $\mathcal{P} \cong \mathcal{Q}$.

An *automorphism* of a polytope $\mathcal{P}$ is an isomorphism of $\mathcal{P}$ onto itself. The set of all automorphisms of a polytope $\mathcal{P}$ together with the composition law forms a group called the *automorphism group* $\Gamma(\mathcal{P})$. Clearly, if $\mathcal{P}$ is finite, then $\Gamma(\mathcal{P})$ is also finite.

A bijection $\varphi : \mathcal{P} \to \mathcal{Q}$ is a *duality* if both φ and φ^{-1} reverse incidence. We then call $\mathcal{P}$ and $\mathcal{Q}$ *duals* of each other.

A polytope $\mathcal{P}$ is *self-dual* if $\mathcal{P} \cong \mathcal{P}^*$ where $\mathcal{P}^*$ is the dual polytope of $\mathcal{P}$.

The automorphisms of a polytope are defined as permutations of its face-set. Sometimes other representations will be used as we will see further in this course.

Proposition 2A4 For each polytope $\mathcal{P}$, the group $\Gamma(\mathcal{P})$ acts freely on the flags of $\mathcal{P}$.

1.3 Abstract regular polytopes

An *abstract regular polytope* is an abstract polytope $(\mathcal{P}, \leq)$ such that $\Gamma(\mathcal{P})$ is transitive on $\mathcal{F}(\mathcal{P})$.

Proposition 2B2 If $\mathcal{P}$ is a finite abstract regular polytope, then $\mid \Gamma(\mathcal{P}) \mid = \mid \mathcal{F}(\mathcal{P}) \mid$.

For $n \geq 2$ and $i = 1, \ldots, n-1$, if F is an $(i-2)$-face and G is an $(i+1)$-face of $\mathcal{P}$ incident with F, then we write $p_i(F, G)$ for the number of i-faces (or $(i-1)$-faces) of $\mathcal{P}$ in the section G/F. Then G/F is isomorphic to a 2-polytope with $p_i(F, G)$ vertices. Observe that in the case of regular polytopes, this number is independent of the choice of F and G so we write it p_i. We call the ordered set $[p_i : i = 1, \ldots, n-1]$ the *Schläfli symbol* of $\mathcal{P}$.

Proposition 2B3 Let $\mathcal{P}$ be a regular n-polytope.

1. All sections of $\mathcal{P}$ are regular polytopes, and any two sections which are defined by faces of the same ranks are isomorphic. In particular $\mathcal{P}$ has isomorphic facets and isomorphic vertex-figures.

2. The group of each section of $\mathcal{P}$ is a subgroup of $\Gamma(\mathcal{P})$. More precisely, if F_j is a j-face and F_k is a k-face with $-1 \leq j < k \leq n$, and $F_j < F_k$, then $\Gamma(F_k/F_j)$ is isomorphic to the stabilizer in $\Gamma(\mathcal{P})$ of any chain of type $\{-1, 0, \ldots, j-1, j, k, k+1, \ldots, n\}$ which includes both F_j and F_k.

Proposition 2B4 An n-polytope $\mathcal{P}$ is regular if and only if, for some flag Φ of $\mathcal{P}$ and each $j = 0, \ldots, n-1$, there exists a (unique) involutory automorphism ρ_j of $\mathcal{P}$ such that $\Phi\rho_j = \Phi^j$.

Observe that such automorphisms exist for one flag if they exist for each flag.

Let $\mathcal{P}$ be a regular n-polytope. We choose a fixed flag

$$\Phi := F_{-1}, F_0, \ldots, F_n$$

of $\mathcal{P}$ and call it the *base flag.*

For $j \in I := \{0, \ldots, n-1\}$, let ρ_j denote the unique involutory automorphism of $\mathcal{P}$ such that $\Phi\rho_j = \Phi^j$. The elements $\rho_0, \ldots, \rho_{n-1}$ are called the *distinguished generators* of $\Gamma(\mathcal{P})$. Subgroups generated by subsets of the distinguished generators are called *distinguished subgroups* or sometimes *parabolic subgroups.*

We write $\Phi_J := \{F_j \in \Phi \mid j \in J\}$ and $\Gamma_J := \{\rho_j \mid j \notin J\}$. In the latter case, we use the shorthand $\Gamma_j := \Gamma_{\{j\}}$. Also $\Gamma_\emptyset = \Gamma(\mathcal{P})$ and $\Gamma_I = \{1\}$, the trivial subgroup.

We write $\Gamma(\mathcal{P}, \Omega)$ for the stabiliser in $\Gamma(\mathcal{P})$ of a chain Ω.

Proposition 2B7 Let Φ be the base flag of $\mathcal{P}$. Let Φ_J be the chain of type J with $\Phi_J \subseteq \Phi$. Then $\Gamma(\mathcal{P}, \Phi_J) = \Gamma_J$.

Proposition 2B8 Let $\mathcal{P}$ be a regular n-polytope and let $\rho_0, \ldots, \rho_{n-1}$ be the distinguished generators of its group with respect to some flag. Then $\Gamma(\mathcal{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$.

Proposition 2B9 Let $\mathcal{P}$ be a regular n-polytope and let $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ be its group. Then

1. If $-1 \leq j \leq k \leq n$, then $\Gamma(F_k/F_j) \cong \langle \rho_{j+1}, \ldots, \rho_{k-1} \rangle$.

2. In particular,
$$\Gamma(F_j/F_{-1}) \cong \langle \rho_0, \ldots, \rho_{j-1} \rangle,$$
$$\Gamma(F_n/F_j) \cong \langle \rho_{j+1}, \ldots, \rho_{n-1} \rangle.$$

3. If $\mathcal{P}$ is of type $[p_1, \ldots, p_{n-1}]$, then for $1 \leq k \leq n-1$,
$$\Gamma(F_{k+1}/F_{k-2}) \cong \langle \rho_{k-1}, \rho_k \rangle \cong D_{2p_k}$$

Proposition 2B10 If $J, K \subseteq I$, then

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle$$

The condition in the above proposition is called the *intersection property* of $\mathcal{P}$ (with respect to the generators $\rho_0, \ldots, \rho_{n-1}$).

Proposition 2B11 If $\mid j - k \mid \geq 2$, then $\rho_j \rho_k = \rho_k \rho_j$.

Propositions 2B8 shows that the automorphism group $\Gamma(\mathcal{P})$ of an abstract regular polytope $\mathcal{P}$ is a group generated by involutions (or *ggi*). Proposition 2B11 says the generators ρ_j and ρ_k commute if $\mid j - k \mid \geq 2$. Hence the *ggi* $\Gamma(\mathcal{P})$ is of a special kind called *string ggi* or *sggi* for short.

Proposition 2B12 If $0 \leq k \leq n-1$, then

$$\Gamma(\mathcal{P}, F_k) = \Gamma_k = \Gamma(F_k/F_{-1}) \times \Gamma(F_n/F_k)$$

Lemma 2B13 Let $0 \leq j, k \leq n-1$ and let G_j be a j-face of $\mathcal{P}$. Then G_j is incident with F_k if and only if $G_j = F_j \gamma$ for some $\gamma \in \Gamma_k$.

Theorem 2B14 Let $0 \leq j, k \leq n-1$ and let $\varphi, \psi \in \Gamma$. Then the following three conditions are equivalent.

1. $F_j \varphi \leq F_k \psi$;
2. $\varphi \psi^{-1} \in \langle \rho_{j+1)}, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{k-1} \rangle$;
3. $\Gamma_j \varphi \cap \Gamma_k \psi \neq \emptyset$.

By Propositions 2B9 and 2B11, if $\mathcal{P}$ is a regular n-polytope of type $[p_1, \ldots, p_{n-1}]$, then the distinguished generators of $\Gamma(\mathcal{P})$ satisfy relations $(\rho_j \rho_k)^{p_{jk}} = 1$ for $0 \leq j \leq k \leq n-1$, where

1. $p_{jk} = 1$ if $j = k$;
2. $p_{jk} = 2$ if $\mid j - k \mid \geq 2$;
3. $p_{jk} = p_k$ if $j = k - 1$.

1.4 Coxeter groups

Here we study only Coxeter groups constructed with finitely many distinguished generators.

Let $M = (m_{ij})_{i,j=1,\dots,k}$ be a $k \times k$ matrix whose entries m_{ij} are positive integers or ∞. The matrix M is called a *Coxeter matrix* if $m_{ii} = 1$ for $i = 1, \dots, k$ and $m_{ij} = m_{ji} \geq 2$ for $1 \leq i < j \leq k$.

Let M be a Coxeter matrix. The *Coxeter group with Coxeter matrix M* is the group $W = W(M)$ with generators $\sigma_1, \dots, \sigma_k$ and presentation

$$(\sigma_i\sigma_j)^{m_{ij}} = 1_W \text{ for all } i, j \text{ with } m_{ij} \neq \infty$$

The set $S := \{\sigma_1, \dots, \sigma_k\}$ is called the set of *distinguished generators* of W.

The *rank* of $W(M)$ is the size of S.

The pair $(W; S)$ is called a *Coxeter system.*

If $k = 1$ then $W \cong C_2$. If $k = 2$ then $W \cong D_{2m_{12}}$.

Let $K := \{1, \dots, k\}$. For $I \subseteq K$, we define the *distinguished subgroup* or *parabolic subgroup* W_I of W by

$$W_I := \langle \sigma_i \mid i \in I \rangle.$$

Then $W_\emptyset = \{1_W\}$ and $W_K = W$.

Theorem 3A2 Let $W = \langle S \rangle$ be a Coxeter group with Coxeter matrix $M = (m_{ij})_{i,j \in K}$. Then the distinguished subgroups have the following properties.

1. Each group W_I with $\emptyset \neq I \subset K$ is (isomorphic to) the Coxeter group with Coxeter matrix $M = (m_{ij})_{i,j \in I}$;

2. If $I, J \subset K$, then $W_I \cap W_J = W_{I \cap J}$;

3. The subgroups W_I with $I \subset K$ are mutually distinct. Equivalently, if $j \notin I$, then $\sigma_j \notin W_I$.

Point 2. of the Theorem above is called the *intersection property.*

If $W = W(M)$ is a Coxeter group with Coxeter matrix M, the *Coxeter diagram* $\mathcal{D} = \mathcal{D}(M)$ is a labelled graph whose vertices represent the generators of W, and, for $i, j \in K$, an edge with label m_{ij} joins the i^{th} and j^{th} vertex, omitting edges when $m_{ij} \leq 2$. Also, if $m_{ij} = 3$, we don't write the label on the corresponding edge.

A *string-Coxeter diagram* is a diagram of the form

with possibly some of the p_i's equal to 2. This group is denoted by $[p_1, \ldots, p_{k-1}]$ if $k \geq 2$, or $[1]$ if $k = 1$. A *string Coxeter group* is a Coxeter group with a string Coxeter diagram.

Proposition 3A4 Let $\mathcal{D}$ be a Coxeter diagram without improper branches and let $\mathcal{D}_1, \ldots, \mathcal{D}_m$ be its connected components. Then

$$W(\mathcal{D}) \cong W(\mathcal{D}_1) \times \ldots \times W(\mathcal{D}_m),$$

but no component $W(\mathcal{D}_i)$ is itself a direct product of non-trivial distinguished subgroups.

A Coxeter group is *irreducible* if its Coxeter diagram is connected. It is *reducible* otherwise.

The symmetric group of a convex regular n-polytope or a regular tessellation in euclidean or hyperbolic $(n-1)$-space is a Coxeter group.

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix, and $W = W(M) = \langle \sigma_1, \ldots, \sigma_k \rangle$ the corresponding Coxeter group. Let $E := \mathbb{R}^k$ and let $\{e_1, \ldots, e_k\}$ be the canonical basis of E. Define the symmetric bilinear form $\langle \cdot, \cdot \rangle_M$ by

$$\langle e_i, e_j \rangle_M := -\cos(\pi / m_{ij}) \quad \text{for } i, j = 1, \ldots, k.$$

This bilinear form is called the *canonical bilinear form.*

For $i = 1, \ldots, k$, consider the linear mapping $S_i : E \to E$ defined for $x \in E$ by

$$xS_i := x - 2\langle e_i, x \rangle_M e_i.$$

Denote by $\mathcal{I}(M)$ the isometry group determined by $\langle \cdot, \cdot \rangle_M$; this is the subgroup of the general linear group $\mathrm{GL}(E)$ comprising the linear transformations which preserve the form $\langle \cdot, \cdot \rangle_M$. By the construction of $\langle \cdot, \cdot \rangle_M$, for $i, j = 1, \ldots, k$, we have

$$(S_i S_j)^{m_{ij}} = 1_{\mathcal{I}(M)}.$$

Hence $\alpha : W(M) \to \mathcal{I}(M) : \sigma_i \to S_i (i = 1, \ldots, k)$ is a homomorphism. This homomorphism is called the *canonical representation* of W.

