1. Introduction

Let Γ be a connected graph and let G be a subgroup of the automorphism group of Γ. The general aim of this chapter is to obtain a way to generate all connected graphs Γ' having a group of automorphisms G' resembling the "local" structure of G. The word "resembling" and "local" should become clear by the end of the first lecture. However it is good to have an example in mind: say Γ is the Petersen graph and G is the alternating group $\mathrm{Alt}(5)$ of degree 5. Recall that the Petersen graph can be thought of as the graph with vertex set the collection of 2-subsets of $\{1,2,3,4,5\}$ where two such subsets x and y are declared to be adjacent if $x \cap y = \emptyset$. (In other words, the Petersen graph is a special case of a Kneser graph.) In particular, the stabilizer G_x of the vertex $x = \{1,2\}$ is the group $\langle (1,2)(3,4), (1,2)(3,5)\rangle$ isomorphic to the symmetric group $\mathrm{Sym}(3)$ of degree 3, and the stabiliser G_e of the edge $e = \{\{1,2\},\{3,4\}\}$ is the group $\langle (1,2)(3,4), (1,3)(1,4)\rangle$ isomorphic to an elementary abelian group of order 4. Moreover, the stabiliser of the arc $a = (\{1,2\},\{3,4\})$ is $G_a = G_x \cap G_e = \langle (1,2)(3,4)\rangle$ is cyclic of order 2. (An arc in a graph is an ordered pair of adjacent vertices.) Broadly speaking, in this example, the local structure of G is the triple (G_x, G_e, G_a) and we are interested in describing a way to obtain all connected graphs Γ' (finite or infinite) admitting a group of automorphisms G' looking locally as (G_x, G_e, G_a), that is, $G_{x'} \cong G_x$, $G_{e'} \cong G_e$ and $G_{a'} \cong G_a$ for some vertex x', for some edge e' and some arc a' of Γ'.

We follow closely and almost verbatim the wonderful book of Dicks and Dunwoody [3]. All the material (and the notation) of this course is taken from [3], the book has too much material to present all here and we advise the reader to consult [3] for a better and deeper understanding of this area of group actions on graphs. Another natural reference for this part is the book of Serre [4] or the "Green book" [2].

2. Group actions

Given a group G, for us a ***G-set*** X is a non-empty set X together with an action of G on X, that is, a function

$$\begin{aligned} G \times X &\to X \\ (g,x) &\mapsto gx, \end{aligned}$$

such that

- **(i):** $1x = x$ for every $x \in X$,
- **(ii):** $g'(gx) = (g'g)x$ for every $x \in X$ and for every $g, g' \in G$.

Given $x \in X$, the ***G-orbit*** of x is $Gx = \{gx \mid g \in G\}$. The ***stabilizer*** of x is the subgroup $G_x = \{g \in G \mid gx = x\}$ of G. For the ***quotient set*** of the G-set X, we mean $X/G = \{Gx \mid x \in X\}$, that is, the set of G-orbits of G on X.

Exercise 2.1. Let G be a group, let H be a subgroup of G and let X be the set of left cosets of H in G. (That is, $X = \{gH \mid g \in G\}$.) Define a function $G \times X \to X$ by $(x, yH) \mapsto xyH$. Prove that this function defines an action of G on X.

A ***G-transversal*** in X is a subset S of X which intersects each G-orbit exactly once, so S is a set of representatives for the action of G on X. (Here recall that G-orbits are pairwise disjoint and their union is the whole of X.)

If X and Y are two G-sets, then a function $\alpha : X \to Y$ is a ***G-map*** if $\alpha(gx) = g\alpha(x)$, for every $x \in X$ and for every $g \in G$. If α is bijective, we say that X and Y are ***isomorphic*** G-sets.

Exercise 2.2. Let X be a G-set and let S be a G-transversal of X. Consider the set

$$Y = \bigcup_{s \in S} G/G_s.$$

Here G/G_s consists of the set of left cosets of G_s in G, that is, $G/G_s = \{gG_s \mid g \in G\}$. Moreover, here the union has to be understood a disjoint union, that is, if s, s' are two distinct elements of S with $G_s = G_{s'}$ then we take two copies of the coset space $G/G_s = G/G_{s'}$, one copy labelled with s and the other copy labelled with s'.

Consider the map $\alpha : X \to Y$ defined by $\alpha(gx) = gG_x$, for every $x \in S$ and $g \in G$. Prove that α is well-defined and that α is an isomorphism of G-sets.

Exercise 2.3. Let G be a group and set $X = G$. Now the multiplication in G defines an action $G \times X \to X$ of G on X via $(g, x) \mapsto gx$. Prove that X is a G-set. This is a special case of Exercise 2.1

The following is more than an exercise and is taken from [1].

Exercise 2.4. Prove that the alternating group on 8 symbols $\mathrm{Alt}(8)$ and the special linear group $\mathrm{SL}_4(2)$ are isomorphic.

The projective special linear group $\mathrm{PSL}_3(2) = \mathrm{SL}_3(2)$ is simple and acts on the Fano plane, that is, on the projective plane over the finite field $\mathbb{F}_2$. Now, under this action, $\mathrm{PSL}_3(2)$ is a subgroup of $\mathrm{Sym}(7)$, and since $\mathrm{PSL}_3(2)$ is simple we actually have $\mathrm{PSL}_3(2) \leq \mathrm{Alt}(7)$. Actually it is important to note that $\mathrm{PSL}_3(2)$ is the automorphism group of the Fano plane, or, in other words, the stabilizer in $\mathrm{Sym}(7)$ of the Fano plane is $\mathrm{PSL}_3(2)$. See Figure 1. Moreover, since $|\mathrm{Sym}(8) :$

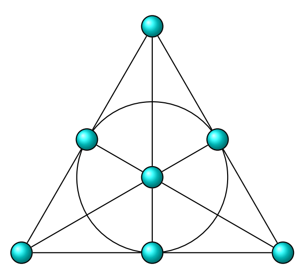

FIGURE 1. Fano plane

$\mathrm{PSL}_3(2)| = 30$, there exist exactly 30 different labellings of the Fano plane, or in other words there are 30 different ways to assign the structure of a projective plane to the set $\{1, 2, 3, 4, 5, 6, 7\}$ of seven points. (Another way to see this is to observe that $\mathrm{PSL}_3(2)$ is the stabilizer of a Fano plane being the automorphism group of a Fano plane.) In particular, as $\mathrm{PSL}_3(2) \leq \mathrm{Alt}(7)$, we see that these 30 Fano planes fall into two orbits, each of size 15.

Let Ω be one of the $\mathrm{Alt}(7)$-orbits. Consider the set

$$\mathcal{S} = \{(\ell, \Pi) \mid \Pi \in \Omega, \ell \subseteq \{1, 2, 3, 4, 5, 6, 7\}, |\ell| = 3,\ \ell \text{ line of } \Pi\}.$$

Each plane contains seven lines and hence there are $15 \times 7 = 105$ pairs (ℓ, Π), where ℓ is a 3-subset of $\{1,2,3,4,5,6,7\}$, $\Pi \in \Omega$ and ℓ is a line of Π. That is $|\mathcal{S}| = 105$. As $\mathrm{Alt}(7)$ acts transitively on 3-subsets of $\{1,2,3,4,5,6,7\}$, we deduce that each of the $\binom{7}{3} = 35$ triples of $\{1,2,3,4,5,6,7\}$ is a line for exactly $\frac{105}{35} = 3$ planes in Ω.

Now we define a new incident structure $\mathcal{G}$: the "points" are the elements of Ω and the "lines" are the triples containing a fixed 3-subset ℓ.

We would like to identify the "line" Π_1, Π_2, Π_3 with the 3–subset ℓ in common to Π_1, Π_2, Π_3. To do this we have to make sure that Π_1, Π_2, Π_3 do not have two 3-subsets ℓ_1 and ℓ_2 in common. If this were the case, then we may assume that $\ell_1 = \{1,2,3\}$ and $\ell_2 = \{1,4,5\}$. Therefore $\ell_3 = \{1,6,7\}$ must also be a line of Π_1, Π_2, Π_3. Relabeling the index set, we may assume that $\ell_4 = \{2,4,6\}$ is a line of Π_1. Now it is easy to see that Π_1 is uniquely determined and consists of

$$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,5,6\},\{3,4,7\}.\}$$

Thus we may assume that ℓ_4 is not a line of Π_2 and Π_3. Therefore the line in Π_2 and Π_3 through 2 and 4 must be $\{2,4,7\}$; again, there exists a unique Fano plane having ℓ_1, ℓ_2, ℓ_3 and $\{2,4,7\}$ as lines and hence $\Pi_2 = \Pi_3$.

This shows that $\mathcal{G}$ has 15 "points" and $\binom{7}{3} = 35$ "lines".

Observe that given any two "points" Π_1 and Π_2 there exists at most one "line" passing through Π_1 and Π_2 (this is simply due to the fact that each 3-set of $\{1,2,3,4,5,6,7\}$ is contained in only three Fano planes, see the counting argument on the set $\mathcal{S}$ above) and hence if there is a "line" passing through Π_1 and Π_2 it must be unique. We claim that any two distinct "points" is in exactly one "line". Consider

$$\mathcal{S}' = \{(\Pi_1, \Pi_2, \ell) \mid \Pi_1, \Pi_2 \in \Omega,\ \Pi_1 \neq \Pi_2,\ \ell \text{ line of } \Pi_1 \text{ and } \Pi_2\}.$$

Take Π_1 and Π_2 in Ω and suppose (by contradiction) that Π_1 and Π_2 have two distinct lines in common, say $\ell_1 = \{1,2,3\}$ and $\ell_2 = \{1,4,5\}$. Arguing as above we see that (up to a relabelling) the lines of Π_1 are

$$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,5,6\},\{3,4,7\}.\}$$

and the lines of Π_2

$$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}.\}.$$

Now $(6,7)$ is an odd permutation mapping Π_1 to Π_2, contradicting $\Pi_1, \Pi_2 \in \Omega$. Therefore two distinct Fano planes in Ω have at most one line in common. Therefore

$$|\mathcal{S}'| = \sum_{\Pi_1,\Pi_2 \in \Omega, \Pi_1 \neq \Pi_2} |\Pi_1 \cap \Pi_2| \leq \sum_{\Pi_1,\Pi_2 \in \Omega, \Pi_1 \neq \Pi_2} 1 = 15 \times 14 = 210.$$

Since each 3-set ℓ is in exactly 3 elements of Ω, we have

$$|\mathcal{S}'| = \binom{7}{3} \times 3 \times 2 = 35 \times 6 = 210.$$

Therefore any two distinct elements of Ω do have a line in common, that is, any two "points" of $\mathcal{G}$ are in exactly one "line" of $\mathcal{G}$. We aim to prove that the incidence structure $\mathcal{G}$ is isomorphic to the projective space $PG(3,2)$. So far we have shown that it contains the right number of points (namely 15), the right number of lines (namely 35) and that any two "points" are in a unique "line".

Fix $A = \{0\} \cup \Omega$, where 0 has to be considered just a symbol. We define a binary operation $+ : A \times A \to A$ by

(i): $0 + \Pi = \Pi + 0 = \Pi$, for each $\Pi \in \Omega$;
(ii): $\Pi + \Pi = 0$, for each $\Pi \in \Omega$;
(iii): $\Pi_1 + \Pi_2 = \Pi_3$, for each $\Pi_1, \Pi_2 \in \Omega$ with $\Pi_1 \neq \Pi_2$, where $\{\Pi_1, \Pi_2, \Pi_3\}$ is a "line" of $\mathcal{G}$.

Clearly, 0 is the identity element of A, $+$ is commutative and each element has an inverse. We claim that $+$ is associative. Obseve that from this it follows that A is isomorphic to $\mathbb{F}_2^4$. Moreover, the definition of $+$ will imply that $\mathcal{G}$ is the projective geometry over A and hence $\mathcal{G}$ is isomorphic to $PG(3,2)$. We postpone this claim until the end of the exercise.