Theorem 3B1 The Coxeter group $W = W(M)$ is finite if and only if the canonical bilinear form $\langle \cdot, \cdot \rangle_M$ is positive definite.

Theorem 3B2 The finite irreducible Coxeter groups are precisely those with a diagram listed in Table 1.1.

1.5 C-groups and string C-groups

Let $n \geq 1$ be an integer. A *C-group of rank n* is a group Γ generated by pairwise distinct involutions $\rho_0, \ldots, \rho_{n-1}$ which satisfy the following property, called the *intersection property*.

$$\forall J, K \subseteq I := \{0, \ldots, n-1\}, \langle \rho_j \mid j \in J \rangle \cap \langle \rho_k \mid k \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle.$$

A C-group $(\Gamma, \{\rho_0, \ldots, \rho_{n-1}\})$ is a *string C-group* if its generators satisfy the following relations.

$$(\rho_j \rho_k)^2 = 1_\Gamma \forall j, k \in \{0, \ldots n-1\} with \mid j - k \mid \geq 2.$$

In that case, the underlying Coxeter diagram is a string diagram.

1.6 From string C-groups to abstract regular polytopes

We explain how to construct a regular n-polytope $\mathcal{P}$ from a string C-group $(\Gamma, \{\rho_0, \ldots, \rho_{n-1}\})$ with $n \geq 1$.

For $i = -1, \ldots, n$, define
$$\Gamma_j := \langle \rho_i \mid i \neq j \rangle.$$

Observe that $\Gamma_{-1} = \Gamma = \Gamma_n$ and that the subgroups $\Gamma_0, \ldots, \Gamma_{n-1}$ are mutually distinct and distinct from Γ.

For $j \in I$, take as set of j-faces of $\mathcal{P}$ the set of all right cosets $\Gamma_j \varphi$ in Γ, with $\varphi \in \Gamma$. As improper faces of $\mathcal{P}$, we choose two copies of Γ, one denoted by Γ_{-1} and the other denoted by Γ.

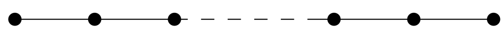
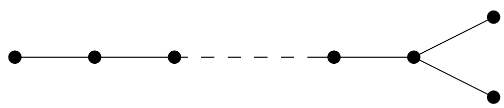
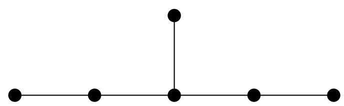
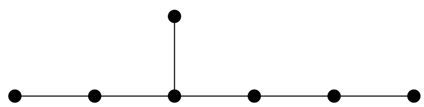
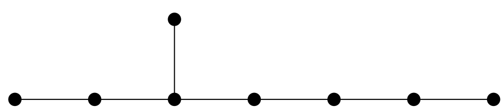
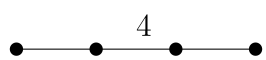
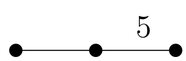
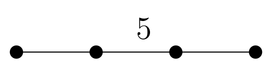
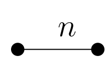
Notation	Diagram	Order
A_n with $n \geq 1$		$(n+1)!$
$B_n = C_n$ with $n \geq 2$		$2^n n!$
D_n with $n \geq 4$		$2^{n-1} n!$
E_6		$72 \cdot 6! = 51840$
E_7		$8 \cdot 9! = 2903040$
E_8		$192 \cdot 10! = 696729600$
F_4		1152
H_3		120
H_4		14400
I_n with $n \geq 1$		$2n$

Table 1.1: Spherical Coxeter groups

Define $\Gamma_j\varphi \leq \Gamma_k\psi$ to mean

2E3 $$-1 \leq j \leq k \leq n \quad \text{and} \quad \varphi\psi^{-1} \in \langle \rho_{j+1}, \ldots, \rho_{n-1}\rangle\langle \rho_0, \ldots, \rho_{k-1}\rangle$$

or equivalently

2E5 $$-1 \leq j \leq k \leq n \quad \text{and} \quad \Gamma_j\varphi \cap \Gamma_k\psi \neq \emptyset.$$

Lemma 2E4 The group Γ acts on $\mathcal{P}$ as a family of order preserving automorphisms.

Lemma 2E6 The condition (2E3) induces a partial order on $\mathcal{P}$.

Lemman 2E8 Γ is transitive on all chains of $\mathcal{P}$ of each given type $K \subseteq I$.

We define the base flag of $\mathcal{P}$ as $\Phi := \{\Gamma_0, \ldots, \Gamma_{n-1}\}$.

Lemma 2E9 If $K \subseteq I$, then the stabiliser of the chain Φ_K of type K in the base flag Φ is $\Gamma_K := \langle \rho_i \mid i \notin K\rangle$.

Corollary 2E10 Γ is simply transitive on $\mathcal{F}(\mathcal{P})$.

Theorem 2E11 Let $n \geq 1$, and let $(\Gamma, \{\rho_0, \ldots, \rho_{n-1}\})$ be a string C-group and $\mathcal{P} := \mathcal{P}(\Gamma)$ the corresponding poset. Then $\mathcal{P}$ is a regular n-polytope such that $\Gamma(\mathcal{P}) = \Gamma$.

Proof of Theorem 2E11

We need to check that (P1), ..., (P4) are satisfied by $\mathcal{P}$.

(P1) : trivial with Γ_{-1} and Γ_n.

(P2) : By Lemma 2E8, every chain Ω in $\mathcal{P}$ of type K can be expressed in the form $\Omega = \Phi_K\varphi$ for some $\varphi \in \Gamma$. Hence $\Omega \subseteq \Phi\varphi$.

(P4) : By Lemma 2E8, taking $K = I\backslash\{j\}$ for any $j \in I$, we see that the stabiliser of $\Phi_{I\backslash\{j\}}$ is $\Gamma_{I\backslash\{j\}} = \langle \rho_j \rangle$. Hence there is exactly one flag apart from Φ which contains $\Phi_{I\backslash\{j\}}$, namely $\Phi^j = \{\Gamma_0, \ldots, \Gamma_{j-1}, \Gamma_j\rho_j, \Gamma_{j+1}, \ldots, \Gamma_{n-1}\}$.

(P3) or (P3)': By Corollary 2E10, Γ is simply transitive on $\mathcal{F}(\mathcal{P})$. It thus suffices to consider the special case where one flag is the base flag Φ. If $\Psi \in \mathcal{F}(\mathcal{P})$ is another flag, let $K \subseteq I$ be such that $\Phi \cap \Psi = \Phi_K$. Since $\Phi_K \subseteq \Psi$, Lemma 2E9 and Corollary 2E10 give $\Psi = \Phi\varphi$ for a unique $\varphi \in \Gamma_K$. This says that
$$\Psi = \Phi_{\rho_{k(1)}\cdots\rho_{k(m)}} = \Phi^{k(1)\ldots k(m)}$$

for some $k(1)\dots k(m) \in I \backslash K$, giving an adjacency sequence of flags which each contain Φ_K. □

Corollary 2E13 The string C-groups are precisely the groups of regular polytopes.

1.7 Permutation representation graphs

In this section we discuss permutation representations of string C-groups. In particular we define permutation representation graphs for regular polytopes, introduced by Pellicer in his PhD thesis (see also [23]).

A *permutation representation graph* of a regular d-polytope $\mathcal{P}$ is a permutation representation of $\Gamma(\mathcal{P}) = \langle \rho_0, \dots, \rho_{d-1} \rangle$ represented on a graph as follows. Let ϕ be an embedding of $\Gamma(\mathcal{P})$ into the symmetric group S_n for some n. The permutation representation graph G of $\mathcal{P}$ determined by ϕ is the multigraph with n vertices, and with edge labels in the set $\{0, \dots, d-1\}$, such that any two vertices v, w are joined by an edge of label j if and only if $(v)(\phi(\rho_j)) = w$. These representations are faithful since ϕ is an embedding.

Since $\Gamma(\mathcal{P})$ has a string diagram, the connected components of the graphs induced by edges with labels i and j for $|i - j| > 1$ must either be single vertices, single edges, double edges, or alternating squares. The value of n for which we choose an embedding into S_n is not unique. For example the regular toroidal polytope $\mathcal{P} = \{4, 4\}_{(2,0)}$ has 32 flags and thus there is an embedding of $\Gamma(\mathcal{P})$ into S_{32}, giving a permutation representation graph isomorphic to the Cayley graph of $\Gamma(\mathcal{P})$. However, the action of $\Gamma(\mathcal{P})$ on the edges of the polytope determines the action on the flags, and thus there is an embedding into S_8 acting on the set of edges. From the two different embeddings, we get the two different permutation representation graphs in Figure 1.1. Here ρ_0, ρ_1, and ρ_2 are represented by the edges coloured blue, red, and green respectively.

1.8 Exercises

1. For each axiom (P1), (P2), (P3) and (P4), find examples of posets that don't satisfy it but satisfy the three others.

2. Given a cube and a base flag, write the distinguished generators of

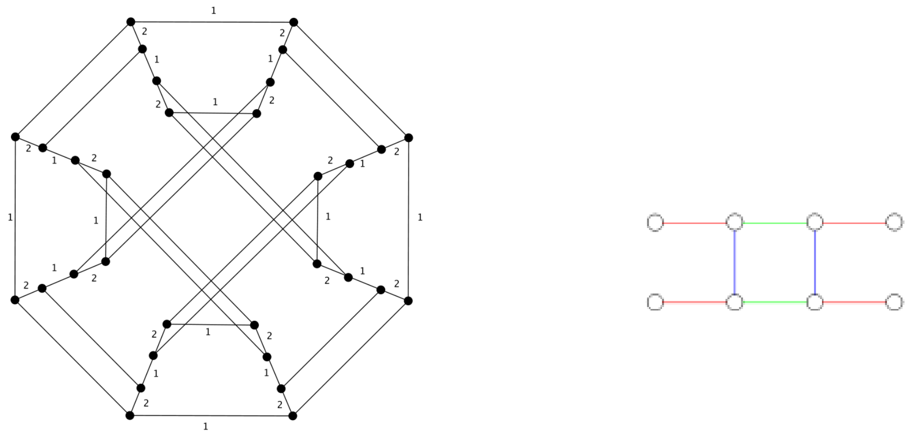


Figure 1.1: Permutation representation graphs of $\{4,4\}_{(2,0)}$

the automorphism group of the cube and show that this group acts regularly on the flags of the cube.

3. Show that these generators satisfy the intersection property.

4. Draw the permutation representation graph corresponding to the generators you wrote for the automorphism group of the cube.

5. Show that the automorphism group of the cube is isomorphic to $Sym(4) \times C_2$.

6. Prove that there exists a unique abstract regular 3-polytope with Schläfli symbol $[3,3]$.

7. Give a string C-group representation for each of the five platonic solids (i.e. the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron).

8. The Cayley graph of a string C-group $(G, \{\rho_0, \ldots, \rho_{n-1}\}$ is a graph whose vertices are the elements of G. Two vertices v, w are joined by an edge with label j provided $v * \rho_j = w$. Draw the Cayley graph of the automorphism group of the cube.

9. Compute the Schläfli symbols of the tesselations of the euclidean plane by equilateral triangles, squares and hexagons. Show that the first and third one are dual of each other and that the second one is self-dual.

10. A *Petrie path* of a polyhedron $\mathcal{P}$ is defined from a vertex v_0 and an incident edge e_0 of $\mathcal{P}$ by taking successive vertices v_i and e_i such that v_i is adjacent to e_{i-1} and e_i, e_i is adjacent to v_{i-1} and v_i (for $i \leq 1$), and no three consecutive edges in the path lie on a same face of $\mathcal{P}$. A *Petrie polygon* is a Petrie path with $v_0 = v_i$ for some $i > 0$. The *length* of a Petrie polygon is the smallest value of i such that $v_0 = v_i$ and $e_0 = e_i$. Compute the lengths of the Petrie polygons of the five platonic solids.

11. The *Petrial* of an abstract regular polyhedron $\mathcal{P}$ is the polyhedron obtained from $\mathcal{P}$ by taking the same vertices and edges and replacing the faces by the Petrie polygons of $\mathcal{P}$. What are the Schläfli symbols of the Petrials of the five platonic solids?