By construction $\mathrm{Alt}(7)$ acts on $\mathcal{G}$ and hence $\mathrm{Alt}(7) \leq \mathrm{Aut}(\mathcal{G}) = \mathrm{PSL}_4(2) = \mathrm{SL}_4(2)$. From an order argument, we see that $\mathrm{Alt}(7)$ has index 8 in $\mathrm{SL}_4(2)$. Thus by considering the action of $\mathrm{SL}_4(2)$ on the left cosets of $\mathrm{Alt}(7)$, we obtain an embedding of $\mathrm{SL}_4(2)$ in $\mathrm{Sym}(8)$. This shows that $\mathrm{SL}_4(2) \cong \mathrm{Alt}(8)$.

It remains to prove that $+$ is associative. By construction $+$ is associative on collinear triples, therefore it remains to establish that $+$ is associative on non-collinear triples (Π_1, Π_2, Π_3). Here we leave some details to check. Given a Fano plane Π' on the other $\mathrm{Alt}(7)$-orbit, we construct an injective mapping

$$f_{\Pi'} : \{1,2,3,4,5,6,7\} \to \Omega.$$

Take $x \in \{1,2,3,4,5,6,7\}$, consider the three lines ℓ_1, ℓ_2, ℓ_3 of Π' containing x. Implicitly we have seen above that there exists only two Fano planes containing ℓ_1, ℓ_2, ℓ_3: one Fano plane per each $\mathrm{Alt}(7)$-orbit, one Fano plane is Π', the other is $f_{\Pi'}(x)$. It is not hard to see that $f_{\Pi'}$ takes lines in Π' in "lines" in $\mathcal{G}$. Therefore the image of $f_{\Pi'}$ forms a subspace of $\mathcal{G}$ isomorphic to Π' and hence to $PG(2,2)$. Clearly, we can construct 15 such subspaces/subplanes, one for each Fano plane not in Ω. Now each triple of non-collinear "points" of $\mathcal{G}$ is in one of these "planes", which are isomorphic to the Fano plane $PG(2,2)$. From this it is easy to deduce that $+$ is associative.

3. Graphs and G-graphs

Definition 3.1. Let G be a group. A G-***graph*** is a 5-tuple (X, V, E, ι, τ), where X is a G-set, $X = V \cup E$, $V \cap E = \emptyset$, and $\iota, \tau : E \to V$ are two G-maps. The elements of V are called ***vertices*** and the elements of E are called ***edges***. If $e \in E$, then ιe is called the initial vertex of e and τe is called the terminal vertex of e. The definition allows the possibility that $\iota e = \tau e$, in this case we say that e is a loop. The definition also allows the possibility that $\iota e = \iota e'$ and $\tau e = \tau e'$ for two distinct edges $e, e' \in E$, in this case in a "geometric" interpretation (for instance as a CW-complex) of the graph we can think of having two multiple edges from ιe to τe, one labelled e and the other labelled e'. (Here, for avoiding cumbersome notation, we write ιe for $\iota(e)$ and similarly, we write τe for $\tau(e)$.)

In most of our examples the function $(\iota, \tau) : E \to V \times V$ defined by $e \mapsto (\iota e, \tau e)$ is injective, but this assumption is not necessary to develop the theory. Observe that when (ι, τ) is injective, we have $G_e = G_{\iota e} \cap G_{\tau e}$ for every $e \in E$.

When the word G is omitted we assume that $G = 1$.

Given a subset Y of X, we write $VY = Y \cap V$ and $EY = Y \cap V$. If Y is non-empty an if, for each $e \in EY$, we have $\iota e, \tau e \in VY$, then Y is a ***subgraph*** of X.

The ***quotient graph*** X/G is the graph $(X/G, V/G, E/G, \bar{\iota}, \bar{\tau})$ where $\bar{\iota} : E/G \to V/G$ and $\bar{\tau} : E/G \to V/G$ are defined by $\bar{\iota}(Ge) = G(\iota e)$ and $\bar{\tau}(Ge) = G(\tau e)$.

Exercise 3.2. Prove that the functions $\bar{\iota}$, $\bar{\tau}$ are well-defined and that the 5-tuple $(X/G, V/G, E/G, \bar{\iota}, \bar{\tau})$ is indeed a graph with the above definition.

Let G be a group. Now G is a G-set via the natural action of G on itself by left multiplication: $G \times G \to G$ defined by $(g, x) \mapsto gx$. (See Exercise 2.3.) Let S be a subset of G, with this terminology and notation, the ***Cayley graph*** $\mathrm{Cay}(G, S)$ of G with connection set S is the G-graph with $V = G$, $E = G \times S$, $\iota : E \to V$ defined by $\iota(g, s) = g$, $\tau : G \times S \to G$ defined by $\tau(g, s) = gs$. The quotient graph X/G consists of a single vertex and with $|S|$ edges (actually, $|S|$ loops).

For practising let us turn the Petersen graph structure into a G-graph, where $G = \mathrm{Alt}(5)$, with this notation and terminology. We have two ways to do this. We give the first way: $V = \{x \subseteq \{1, 2, 3, 4, 5\} \mid |x| = 2\}$, $E = \{(x, y) \mid x, y \in V, x \cap y = \emptyset\}$, $\iota : E \to V$ is defined by $\iota(x, y) = x$ and $\tau(x, y) = y$. Now, X/G has only one vertex and only one edge/loop.

Exercise 3.3. In the construction of the Petersen graph above use $G = \langle (1, 2, 3, 4, 5)\rangle$ and describe X/G. Same question with $G = \langle (1, 2, 3, 4, 5), (2, 3, 5, 4)\rangle$ and $G = \langle (1, 2, 3, 4, 5), (2, 5)(3, 4)\rangle$.

Another (more standard) way to encode the Petersen graph structure through G-graphs is via the idea of subdivision graph. Here

$$V = \{x \subseteq \{1, 2, 3, 4, 5\} \mid |x| = 2\} \cup \{\{x, y\} \mid x, y \subseteq \{1, 2, 3, 4, 5\}, |x| = |y| = 2, x \cap y = \emptyset\},$$
$$E = \{(x, \{x, y\}) \mid x, y \subseteq \{1, 2, 3, 4, 5\}, |x| = |y| = 2, x \cap y = \emptyset\},$$

$\iota : E \to V$ is defined by $\iota(x, \{x, y\}) = x$ and $\tau : E \to V$ is defined by $\tau(x, \{x, y\}) = \{x, y\}$. When $G = \mathrm{Alt}(5)$, X/G consists of two vertices (namely $\{x \subseteq \{1, 2, 3, 4, 5\} \mid |x| = 2\}$ and $\{\{x, y\} \mid x, y \subseteq \{1, 2, 3, 4, 5\}, |x| = |y| = 2, x \cap y = \emptyset\}$ and with one edge directed from the first G-orbit to the second G-orbit.

4. Trees and fundamental G-transversals

Let $X = (X, V, E, \iota, \tau)$ be a graph. We write $EX^+ = \{(e, 1) \mid e \in E\}$ and $EX^- = \{(e, -1) \mid e \in E\}$. (For simplicity we write $e^1 = (e, 1)$ and $e^{-1} = (e, -1)$.) The incident functions ι and τ can be extended to the set $EX^+ \cup EX^-$ by setting $\iota e^1 = \tau e^{-1} = \iota e$ and $\iota e^{-1} = \tau e^1 = \tau e$, for every $e \in EX$. In a geometric interpretation of e^1 and e^{-1}, we can think of e^1 and e^{-1} as travelling along e in the "right" way (for e^1) and in the "wrong" way (for e^{-1}).

A ***path*** in X is a finite sequence

$$v_0, e_1^{\varepsilon_1}, v_1, e_2^{\varepsilon_2}, \dots, e_n^{\varepsilon_n}, v_n$$

such that

- **(i):** $n \in \mathbb{N}$;
- **(ii):** $v_i \in VX$ for every $i \in \{0, 1, \dots, n\}$;
- **(iii):** $e_i^{\varepsilon_i} \in EX^+ \cup EX^-$, $\iota e_i^{\varepsilon_i} = v_{i-1}$ and $\tau e_i^{\varepsilon_i} = v_i$ for every $i \in \{1, \dots, n\}$.

The integer n is called the ***length*** of the path p. Moreover, the functions ι, τ can be extended to the set of all paths by setting $\iota p = v_0$ and $\tau p = v_n$. Actually we can simplify the notation here, slightly. We can abbreviate the path p by the string $e_1^{\varepsilon_1}, \dots, e_n^{\varepsilon_n}$. Here we should take some care: when $n = 0$, the path is empty and hence to recover the original information we need to specify v_0 (or v_0 should be clear from the context); when $n \geq 1$, the vertices on the path p can be recovered from the abbreviated notation by computing $\iota e_i^{\varepsilon_i}$ and $\tau e_i^{\varepsilon_i}$.

If $n \geq 1$ and, for each $i \in \{1, \ldots, n-1\}$, we have $e_{i+1}^{\varepsilon_{i+1}} \neq e_i^{-\varepsilon_i}$, then we say that p is ***reduced***. If p is not reduced and $e_{i+1}^{\varepsilon_{i+1}} = e_i^{-\varepsilon_i}$ (for some $i \in \{1, \ldots, n-1\}$), then $e_1^{\varepsilon_1}, \ldots, e_{i-1}^{\varepsilon_{i-1}}, e_{i+2}^{\varepsilon_{i+2}}, \ldots, e_n^{\varepsilon_n}$ is a path of length $n-2$ in X (still) from v_0 to v_n.

A graph X is a ***tree*** if, for every $v, w \in V$, there exists a unique reduced path from v to w. If Y is a subgraph of X and Y is a tree, then we say that Y is a ***subtree*** of X.

A path p is ***closed*** if $\iota p = \tau p$; in particular, paths of length zero are closed. Moreover, p is ***simple closed*** if p is closed of length > 0 and there are no other repetitions of vertices other then $\iota p = \tau p$. A graph with no simple closed paths is said to be a ***forest***.

Two elements of X are ***connected*** if they both occur in a path of X. Connectedness is an equivalence relation (via concatenation and inversion of paths). The equivalence classes under this relation are called ***components*** and are subgraphs of X. A graph is ***connected*** if it has only one component.

The following is a rather easy exercise but it helps to practise with the above notation and terminology.

Exercise 4.1. Prove that X is a tree if and only if it is a connected forest.

Proposition 4.2. *Let X be a G-graph with X/G connected. Then there exist two subsets Y_0 and Y of X with $Y_0 \subseteq Y \subseteq X$ and such that*

- **(i):** *Y is a G-transversal of X;*
- **(ii):** *Y_0 is a subtree of X;*
- **(iii):** *$VY = VY_0$ and for each $e \in EY$, $\iota e \in VY = VY_0$.*

Proof. Let $\bar{X} := X/G$ be the quotient graph. We adopt the "bar" notation for the projection $- : X \to \bar{X}$ mapping x to $\bar{x} := Gx$, for all $x \in X$.

Choose $v_0 \in VX$. Let Y_0 be a maximal subtree of X with $v_0 \in VY_0$ and such that the restriction of $-$ to Y_0 is injective. In other words, Y_0 is a maximal subtree of X with the property that distinct elements of Y_0 are in distinct G-orbits. Since $v_0 \in Y_0$, we have $Y_0 \neq \emptyset$ and hence the existence of such a subtree Y_0 is obvious if X is finite, and it relies on Zorn's lemma if X is infinite.

We claim that $V\bar{Y}_0 = V\bar{X}$. (This yields that Y_0 contains a representative of VX for each G-orbit.) We argue by contradiction and we assume that $V\bar{Y}_0 \subsetneq V\bar{X}$. Since $X/G = \bar{X}$ is connected, every vertex of $\bar{Y}_0$ is connected to a vertex in $\bar{X}$; in particular, there exists an edge $Ge = \bar{e} \in E\bar{X}$ with one end $Gv = \bar{v}$ in $\bar{Y}_0$ and the other end in $\bar{X} \setminus \bar{Y}_0$. Replacing v by some other element in $V\bar{Y}_0$, we may assume that $v \in VY_0$. Clearly, $e \in EX$. We have either $\bar{v} = \overline{\iota e}$ or $\bar{v} = \bar{\tau}\bar{e}$. In both cases, we see that v is in the same G-orbit of either ιe or τe. That is, there exists $g \in G$, with $v = \iota(ge)$ or $v = \tau(ge)$. In particular, replacing e by ge if necessary, we may assume that $v \in \{\iota e, \tau e\}$. Write w of the element in $\{\iota e, \tau e\} \setminus \{v\}$. Since $\bar{e}, \bar{w} \notin \bar{Y}_0$,we get $e, w \notin Y_0$. However, a moment's thought gives that $Y_0 \cup \{e, w\}$ is a tree that contradicts the maximality of Y_0. Therefore $V\bar{Y}_0 = V\bar{X}$.

Since the restriction of $-$ to Y_0 is injective and since $V\bar{Y}_0 = V\bar{X}$, to find Y it suffices to add to Y_0 some edges of X. Indeed, for each $\bar{e} \in E\bar{X} \setminus E\bar{Y}_0$, $\overline{\iota e}$ comes from a unique vertex of Y_0, and replacing e by ge (for some $g \in G$), we may assume that $\iota e \in VY_0$. Adjoining these edges e to Y_0, we obtain a subset Y of X such that Y is a G-transversal of X and such that $\iota e \in Y$ for every $e \in EY$.

By construction Y is a G-transversal of X and hence (i) holds. Also by construction Y_0 is a subtree of X and, Y and Y_0 satisfy also (iii). $\square$

Corollary 4.3. *If X is a connected graph then X has a maximal subtree. Any maximal subtree of X has vertex set all of VX.*

Before moving on let us see a few examples. In the Cayley graph $\mathrm{Cay}(G, S)$, we may take $Y_0 = \{1_G\}$ and for Y the subset consisting of the vertex 1 and the edges $\{(1, s) \mid s \in S\}$. In the first version of the Petersen graph, when $G = \mathrm{Alt}(5)$, the subtree Y_0 consists only of the vertex $\{1, 2\}$ and Y consists of the vertex $\{1, 2\}$ together with the edge $(\{1, 2\}, \{3, 4\})$. In the second version of the Petersen graph, when $G = \mathrm{Alt}(5)$, we have $Y_0 = Y$, the vertices of Y_0 are $\{1, 2\}$ and $\{\{1, 2\}, \{3, 4\}\}$ and we have only one edge from $\{1, 2\}$ to $\{\{1, 2\}, \{3, 4\}\}$.

In what follows we apply Proposition 4.2 only when X connected. In this case, clearly X/G is connected.

Exercise 4.4. Let X, Y, Y_0 and $\bar{X} = X/G$ be as in Proposition 4.2. Observe that, for each $e \in EY$, there exist (and are unique) $v, w \in VY = VY_0$ with $\bar{v} = \overline{\iota e}$ and $\bar{w} = \bar{\tau}\bar{e}$. This allow us to define two functions $\tilde{\iota} : EY \to VY = VY_0$ and $\tilde{\tau} : VE \to VY = VY_0$ by $\tilde{\iota}e = v$ and $\tilde{\tau}e = w$. Observe that since $\iota e \in VY = VY_0$ by Proposition 4.2 (iii), $\tilde{\iota}$ is the restriction of ι to the subset EY of EX.

Prove that the 5-tuple $(Y, EY, VY, \tilde{\iota}, \tilde{\tau})$ defines a structure of graph on Y. Prove that, with this structure, $Y \cong \bar{X} = X/G$ as graphs.

Observe that this isomorphism allows to think of Y_0 simultaneously as a subtree of X and as a spanning subtree of $\bar{X} = X/G$.

Implicitly we use Exercise 4.4 very often later. Before concluding this section we warn the reader that, although $\tilde{\iota}$ is the restriction of ι to EY, in general $\tilde{\tau}$ is not the restriction of τ to EY and hence we cannot think of Y as a subgraph of X. See the examples above, when X is a Cayley graph.

5. Graph of groups

Definition 5.1. A ***graph of groups*** is a pair $(G(-), Y)$, where $(Y, E, V, \bar{\iota}, \bar{\tau})$ is a connected graph and $G(-)$ is a function which assigns:

- **(i):** a group $G(v)$ for each $v \in V$;
- **(ii):** a subgroup $G(e)$ of $G(\iota)$ for each $e \in E$;
- **(iii):** an injective group homomorphism $t_e : G(e) \to G(\bar{\tau}e)$ for each $e \in E$.

There is one case that the reader should keep in mind and that motivates our interest. Let X be a G-graph with X/G connected and choose a fundamental G-transversal Y for X with subtree Y_0. Since each element of X lies in the same G-orbit as a unique element of Y, for each $e \in EY$ there are unique $\bar{\iota}e, \bar{\tau}e \in VY$ lying in the same G orbits as $\iota e, \tau e$ respectively, and in fact $\bar{\iota}e = \iota e$. Using the incident functions $\bar{\iota}, \bar{\tau} : EY \to EV$ we turn Y into a graph, and by construction the graph Y is isomorphic to the graph X/G. See Exercize 4.4.

Now Y_0 is a maximal subtree of Y and a subtree of X. However, Y is not a subgraph of X unless $\bar{\tau}$ equals τ; in particular, an arbitrary maximal subtree of Y need not be a subgraph of X.

For each $e \in EY$, the vertices τe and $\bar{\tau} e$ lie in the same G-orbit in EX, therefore we can choose an element $t_e \in G$ with $t_e(\bar{\tau} e) = \tau e$; if $e \in EY_0$, then $\bar{\tau} e = \tau e$ and we can take $t_e = 1$ (actually we do take $t_e = 1$). We call the collection $(t_e \mid e \in EY)$ a ***family of connecting elements***.

Observe that $G_e \leq G_{\bar{\iota} e}$ and $G_e \leq G_{\tau e} = t_e G_{\bar{\tau} e} t_e^{-1}$ and hence there is an embedding $t_e : G_e \to G_{\bar{\tau} e}$ defined by $g \mapsto g^{t_e} = t_e^{-1} g t_e$.

In particular this shows that X gives rise to a graph of groups, this graph of groups is called associated to X, with respect to the fundamental G-transversal Y, the maximal subtree Y_0, and the family of connecting elements $(t_e \mid e \in EY)$.

We now show how this construction is reversible, that is, we show that a graph of groups determines a group acting on a graph with connected quotient graph. Actually something deaper will be true. For each graph of groups $(G(-), Y)$ there exists a G-graph T, such that the graph of groups associated to G and T is $(G(-), Y)$. We will show that T can be chosen to be a tree and the group G can be chosen to be in some sense "universal". (This "sense" can be made topologically precise: see later in the text.)

Definition 5.2. Let $(G(-), Y)$ be a graph of groups. Choose a maximal subtree Y_0 of Y, so $VY_0 = VY$ by Corollary 4.3. The associated ***fundamental group*** $\pi(G(-), Y, Y_0)$ is the group presented with

generating set: $\{t_e \mid e \in E\} \cup \bigcup_{v \in V} G(v)$ (here the union has to be understood as a disjoint union);

relations: (1) the relations for $G(v)$, for each $v \in V$,

(2) $t_e^{-1} g t_e = g^{t_e}$ for each $e \in E$, $g \in G(e) \leq G(\bar{\iota} e)$ (thus $g^{t_e} \in G(\bar{\tau} e)$),

(3) $t_e = 1$ for each $e \in EY_0$.

To avoid some possible confusion that might arise using the symbol t_e, in the sequel we write g^{t_e} when we think of t_e as a group homomorphism $t_e : G(e) \to G(\bar{\tau} e)$, and we write $t_e^{-1} g t_e$ when we think of t_e as conjugation in $\pi(G(-), Y, Y_0)$.

It is not at all clear from Definition 5.2 whether $G(y)$ are genuine subgroups of $\pi(G(-), Y, Y_0)$, that is, from the definition it is not clear whether $G(y)$ does embed isomorphically into $\pi(G(-), Y, Y_0)$. It is only clear that $G(y)$ in $\pi(G(-), Y, Y_0)$ is a projection of the original $G(y)$.

Definition 5.3. Let $G = \pi(G(-), Y, Y_0)$. Let T be the G-set presented with generating set Y and relations saying that each $y \in Y$ is $G(y)$-stable: see Exercise 2.2. Formally, we may think of T as the set

$$T = \{(gG(y), y) \mid g \in G, y \in Y\} = \bigcup_{y \in Y} \frac{G}{G(y)} \times \{y\}.$$

Then T has G subsets $VT = GV$, $ET = GE$ and here $ET = T \setminus VT$. There are also two maps $\iota, \tau : ET \to VT$ defined by $\iota(ge) = g(\bar{\iota} e)$ and $\tau(ge) = g(\bar{\tau} e)$ for all $g \in G$ and $e \in E$. (With the notation above we have $\iota : (gG(e), e) \mapsto (gG(\bar{\iota} e), \bar{\iota} e)$ and $\tau : (gG(e), e) \mapsto (gG(\bar{\tau} e), \bar{\tau} e)$.) In particular, T is a G-graph. We call T the ***standard graph*** of the graph of groups.

By construction Y is a fundamental G-transversal of T. Moreover, once we know that $G(v)$ embeds into G, for each $v \in V$, we have

$$\begin{aligned}\iota^{-1}(v) &= \bigcup_{e \in \bar{\iota}^{-1}(v)} G(v)e \cong \bigcup_{e \in \bar{\iota}^{-1}(v)} \frac{G(v)}{G(e)}\\ \tau^{-1}(v) &= \bigcup_{e \in \bar{\tau}^{-1}(v)} G(v)t_e^{-1}e \cong \bigcup_{e \in \bar{\tau}^{-1}(v)} \frac{G(v)}{G(e)^{t_e}}.\end{aligned}$$

This proves that in T, the vertex v has $\sum_{e \in \bar{\iota}^{-1}(v)} |G(v) : G(e)|$ edges going out and $\sum_{e \in \bar{\tau}^{-1}(v)} |G(v) : t_e(G(e))|$ edges going in.

Exercise 5.4. Prove that the maps ι, τ above are well-defined. Fill in the details to prove that $T = (T, GV, GE, \iota, \tau)$ is a G-graph.

The main results that will follow show that T is a tree and that the vertex groups embed as subgroups of $\pi(G(-), Y, Y_0)$. In particular, it follows that we can recover the graph of groups $(G(-), Y)$ from the action of the fundamental group $\pi(G(-), Y, Y_0)$ on the fundamental graph (tree) $T(G(-), Y, Y_0)$.

Example 5.5. Let $(G(-), Y)$ be a graph of groups.

(i): If $G(y) = 1$ for each $y \in Y$, then $\pi(G(-), Y, Y_0)$ is the group generated by $\{t_e \mid e \in EY\}$ subject to the only relations $t_e = 1$ for all $e \in EY_0$. In particular $\pi(G(-), Y, Y_0)$ is a free group of rank $|EY \setminus EY_0|$.

In particular, the isomorphism class of the fundamental group $\pi(G(-), Y, Y_0)$ does not depend on Y_0 and hence we will simply write $\pi(Y)$ for $\pi(G(-), Y, Y_0)$: this is the usual fundamental group of the graph Y. See Definition 10.1.

(ii): Suppose that Y consists of only one edge e and two distinct vertices ιe and τe. Let $A = G(\iota e)$, $B = G(\tau e)$ and $C = A \cap B = G(e)$. In particular, C is a subgroup of A, and there is specified an embedding $t_e : C \to B$. Here $Y_0 = Y$ and the fundamental group is called the ***free product*** of A and B ***amalgamating*** C, denoted $A *_C B$. This is the group presented on the generating set $A \cup B$ (here the union has to be understood disjoint), together with the relations of A and B, and with relations saying that $c = c^{t_e}$ for all $c \in C$. (To see this observe that $e \in EY_0$ and hence $t_e = 1$.)