Chapter 2

Algorithms and Atlases

2.1 Algorithms

In [20] Vauthier and I designed a series of algorithms to classify all string-C-group representations of a given group up to isomorphism and duality. In this section, we present two possible search algorithms, one depth-first search (which we denote by the letter 'D' in Table 2.1) and one breadth-first search (denoted by the letter 'B' in Table 2.1), which outperform the Leemans-Vauthier algorithm. This section is mainly taken from [17].

In both algorithms, we will be concerned with classifying all regular polytopes with automorphism group G up to isomorphism. In other words, we will be looking for all nonisomorphic ways of representing G as a string C-group $\langle \rho_0, \ldots, \rho_r \rangle$.

Both the breadth search and the depth search start out the same way.

Step 0: Find the automorphism group of G. Represent this group $Aut(G)$ as a permutation group acting on the set L of all involutions of G, as these are the possible generators in the string C group. Construct a list L_0 of all conjugacy classes of involutions. This gives the candidates for ρ_0.

2.1.1 Breadth-first algorithm for classifying polytopes

In [12] a very similar algorithm is provided and was implemented in GAP.

1. Given L_k construct L_{k+1} as follows. Let K_t be the stabilizer of a tuple t in L_k under the action of $Aut(G)$.

2. Pick a set R of representatives of the orbits of the action of K_t on L.
3. For each element r of R, check if the group $\langle t, r\rangle$ is a string C-group.
   (a) If it is a string C group and generates the whole group, add it to the list P.
   (b) If it is a string C-group but generates a proper subgroup of G, then add $[t, r]$ to the set L_{k+1}.
4. Stop when L_{k+1} is empty.

2.1.2 Depth-first algorithm for classifying polytopes

This algorithm gives a recursive approach for finding all polytopes up to isomorphism.

1. For each element $t_i = [r_0, \ldots, r_i]$ of L_i, find the stabilizer S_{t_i} of t_i under the action of $S_{[r_0,\ldots,r_{i-1}]}$ on L.
2. Construct a list R_{t,r_i} of representatives of the orbits of this action. This will give you candidates for ρ_{i+1}.
3. For each element r_{i+1} of R_{t,r_i}, check if the group $\langle t, r_{i+1}\rangle$ is a string C-group.
   (a) If it is a string C group and generates the whole group then add it to the list P.
   (b) If it is a string C-group but generates a proper subgroup of G then repeat the algorithm on $t_{i+1} = [r_0, \ldots, r_i, r_{i+1}]$.

2.1.3 Comparison of algorithms

In general the depth-first search algorithm has two clear advantages. First, we saw in the breadth-first search algorithm that the entire list L_1 is constructed, then the entire list L_2 is constructed before calculating L_3, and so on. These lists can be very large, so storing them can become very memory intensive for large groups. For this reason, the depth first search is usually less memory intensive.

Second, in the breadth-first search algorithm, you are computing the stabilizer of a tuple of elements $t_k = [r_0, \ldots, r_k]$ under the action of a large group

Group	Time 2	Time B	Time D	Memory 2	Memory B	Memory D
$Alt(5)$	0.18s	0.16s	0.17s	0.01MB	0.01MB	0.01MB
$Alt(5) \times C_2$	0.29s	0.19s	0.17s	0.01MB	0.01MB	0.01MB
$P\Gamma L(2,9)$	1.86s	0.35s	0.29s	0.01MB	0.01MB	0.01MB
$Sym(7)$	5.95s	0.66s	0.45s	1.04MB	0.01MB	0.01MB
$PSL(2,25)$	0.39 s	0.27s	0.22s	0.01MB	0.01MB	0.01MB
$P\Sigma U(3,3)$	4.62 s	0.63s	0.35s	1.29MB	1.29MB	1.29MB
$PGL(2,27)$	4.38s	1.27s	0.36s	1.04MB	1.04MB	0.01MB
$Sz(8)$	0.44s	0.26s	0.26s	1.29MB	1.29MB	1.29MB
M_{12}	12.19s	1.49s	1.02s	2.32MB	2.32MB	1.29MB
J_1	54.36s	9.04s	2.78s	3.95MB	27.32MB	4.82MB
$Alt(9)$	37.12s	2.86s	1.68s	2.32MB	1.82MB	0.01MB
$Sym(9)$	2193.75s	49.42s	26.77s	10.26MB	4.54MB	1.45MB
$Alt(10)$	851.78s	41.53s	19.57s	16.36MB	8.07MB	1.57MB
HS		301.86s	97.75s		90.45MB	16.39MB

Table 2.1: Comparison of algorithms

$Aut(G)$. By comparison, in the depth-first search algorithm, you simply consider the stabilizer of the r_k under the action of a much smaller group which is $S_{t_{i-1}}$ which you have stored from the previous step. These two approaches yield the same stabilizing subgroup, but dealing with the action of a smaller group will usually make the depth first search faster.

We also note that there are many more ways that could be used to improve the memory use of our algorithms. For example, currently we consider $Aut(G)$ as a permutation group acting on the list L of all involutions. Instead of acting on L, one could instead construct $Aut(G)$ to act on the set of indices of each involution in this list. It is easy to convert back and forth from an index to an involution, and storing and working with these indices can be more efficient than working in the original group. This approach was in fact used in [20].

We conclude this section by giving a quantitative comparison of the 3 algorithms in the form of the following table comparing time and memory usage in the classification of all non-isomorphic polytopes for a given group.

G	Aut(G)	# G	# of involutions	Number of Polytopes
M_{11}	M_{11}	7920	165	0
M_{12}	$M_{12}:2$	95040	891 = 396+495	37 = 23+14
$M_{12}:2$	$M_{12}:2$	190080	1683 = 396+495+792	266 = 223+43
J_1	J_1	175560	1463	150 = 148+2
M_{22}	$M_{22}:2$	443510	1155	0
$M_{22}:2$	$M_{22}:2$	887040	2871 = 330+1155+1386	195 = 133+62
J_2	$J_2:2$	604800	2835 =315+2520	154 = 137+17
$J_2:2$	$J_2:2$	1209600	4635 = 315+1800+2520	452 = 368+82+2

Table 2.2: Sporadic groups and their automorphism groups

2.2 Atlases

Several papers have been published announcing classifications of string-C-group representations for groups.

2.2.1 The Leemans-Vauthier atlas

This atlas [20], available online at http://www.auckland.ac.nz/∼dleemans/polytopes/" contains classifications for all the finite almost simple groups appearing in the Atlas of Finite Groups [8].

2.2.2 Hartley's atlas

This atlas, also available online at http://www.abstract-polytopes.com/atlas/ contains all abstract regular polytopes with at most 2000 flags (except for 1024 and 1536).

2.2.3 Conder's atlas

This atlas, available on Marston Conder's website, contains abstract regular polytopes with at most 2000 flags excluding those of rank two and those that have a "2" in their Schläfli symbol.

G	Aut(G)	# G	# of involutions	Number of Polytopes
$Alt(5)$	$Sym(5)$	60	15	2
$Sym(5)$	$Sym(5)$	120	25 = 10+15	5 = 4+1
$Alt(6)$	$P\Gamma L(2,9)$	360	45	0
$PGL(2,9)$	$P\Gamma L(2,9)$	720	81 =36+45	14
$P\Sigma L(2,9)$	$P\Gamma L(2,9)$	720	75 = 15+15+45	7 = 2+4+1
M_{10}	$P\Gamma L(2,9)$	720	45	0
$P\Gamma L(2,9)$	$P\Gamma L(2,9)$	1440	111 = 30+36+45	12
$Alt(7)$	$Sym(7)$	2520	105	0
$Sym(7)$	$Sym(7)$	5040	231 = 21+105+105	44 = 35+7+1+1
$Alt(8)$	$Sym(8)$	20160	315 = 105+210	0
$Sym(8)$	$Sym(8)$	40320	763 = 28+105+210+420	117 = 68+36+11+1+1
$Alt(9)$	$Sym(9)$	181440	1323 = 378+945	47 = 41+6
$Sym(9)$	$Sym(9)$	362880	2619 = 36+378+945+1260	182 = 129+37+7+7+1+1

Table 2.3: Alternating groups and their automorphism groups

2.2.4 Hartley-Hulpke and sporadic groups

In [12], Hartley and Hulpke pushed further the computations started in [**?**] by writing a new algorithm that permitted them to classify all abstract regular polytopes for sporadic groups up to the Held group (of order 4,030,387,200).

2.2.5 Leemans and Mixer

With Mark Mixer [17], we designed a new series of algorithms (described before) that permitted us to go even further, classifying all abstract regular polytopes for Co_3 (of order 495,766,656,000). We also gathered more data on the alternating and symmetric groups at the time.

2.2.6 Connor, Leemans and Mixer

With Connor and Mixer [7], we pushed even further by designing a complete new approach for polytopes of rank at least 4. We managed to get a complete classification of abstract regular polytopes of rank at least 4 for the O'Nan group (whose size is roughly the same as Co_3 but which has a much larger permutation representation degree). With Connor, we attacked the rank 3 using character theory [5].

G	**Aut(G)**	**# G**	**# invols**	**# of Polytopes**
$PSL(2,4) = PSL(2,5)$	$Sym(5)$	60	15	2
$Sym(5)$	$Sym(5)$	120	25 = 10+15	5 = 4+1
$PSL(3,2) = PSL(2,7)$	$P\Gamma L(2,7)$	168	21	0
$PGL(2,7) = P\Gamma L(2,7)$	$P\Gamma L(2,7)$	336	49 = 21+28	16
$Alt(6) = PSL(2,9)$	$P\Gamma L(2,9)$	360	45	0
$PGL(2,9)$	$P\Gamma L(2,9)$	720	81 = 36+45	14
$P\Sigma L(2,9)$	$P\Gamma L(2,9)$	720	75 = 15+15+45	7 = 2+4+1
M_{10}	$P\Gamma L(2,9)$	720	45	0
$P\Gamma L(2,9)$	$P\Gamma L(2,9)$	1440	111 = 30+36+45	12
$PSL(2,8)$	$P\Gamma L(2,8)$	504	63	7
$P\Gamma L(2,8)$	$P\Gamma L(2,8)$	1512	63	0
$PSL(2,11)$	$PGL(2,11)$	660	55	4 = 3+1
$PGL(2,11)$	$PGL(2,11)$	1320	121 = 55+66	42
$PSL(2,13)$	$PGL(2,13)$	1092	91	11
$PGL(2,13)$	$PGL(2,13)$	2184	169 = 78+91	59
$PSL(2,17)$	$PGL(2,17)$	2448	153	16
$PGL(2,17)$	$PGL(2,17)$	4896	289 = 136+153	110
$PSL(2,19)$	$PGL(2,19)$	3420	171	18 = 17+1
$PGL(2,19)$	$PGL(2,19)$	6840	361 = 171+190	140
$PSL(2,16)$	$P\Gamma L(2,16)$	4080	255	27
$PSL(2,16):2$	$P\Gamma L(2,16)$	8160	323 = 68+255	26 = 21+5
$P\Gamma L(2,16)$	$P\Gamma L(2,16)$	16320	323 = 68+255	0
$PSL(2,23)$	$PGL(2,23)$	6072	253	28
$PGL(2,23)$	$PGL(2,23)$	12144	529 = 253+276	212
$PSL(2,25)$	$P\Gamma L(2,25)$	7800	325	17
$PGL(2,25)$	$P\Gamma L(2,25)$	15600	625 = 300+325	127
$P\Sigma L(2,25)$	$P\Gamma L(2,25)$	15600	455 =65+65+325	51 = 34+17
$PSL(2,25).2$	$P\Gamma L(2,25)$	7800	325	0
$P\Gamma L(2,25)$	$P\Gamma L(2,25)$	31200	755 = 130+300+325	64
$PSL(2,27)$	$P\Gamma L(2,27)$	9828	351	14
$PGL(2,27)$	$P\Gamma L(2,27)$	19656	729 = 351+378	98
$P\Sigma L(2,27)$	$P\Gamma L(2,27)$	29484	351	0
$P\Gamma L(2,27)$	$P\Gamma L(2,27)$	58968	729 = 351+378	0
$PSL(2,29)$	$PGL(2,29)$	12180	435	50
$PGL(2,29)$	$PGL(2,29)$	24360	841 = 406+435	337
$PSL(2,31)$	$PGL(2,31)$	14880	465	51
$PGL(2,31)$	$PGL(2,31)$	29760	961 = 465+496	394
$PSL(2,32)$	$P\Gamma L(2,32)$	32736	1023	93
$P\Gamma L(2,32)$	$P\Gamma L(2,32)$	163680	1023	0