In the case $C = 1$, we write simply $A * B$ and call it the ***free product*** of A and B.

(iii): Suppose that Y has one edge e and one vertex $v = \iota e = \tau e$. Let $A = G(v)$, $C = G(e)$; so C is a subgroup of A and there is specified an embedding $t_e : C \to A$. Here Y_0 consists of the single vertex v and the fundamental group is called the ***HNN extension*** of A by $t_e : C \to A$, denoted by $A *_C t_e$. This group is formed by adjoining to A an indeterminate t_e satisfying the relations $t_e^{-1} c t_e = c^{t_e}$ for all $c \in C$.

Observe that case (i) of Example 5.5 arises (for instance) from the graph of groups associated to a Cayley graph. Whilst cases (ii) and (iii) of Example 5.5 arise, for example, for the graph of groups associated to the two constructions we gave of the Petersen graph as a G-graph.

Exercise 5.6. In this exercise we use the HNN extension to show the existence of an infinite countable group G with the property that any two of its non-identity elements are G-conjugate. (Incidentally show that C_1 and C_2 are the only finite groups with this property.) For this exercise we need some material that will be proved later but that we quote here.

Fact. Let G be a group, let A and B be two subgroups of G and let $\phi : A \to B$ be an isomorphism. Then the group

$$\bar{G} = \langle G, t \mid t^{-1}at = \phi(a) \text{ for all } a \in A\rangle$$

has the following properties:

(i): G embeds isomorphically into $\bar{G}$;
(ii): any element of finite order of $\bar{G}$ lies in a $\bar{G}$-conjugate of G, in particular if G is torsion-free then so is $\bar{G}$;
(iii): $t^{-1}Gt \cap G = B$ and $tGt^{-1} \cap G = A$.

Observe that $\bar{G}$ is indeed the fundamental group of a graph of groups arising from a HNN extension.

Here we start the construction. Start with G_0 an infinite countable torsion free group; $\mathbb{Z}$ will do. Let a and b be two arbitrary non-identity elements of G_0 and observe that $A = \langle a\rangle$ and $B = \langle b\rangle$ are isomorphic (being both isomorphic to $\mathbb{Z}$). Therefore from the construction above, there exists a torsion free group G' containing G_0 and such that a and b are G'-conjugate.

Repeating the construction in the paragraph above for each pair of non-identity elements of G_0, we obtain an infinite countable torsion free group G_1 with $G_0 \le G_1$ and such that any two non-identity elements of G_0 are G_1-conjugate.

Now we construct inductively a chain of infinite countable torsion free groups $(G_n)_{n\in\mathbb{N}}$. The group G_{n+1} is obtained applying the previous paragraph to G_n in place of G_0.

Set $G = \bigcup_{n\in\mathbb{N}} G_n$. By construction, G is infinite countable, torsion free and any two of its non-identity elements are conjugate in G.

Here we give another application of the HNN extension.

Exercise 5.7. Before stating the exercise we give some context to it. Let $X = (X, V, E, \iota, \tau)$ be a G-graph with G acting transitively on E. Fix $v \in V$ and let G_v be the stabilizer of v in G. Now consider $X^+(v) = \{\tau e \mid e \in E \text{ with } \iota e = v\}$, that is, $X^+(v)$ is the set of out-neighbours of v; similarly, consider $X^-(v) = \{\iota e \mid e \in E \text{ with } \tau e = v\}$, that is, $X^-(v)$ is the set of in-neighbours of v. Now, G_v acts transitvely on both $X^+(v)$ and $X^-(v)$ and hence induces two transitive permutation groups $G_v^{X^+(v)}$ and $G_v^{X^-(v)}$. A natural question that one might ask, is whether there is any relation between $G_v^{X^+(v)}$ and $G_v^{X^-(v)}$. For finite graphs, these two groups do share very many properties (and we do not go into this here, I am sure you know more than I do in this.) Here we are intererested into the infinite world.

Show that given any two transitive permutation groups A and B, there exists an infinite transitive pemutation group G acting on a graph $X = (X, V, E, \iota, \tau)$, such that given a vertex v of Γ, the group A is permutation isomorphic to the permutation group induced by G_v on the in-neighbours of v (that is, isomorphic to $G_v^{X^+(v)}$), and the group B is permutation isomorphic to the permutation group induced by G_v on the out-neighbours (that is, isomorphic to $G_v^{X-(v)}$).

Here is a hint. Let Δ be the doman of A and let Γ be the domain of B. Fix $\delta \in \Delta$ and $\gamma \in \Gamma$. Here, given a group T, the set $T^{(\mathbb{N})}$ denotes the set of functions

$f : \mathbb{N} \to T$ with finite support (that is, $\{n \in \mathbb{N} \mid f(n) \neq 1\}$ is finite). Clearly, $T^{(\mathbb{N})}$ with the usual point-wise multiplication is a group. Take

$$\begin{aligned} \bar{G} &= & H^{(\mathbb{N})} \times H_\delta^{(\mathbb{N})} \times K^{(\mathbb{N})} \times K_\gamma^{(\mathbb{N})}, \\ \bar{A} &= & H_\delta \times H^{(\mathbb{N})} \times H_\delta^{(\mathbb{N})} \times K^{(\mathbb{N})} \times K_\gamma^{(\mathbb{N})}, \\ \bar{B} &= & H^{(\mathbb{N})} \times H_\delta^{(\mathbb{N})} \times K_\gamma \times K^{(\mathbb{N})} \times K_\gamma^{(\mathbb{N})}. \end{aligned}$$

Observe that $\bar{A}$ and $\bar{B}$ are isomorphic subgroups of $\bar{G}$. Fix $\phi : A \to B$ an isomorphism. Observe that the permutation group induced by $\bar{G}$ on the left cosets of $\bar{A}$ in $\bar{G}$ is permutation isomorphic to A, and that the permutation group induced by $\bar{G}$ on the left cosets of $\bar{B}$ in $\bar{G}$ is permutation isomorphic to B.

Let $G = \langle \bar{G}, t \mid t^{-1}at = \phi(a) \text{ for all } a \in \bar{A}\rangle$ be the HNN extension with respect to $\bar{G}, \bar{A}, \bar{B}$ and the isomorphism ϕ. Consider then the set V consisting of the left cosets of $\bar{G}$ in G and $E = \{(x\bar{G}, xt\bar{G}) \mid x \in G\}$. End of hint.

6. Groups acting on trees

For all of this section T is a G-tree. We prove a theorem describing the algebraic and the group action structure of G.

Theorem 6.1. *Let Y be a G-transversal of T with fundamental subtree Y_0 and denote the incidence functions with $\bar{\iota}$ and $\bar{\tau}$; for each $e \in EY$, choose $t_e \in G$ with $t_e\bar{\tau}e = \tau e$, and with $t_e = 1$ when $e \in EY_0$; consider then the graph of groups $(G(-), Y)$. Then G is "naturally" isomorphic to the fundamental group $\pi(G(-), Y, Y_0)$. In other words, G has a presentation with generating set*

$$\{t_e \mid e \in EY\} \cup \bigcup_{v \in VY} G_v,$$

where the union has to be understood disjoint; and with relations:

1. *the relations for G_v, for each $v \in VY$;*
2. *$t_e^{-1}gt_e = g^{t_e}$ for all $e \in EY$, $g \in G_e \leq G_{\bar{\iota}e}$ (and hence $g^{t_e} \in G_{\bar{\tau}e}$);*
3. *$t_e = 1$ for all $e \in EY_0$.*

Among other things, the main point of this theorem is that once we have the "local" structure of G and we know that T is a tree, the entire structure of G is also known. A similar result will hold for general groups acting on graphs.

Proof. Let $P = \pi(G(-), Y, Y_0)$. Since P is the group with presentation satisfying the conditions of this theorem and since G is a group that does also satisfy the same conditions, we have a natural homomorphism $\pi : P \to G$ and the point of the theorem and of the proof is to show that π is actually an isomorphism. Observe that the action of G on T and the homomorphism π allow to define a P-action on T, formally $pt = \pi(p)t$ for every $t \in T$ and for every $p \in P$.

`Claim.` Let $v \in VY$. The paths of length 1 in T starting at v are the sequences of the form $v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^\varepsilon, gt_e^\varepsilon w$ where v, e^ε, w is a path in Y and $g \in G_v$.

Let v, e^ε, w be a path in Y. Suppose $\varepsilon = 1$. Then $v = \bar{\iota}e = \iota e$ and $w = \bar{\tau}e$. Then the definition of t_e gives that $\tau e = t_e\bar{\tau}e = t_ew$ and hence v, e, t_ew is a path in T. Clearly, $v = gv, ge, gt_ew$ is a path in T for every $g \in G_v$. Suppose that $\varepsilon = -1$. Then $v = \bar{\iota}e^{-1} = \bar{\tau}e = t_e^{-1}\tau e = \tau(t_e^{-1}e) = \iota(t_e^{-1}e^{-1})$ and $\tau(t_e^{-1}e^{-1}) = \iota(t_e^{-1}e) =$

$t_e^{-1}\iota e = t_e^{-1}\bar{\iota}e = t_e^{-1}\bar{\tau}e^{-1} = t_e^{-1}w$. Thus $v, t_e^{-1}e^{-1}, t_e^{-1}w$ is a path in Y. Clearly, $v, gt_e^{-1}e^{-1}, gt_e^{-1}w$ is also a path in T for every $g \in G_v$.

We prove the converse. Any edge of T incident to v can be written in the form ge, for some $e \in EY$ and $g \in G$. If $v = \iota(ge) = g\iota e$, then $v = \iota(ge) = g\bar{\iota}e$. Since Y is a G-transversal, we get $g \in G_v$. Also, $\tau(ge) = g\tau e = gt_e\bar{\tau}e$. Since $\varepsilon = 1$, the path $v, ge, \tau(ge)$ is indeed of the form $v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^\varepsilon, gt_e^\varepsilon\bar{\tau}e$. If $v = \tau(ge)$, then $v = \tau(ge) = g\tau e = gt_e\bar{\tau}e$. Since Y is a G-transversal, we get $v = \bar{\tau}e = \bar{\iota}e^{-1}$ and $gt_e \in G_v$. Write $h = gt_e$ with $h \in G_v$. Thus $g = ht_e^{-1}$. Write $w = \bar{\tau}e^{-1}$. Now, v, e^{-1}, w is a path in Y. We look at $v, ht_e^{-1}e^{-1}, ht_e^{-1}w$ and observe that this is of the form $v, ht_e^{\frac{1}{2}(\varepsilon-1)}e^\varepsilon, ht_e^\varepsilon\bar{\tau}e$ with $\varepsilon = -1$. ∎

With this claim and with the connectedness of T it is easy to show that π is surjetive. (To emphasise the proof we use π.) As P contains all elements t_e (with $e \in EY$) and all elements g (with $g \in G_v$), we get that PY contains all edges adjacent to elements of Y and also their end vertices. Therefore all edges and all end vertices of T adjacent to PY also lie in PY. As T is connected, we get $T = PY$. Let $v_0 \in VY$. For every $g \in G$, we have $gv_0 \in T = PY$ and hence there exists $p \in P$ and $v \in Y$ with $gv_0 = \pi(p)v$. As Y is a G-transversal of T, we get $v_0 = v$. Therefore $\pi(p^{-1})gv_0 = v_0$. Write $h = \pi(p^{-1})g \in G_v$. As $h \in G_{v_0} \leq P$, we deduce $g = \pi(p)h$ and hence $g \in \pi(P)$. Therefore π is surjective.