Table 2.4: $PSL(2,q)$ groups and their automorphism groups

G	Aut(G)	# G	# invols	# of Polytopes
$PSL(3,2) = PSL(2,7)$	$P\Gamma L(2,7)$	168	21	0
$PGL(2,7) = P\Gamma L(2,7)$	$P\Gamma L(2,7)$	336	49 = 21+28	16
$PSL(3,3)$	$PSL(3,3):2$	5616	117	0
$PSL(3,3):2$	$PSL(3,3):2$	11232	351 = 117+234	68 = 67+1
$PSL(3,4)$	$PSL(3,4).D_{12}$	20160	315	0
$PSL(3,4).2_1$	$PSL(3,4).D_{12}$	40320	595 = 280+315	4
$PSL(3,4).3 = PGL(3,4)$	$PSL(3,4).D_{12}$	60480	315	0
$PSL(3,4).3.2_3$	$PSL(3,4).D_{12}$	120960	1323 = 315+1008	52 = 50+2
$PSL(3,4).3.2_2 = P\Gamma L(3,4)$	$PSL(3,4).D_{12}$	120960	675 = 315+360	0
$PSL(3,4).6$	$PSL(3,4).D_{12}$	120960	595 = 280+315	0
$PSL(3,4).D_{12}$	$PSL(3,4).D_{12}$	241920	1963 = 280+ 315+360+1008	119 = 100+16+3
$PSL(3,4).2_3$	$PSL(3,4).2^2$	40320	651 = 315+336	53 = 44+9
$PSL(3,4).2_2 = P\Sigma L(3,4)$	$PSL(3,4).2^2$	40320	435 = 120+315	0
$PSL(3,4).2^2$	$PSL(3,4).2^2$	80640	1051 = 120+ 280+315+336	147 = 88+59
$PSL(3,5)$	$PSL(3,5):2$	372000	775	0
$PSL(3,5):2$	$PSL(3,5):2$	744000	3875 =775+3100	498 = 496+2

Table 2.5: Other linear groups and their automorphism groups

G	Aut(G)	# G	# invols	# of Polytopes
$PSU(3,3)$	$P\Gamma L(3,3)$	6048	63	0
$P\Gamma U(3,3)$	$P\Gamma U(3,3)$	12096	315 = 62+252	31 = 25+6
$PSU(4,2)$	$P\Gamma U(4,2)$	25920	315 = 45+270	0
$P\Gamma U(4,2)$	$P\Gamma U(4,2)$	51840	891 = 36+45+270+540	147 = 87+50+10
$PSU(3,4)$	$P\Gamma U(3,4)$	62400	195	0
$PSU(3,4):2$	$P\Gamma U(3,4)$	124800	1235 =195+1040	80 = 78+2
$P\Gamma U(3,4)$	$P\Gamma U(3,4)$	249600	1235 = 195+1040	0
$PSU(3,5)$	$P\Gamma U(3,5)$	126000	525	0
$PGU(3,5)$	$P\Gamma U(3,5)$	378000	525	0
$P\Gamma U(3,5)$	$P\Gamma U(3,5)$	756000	3675 = 525+3150	247 = 237+10
$P\Sigma U(3,5)$	$P\Sigma U(3,5)$	252000	1575 = 525+1050	116 = 105+11

Table 2.6: Unitary groups and their automorphism groups

G	**Aut(G)**	**# G**	**# invols**	**# of Polytopes**
$Sz(8)$	$Sz(8):3$	29120	455	7
$Sz(8):3$	$Sz(8):3$	87360	455	0

Table 2.7: Suzuki groups and their automorphism groups

Chapter 3

Suzuki groups

3.1 Basic facts

Looking at the data obtained in the Leemans-Vauthier atlas, some conjectures arose and several have now been proven for the Suzuki groups, the groups $PSL(n,q)$ with $n \leq 4$, the symmetric and alternating groups, etc. The Suzuki groups were the first ones studied in this vein. We refer to [25, 26] for the basic properties of Suzuki groups. Throughout this section, $q = 2^{2e+1}$ and $e > 0$ is an integer.

In the projective 3-space $\mathrm{PG}(3,q)$ over the finite field $\mathrm{GF}(q)$, an *ovoid* $\mathcal{D}$ is a set of q^2+1 points satisfying the following axioms:

1. no three points are collinear;
2. for every $p \in \mathcal{D}$, there exists a hyperplane E of $\mathrm{PG}(3,q)$ such that $\mathcal{D} \cap E = \{p\}$;
3. for each such $p \in \mathcal{D}$ and E, for every line ℓ of $\mathrm{PG}(3,q)$ through p that is not contained in E, there exists a point $p' \in \mathcal{D} \cap \ell$ with $p' \neq p$.

For instance, quadrics are ovoids in $\mathrm{PG}(3,q)$. Jacques Tits exhibited a class of ovoids that are not quadrics, but occur as the fixed points of an involutary automorphism of $\mathrm{PSp}(4,q)$ [26]. Those ovoids are now called Tits ovoids or Suzuki ovoids. The *Suzuki group* $\mathrm{Sz}(q)$ is defined as the subgroup of the collineations of $\mathrm{PG}(3,q)$ that leave a Suzuki-Tits ovoid invariant. Tits showed that the choice of different Suzuki-Tits ovoids $\mathcal{D}$ (of the same projective space $\mathrm{PG}(3,q)$) gives rise to conjugate groups in the group of all collineations of $\mathrm{PG}(3,q)$. Suzuki showed that such a group is simple.

We shall use several properties of Suzuki groups, their subgroups and their elements. Given q, the maximal subgroups of $\mathrm{Sz}(q)$ have one of the following structures [25]:

$$(E_q \cdot E_q) : C_{q-1}, \quad D_{2(q-1)}, \quad C_{\alpha_q} : C_4, \quad C_{\beta_q} : C_4, \quad \mathrm{Sz}(q'),$$

where the symbol ":" stands for a split extension (also called semi-direct product) and the symbol "·" stands for a non-split extension, E_q denotes an elementary abelian group of order q, $\alpha_q := q + \sqrt{2q} + 1$, $\beta_q := q - \sqrt{2q} + 1$ and $q' := 2^{2e'+1}$ such that $2e'+1 \mid 2e+1$. Clearly, if $2e+1$ is a prime number, then $Sz(q)$ has no proper subgroups of Suzuki type.

Proposition 3.1.1. *[26, 25] Let $\mathcal{D}$ be an ovoid of $\mathrm{PG}(3, q)$ and let $\mathrm{Sz}(q)$ be its Suzuki group. Then:*

1. *The order of $\mathrm{Sz}(q)$ is $(q^2+1)q^2(q-1)$ and $\mathrm{Sz}(q)$ acts 2-transitively on $\mathcal{D}$.*

2. *The order of $\mathrm{Sz}(q)$ is not divisible by 3.*

3. *The group $\mathrm{Sz}(q)$ has no dihedral subgroup of order 8.*

4. *$\mathrm{Aut}(\mathrm{Sz}(q)) \cong \mathrm{Sz}(q) : C_{2e+1}$; hence, for every involution ρ of $\mathrm{Aut}(\mathrm{Sz}(q))$ we have that $\rho \in \mathrm{Sz}(q)$. In particular, if $\mathrm{Sz}(q) < G \leq \mathrm{Aut}(\mathrm{Sz}(q))$, then G cannot be generated by involutions.*

3.2 Abstract regular polytopes

Theorem 3.2.1. *[16] Let $Sz(q) \leq G \leq Aut(Sz(q))$ with $q = 2^{2e+1}$ and $e > 0$ a positive integer. Then G is a C-group if and only if $G = Sz(q)$. Moreover, if $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ is a string C-group, then $n = 3$.*

We may translate this theorem in abstract regular polytopes theory. It means that

- if $Sz(q) < G \leq Aut(Sz(q))$, then G is not the automorphism group of an abstract regular polytope;

- if $G = Sz(q)$, there exists an abstract regular polytope $\mathcal{P}$ such that $G = Aut(\mathcal{P})$. Moreover, if $\mathcal{P}$ is an abstract regular polytope such that $G = Aut(\mathcal{P})$, then $\mathcal{P}$ must be an abstract polyhedron, i.e. an rank three polytope.

Chapter 4

Dihedral groups

4.1 Basic facts and classification theorem

Dihedral groups are usually defined as a finitely presented group with two generators a and b satisfying

$$a^2 = b^2 = (a * b)^n = 1$$

(with n a positive integer or infinity which we won't allow here). The generators a and b are involutions (i.e. elements of order 2). Observe that when $n = 1$, D_2 is a cyclic group of order 2. Hence, when we talk about dihedral (sub)groups, we assume $n > 1$. All finite groups generated by two involutions are dihedral groups. They also appear as automorphism groups of regular n-gons. Dihedral groups are used as "stones" to build C-groups of rank at least three. They are parabolic subgroups of rank two (and, as we will see, sometimes rank three) of C-groups.

We write C_n to denote a cyclic group of order n and D_{2n} to denote a dihedral group of order $2n$. We say that D_{2n} is *odd*, *simply even* or *doubly even* provided that n is odd, or congruent to 2 or 0 modulo 4, respectively.

Classification Theorem 4.1.1. *[4] Let $\mathcal{D} := (D_{2n}, \{\rho_0, \ldots, \rho_{r-1}\})$ be a C-group (with n a positive integer). Then $r \leq 3$. Moreover,*

1. *$r = 1$ if and only if $n = 1$; and there exists a unique choice of ρ_0 in $D_2 \cong C_2$ such that $\mathcal{D}$ is a C-group.*
2. *if $r = 2$, there exists a unique choice of ρ_0, ρ_1 up to isomorphism such that $\mathcal{D}$ is a C-group.*

3. *if $r = 3$, then $n > 2$ and $n \equiv 2 \pmod 4$. Also, there exists a unique choice of ρ_0, ρ_1, ρ_2 up to isomorphism such that $(D_{2n}, \{\rho_0, \rho_1, \rho_2\})$ is a C-group.*

As an application of the Classification Theorem 4.1.1, we have the following obvious corollary.

Corollary 4.1.2. *[4] All C-groups of D_{2n} are string C-groups.*

We recall here what the normalizer of a maximal dihedral subgroup of a dihedral group is.

Lemma 4.1.3. *[15] Let m be a maximal divisor of n. If $\frac{n}{m} \neq 2$ and $m \neq 1$, then D_{2m} is self-normalizing in D_{2n}. If $\frac{n}{m} = 2$ then D_{2m} is a normal subgroup of D_{2n}. Moreover there are two subgroups isomorphic to D_{2m} in D_{2n} which are fused under the action of $\mathrm{Aut}(D_{2n})$.*

By applying Lemma 4.1.3 recursively for maximal subgroups we obtain the following corollary.

Corollary 4.1.4. *[4] Let n be an odd integer. Let $G \cong D_{2n}$ and let $D_{2m} \cong H, K < G$. Then H and K are self-normalizing in G. Moreover H and K are conjugate in G. The length of their conjugacy class is $\frac{n}{m}$.*

Proof. The first part is an application of Lemma 4.1.3 and is clear. It remains to prove the second part of the statement. Observe that $H = \langle \sigma_H, \tau_H \rangle$ and $K = \langle \sigma_K, \tau_K \rangle$ where σ_H and σ_K are involutions of G, and τ_H and τ_K are cycles of length m of G. Moreover σ_H and σ_K are conjugate in G, while $\langle \tau_H \rangle = \langle \tau_K \rangle$. Hence there exists $g \in G$ such that $\sigma_H^g = \sigma_K$ and which normalizes $\langle \tau_K \rangle$. Therefore $H^g = K$. The third part of the statement is clear at this point. □

The following lemma determines when two dihedral subgroups of a dihedral group can intersect in a cyclic group of order 2.

Lemma 4.1.5. *[4] Let $G = D_{2n}$, and let D_{2p}, D_{2q} be subgroups of G such that $D_{2p} \cap D_{2q} = C_2$. The following statements hold:*

- *if n is odd then $(p, q) = 1$;*
- *if n is even then $(p, q) = 1$ or 2.*

Proof. Set $k = (p, q)$. Observe that D_{2p} has a unique subgroup C_p and that D_{2q} has a unique subgroup C_q. Moreover there exists a unique cyclic group C_l of order l in G for all $2 \neq l \mid n$. Since k is the greatest common divisor of p and q, we conclude that $C_p \cap C_q = C_k$. In particular, $D_{2p} \cap D_{2q} = C_2$ contains C_k. Therefore we see that $k \leq 2$. This shows that if n is odd, then $(p, q) = 1$.

Now assume that n is even and that $k = 2$. Then $C_p \cap C_q = D_{2p} \cap D_{2q}$ and therefore $D_{2p} \cap D_{2q} = Z(G) \cong C_2$. □

4.2 Proof of the Classification Theorem 4.1.1

Lemma 4.2.1. *[4] Let n be odd. Then D_{2n} is not a C-group of rank 3.*

Proof. By way of contradiction, suppose that $\mathcal{D} = (D_{2n}, \{\rho_0, \rho_1, \rho_2\})$ is a C-group of rank 3. Since n is odd, the maximal parabolic subgroups of $\mathcal{D}$ are odd dihedral groups, say $G_0 \cong D_{2p}$, $G_1 \cong D_{2q}$ and $G_2 \cong D_{2r}$, while the minimal parabolic subgroups of $\mathcal{D}$ are cyclic groups of order 2. By Lemma 4.1.5, we see that p, q and r are coprime and pqr must divide n. But then, $\langle G_0, G_1\rangle$ is a dihedral group whose cyclic subgroup H of index 2 is generated by the cyclic subgroups of G_0 and G_1. The subgroup H is of order pq and as $\langle G_0, G_1\rangle$ contains ρ_0, ρ_1 and ρ_2, we have that $\langle \rho_0, \rho_1, \rho_2\rangle \neq G$, a contradiction. □

Lemma 4.2.2. *[4] Let n be odd. Then D_{2n} is not a C-group of rank r for $r \geq 3$.*

Proof. By Lemma 4.2.1, we already know that D_{2n} is not a C-group of rank 3. Suppose that $\mathcal{D} = (D_{2n}, \{\rho_0, \dots, \rho_{r-1}\})$ is a C-group of rank $r \geq 4$. Observe that each subgroup of G is either a cyclic group or an odd dihedral group. Therefore the minimal parabolic subgroups of $\mathcal{D}$ are cyclic groups of order 2 and the remaining parabolic subgroups must be odd dihedral groups. In particular, any triple $\{\rho_i, \rho_j, \rho_k\}$ of involutions of $\mathcal{D}$ provides a C-group of rank 3 for an odd dihedral group, a contradiction in view of Lemma 4.2.1. □

Lemma 4.2.3. *[4] Let n be odd. Up to isomorphism, there exists a unique C-group $(D_{2n}, \{\rho_0, \rho_1\})$ of rank 2.*

Proof. Since n is odd, the group D_{2n} has a unique conjugacy class of involutions. Each of them is a reflection through a vertex and the middle of the opposite edge in the permutation representation of D_{2n} as the automorphism group of a regular n-gon in a Euclidean space. Let ρ_0 and ρ_1 be any of them. The angle formed by the corresponding reflection axes is $\frac{2k\pi}{n}$ for some positive integer $1 \neq k < n$. If $(k, n) = 1$, it is clear that $\langle \rho_0, \rho_1 \rangle = D_{2n}$. Up to isomorphism, there is obviously a unique pair of generating involutions in a D_{2n} as they give exactly the same presentation for this group. □

Lemma 4.2.4. *[4] Let n be even. Up to isomorphism, there exists a unique C-group of rank 2 $(D_{2n}, \{\rho_0, \rho_1\})$. Moreover, up to isomorphism, there exists also a unique C-group of rank 3 $(D_{2n}, \{\rho_0, \rho_1, \rho_2\})$ if and only if $n \equiv 2 \ (mod\ 4)$. Finally, there does not exist a C-group $(D_{2n}, \{\rho_0, \dots, \rho_{r-1}\})$ for $r \geq 4$.*

Proof. Let $\mathcal{D} := (D_{2n}, \{\rho_0, \dots, \rho_{r-1}\})$ be a C-group of rank r. We divide our proof in two cases.

First case: suppose without loss of generality that ρ_0 is the central involution of D_{2n}. In that case, ρ_0 commutes with ρ_i for all $i = 1, \dots, r-1$. In particular, $D_{2n} = \langle \rho_0 \rangle \times \langle \rho_1, \dots, \rho_{r-1} \rangle$. However a doubly even dihedral group cannot be written as a direct product of a cyclic group of order 2 and a dihedral group of order n and therefore $n \equiv 2 \ (mod\ 4)$. We can write $n = 2m$ with m odd and $(D_{2m}, \{\rho_1, \dots, \rho_{r-1}\})$ is a C-group. By Lemmas 4.2.1 and 4.2.2, we have $r - 1 = 2$. We conclude that the only rank 3 C-group $(D_{2n}, \{\rho_0, \rho_1, \rho_2\})$ for $n \equiv 2 \ (mod\ 4)$ is such that ρ_0 commutes with both ρ_1 and ρ_2, with $\langle \rho_1, \rho_2 \rangle \cong D_n$.

Second case: suppose that none of the involutions $\rho_0, \dots, \rho_{r-1}$ is central. The group D_{2n} has three conjugacy classes of involutions: the class of the central involution and two classes of length $\frac{n}{4}$ each. The ρ_i's belong to those two classes by assumption. By Lemma 4.1.5, the orders of the products $\rho_i\rho_j$ are pairwise coprime and thus at most one of them is even. Set $p = o(\rho_0\rho_1)$ and $q = o(\rho_0\rho_{r-1})$. Using similar arguments as in the proof of Lemma 4.2.1, we can conclude that $\mathcal{D}$ does not satisfy the intersection property or that $\langle \rho_0, \dots, \rho_{r-1} \rangle < D_{2n}$. Therefore $\mathcal{D}$ is not a C-group, a contradiction. In conclusion, r cannot be larger than 2 when none of the ρ_i's is central in D_{2n}. By applying the same kind of argument as in the proof of Lemma 4.2.3, it is easily seen that, up to isomorphism, there is a unique C-group $(D_{2n}, \{\rho_0, \rho_1\})$ of rank 2. □

Proof of Classification Theorem 4.1.1. The fact that $r < 4$ as well as (3) are consequences of Lemma 4.2.2 and the second part of Lemma 4.2.4. (1) is straightforward. (2) is a combination of Lemma 4.2.3 and the first part of Lemma 4.2.4. □

Chapter 5

Almost simple groups with socle $\mathrm{PSL}(2, q)$

The following definitions (restricted to the finite case) are borrowed from Carter [3]. Let V be a vector space of dimension n over a finite field $\mathrm{GF}(q)$. The group of all non-singular linear transformations of V onto itself is called the *general linear group* $\mathrm{GL}(n, q)$. The transformations of determinant 1 form a normal subgroup $\mathrm{SL}(n, q)$, the *special linear group*. The factor group $\mathrm{GL}(n, q)/\mathrm{SL}(n, q)$ is isomorphic to the multiplicative group of non-zero elements of $\mathrm{GF}(q)$. The centre Z of $\mathrm{GL}(n, q)$ consists of all transformations of the form $T(x) = \lambda x$ for $\lambda \in \mathrm{GF}(q)$ with $\lambda \neq 0$. The factor group $\mathrm{GL}(n, q)/\mathrm{Z}$ is the *projective general linear group* $\mathrm{PGL}(n, q)$. It operates on the projective space of dimension $n - 1$ associated with V. The centre of $\mathrm{SL}(n, q)$ is the subgroup $\mathrm{Z} \cap \mathrm{SL}(n, q)$, and the factor group $\mathrm{SL}(n, q)/(\mathrm{Z} \cap \mathrm{SL}(n, q))$ is the *projective special linear group* $\mathrm{PSL}(n, q)$. The projective special linear groups are generally simple. In fact, $\mathrm{PSL}(n, q)$ is a simple group for all $n \geq 2$, except for the groups $\mathrm{PSL}(2, 2) \cong \mathrm{S}_3$ and $\mathrm{PSL}(2, 3) \cong \mathrm{A}_4$. The order of $\mathrm{PSL}(2, q)$ is given by

$$|\mathrm{PSL}(2, q)| = \frac{q(q^2 - 1)}{(2, q - 1)}$$

where $(2, q - 1)$ denotes the greatest common divisor of 2 and $q - 1$.

We denote the automorphism group of $\mathrm{PSL}(2, q)$ by $\mathrm{P\Gamma L}(2, q)$. It is obtained by adjoining field automorphisms to the transformations of $\mathrm{PGL}(2, q)$. Hence $\mathrm{P\Gamma L}(2, q) = \mathrm{PGL}(2, q) \rtimes \mathrm{Gal}(\mathrm{GF}(q))$ and thus $\mathrm{P\Gamma L}(2, q) = \mathrm{PSL}(2, q) \rtimes (\mathrm{C}_n \times 2)$. Adjoining the field automorphisms to $\mathrm{PSL}(2, q)$ yields a subgroup of $\mathrm{P\Gamma L}(2,q)$ denoted by $\mathrm{P\Sigma L}(2,q)$. Naturally $\mathrm{P\Sigma L}(2,q) = \mathrm{PSL}(2,q) \rtimes \mathrm{Gal}(\mathrm{GF}(q))$.

Observe that $\mathrm{PGL}(2,q)$ and $\mathrm{PSL}(2,q)$ coincide in even characteristic and so do $\mathrm{P\Gamma L}(2,q)$ and $\mathrm{P\Sigma L}(2,q)$.

5.1 The groups PSL(2, q) and PGL(2, q)

These groups were investigated by Leemans and Schulte in [18] and [19] respectively. The subgroup structure of $\mathrm{PSL}(2,q)$ (and hence of $\mathrm{PGL}(2,q)$ as $\mathrm{PGL}(2,q)$ may be seen as a subgroup of $\mathrm{PSL}(2,q^2)$) may be found in Dickson [10] or Huppert [13]. It was first obtained in papers by Moore [22] and Wiman [27].

Theorem 5.1.1. *The group* $\mathrm{PSL}(2,q)$ *of order* $\frac{q(q^2-1)}{(2,q-1)}$, *where* $q = p^r$ *with* p *a prime, contains only:*

1. $q+1$ *conjugate elementary abelian subgroups of order* q, *denoted by* E_q.

2. $\frac{q(q\mp1)}{2}$ *conjugate cyclic subgroups of order* d, *denoted by* d, *for all divisors* d *of* $\frac{(q\pm1)}{(2,q-1)}$.

3. $\frac{q(q^2-1)}{2d(2,q-1)}$ *dihedral groups of order* $2d$, *denoted by* D_{2d}, *for all divisors* d *of* $\frac{(q\pm1)}{(2,q-1)}$ *with* $d > 2$. *The number of conjugacy classes of these subgroups is one if* $\frac{(q\pm1)}{d(2,q-1)}$ *is odd, and two if it is even.*

4. *For* q *odd,* $\frac{q(q^2-1)}{12(2,q-1)}$ *dihedral groups of order 4, denoted by* 2^2. *The number of conjugacy classes of these groups is one if* $q \equiv \pm3(8)$ *and two if* $q \equiv \pm1(8)$. *For* q *even, the groups* 2^2 *are listed under family 5.*

5. $\frac{(2,1,1)(p^k-1)(p^r-1)(p^r-p)\cdots(p^r-p^{s-1})}{(q-1)(p^s-1)(p^s-p)\cdots(p^s-p^{s-1})}$ *sets, each of* $\frac{q^2-1}{(2,1,1)(p^k-1)}$ *conjugate elementary abelian subgroups of order* p^s, *denoted by* E_{p^s}, *for all natural number* s *such that* $1 \le s \le r-1$, *where* $k = (r,s)$ *and* $(2,1,1)$ *is defined as 2, 1 or 1 according as* $p > 2$ *and* $\frac{r}{k}$ *is even,* $p > 2$ *and* $\frac{r}{k}$ *is odd, or* $p = 2$.

6. $\frac{(2,1,1)(p^k-1)(p^r-p)\cdots(p^r-p^{s-1})}{p^{r-s}(q-1)(p^s-1)(p^s-p)\cdots(p^s-p^{s-1})}$ *sets of* $\frac{(q^2-1)p^{r-s}}{(2,1,1)(p^k-1)}$ *subgroups* $E_{p^s} : h$, *each a semidirect product of an elementary abelian group* E_{p^s} *and a cyclic*

group of order h, for all natural numbers s such that $1 \le s \le r$ and all divisors h of $\frac{p^k-1}{(2,1,1)}$, where again $k=(r,s)$ and $(2,1,1)$ is defined as 2, 1 or 1 according as $p>2$ and $\frac{r}{k}$ is even, $p>2$ and $\frac{r}{k}$ is odd, or $p=2$.