It remains to prove that π is injective. Here we omit the symbol π. Let $p \in P$. Using the given generating set of P, we may write

$$p = g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} g_n, \tag{1}$$

with $n \geq 0$, $\varepsilon_i \in \{1, -1\}$ and $g_i \in G_{v_i}$. Observe that we may write the element p in this form and simultaneosly assume that $v_0, e_1^{\varepsilon_1}, v_1, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n = v_0$ is a path in Y. In fact, we may express p as a product of the given generators and their inverses, then using the relations for the G_v to collect together generators from the same G_v into single expressions, and finally inserting 1's as dictated by paths in the maximal subtree Y_0 to obtain an expression as in (1).

Using the claim above it follows that

$$\begin{aligned} &v_0,\ g_0 t_1^{\frac{1}{2}(\varepsilon_1-1)} e_1^{\varepsilon_1},\ g_0 t_{e_1}^{\varepsilon_1} v_1,\ g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\frac{1}{2}(\varepsilon_2-1)} e_2^{\varepsilon_2},\ g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} v_2, \\ &g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 t_{e_3}^{\frac{1}{2}(\varepsilon_3-1)} e_3^{\varepsilon_3},\ g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 t_{e_3}^{\varepsilon_3} v_3, \ldots, \\ &g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\frac{1}{2}(\varepsilon_n-1)} e_{n-1}^{\varepsilon_{n-1}},\ g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} v_n = pv_n \end{aligned} \tag{2}$$

is a path in the tree T. Observe that via the projection $\pi : P \to G$, we can think of these expressions as both elements of P and elements of G.

We prove by induction on n that if $\pi(p) = 1$, then $p = 1$ in P. Since the composition $G_{v_0} \leq P \to G$ is injective, we may assume that $n \geq 1$. Since T is a tree and $n \geq 1$, the path in (2) is not reduced. In particular, there exists $i \in \{0, \ldots, n-1\}$ with the ith edge and the $(i+1)$th edge one inverse of the other. In particular,

$$\varepsilon_{i+1} = -\varepsilon_i \text{ and } t_{e_i}^{\frac{1}{2}(\varepsilon_i-1)} e_i = t_{e_i}^{\varepsilon_i} g_i t_{e_{i+1}}^{\frac{1}{2}(\varepsilon_{i+1}-1)} e_{i+1}.$$

As $e_i, e_{i+1} \in Y$ and Y is a G-transversal, we get $e_i = e_{i+1}$ and hence

$$e_i = t_{e_i}^{\frac{1}{2}(\varepsilon_i+1)} g_i t_{e_i}^{-\frac{1}{2}(\varepsilon_i+1)} e_i.$$

Write $h = t_{e_i}^{\frac{1}{2}(\varepsilon_i+1)} g_i t_{e_i}^{-\frac{1}{2}(\varepsilon_i+1)}$. Observe that $h \in G_{e_i} \leq G_{\bar{\iota} e_i}$ and

$$h^{t_{e_i}} = t_{e_i}^{\frac{1}{2}(\varepsilon_i-1)} g_i t_{e_i}^{-\frac{1}{2}(\varepsilon_i-1)} \in G_{\bar{\tau} e_i}.$$

Recall that in P we also have the equation

$$t_{e_i}^{-1} h t_{e_i} = h^{t_{e_i}}.$$

Suppose $\varepsilon_i = 1$. Then $\bar{\iota} e_i = v_{i-1} = v_{i+1}$, $\bar{\tau} e_i = v_i$. Therefore $h \in G_{\bar{\iota} e_i} = G_{v_{i-1}}$ and $h^{t_{e_i}} = g_i \in G_{v_{i-1}}$. Thus in (1), we may replace $g_{i-1} t_{e_i} g_i t_{e_i}^{-1} g_{i+1}$ with the single generator $g_{i-1} h g_{i+1} \in G_{v_{i+1}} = G_{v_{i-1}}$ and delete e_i, v_i, e_i^{-1} from the path in Y. Therefore we reduced the path by 2.

Suppose $\varepsilon_i = -1$. Then $\bar{\iota} e_i = v_i$, $\bar{\tau} e_i = v_{i-1} = v_{i+1}$. Therefore $h^{t_{e_i}} \in G_{\bar{\tau} e_i} = G_{v_{i-1}}$ and $h = g_i$. We also have the equation $h^{t_{e_i}} = t_{e_i}^{-1} g_i t_{e_i}$ in P. Therefore in (2), we can repance $g_{i-1} t_{e_i}^{-1} g_i t_{e_i} g_{i+1}$ by the single generator $g_{i-1} h^{t_{e_i}} g_{i+1} \in G_{v_{i-1}}$ and we may delete e_i^{-1}, v_i, e_i from the path in Y. Also in this case we reduced the path by 2.

Arguing by induction we obtain that $p = 1$ and hence $\pi : P \to G$ is injective. $\square$

Corollary 6.2. *If $G_y = 1$ for each $y \in Y$, then G is a free group (or rank $|EY \setminus EY_0|$). In fact, $G \cong \pi(T/G)$.*

Proof. If $G_y = 1$ for each $y \in Y$, then G is generated by $\{t_e \mid e \in EY \setminus EY_0\}$ with no relations. Also, $G = \pi(G(-), Y, Y_0) = \pi(Y) = \pi(T/G)$ as $Y \cong T/G$. $\square$

Exercise 6.3. If N is a normal subgroup of G, then T/N is a connected G/N-graph and, for each $Nt \in T/N$, the vertex stabilizer $(G/N)_{Nt}$ equals NG_t/N.

Proposition 6.4. *If N is the subgroup of G generated by the G_v, $v \in VT$, then N is normal and G/N is free. Moreover, T/N is a G/N-free G/N-tree and $G/N \cong \pi(T/N)$.*

Proof. Theorem 6.1 gives $G \cong \pi(G(-), Y, Y_0)$ and the definition of N implies that $G/N \cong \pi(T/G)$.

We now apply this theorem again with G replaced by N. Since N is generated by the vertex stabilizers $N_v = G_v$, $v \in VT$, we see that $\pi(T/N) = N/N = 1$. Hence T/N is a tree. (It is the only possibility for a graph Z to have $\pi(Z) = 1$.) By Exercise 6.3, G/N acts freely on VT/N, and hence on T/N, so $G/N \cong \pi(T/G)$. $\square$

Exercise 6.5. Let H be a subgroup of G with $H \cap G_t = 1$ for each $t \in T$. Prove that H is a free group. In particular, suppose that $\pi : G \to A$ is a group homomorphism such that the restriction of π to G_t is injective, for each $t \in T$. Then the kernel of π is free.

Proposition 6.6. *If $G \to A$ is a homomorphism of groups which is injective on each vertex stabilizer then the kernel N is free. In fact, $N \cong \pi(X)$, where X is the connected G/N-graph T/N.*

If the homomorphism $G \to A$ is surjective then X is a connected A-graph. In Theorem 11.1, we shall see that all group actions on connected graphs can be realised in this way.

Exercise 6.7. Prove that $C_2 * C_2$ is an infinite dihedral group.

Proposition 6.8. *Let v be a given vertex of T. Then G stabilizes a vertex of G if and only if there is an integer N such that the distance from v to each element of Gv is at most N.*

Proof. Suppose that G stabilizers the vertex v_0 of T and let n be the distance from v_0 to v. Then clearly, the distance between v and gv is at most $d(v, v_0)+d(v_0, gv) = 2n$, for every $g \in G$.

We prove the converse. Suppose that there is an integer N such that, for each $g \in G$, the T-geodesic from v to gv has length at most N. Let T' be the subtree generated by $Gv = \{gv \mid g \in G\}$.

It is not hard to see that T' is a G-subtree of T and no reduced path has length greater than $2N$.

If T' has at most one edge then every element of T' is G-invariant, and we have the desired G-invariant vertex. Thus we may assume that T' has at least two edges, and hence some vertex of T' has valency at least two. Now delte from T' all vertices of valency one, and their incident edges. This leaves a G-subtree T'' in which no reduced path has length greater than $2N - 2$. Therefore, arguing by induction, we deduce that G stabilizes a vertex in T''. $\square$

Exercise 6.9. Use Proposition 6.8 to show that if $H \leq G$ is finite, then H must by conjugate to G_v for some v. (That is, H stabilizes some vertex of T.)

Fix a vertex v of T. In fact, applying Proposition 6.8 to the group H, we see that $Hv = \{hv \mid h \in H\}$ is finite and hence the distance from v to any element of Hv is bounded above by an absolute constant. Therefore H fixes a vertex. Thus H is conjugate to G_{v_0}, for some v_0.

7. The special linear group $\mathrm{SL}_2(\mathbb{Z})$ acting on the hyperbolic plane

Here we consider the group $G = \mathrm{SL}_2(\mathbb{Z})$, the group of 2×2 matrices with integer coefficients and determinant 1. The group G has a natural action on the hyperbolic plane $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} = \{x + iy \mid x, y \in \mathbb{R}, y > 0\}$. Indeed, given

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $z = x + iy \in \mathcal{H}$, we define

$$gz = \frac{az + b}{cz + d}.$$

By expanding, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (x + iy) = \frac{(ax + b)(cx + d) + acy^2 + iy}{(cx + d)^2 + c^2y^2}$$

and this shows that gz is indeed an element of $\mathcal{H}$.

Exercise 7.1. Show that this does define an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. Compute the stabilizer G_i of the imaginary point i and the stabilizer G_z of the point $z = \frac{1}{2}(1 + \sqrt{3}i)$. Deduce that G has more than one orbit on $\mathcal{H}$: this fact is also clear because G is countable and $\mathcal{H}$ is not.

Let $Y = \{\cos\theta + i \sin\theta \mid \pi/3 \leq \theta \leq \pi/2\}$. We think of Y as a drawing of a graph with only one edge e having $\iota e = i$ and $\tau e = \frac{1}{2}(1 + i\sqrt{3})$.

Define $T = GY$. So far T is defined as a subset of $\mathcal{H}$. We aim to give a graph structure to T.

Lemma 7.2. *Let* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ *and let* $z = x + iy \in Y$. *Suppose that* $gz \in Y$. *Then one of the following holds:*

(i): *each point of* Y *is* g*-invariant;*
(ii): z *is either* ιe *or* τe *and* z *is* g*-invariant.*

Moreover, if $\mathrm{Re}(gz) = 0$, *then* $gz = \iota e = i$.

Proof. Suppose that $c^2 \neq d^2$ and observe that $|z| = 1$ because $z \in Y$. Using $|z| = 1$, we deduce

$$|(c^2 - d^2)gz - (ac - bd)| = \left|(c^2 - d^2)\frac{az + b}{cz + d} - (ac - bd)\right| = \left|\frac{dz + c}{cz + d}\right| = 1.$$

(Only in the last equality we need to use $|z| = 1$.) This shows that gz is on a circle with center $\frac{ac-bd}{c^2-d^2}$ and radius $\frac{1}{|c^2-d^2|}$.

If $gz \in Y$, then $\mathrm{Im}(gz) \geq \frac{\sqrt{3}}{2} \geq \frac{1}{2}$ and hence $|c^2 - d^2| \leq 2$. In particular, since $c, d \in \mathbb{Z}$, we obtain $|c^2 - d^2| = 1$. Therefore $1 = |c^2 - d^2| = |c + d||c - d|$ and hence $|c + d| = 1 = |c - d|$. This is possible if and only if

$$c = 0 \text{ and } d \in \{-1, 1\}, \text{ or } c \in \{1, -1\} \text{ and } d = 0. \tag{3}$$

Thus gz is on a circle of centre $\frac{ac-bd}{c^2-d^2}$ and radius 1. The centre $\frac{ac-bd}{c^2-d^2} \in \mathbb{Z}$ and has to be between 0 and 1 to make sure that it does intersect Y. Thus we either have $ac - bd = 0$ or $ac - bd = 1$.

Suppose that $ac - bd = 0$, that is, $ac = bd$. Now Eq. (3) easily implies that g is one of the following four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The first two matrices fix point-wise the whole of $\mathcal{H}$. The third and the fourth matrix map z to $-\frac{1}{z} = -\bar{z}$ and hence g fixes the end point i of Y.