7. *For q odd or $q=4^m$, $\frac{q(q^2-1)}{12(2,q-1)}$ alternating groups A_4, of order 12. The number of conjugacy classes of these groups is one if $q \equiv \pm 3(8)$ and two if $q \equiv \pm 1(8)$.*

8. *For $q \equiv \pm 1(8)$, two conjugacy classes of $\frac{q(q^2-1)}{24(2,q-1)}$ symmetric groups S_4, of order 24.*

9. *For $q \equiv \pm 1(5)$, two conjugacy classes of $\frac{q(q^2-1)}{60(2,q-1)}$ alternating groups A_5, of order 60; and for $q=4^m$, one conjugacy class of $\frac{q(q^2-1)}{60(2,q-1)}$ alternating groups A_5. For $q \equiv 0(5)$, the groups A_5 are listed under family 10.*

10. *$\frac{q(q^2-1)}{p^w(p^{2w}-1)}$ groups $\mathrm{PSL}(2,p^w)$, for all divisors w of r. The number of conjugacy classes of these groups is two, one or one according as $p>2$ and $\frac{r}{w}$ is even, $p>2$ and $\frac{r}{w}$ is odd, or $p=2$.*

11. *Two conjugacy classes of $\frac{q(q^2-1)}{2p^w(p^{2w}-1)}$ groups $\mathrm{PGL}(2,p^w)$, for all w such that $2w$ is a divisor of r.*

Observe that when q is even, family 11 of Theorem 5.1.1 is a subfamily of family 10.

Suppose Γ is a string C-group representation of rank r of $\mathrm{PSL}(2,q)$ with distinguished generators $\rho_0,\ldots,\rho_{r-1}$.

The following theorem, due to Sjerve and Cherkassoff, shows when a group $\mathrm{PSL}(2,q)$ or $\mathrm{PGL}(2,q)$ has a string C-group representation of rank 3.

Theorem 5.1.2. *[24] The group $\mathrm{PSL}(2,q)$ can be generated by three involutions, two of which commute, if and only if $q \neq 2,3,7$ or 9.*

The group $\mathrm{PGL}(2,q)$ can be generated by three involutions, two of which commute, if and only if $q \neq 2$.

We now bound the rank.

Theorem 5.1.3. *If q is even, the maximum rank of a string C-group representation for $\mathrm{PSL}(2,q)=\mathrm{PGL}(2,q)$ is 3. If q is odd, the maximum rank of a string C-group representation for $\mathrm{PSL}(2,q)$ and for $\mathrm{PGL}(2,q)$ is 4.*

Proof. If q is even, the centralizer of an involution is an elementary abelian group of order $q = 2^d$ for some d. Therefore, the subgroup $\Gamma_1 = \langle \rho_0 \rangle \times \langle \rho_2, \ldots, \rho_{r-1} \rangle$ is also elementary abelian. This implies that the diagram of Γ is connected if and only if $r = 3$. If Γ is $\mathrm{PSL}(2, q)$ we then readily see that the rank of Γ is at most 3 as Γ is simple. If Γ is $\mathrm{PGL}(2, q)$ then, since we know $\mathrm{PGL}(2, q)$ cannot be written as the direct product of two smaller groups, we also have $r = 3$.

If q is odd, the centralizer of an involution in $\mathrm{PSL}(2, q)$ is a dihedral group D_{q-1} when $q \equiv 1 \mathrm{mod} 4$ or D_{q+1} when $q \equiv 3 \mathrm{mod} 4$. In both cases, the centre is a cyclic group of order 2. Hence the subgroup Γ_1 must be a subgroup of a dihedral group and referring to Classification Theorem 4.1.1, we get that r is at most 4. □

So it remains to consider the rank four with q odd. We assume that G is a group isomorphic to $\mathrm{PSL}(2, q)$ or $\mathrm{PGL}(2, q)$, with $q = p^r$, p an odd prime and r a positive integer. Moreover, we assume that $(G, \{\rho_0, \ldots, \rho_3\})$ is a string C-group of type $\{t, l, s\}$ (i.e. the orders of $\rho_0\rho_1$, $\rho_1\rho_2$ and $\rho_2\rho_3$ are t, l and s, respectively). Clearly, $t, l, s \geq 3$, since G is not a direct product of two non-trivial groups. As before we set

$$G_i = \langle \rho_j \mid j \in \{0, \ldots, 3\} \backslash \{i\} \rangle,$$

for $i = 0, \ldots, 3$.

We say that a subgroup H of G is an (irreducible) *rank* 3 *subgroup* of G if H is a rank 3 string C-group with a connected Coxeter diagram.

By Theorem 5.1.1, the rank 3 subgroups of G are isomorphic to S_4, A_5, or $\mathrm{PSL}(2, q')$ or $\mathrm{PGL}(2, q')$ for some q'. These are the only possible types of subgroups for G_0 and G_3.

We begin with a sequence of lemmas aimed at eliminating $\mathrm{PSL}(2, q')$ and $\mathrm{PGL}(2, q')$ as possibilities.

Lemma 5.1.4. *The orders t of $\rho_0\rho_1$ and s of $\rho_2\rho_3$ must be odd.*

Proof: This is due to the fact that the centralizer of an involution is a dihedral group and its centre is a cyclic group of order 2. □

Lemma 5.1.5. *[18] Let H and K be two subgroups of type* $\mathrm{PSL}(2, q')$ *in* $\mathrm{PSL}(2, q)$*, with ${q'}^m = q$ for some positive integer m. Then $H \cap K$ cannot be a dihedral group D_{2k} with $k > 2$ (and k a divisor of $\frac{(q' \pm 1)}{2}$).*

Proof: Let $k > 2$, and let k be a divisor of $\frac{q'\pm1}{2}$. By Theorem 5.1.1, we know that

- in PSL$(2, q)$, there are $\frac{q(q^2-1)}{q'(q'^2-1)}$ subgroups isomorphic to PSL$(2, q')$;
- in PSL$(2, q)$, there are $\frac{q(q^2-1)}{4k}$ subgroups isomorphic to D_{2k};
- in PSL$(2, q')$, there are $\frac{q'(q'^2-1)}{4k}$ subgroups isomorphic to D_{2k}.

Let $n := \frac{(q'-1)}{2}$ if $k \mid \frac{(q'-1)}{2}$ and $n := \frac{(q'+1)}{2}$ if $k \mid \frac{(q'+1)}{2}$. By Theorem 5.1.1, there are $\frac{q(q^2-1)}{4n}$ subgroups D_{2n} in PSL$(2, q)$. Each subgroup D_{2n} contains $\frac{n}{k}$ subgroups D_{2k}. Therefore, each subgroup D_{2k} is contained in exactly one subgroup D_{2n}. The same kind of arguments show that each D_{2n} is contained in exactly one PSL$(2, q')$. Therefore, each subgroup D_{2k} of PSL$(2, q)$ (with $k > 2$ and k a divisor of $\frac{(q'\pm1)}{2}$) is contained in a subgroup PSL$(2, q')$ of PSL$(2, q)$, and the number of subgroups PSL$(2, q')$ containing a given subgroup D_{2k} is precisely one. Now the lemma follows. □

Lemma 5.1.6. *[18] The subgroups G_0 and G_3 of G cannot be isomorphic to* PSL$(2, q')$*, with ${q'}^m = q$ for some positive integer m.*

Proof: Suppose, without loss of generality, that $G_3 \cong$ PSL$(2, q')$. Then ρ_3 conjugates two subgroups PSL$(2, q')$ of G whose intersection contains a dihedral group of order $2t$; in terms of the underlying polytope, of which G is the automorphism group, these two subgroups are the stabilizers of the two facets which share the 2-face in the base flag. Then this intersection itself, being a subgroup of PSL$(2, q)$, must be a dihedral group. However, this contradicts Lemma 5.1.5. □

Lemma 5.1.7. *The subgroups G_0 and G_3 of G cannot be isomorphic to* PGL$(2, q')$*, with ${q'}^m = q$ for some positive integer m.*

Proof: Suppose that $G_0 \cong$ PGL$(2, q')$. Then $G_0 \cap G_0^{\rho_0} \geq D_{2s}$ with s odd. The normalisers $N_G(D_{2s}) = D_{4s}$ and $N_{G_0}(D_{2s}) = D_{4s}$ as n must divide one of $q' \pm 1$ and the normaliser is twice bigger if $(q' \pm 1)/n$ is even. But then there is a unique subgroup PGL$(2, q')$ containing D_{2s} implying that $G_0 = G_0^{\rho_0}$. But then G_0 is normalised by G, a contradiction. □

We finally have all the tools to prove the following theorem.

Facet	Vertex-figure	Order of G	Structure of G
$\{5,5\}_3$	$\{5,5\}_3$	1	
$\{5,5\}_3$	$\{5,3\}_5$	1	
$\{5,3\}_5$	$\{3,5\}_5$	3420	PSL(2, 19)
$\{5,3\}_5$	$\{3,4\}_3$	60	$A_5 \cong$ PSL(2, 5)
$\{5,3\}_5$	$\{3,3\}_4$	1	
$\{4,3\}_3$	$\{3,4\}_3$	96	$2^4 : S_3$
$\{4,3\}_3$	$\{3,3\}_4$	24	$S_4 =$ PGL(2, 3)
$\{3,5\}_5$	$\{5,3\}_5$	660	PSL(2, 11)
$\{3,4\}_3$	$\{4,3\}_3$	1	
$\{3,3\}_4$	$\{3,3\}_4$	120	$S_5 \cong$ PGL(2, 5)

Table 5.1: Combinations of rank 3 polytopes

Theorem 5.1.8. *[18] Let* $(G, \{\rho_0, \ldots, \rho_3\})$ *be a string C-group. If* $G \cong$ PSL(2, q)*, then* $q = 11$ *or* 19*. If* $G \cong$ PGL(2, q) *then* $q = 5$.

Proof: Lemmas 5.1.5 and 5.1.7 reduce the possible subgroups for G_0 and G_3 to only two kinds, S_4 and A_5.

The only rank 3 polytopes with group S_4 are $\{3,3\}$ $(= \{3,3\}_4)$, $\{3,4\}_3$ and $\{4,3\}_3$, and those with group A_5 are $\{3,5\}_5$, $\{5,3\}_5$ and $\{5,5\}_3$. Recall here from [9] that $\{m,n\}_k$ is obtained from the regular tessellation $\{m,n\}$ by identifying any two vertices that are separated by k steps along a Petrie polygon of $\{m,n\}$. Its group $\langle \tau_0, \tau_1, \tau_2 \rangle$ has a presentation consisting of the standard Coxeter type relations for $\{m,n\}$ and the single extra relation $(\tau_0\tau_1\tau_2)^k = 1$.)

We can now check which pairs of polyhedra can be combined to form the facets and vertex-figures, respectively, of a regular rank 4 polytope. Table 5.1 gives the possible combinations and the structure of the corresponding "universal" groups; these groups are obtained by taking as defining relations just those of the facet group and vertex-figure group as well as $(\rho_0\rho_3)^2 = 1$. Only one from a pair of dual combinations is listed, since dual combinations yield the same groups (with the orders of the generators reversed). The results in this table can easily be obtained using a Computational Algebra package like Magma [1] (or, if necessary, by hand). Finally, by inspection we readily see that the only possibilities for $(G, \{\rho_0, \ldots, \rho_3\})$ to be a string C-group of rank 4 occur when $q = 11$ or $q = 19$. □

Note that the groups occurring in rows 1, 2, 5 and 9 of Table 5.1 are trivial.

In row 4, the group actually is a group $\mathrm{PSL}(2,q)$ but is too small to be a C-group of rank 4. In row 7 we also do not have a C-group of rank 4. Finally, in row 6 we obtain a C-group of rank 4 isomorphic to $2^4 : S_3$, namely the group of the universal locally projective regular 4-polytope $\{\{4,3\}_3, \{3,4\}_3\}$.

In terms of regular polytopes Theorem 5.1.8 can be rephrased as follows.

Theorem 5.1.9. *The only regular polytopes of rank* 4 *with automorphism groups of type* $\mathrm{PSL}(2,q)$ *are the* 11*-cell* $\{\{3,5\}_5, \{5,3\}_5\}$ *with group* $\mathrm{PSL}(2,11)$ *of order* 660*, and the* 57*-cell* $\{\{5,3\}_5, \{3,5\}_5\}$ *with group* $\mathrm{PSL}(2,19)$ *of order* 3420*. The only regular polytope of rank* 4 *with automorphism groups of type* $\mathrm{PGL}(2,q)$ *is the 4-simplex* $\{\{3,3\}_4, \{3,3\}_4\}$ *with group* $\mathrm{PGL}(2,5)$ *of order* 120*.*

5.2 The groups $\Gamma(2,q)$ and $\Sigma(2,q)$

The subgroup structure of $\mathrm{P\Gamma L}(2,q)$ has been extensively studied for instance in [11] or [2].