Now, if $ac - bd = 1$, then Eq. (3) yields that g is one of the following four matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The first two matrices act on the whole of $\mathcal{H}$ by the translation $z \mapsto z - 1$, and a computation shows that there is no point z of Y with $gz \in Y$. The third and the fourth matrices map z to $\frac{z-1}{z} = 1 - \frac{1}{z} = 1 - \bar{z}$ and hence g fixes the end point $\frac{1}{2}(1 + i\sqrt{3})$ of Y.

Finally, suppose that $c^2 = d^2$. Then $d = \varepsilon c$, where $\varepsilon \in \{1, -1\}$. As $ad - bc = 1$, we get $(a\varepsilon - b)c = 1$. Thus $c, d \in \{1, -1\}$ and, a and b cannot be both odd or both even (that is, a and b have opposite parity). A computation, with all possible combinations of $c, d \in \{1, -1\}$, shows that

$$2\mathrm{Re}(gz) = ca + bd,$$

and hence $2\mathrm{Re}(gz)$ is an odd integer. In particular, $2\mathrm{Re}(gz) \neq 0$.

If $gz \in Y$, we must have $ac + bd = 1$ and $gz = \frac{1}{2}(1 + \sqrt{3}i) = \tau e$. Another computation easily shows that g fixes τe. $\square$

Exercise 7.3. Fill in the missing computations in the proof of Lemma 7.2.

Lemma 7.2 shows that, for every $g \in G$, either $Y = gY$, $Y \cap gY = \emptyset$ or $Y \cap gY$ is either $\{\iota e\}$ or $\{\tau e\}$, that is, an end point of Y. This allows us to think of T as a drawing of a G-graph on the hyperbolic space. A computation gives

$$G_{\iota e} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

and hence $G_{\iota e} \cong C_4$. A similar computation gives

$$G_e = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and hence $G_e \cong C_2$. Finally,

$$G_{\tau e} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$$

and hence $G_{\tau e} \cong C_6$. See also Exercise 7.1.

From now on we think of T has a graph or the drawing in $\mathcal{H}$ of a graph.

Lemma 7.4. *The set T contains no simple closed path.*

Proof. We argue by contradiction. Suppose that T has a simple closed path. Then, the definition of $T = GY$ gives that there is a simple closed path from ιe to τe which does not use e. The only edges adjacent to $\iota e = i$ are e itself and

$$e' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e = \{\cos\theta + i \sin\theta \mid \pi/2 \le \theta \le 2\pi/3\}.$$

In particular, there is a path in T from $\frac{1}{2}(1 + \sqrt{3}i) = \tau e$ to $\tau e' = \frac{1}{2}(-1 + \sqrt{3}i)$. Now as G acts as a topological group on $\mathcal{H}$, our path must intersect the imaginary axes. This is a contradiction, because the only point along the imaginary axes is i by Lemma 7.2. $\square$

From Theorem 6.1, we deduce $\mathrm{SL}_2(\mathbb{Z}) = G \cong C_4 *_{C_2} C_6$. In particular, $\mathrm{SL}_2(\mathbb{Z})$ has presentation:

$$\langle x, a \mid x^4 = a^6 = 1, x^2 = a^3 \rangle.$$

We also deduce the famous isomorphism $\mathrm{PSL}_2(\mathbb{Z}) \cong C_2 * C_3$ and hence $\mathrm{PSL}_2(\mathbb{Z})$ has presentation:

$$\langle x, a \mid x^2 = a^3 = 1 \rangle.$$

The method presented in this section for determining a presentation of $\mathrm{SL}_2(\mathbb{Z})$ can be used for many other interesting arithmetic groups or, more generally, automorphism groups of other algebraic structures (other than the lattice $\mathbb{Z}^2$).

Exercise 7.5. With a little more work we could have shown that $\mathrm{GL}_2(\mathbb{Z}) = A' *_{C'} B'$, where A' is a dihedral group of order 8, B' is a dihedral group of order 12 and C' is an elementary abelian group of order 4.

Now consider $F_2 = \langle x, y \rangle$ the free group of rank 2 on the generators x and y. We can encode each automorphism $\phi \in \mathrm{Aut}(F_2)$ of F_2 with the pair $(\phi(x), \phi(y))$. Consider $q = (xy, y^{-1})$, $r = (x, y^{-1})$ and $t = (y, x)$. Prove that

$$q^2 = r^2 = t^2 = (tr)^4 = 1$$

and that $(tr)^2(tq)^3$ is the conjugation by xy.

Observe now that each $\phi \in \mathrm{Aut}(F_2)$ deternines an isomorphism of the quotient $F_2/[F_2, F_2] \cong \mathbb{Z}^2$, and hence there is a natural homomorphism $\mathrm{Aut}(F_2) \to \mathrm{GL}_2(\mathbb{Z})$. Deduce from above that this homomorphism is surjective with kernel F_2.

It is possible to prove that $\mathrm{Aut}(F_2)$ is generated by q, r and t. Deduce that $\mathrm{Aut}(F_2) = A *_C B$, where A is an extension of F_2 by A', B is an extension of F_2 by B' and C is an extension of F_2 by C'.

8. The exact sequence of a tree

Let V be a G-set. We write $\mathbb{Z}V$ for the free abelian group spanned by V. Thus the elements of $\mathbb{Z}V$ are formal sums $\sum_{v\in V} n_v v$, with $n_v \in \mathbb{Z}$ being zero for all but finitely many $v \in V$. Clearly, the action of G on V turns $\mathbb{Z}V$ into a $\mathbb{Z}G$-module (the permutation $\mathbb{Z}G$-module on the set V):

$$g\left(\sum_{v\in V} n_v v\right) = \sum_{v\in V} n_v(gv).$$

Let $\varepsilon : \mathbb{Z}V \to \mathbb{Z}$ be the map defined by

$$\varepsilon\left(\sum_{v\in V} n_v v\right) = \sum_{v\in V} n_v.$$

It is easy to see that ε is a G-module homomorphism, this mapping is called the ***augmentation function*** of V and the kernel of ε is called the ***augmentation module*** of V.

Definition 8.1. Let $X = (X, V, E, \iota, \tau)$ be a G-graph. The ***boundary map*** is the G-linear map $\partial : \mathbb{Z}E \to \mathbb{Z}V$ defined by

$$\partial\left(\sum_{v\in V} n_v v\right) = \sum_{v\in V} n_v \tau v - \sum_{v\in V} n_v \iota v.$$

In particular, $\partial(v) = \tau v - \iota v$. The sequence

$$0 \to \mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a complex, that is, the composition of two consecutive maps is zero.

Recall that a sequence of G-modules $A \xrightarrow{f} B \xrightarrow{g} C$ is said to be exact if $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Exercise 8.2. Show that the graph X is connected if and only if $\mathbb{Z}E \xrightarrow{\partial} \mathbb{Z}V \xrightarrow{\varepsilon} 0$ is exact. This requires to prove two things: that $\mathrm{Im}(\partial) = \mathrm{Ker}(\varepsilon)$ and that ε is surjective.

Exercise 8.3. Show that the graph X is a forest if and only if $0 \to \mathbb{Z}V \xrightarrow{\partial} \mathbb{Z}E$ is exact. As for Exercise 8.2, this requires to prove that ∂ is injective.

Putting Exercises 8.2 and 8.3 together, we have that when X is a tree, the sequence $0 \to \mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{\varepsilon} 0$ is exact. Actually, we can be more precise than this and we can contruct an explicit function $\mathbb{Z}VT \to \mathbb{Z}ET$ partially inverting ∂.

Definition 8.4. Let $T = (T, V, E, \iota, \tau)$ be a G-tree. Given two vertices $v, w \in V$, consider the ***geodesic*** $v = v_0, e_1^{\varepsilon_1}, \dots, e_n^{\varepsilon_n}, v_n = w$ of T from v to w. Define

$$T[v, w] = \varepsilon_1 e_1 + \cdots + \varepsilon_n e_n \in \mathbb{Z}E.$$

This definition is well posed because T is a tree and hence there exists a unique path (a geodesic) from v to w. Observe that $T[w, v] = -T[v, w]$ and that $T[v, w] = T[v, u] + T[u, w]$ for every $v, w, u \in V$. Fix $v \in V$. Now define $T[v, -] : V \to \mathbb{Z}E$

by $w \mapsto T[v, w]$. Since $\mathbb{Z}V$ is a free abelian group on V, the map $T[v, -]$ can be extended to a $\mathbb{Z}$-linear map $T[v, -] : \mathbb{Z}V \to \mathbb{Z}E$.

Theorem 8.5. *If T is a G-tree then $0 \to \mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is an exact sequence of G-modules; therefore the augmentation module of V is isomorphic to $\mathbb{Z}ET$.*

Proof. See the previous exercises or argue as follows. Fix $v \in V$. For every $e \in E$, we have

$$T[v, \partial(e)] = -T[v, \iota e] + T[v, \tau e] = T[\iota e, v] + T[v, \tau e] = T[\iota e, \tau e] = e.$$

Thus

$$\mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{T[v,-]} \mathbb{Z}ET$$

is the identity. The rest of the proof is easy. □

We recall here a rather concrete definition of ***semidirect*** product of groups. Let M be a G-module. The semidirect product, denoted by either $M \rtimes G$ or $G \ltimes M$, is the group defined on the set $G \times M$ with product

$$(g, m)(g', m') = (gg', m + gm').$$

Exercise 8.6. Let $\beta : G \to G \ltimes M$ be a function of the form $\beta = (id_G, d)$, for some function $d : G \to M$, that is, $\beta(g) = (g, d(g))$, for every $g \in G$. Prove that β is a group homomorphism if and only if

$$d(xy) = d(x) + xd(y),$$

for each $x, y \in G$. A map $d : G \to M$ with the properties above is called a derivation.

We will see homomorphisms $\beta : G \to G \ltimes M$ as in Exercise 8.6 in the proof of Theorem 11.1. These homormorphisms arise naturally in the following way. Fix $m \in M$ and let $\beta_m : G \to G \ltimes M$ be the automorphism defined by

$$g \mapsto d_m(g) = [g, m] = (g, 0)(1, m)(g^{-1}, 0)(1, -m) = (1, gm - m).$$

Then $d : G \to M$ is given by $m \mapsto gm - m$. In general not every homomorphism $\beta : G \to G \ltimes M$ satisfying $\beta = (id_G, d)$ (for some $d : G \to M$) is of the form $\beta = \beta_m$, for some $m \in M$. In fact, the existence of homomorphisms $\beta : G \to G \ltimes M$ satisfying $\beta = (id_G, d)$ (for some $d : G \to M$) not of the form $\beta = \beta_m$ depends on the homology of G with coefficients in M and in particular on the non-vanishing of the first homology group.

Exercise 8.7. Let G be a group and let M be a G-module. Consider $E = G \ltimes M$ and we think of G and M as subgroups of E. A subgroup C of E is said to be a complement of M in E if $C \cap M = 1$ and $E = CM$. Show that there exists a one-to-one correspondence between complements C of M in G and derivations $d : G \to M$. (Hint: given a derivation $d : G \to M$ consider $C = \{xd(x) \mid x \in G\}$. Conversely, given a complement C of M in E observe that, for each $x \in G$, there exists a unique $m_x \in M$ with $xm_x \in C$.)

9. The fundamental graph of a graph of groups is a tree

We now prove another main result on this theory. We set some notation that we use in the whole of this section.

Let $(G(-), Y)$ be a graph of groups with connected graph $Y = (Y, V, E, \bar{\iota}, \bar{\tau})$. We use the same notation for Y_0, G, T as for Definitions 5.2 and 5.3. Moreover we fix once and for all a vertex v_0 of Y_0.

Lemma 9.1. *Let H be a group and suppose that for every $v \in VY$, $\alpha_v : G(v) \to H$ is a group homomorphism. Let $\alpha : E \to H$ be a function such that*

$$\alpha_{\bar{\iota}e}(g)\alpha(e) = \alpha(e)\alpha_{\bar{\tau}e}(g^{t_e}), \quad \text{for all } e \in E, g \in G(e) \le G(\bar{\iota}e).$$

For $v, w \in VY$, let

$$\alpha(v, w) = \alpha(e_1)^{\varepsilon_1} \cdots \alpha(e_n)^{\varepsilon_n} \in H$$

where $e_1^{\varepsilon_1}, \dots, e_n^{\varepsilon_n}$ is the Y_0-geodesic from v to w.

Then there exists a group homomorphism $\beta : G \to H$ defined on the generating set of G by

$$\begin{aligned} \beta(g) &= \alpha(v_0, v)\alpha_v(g)\alpha(v, v_0) && \text{for all } g \in G(v),\ v \in V, \\ \beta(t_e) &= \alpha(v_0, \bar{\iota}e)\alpha(e)\alpha(\bar{\tau}e, v_0) && \text{for all } e \in E. \end{aligned}$$

Proof. To prove that β defines an homomorphism $\beta : G \to H$, directly from the definition of group presentation, it suffices to show that β respects the relations defining G.

Observe that the definition of α gives

$$\alpha(u, w) = \alpha(u, v)\alpha(v, w) \quad \text{and} \quad \alpha(w, u) = \alpha(u, w)^{-1},$$

for every $u, v, w \in V$.

Fix $v \in V$. We study first the restriction $\beta \restriction G(v)$ of β to the subset $G(v)$ of G. As $\alpha(v_0, v) = \alpha(v, v_0)^{-1}$, by definition $\beta \restriction G(v)$ is the composition of α_v (which is a group homomorphism by hypothesis) with the conjugation by $\alpha(v_0, v)$. In particular $\beta \restriction G(v)$ is a group homomorphism and hence the relations of G (with respect to $G(v)$) are respected.

Fix $e \in E$ and $g \in G(e) \le G(\bar{\iota}e)$. We have

$$\begin{aligned} \beta(g)\beta(t_e) &= \alpha(v_0, \bar{\iota}e)\alpha_{\bar{\iota}e}(g)\alpha(\bar{\iota}e, v_0)\alpha(v_0, \bar{\iota}e)\alpha(e)\alpha(\bar{\tau}e, v_0) \\ &= \alpha(v_0, \bar{\iota}e)\alpha_{\bar{\iota}e}(g)\alpha(e)\alpha(\bar{\tau}e, v_0) \\ &= \alpha(v_0, \bar{\iota}e)\alpha_{\iota e}(g)\alpha(e)\alpha(\bar{\tau}e, v_0) \\ &= \alpha(v_0, \bar{\iota}e)\alpha(e)\alpha_{\bar{\tau}e}(g^{t_e})\alpha(\bar{\tau}e, v_0) \\ &= \alpha(v_0, \bar{\iota}e)\alpha(e)\alpha(\bar{\tau}e, v_0)\alpha(v_0, \bar{\tau}e)\alpha_{\bar{\tau}e}(g^{t_e})\alpha(\bar{\tau}e, v_0) \\ &= \beta(t_e)\beta(g^{t_e}). \end{aligned}$$

In particular,

$$\beta(t_e)^{-1}\beta(g)\beta(t_e) = \beta(g^{t_e}) = \beta(t_e^{-1} g t_e),$$

and hence β respects this type of relations.

Finally, for each $e \in EY_0$, we have $t_e = 1$ and also

$$\begin{aligned} \beta(1) &= \beta(t_e) = \alpha(v_0, \bar{\iota}e)\alpha(e)\alpha(\bar{\tau}e, v_0) \\ &= \alpha(v_0, \bar{\iota}e)\alpha(\bar{\iota}e, \bar{\tau}e)\alpha(\bar{\tau}e, v_0) = \alpha(v_0, v_0) = 1. \end{aligned}$$

Thus β respects all relations defining G and hence $\beta : G \to H$ is a group homomorphism. $\square$

Using Lemma 9.1 we can now give a more standard definition of fundamental group.

Definition 9.2. Let P be the group with

generating set: $\{u_e \mid e \in E\} \cup \bigcup_{v\in V} G(v)$;

relations: (1) the relations for $G(v)$, for every $v \in V$,

(2) $gu_e = u_e g^{t_e}$, for every $e \in E$ and for every $g \in G(e) \leq G(\bar{\iota} e)$ (thus $g^{t_e} \in G(\bar{\tau} e)$).

This group resembles very much our fundamental group $G = \pi(G(-), Y, Y_0)$: the main difference is that in G we also have the relations $t_e = 1$ when $\bar{\tau} e \in Y_0$.

The ***fundamental group*** of $(G(-), Y)$ with respect to v_0 is the subgroup (denoted by $\pi(G(-), Y, v_0)$) of P consisting of all elements p which can be written as a product

$$p = g_0 u_{e_1}^{\varepsilon_1} g_1 \cdots g_{n-1} u_{e_n}^{\varepsilon_n} g_n,$$

where $v_0, e_1^{\varepsilon_1}, v_1, \ldots, v_{n-1}, e_{n-1}^{\varepsilon_{n-1}}, v_n = v_0$ is a closed path in Y with $g_i \in G(v_i)$ for every $i \in \{0, \ldots, n\}$.

As we mentioned above, there exists a natural epimorphism $\pi : P \to G$ sending $G(v)$ identically to itself for each $v \in Y$ and sending u_e to t_e. Now, by using the function $E \to P$ defined by $e \mapsto u_e$, we see that Lemma 9.1 yields a group homomorphism $\beta : G \to P$. Let us consider the composite: $G \xrightarrow{\beta} P \xrightarrow{\pi} G$; this function maps $G(v)$ identically into $G(v)$ and maps $t_e \mapsto t_e$ (see how $\beta(t_e)$ is defined in Lemma 9.1). Therefore the composition is the identity. Therefore β is injective. Moreover,

$$\beta G = \pi(G(-), Y, v_0)$$

and hence $G \cong \pi(G(-), Y, v_0)$.

Among other things, this shows that the isomorphism class of $G = \pi(G(-), Y, Y_0)$ does not depend upon Y_0 and that also G is isomorphic to the usual definition of fundamental group. Sometimes later we write simply $\pi(G(-), Y)$ for G: actually we do this when we want to describe abstract algebraic properties (and not group actions) of $\pi(G(-), Y, Y_0)$.

Theorem 9.3. *Suppose that $|G(v)|$ is finite for each $v \in V$. Let $n \in \mathbb{N} \setminus \{0\}$ be divisible by $|G(v)|$ for each $v \in V$. Then there exists a group homomorphism $G \to \mathrm{Sym}(n)$ such that the composition $G(v) \to G \to \mathrm{Sym}(n)$ is injective for each $v \in V$.*

Proof. Consider the set $\Omega = \{1, \ldots, n\}$. Since $|G(v)|$ divides n, we can partition Ω into $n/|G(v)|$ sets each having cardinality $|G(v)|$. Letting $G(v)$ act on each of these sets via its left regular representation, we obtain an action of $G(v)$ on Ω. Let $\alpha_v : G(v) \to \mathrm{Sym}(\Omega)$ be the embedding resulting from this faithful action.

Observe that the only element of $G(v)$ fixing some element of Ω is the identity, that is, $G(v)$ acts freely on Ω.

Let $e \in E$. Now both $G(\bar{\iota} e)$ and $G(\bar{\tau} e)$ act freely on Ω. Moreover, the emdeddings $G(e) \leq G(\bar{\iota} e)$ and $t_e : G(e) \to G(\bar{\tau} e)$, determine two $G(e)$-actions on Ω (inhereded by the actions of $G(\bar{\iota} e)$ and $G(\bar{\tau} e)$). Let us denote these two $G(e)$-sets by $\Omega_{\bar{\iota} e}$ and $\Omega_{\bar{\tau} e}$. For both actions $G(e)$ acts freeely and hence the two actions are isomorphic. Let $\alpha(e) : \Omega \to \Omega$ be any $G(e)$-isomorphism. This means that, for each $g \in G(e)$ and for each $\omega \in \Omega$, we have

$$\alpha(e)(\alpha_{\bar{\tau} e}(g^{t_e})\omega) = \alpha_{\bar{\iota} e}(g)(\alpha(e)(\omega)).$$

This equation shows that $\alpha(e) \in \mathrm{Sym}(\Omega)$ has the property that

$$\alpha(e)\alpha_{\bar{\tau}e}(g^{t_e}) = \alpha_{\bar{\iota}e}(g)\alpha(e),$$

for every $g \in G(e)$.

We are in the position to apply Lemma 9.1. Therefore there exists a group homomorphism $\beta : G \to \mathrm{Sym}(\Omega)$ and the restriction of β to $G(v)$ is conjugate (via an element of $\mathrm{Sym}(\Omega)$) to the injective homomorphism α_v. Since α_v is injective, we deduce that also the restriction $\beta \restriction G(v)$ is injective, which is what we wanted. □

The main point of Theorem 9.3 is that $G(v)$ is a genuine subgroup of the fundamental group $G(v)$, no "deformations" are possible. This is essentially due to the fact that we have chosen t_e to be an injective homomorphisms.

Exercise 9.4. Here we make a short digression and see what can happen if the connecting homomorphisms are not injective. We do this in an example and we start with a formal definition. Let C, A_1 and A_2 be three groups and let $f_1 : C \to A_1$ and $f_2 : C \to A_2$ be group homomorphisms. Just for this exercise, we call the amalgamated product of A_1 and A_2 on C the group $A_1 *_C A_2$ having the following universal property:

- **(i):** There exist two group homomorphisms $g_1 : A_1 \to G$ and $g_2 : A_2 \to G$ with $g_1 \circ f_1 = g_2 \circ f_2$.
- **(ii):** If H is a group and if $h_1 : A_1 \to H$ and $h_2 : A_2 \to H$ are two group homomorphisms with $h_1 \circ f_1 = h_2 \circ f_2$, then exists a unique group homomorphism $h : G \to H$ such that $h_1 = h \circ g_1$ and $h_2 = h \circ g_2$.

This definition captures and externds the definition of almalgamated product defined earliear.

Let $A_1 = \mathrm{PSL}_2(\mathbb{Q})$, $A_2 = \mathbb{Z}/2\mathbb{Z}$ and $C = \mathbb{Z}$; let $f_1 : C \to A_1$ be any injective homomorphism and let $f_2 : C \to A_2$ be any surjective homomorphism. Prove that $A_1 *_C A_2 = 1$.

Corollary 9.5. *If $G(v)$ is finite for each $v \in VY$, then the inclusion function $G(v) \to G$ is injective for each $v \in VY$.*

Proof. Apply Theorem 9.3. □

In view of Corollary 9.5, from now on, we treat $G(y)$ as a subgroup of G, for every $y \in Y$.

The main result of this course is the following.

Theorem 9.6. *The graph T is a tree.*

Proof. Let CT be the set of all connected components of T. Now, since CT is a partition of VT, there is a natural map $[-] : VT \to CT$ defined by $v \mapsto [v]$, where $[v]$ is the connected component of T containing v. To show that T is a tree we first need to convince ourselves that CT consists of the single element $[v_0]$. Observe that the G-set VT and the map $[-]$ determine a structure of a G-set on CT. Moreover $[\iota e] = [\tau e]$, for each $e \in ET$. Recall that by definition, VT is the G-set generated by V and with relations, the condition that v is $G(v)$-invariant, for each $v \in V$. In particular, CT is also a G-set generated by V and having at least the relations saying that $[v] \in CT$ is $G(v)$-invariant, for every $v \in VT$. However, there are additional relations. Namely, for each $e \in E$, $[\iota e] = [\tau e]$; therefore,

$$[\bar{\iota}e] = [\iota e] = [\tau e] = [t_e \bar{\tau} e] = t_e[\bar{\tau}e].$$

When we specify this relation with $e \in EY_0$, the above relation becomes $[\overline{\iota}e] = [\overline{\tau}e]$, for all $e \in EY_0$. Therefore $[v] = [v_0]$, for every $v \in V$, because Y_0 is a spanning tree of Y. So CT consists of a single G-orbit. Now $[v_0]$ is $G(v)$-invariant for each $v \in V$ because $[v] = [v_0]$; moreover, $[v_0]$ is t_e-invariant for each $e \in EY$ because $[v_0] = [\overline{\iota}e] = t_e[\overline{\tau}e] = t_e[v_0]$. Thus $[v_0]$ is G-invariant. Therefore $CT = \{[v_0]\}$ and T is connected.