We first state a lemma that sums up the possible factor groups of an almost simple groups with socle isomorphic to $\mathrm{PSL}(2,q)$.

Lemma 5.2.1. *Let* $q = p^d$ *be a prime power. Then*

$$\mathrm{P\Gamma L}(2,q)/\mathrm{PSL}(2,q) \cong \begin{cases} \mathrm{C}_d & \text{if } p = 2 \\ \mathrm{C}_d \times 2 & \text{if } p \neq 2 \end{cases}$$

As suggested by Lemma 5.2.1, we divide our discussion in two cases according to the parity of d, the power of the prime p.

Case 1: $q = p^{2k+1}$

We can assume without loss of generality that $k \geq 1$ for if $k = 0$ then $q = p$ in which case $\mathrm{P\Gamma L}(2,q) = \mathrm{PGL}(2,q)$. Let $\mathrm{PSL}(2,p^{2k+1}) \leq G \leq \mathrm{P\Gamma L}(2,p^{2k+1})$ and suppose that G is the automorphism group of an abstract regular polytope. It is obvious from Lemma 5.2.1 that all involutions of G lie in the normal subgroup isomorphic to $\mathrm{PSL}(2,p^{2k+1})$ or $\mathrm{PGL}(2,p^{2k+1})$ of G. Hence involutions generate G if and only if G is isomorphic to $\mathrm{PSL}(2,p^{2k+1})$ or $\mathrm{PGL}(2,p^{2k+1})$.

Case 2: $q = p^{2d}$

Lemma 5.2.1 shows that there exist involutions in G that do not lie in $\mathrm{PGL}(2,q)$. They are so-called Baer involutions and will be detailed later.

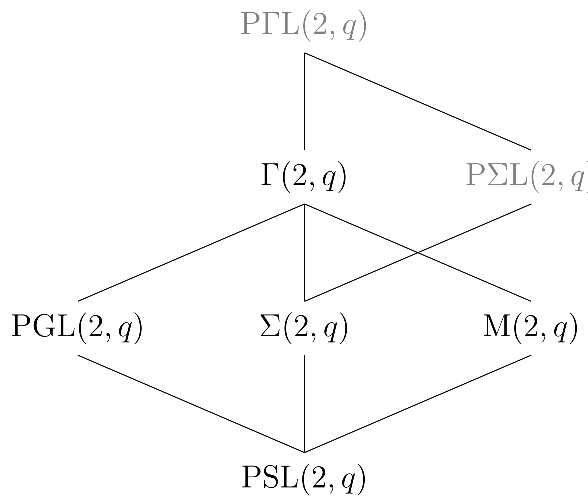


Figure 5.1: $\mathrm{PSL}(2, p^{2d})$ and some overgroups

Moreover it is obvious that the subgroup generated by all involutions of $\mathrm{P\Gamma L}(2, q)$ is a group four times larger than $\mathrm{PSL}(2, q)$in odd characteristic and twice larger in even characteristic.

We make use of the following notation. Let γ be a PGL-involution (i.e. an involution of $\mathrm{PGL}(2, q)$ that is not in $\mathrm{PSL}(2, q)$) of $\mathrm{P\Gamma L}(2, p^{2d})$ and let β be a Baer involution. Then $\Gamma(2, p^{2d}) := \mathrm{PSL}(2, p^{2d}) \rtimes \langle \gamma, \beta \rangle$, $\Sigma(2, p^{2d}) := \mathrm{PSL}(2, p^{2d}) \rtimes \langle \beta \rangle$ and $\mathrm{M}(2, p^{2d}) := \mathrm{PSL}(2, p^{2d}) \cdot \langle \beta\gamma \rangle$. In the latter case, the extension of $\mathrm{PSL}(2, q)$ by $\langle \beta\gamma \rangle$ is non split because this group has a unique conjugacy class of involutions, all in the normal subgroup isomorphic to $\mathrm{PSL}(2, q)$. Hence $\mathrm{M}(2, q)$ is not generated by involutions.

Observe that the notation $\mathrm{M}(2, q)$ comes from the fact that, in $\mathrm{P\Gamma L}(2, 9)$, the group $\mathrm{M}(2, 9)$ is isomorphic to the stabiliser of a point in the degree 11 permutation representation of the Mathieu group M_{11}.

We now assume that $q = p^{2d}$ is a prime power with $d \geq 1$, as we showed that no group $\mathrm{PSL}(2, p^{2k+1}) \leq G \leq \mathrm{P\Gamma L}(2, p^{2k+1})$ other that $\mathrm{PSL}(2, p^{2k+1})$ or $\mathrm{PGL}(2, p^{2k+1})$ is generated by involutions. Consider the group $\mathrm{PSL}(2, q)$ in its natural action on $q + 1$ points.

Let us first assume that p is odd. The group $\mathrm{PSL}(2, q)$ has a unique conjugacy class of involutions. As $q = p^{2d}$, $q - 1$ is divisible by 4 and therefore any involution of $\mathrm{PSL}(2,q)$ lies in the stabilizer of a point, hence fixes at least one point and thus at least two because $q+1$ is even. Since $\mathrm{PSL}(2,q)$ is a subgroup of $\mathrm{PGL}(2,q)$ which is strictly 3-transitive, each involution of $\mathrm{PSL}(2,q)$ fixes exactly two points. The group $\mathrm{PGL}(2,q)$ has two conjugacy classes of involutions, namely those that are in the subgroup $\mathrm{PSL}(2,q)$ of $\mathrm{PGL}(2,q)$ and those that are not. We call involutions of the first kind PSL-involutions and involutions of the latter kind PGL-involutions. The centralizer in $\mathrm{PGL}(2,q)$ of a PGL-involution is isomorphic to $\mathrm{D}_{2(q+1)}$ and has no fixed point. Thus a PGL-involution has no fixed point.

The group $G \cong \mathrm{P\Gamma L}(2,q)$ (with $q = p^{2d}$) has three conjugacy classes of involutions: PSL-involutions, PGL-involutions, and Baer involutions induced by the field automorphism or order 2 which is the composition of d Frobenius automorphisms $x \mapsto x^p$. Consider $\Gamma(2,q)$ the subgroup of $\mathrm{P\Gamma L}(2,q)$ generated by all involutions of $\mathrm{P\Gamma L}(2,q)$. It is a group of order $2q(q^2-1)$ and contains $\mathrm{PGL}(2,q)$ as an index two normal subgroup as shown by Lemma 5.2.1. Since $\mathrm{PGL}(2,q)$ is strictly 3-transitive on $q+1$ points, the pointwise stabilizer in $\mathrm{PGL}(2,q)$ of any three points is trivial. Therefore the pointwise stabilizer in $\Gamma(2,q)$ of three points is of order 2 and yields a Baer involution β. Moreover three points yield a subfield $\mathrm{GF}(\sqrt{q})$. Hence the elements commuting with a Baer involution β act transitively on the $\sqrt{q}+1$ points fixed by β and this action corresponds to the action of $\mathrm{PGL}(2,\sqrt{q})$ on $\sqrt{q}+1$ points. Therefore the centralizer of β in $\mathrm{P\Gamma L}(2,q)$ is $\mathrm{C}_{\mathrm{P\Gamma L}(2,q)}(\beta) \cong \langle\beta\rangle \times \mathrm{PGL}(2,\sqrt{q})$. Obviously $\mathrm{PGL}(2,\sqrt{q}) < \mathrm{PSL}(2,q)$.

In even characteristic, it is well known that $\mathrm{PSL}(2,q) = \mathrm{PGL}(2,q)$. Therefore $\mathrm{P\Sigma L}(2,q) = \mathrm{P\Gamma L}(2,q)$ has two conjugacy classes of involutions: PSL-involutions and Baer involutions. The former ones have exactly one fixed point and the latter ones have $q+1$ fixed points.

Theorem 5.2.2. *[6] Let $q := p^{2d}$ be a prime power with p odd and let $G \cong \Gamma(2,q)$. Then G acts regularly on polytopes of rank 3 only.*

Proof. Let us first show by induction that G does not act regularly on polytopes of rank ≥ 4. By way of contradiction, suppose that G acts regularly on a polytope $\mathcal{P} = \{\rho_0, \ldots, \rho_3\}$ of rank 4. In order to have $\Gamma(2,q) \cong \langle \rho_0, \ldots, \rho_3\rangle$, one of the involutions of $\mathcal{P}$ must be a Baer involution and another must be a PGL-involution. Let us first assume that ρ_0 is a Baer involution β. Since $\mathrm{C}_G(\beta) = \langle\beta\rangle \times \mathrm{PGL}(2,\sqrt{q})$, it is clear that every involution commuting with β must be a PSL-involution or a Baer involution since $\mathrm{PGL}(2,\sqrt{q}) < \mathrm{PSL}(2,q)$.

Hence ρ_1 has to be a PGL-involution γ and ρ_3 has to be a PSL-involution, say σ.

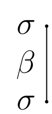
Observe that $\beta\gamma$ is of order divisible by 4. Indeed, $\beta\gamma$ belongs to the left coset $\mathrm{PGL}(2, q)\beta$ in G as does any product of an odd number of $\beta\gamma$, while $(\beta\gamma)^{o(\beta\gamma)} = 1_G$. Hence $o(\beta\gamma)$ is even. Consider the involution

$$\delta := \underbrace{(\beta\gamma)(\beta\gamma)\cdots(\beta\gamma)}_{\frac{o(\beta\gamma)}{2}} = \underbrace{(\gamma\beta)(\gamma\beta)\cdots(\gamma\beta)}_{\frac{o(\beta\gamma)}{2}}.$$

Observe that $\beta\delta = \delta\beta$ and $\gamma\delta = \delta\gamma$, i.e. δ is the central involution of $\langle \beta, \gamma\rangle$. Therefore $\gamma\delta$ is an element of $\mathrm{PGL}(2, q)$. Thus $o(\beta\gamma)/2$ must be even since the product of γ with an odd number of $\beta\gamma$ lies in the left coset $\mathrm{PGL}(2, q)\beta$.

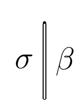
Observe moreover that δ is a PSL-involution since it commutes simultaneously with a Baer involution and with a PGL-involution. Hence δ has exactly two fixed points. We now divide our proof in two cases: in the first case we assume that β and σ do not fix common points; in the second case we assume that β fixes the two points fixed by σ.

Case 1. The CPR graph of $\mathcal{P}$, when restricted to β and σ, is composed with the following connected components, when restricted to β and σ. Since β does not fix the points fixed by σ but commute with σ, it has to swap them. Hence there is exactly one connected component (where loops are not drawn)

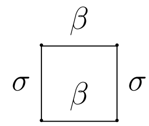


The $\sqrt{q} + 1$ points fixed by β are swapped pairwise by σ. Thus there are $\frac{\sqrt{q}+1}{2}$ connected components

There remains $q - \sqrt{q} - 2$ points. Since β and σ commute, the product $\beta\sigma$ is a Baer involution and thus $\beta\sigma$ has $\sqrt{q} + 1$ fixed points. Hence there are $\frac{\sqrt{q}+1}{2}$ connected components



The remaining $q - 2\sqrt{q} - 3$ points are partitioned into $\frac{q-2\sqrt{q}-3}{4}$ squares



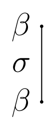
Now γ commutes with σ, hence γ swaps the two points fixed by σ. Thus the PSL-involution δ fixes the points fixed by σ. Since the pointwise stabilizer of two points in $\mathrm{PSL}(2, q)$ is a cyclic group, it has a unique involution, and therefore $\delta = \sigma$. Hence $\langle \beta, \sigma \rangle \leq \langle \beta, \gamma \rangle$ and the intersection property fails. Thus β and σ must have two common fixed points.

Case 2. Assume that β fixes the two points fixed by σ. The CPR graph of $\mathcal{P}$, when restricted to β and σ, is composed with the following connected components. By assumption, there are two double loops suggested by

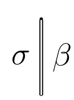
$$\sigma, \beta \cdot$$

$$\sigma, \beta \cdot$$

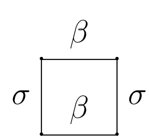
The remaining $\sqrt{q} - 1$ points fixed by β are swapped pairwise by σ. Thus there are $\frac{\sqrt{q}-1}{2}$ connected components



Since β and σ commute, the product $\beta\sigma$ is a Baer involution and thus $\beta\sigma$ has $\sqrt{q} + 1$ fixed points. Hence there are $\frac{\sqrt{q}-1}{2}$ double edges



The remaining $q - 2\sqrt{q} + 1$ points are partitioned into $\frac{q-2\sqrt{q}+1}{4}$ squares



Now γ swaps the two points fixed by σ (that are also fixed by β). Since $o(\beta\gamma) \equiv 0 \bmod 4$, δ fixes the two points fixed by σ. As in Case 1, we conclude that $\delta = \sigma$ and $\langle \beta, \sigma \rangle \leq \langle \beta, \gamma \rangle$ which contradicts the intersection property.

Notice that these arguments apply also if $\gamma = \rho_0$ and $\beta = \rho_1$ to get to a contradiction. Whence if a polytope of rank 4 exists for G, say $\mathcal{P} = \{\rho_0, \ldots, \rho_3\}$, the conjugacy classes to which the involutions ρ_i's belong are completely determined up to duality: $\tau := \rho_0$ and $\sigma := \rho_3$ must be PSL-involutions, $\gamma := \rho_1$ must be a PGL-involution and ρ_2 must be a Baer involution. Consider the subgroup $\langle \tau, \gamma \rangle$. Obviously $\tau\gamma$ has even order and we define the involution

$$\delta := \underbrace{(\tau\gamma)(\tau\gamma)\cdots(\tau\gamma)}_{\frac{o(\tau\gamma)}{2}}.$$

Since both τ and γ commute with σ, they must swap the two points fixed by σ. Hence δ fixes those two points and we conclude that $\delta = \sigma$. Therefore $\langle \gamma, \sigma \rangle \leq \langle \tau, \gamma \rangle$ and the intersection property is violated.

This shows that there does not exist string C-group representations of G of rank 4. Using these arguments, it is now trivial to show by induction that G does not act on abstract regular polytopes of rank ≥ 4.

Finally we have to show that there exist string C-group representations of G of rank 3. Let γ be a PGL-involution and σ a PSL-involution that generate a subgroup isomorphic to $\mathrm{D}_{2(q+1)}$ which is a maximal subgroup of $\mathrm{PGL}(2, q) < G$. Let $\beta \in \mathrm{C}_G(\sigma)$ be a Baer involution. It obviously does not commute with γ. Moreover $\langle \beta, \gamma, \sigma \rangle$ cannot be a proper subgroup of G for otherwise it has to be $\mathrm{N}_G(\langle \gamma, \sigma \rangle)$ and therefore β normalizes $\langle \gamma, \sigma \rangle$, a contradiction. Whence β, γ and σ generate the whole of G. Since $q - 1 \equiv 0 \mathrm{mod} 4$, $\frac{q+1}{2}$ is odd. Suppose that $\chi \in \langle \gamma\sigma \rangle \cap \langle \beta\gamma \rangle$. Observe that χ belongs to $\mathrm{PSL}(2, q)$ since any element $(\gamma\sigma)\ldots(\gamma\sigma)$ belongs to $\mathrm{PSL}(2, q)$ or its coset $\mathrm{PSL}(2, q)\gamma$ in $\mathrm{PGL}(2, q)$, and similarly any element $(\gamma\beta)\ldots(\gamma\beta)$ belongs to $\mathrm{PSL}(2, q)$ or its coset $\mathrm{PSL}(2, q)\beta$ in $\mathrm{PSL}(2, q) \cdot \langle \beta\gamma \rangle$. Now we make use of Theorem 1.5 of [11] to observe that $\langle \beta, \gamma \rangle$ is a subgroup of $\mathrm{N}_G(\mathrm{D}_{2(q-1)})$. Thus $o(\gamma\beta) \mid q - 1$ and $o(\gamma\beta) \nmid q + 1$ since $4 \mid o(\gamma\beta)$. Hence $(o(\gamma\beta), o(\gamma\sigma)) = 2$ and therefore $\chi = 1_G$ or χ is an involution. If χ is an involution, then

$$\chi = \underbrace{(\gamma\sigma)\ldots(\gamma\sigma)}_{\frac{q+1}{2}}$$

but since $\frac{q+1}{2}$ is odd, χ belongs to the coset $\mathrm{PSL}(2,q)\gamma$, a contradiction. Thus $\chi = 1_G$. This shows $\langle\gamma,\sigma\rangle \cap \langle\gamma,\beta\rangle = \langle\gamma\rangle$. By Proposition 2E16 of [21], this is the only intersection condition to check in order to ensure that $\mathcal{P} := (G, \{\beta,\gamma,\sigma\})$ is an abstract regular polytope as the remaining ones hold trivially. □

Theorem 5.2.3. *[6] Let $q = p^{2d}$ be a prime power and let $G \cong \Sigma(2,q)$.*

- *If $q = 4$ then $G \cong \mathrm{S}_5$ has polytopes of ranks 3 and 4.*
- *If $q = 9$ then $G \cong \mathrm{S}_6$ has polytopes of ranks 3, 4 and 5.*
- *If $q \geq 16$ then G has polytopes of ranks 3 and 4.*

Proof. In case $q = 4$ or 9, it is straightforward to check the result with Magma [1]. We may also refer to [20]. Let us assume from now on that $q \geq 16$. It follows from Giudici [11] that G has no subgroup isomorphic to a direct product of two dihedral subgroups of orders at least 6. Hence the bound of the rank of an abstract regular polytope for G is 4. Let us show that this bound is sharp. Consider a Baer involution β of G and its centralizer $\mathrm{C}_G(\beta) \cong \mathrm{C}_2 \times \mathrm{PGL}(2,\sqrt{q})$. It follows from Conder, et al. [**?**, Theorem 6.1] that $\mathrm{C}_2 \times \mathrm{PGL}(2,q')$ always has polytopes of rank 3 with Schläfli symbols $\{q'+1, q'-1\}$ for any $q' \geq 5$. Hence there is a polytope $\{\beta\sigma_0, \beta\sigma_1, \beta\sigma_2\}$ for $\mathrm{C}_G(\beta)$ with Schläfli symbols $\{\sqrt{q}+1, \sqrt{q}-1\}$ where the σ_i's are involutions of $\mathrm{PGL}(2,\sqrt{q})$. Since $\langle\beta\sigma_1, \beta\sigma_2\rangle \cong \langle\sigma_1,\sigma_2\rangle \cong \mathrm{D}_{2(\sqrt{q}-1)}$, the involutions σ_1 and σ_2 have two fixed points in their permutation representations on $\sqrt{q}+1$ points. Hence they are PSL-involutions in $\mathrm{PGL}(2,\sqrt{q})$ and thus are conjugate by an element $\chi \in \mathrm{PGL}(2,\sqrt{q})$, i.e. $\sigma_1^\chi = \sigma_2$. Moreover χ and β commute and the following holds:

$$\begin{aligned}(\beta\sigma_1)^{\chi\beta} &= \chi\sigma_1\beta\chi^{-1}\\ &= \beta\chi\sigma_1\chi^{-1}\\ &= \beta\sigma_2\end{aligned}$$

Therefore $\chi\beta$ normalizes $\langle\beta\sigma_1, \beta\sigma_2\rangle$ and swaps $\beta\sigma_1$ and $\beta\sigma_2$ by conjugation. Set $\rho_i := \beta\sigma_i$ for $i \in \{0,1,2\}$ and $\rho_3 := \rho_0^{\chi\beta}$. Then ρ_3 is a Baer involution different from ρ_0. The 4-tuple $\{\rho_0,\rho_1,\rho_2,\rho_3\}$ yields an abstract regular polytope for $\langle\rho_0,\rho_1,\rho_2,\rho_3\rangle = G$ (because $\mathrm{C}_G(\beta)$ is a maximal subgroup of G) and

$$\langle\rho_0,\rho_1,\rho_2\rangle \cap \langle\rho_1,\rho_2,\rho_3\rangle = \langle\rho_1,\rho_2\rangle.$$

Again, by Proposition 2E16 of [21], the intersection property therefore holds. The Schläfli symbols of this polytopes are $\{\sqrt{q}+1, \sqrt{q}-1, \sqrt{q}+1\}$. Observe that this polytope is self-dual.

Now let us show that G always has string C-group representations of rank 3. Let us assume first that q is odd. Let σ, τ be two PSL-involutions that generate a subgroup isomorphic to D_{q+1}. Let β be a Baer involution that commutes with τ but not with σ. Obviously, $\langle\beta,\sigma,\tau\rangle = \Sigma(2,q)$, for otherwise it has to generate $\mathrm{N}_G(\mathrm{D}_{q+1})$, a contradiction. Moreover, $\langle\beta,\sigma\rangle\cap\langle\sigma,\tau\rangle = \langle\sigma\rangle$. Indeed, let χ belong to $\langle\beta\sigma\rangle \cap \langle\sigma\tau\rangle$. By assumption, $o(\sigma\tau) = \frac{q+1}{2}$ is odd, while $o(\beta\sigma)$ is even. Moreover either $o(\beta\sigma)$ divides $q-1$ (in case β and σ do not fix common points) or p divides $o(\beta\sigma)$ (in case β fixes the two fixed points of σ) as it is clear by looking at Theorem 5 in [11]. In either case we conclude that $(o(\beta\sigma), o(\sigma\tau)) = 1$ and thus $\chi = 1_G$. Therefore, by Proposition 2E16 of [**?**], the intersection property is satisfied. In conclusion, $\{\beta,\sigma,\tau\}$ yields an abstract regular polytope of rank 3 with automorphism group isomorphic to $\Sigma(2,q)$.

Assume now that q is even and let σ and τ be two PSL-involutions that generate a subgroup isomorphic to $\mathrm{D}_{2(q+1)}$. We then play the same game as before by taking a Baer involution β that commutes with σ and not with τ. □

We can now state our classification theorem for almost simple groups with socle $\mathrm{PSL}(2,q)$.

Theorem 5.2.4. *[6] Let* $\mathrm{PSL}(2,q) \le G \le \mathrm{P\Gamma L}(2,q)$ *be an almost simple group acting regularly on abstract polytopes. Then*

1. *if* $q = 2$ *then* $G \cong \mathrm{PSL}(2,2) \cong S_3$ *and* G *has a unique rank* 2 *abstract regular polytope, namely the triangle;*

2. *if* $q = 3$ *then* $G \cong \mathrm{PGL}(2,3) \cong \mathrm{S}_4$ *and* G *acts on polytopes of rank* 3 *only;*

3. *if* $q = 4$ *or* 5 *then either* $G \cong \mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong \mathrm{A}_5$ *and* G *acts on polytopes of rank* 3 *only, or* $G \cong \mathrm{PGL}(2,5) \cong \mathrm{S}_5$ *and* G *acts on polytopes of ranks* 3 *and* 4*;*

4. *if* $q = 7$ *then* $G \cong \mathrm{PGL}(2,7)$ *and* G *acts on polytopes of rank* 3 *only;*

5. *if* $q \ge 8$ *then*

(a) *if* $q = 2^{2k+1}$, $k \geq 1$, *then* $G \cong \mathrm{PSL}(2, 2^{2k+1})$ *and* G *acts on polytopes of rank* 3 *only;*

(b) *if* $q = 9$ *then either* $G \cong \mathrm{PGL}(2,9)$, *or* $G \cong \mathrm{P\Sigma L}(2,9) \cong \mathrm{S}_6$, *or* $G \cong \mathrm{P\Gamma L}(2,9)$, *and* G *acts on polytopes of rank* 3*; moreover* $\mathrm{P\Sigma L}(2,9)$ *acts on polytopes of ranks* 3*,* 4 *and* 5*;*

(c) *if* $q = p^{2k+1} \geq 11$, p *an odd prime and* $k \geq 0$, *then* $G \cong \mathrm{PSL}(2, p^{2k+1})$ *or* $G \cong \mathrm{PGL}(2, p^{2k+1})$*; in either case,* G *acts on polytopes of rank* 3*; if moreover* $q = 11$ *or* 19 *then* $G \cong \mathrm{PSL}(2,q)$ *acts on polytopes of rank* 4*;*

(d) *if* $q = p^{2k} \geq 16$, p *any prime and* $k \geq 1$ *then either* $G \cong \mathrm{PSL}(2, p^{2k})$ *or* $G \cong \mathrm{PGL}(2, p^{2k})$ *or* $G \cong \mathrm{PSL}(2, p^{2k}) \rtimes \langle \beta \rangle$ *or* $G \cong \mathrm{PGL}(2, p^{2k}) \rtimes \langle \beta \rangle$, *where* β *is a Baer involution of* $\mathrm{P\Gamma L}(2, p^{2k})$*; in all four cases,* G *acts on polytopes of rank* 3*; moreover,* $\mathrm{PSL}(2, p^{2k}) \rtimes \langle \beta \rangle$ *acts on polytopes of rank* 4*.*

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