In view of Exercise 8.3, to show that T is a tree it suffices to show that the boundary map $\partial : \mathbb{Z}ET \to \mathbb{Z}VT$ is injective.

For each $v \in G$, let $\alpha_v : G(v) \to G \ltimes \mathbb{Z}ET$ be the injective homomorphism defined by $g \mapsto (g, 0)$. (Here $G \ltimes \mathbb{Z}ET$ is the semidirect product of G via the G-module $\mathbb{Z}ET$.) Let $\alpha : E \to G \ltimes \mathbb{Z}ET$ be the function defined by $e \mapsto (t_e, e)$.

Let $e \in E$ and let $g \in G(e)$. Then

$$\alpha_{\overline{\iota}e}(g)\alpha(e) = (g,0)(t_e,e) = (gt_e, ge) = (t_e g^{t_e}, e) = (t_e, e)(g^{t_e}, 0) = \alpha(e)\alpha_{\overline{\tau}e}(g^{t_e}),$$

and hence the hypothesis of Lemma 9.1 are satisfied.

Recall the definition of the functions $\alpha(v, w)$ in Lemma 9.1. Given $v, w \in V$, let $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ be the Y_0-geodesic from v to w. Then

$$\begin{aligned} \alpha(v,w) &= \alpha(e_1)^{\varepsilon_1} \cdots \alpha(e_n)^{\varepsilon_n} = (1, e_1^{\varepsilon_1}) \cdots (1, e_n^{\varepsilon_n}) \\ &= (1, \varepsilon_1 e_1 + \cdots + \varepsilon_n e_n) = (1, Y_0[v,w]), \end{aligned}$$

where $Y_0[v,w] = \varepsilon_1 e_1 + \cdots + \varepsilon_n e_n \in \mathbb{Z}EY_0 \subseteq \mathbb{Z}ET$. (See also Definition 8.4.) By Lemma 9.1, there exists a group homomorphism $\beta : G \to G \ltimes \mathbb{Z}ET$ such that

(1) for each $v \in V$ and for each $g \in G(v)$,

$$\begin{aligned} \beta(g) &= \alpha(v_0, v)\alpha_v(g)\alpha(v, v_0) \\ &= (1, Y_0[v_0,v])(g,0)(1, Y_0[v,v_0]) = (g, Y_0[v_0,v] + gY_0[v,v_0]); \end{aligned}$$

(2) for each $e \in E$,

$$\begin{aligned} \beta(t_e) &= \alpha(v_0, \overline{\iota}e)\alpha(e)\alpha(\overline{\tau}e, v_0) \\ &= (1, Y_0[v_0, \overline{\iota}e])(t_e, e)(1, Y_0[\overline{\tau}e, v_0]) \\ &= (t_e, Y_0[v_0 + \overline{\iota}e] + e + t_e Y_0[\overline{\tau}e, v_0]). \end{aligned}$$

This shows that the group homomorphism β is actually of the form (id_G, d), where $d : G \to \mathbb{Z}ET$ is a function. Recall now Exercise 8.6.

We highlight here some properties of d that we need:

(4) $$d(xy) = d(x) + xd(y), \qquad \text{for every } x, y \in G;$$

(5) $$d(g) = Y_0[v_0, v] + gY_0[v, v_0], \qquad \text{for every } v \in V, g \in G(v);$$

(6) $$d(t_e) = Y_0[v_0, \overline{\iota}e] + e + t_e Y_0[\overline{\tau}e, v_0], \qquad \text{for every } e \in E.$$

We claim that there exists a $\mathbb{Z}$-linear map $T[v_0, -] : \mathbb{Z}VT \to \mathbb{Z}ET$ with

(7) $$T[v_0, gv] = d(g) + gY_0[v_0, v], \qquad \text{for all } g \in G, v \in V.$$

To make sure that such a map $T[v_0, -]$ exists we have to make sure that the definition is consistent, that is, if $v, v' \in V$ and $g, g' \in G$ with $gv = g'v'$, then

$T[v_0, gv] = T[v_0, g'v']$, that is, $d(g) + gY_0[v_0, v] = d(g') + g'Y_0[v_0, v']$. Now, as Y is a G-transversal, we have $v = v'$ and $g' = gh$ for some $h \in G(v)$. Therefore,

$$\begin{aligned} T[v_0, g'v'] &= d(g') + g'Y_0[v_0, v] = d(gh) + ghY_0[v_0, v] \\ &= d(g) + gd(h) + ghY_0[v_0, v] \\ &= d(g) + g(Y_0[v_0, v] + hY_0[v, v_0]) + ghY_0[v_0, v] \\ &= d(g) + gY_0[v_0, v] + ghY_0[v, v] = d(g) + gY_0[v_0, v] \\ &= T[v_0, gv]. \end{aligned}$$

This shows that $T[v_0, -]$ is well-defined.

For every $g \in G$ and for every $e \in E$, we have

$$\begin{aligned} T[v_0, \partial(ge)] &= -T[v_0, g\iota e] + T[v_0, g\tau e] = -T[v_0, g\bar{\iota} e] + T[v_0, gt_e\bar{\tau} e] \\ &= (gY_0[\bar{\iota} e, v_0] - d(g)) + (d(gt_e) + gt_eY_0[v_0, \bar{\tau} e]) \\ &= gY_0[\bar{\iota} e, v_0] - d(g) + (d(g) + gd(t_e)) + gt_eY_0[v_0, \bar{\tau} e] \\ &= gY_0[\bar{\iota} e, v_0] + g(Y_0[v_0, \bar{\iota} e] + e + t_eY_0[\bar{\tau} e, v_0]) + gt_eY_0[v_0, \bar{\tau} e] \\ &= ge. \end{aligned}$$

This shows that the composition $\mathbb{Z}ET \xrightarrow{\partial} \mathbb{Z}VT \xrightarrow{T[v_0,-]} \mathbb{Z}ET$ is the identity map and hence ∂ is injective. □

From the fact that T is a tree and from the results we proved already for actions on trees we can deduce some information on the subgroups of G. In fact, from $H \leq G$, we deduce that T is an H-tree and hence H is isomorphic to the fundamental group of the graph of groups associated with T with respect to a fundamental H-transversal and connecting elements.

Proposition 9.7. *Let H be a subgroup of G such that $H \cap g^{-1}G(y)g = 1$, for every $y \in Y$ and for every $g \in G$. Then H is free.*

In particular, if H is torsion-free and the vertex groups are torsion groups (for instance finite), then H is free.

10. More on Cayley graphs and free groups

Definition 10.1. Let Y be a connected graph, let Y_0 be a maximal subtree of Y and let v_0 be a vertex of Y.

Here we look a little closer at the graph of groups $(G(-), Y)$ with $G(y) = 1$ for all $y \in Y$. As we observed in Example 5.5 (ii) this graph of groups corresponds to Cayley graphs.

The fundamental group of Y with respect to v_0 is $\pi(Y, v_0) = \pi(G(-), Y, v_0)$ (see Definition 9.2). In our case, this is the subgroup of the free group on EY consisting of all elements $e_1^{\varepsilon_1} e_2^{\varepsilon_2} \cdots e_n^{\varepsilon_n}$, where $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ is a closed path in Y starting at v_0. This is now the standard topological definition of fundamental group of Y at v_0 consisting of homotopy classes.

For this case, since the isomorphism type does not depend upon Y_0 or v_0, we simply say fundamental group of Y and write $\pi(Y)$: it is the free group of rank $|EY \setminus EY_0|$.

Let $G = \pi(G(-), Y)$ and let $T = T(G(-), Y, Y_0)$. We think of T as having a distinguisted vertex v_0. By Theorem 9.6, T is a tree and by construction T is a G-free G-tree with $T/G \cong Y$.

Actually, the projection map $T \to G$ is a local isomorphism, that is, it is an isomorphism when restricted to the neighbourhood of each vertex. Any tree with this property is called universal covering tree of Y. Yet again, this is in line with the usual definition of universal coverings in topology.

Define $\operatorname{rank} Y = |EY \setminus EY_0| = \operatorname{rank} G$ (this is the minimal number of generators for the free group G). Moreover the Euler characteristic of a free group F is $\chi(F) = 1 - \operatorname{rank} F$. This is in line with the usual definition of Euler characteristic of a graph (as we now show):

$$\begin{aligned}\chi(Y) &= |VY| - |EY| = |VY_0| - |EY| = (1 + |EY_0|) - |EY| \\ &= 1 - |EY \setminus EY_0| = 1 - \operatorname{rank} Y = 1 - \operatorname{rank} G = \chi(G).\end{aligned}$$

We sum up some of the remaks above into theorems.

Theorem 10.2. *The group G is freely generated by a subset S if and only if the Cayley graph* $\operatorname{Cay}(G, S)$ *is a G-tree.*

Theorem 10.3. *There exists a G-free G-tree if and only if G is a free group.*

Clearly the property characterising free groups in Theorem 10.3 is inherited by subgroups and hence we have the famous Nielsen-Schreier theorem:

Theorem 10.4. *Every subgroup of a free group is free.*

We conclude with a famous result of Schreier.

Theorem 10.5. *If G is a free group of finite rank r and H is a subgroup of G of finite index n, then H is free of rank $1 + n(r - 1)$. Writing this in terms of Euler characteristics, we have $\chi(H) = |G : H|\chi(G)$.*

Proof. Let S be a free generating set of G and let $T = \operatorname{Cay}(G, S)$. Thus T is a G-free G-tree. As H is a subgroup of G, the graph T is also a H-free H-tree. From Corollary 6.2, we have $G \cong \pi(T/G)$ and $H \cong \pi(T/H)$. Since T/G is a finite graph, we have $\chi(G) = \chi(T/G)$. Since ET is G-free, it consists of $|ET/G|$ copies of G, and hence it consists of $|G : H|(ET/G)$ copies of H. Therefore $|ET/H| = |G : H||ET/G|$. A similar argument gives $|VT/H| = |G : H||VT/G|$. From this we obtain $\chi(T/H) = |G : H|\chi(T/G)$. Therefore $\chi(H) = \chi(T/H) = |G : H|\chi(T/G) = |G : H|\chi(G)$. $\square$

11. Structure theorem for groups acting on connected graphs

In this secton X is a connected G-graph. Let Y be a fundamental G-transversal in X with subtree Y_0 and let $\bar\iota, \bar\tau$ be the corresponding incidence functions for the quotient graph. Choose $v_0 \in VY$, and for each $e \in EY$ choose an element $t_e \in G$ with $t_e \bar\tau e = \tau e$, with $t_e = 1$ if $e \in EY_0$. Let $(G(-), Y)$ be the resulting graph of groups and write $P = \pi(G(-), Y, Y_0)$, $T = T(G(-), Y, Y_0)$. We treat v_0 as an element of Y_0, Y, X and T. Observe that we know that T is a tree.

Theorem 11.1. *There is a natural extension $1 \to \pi(X) \to P \to G \to 1$. Further, $\pi(X)$ acts freely on T, and there is a natural isomorphism of G-graphs $T/\pi(X) \cong X$. In particular, T is the universal covering tree of X.*

The action of P on $\pi(X)$ by left conjugation induces a natural G-module structure on $\pi(X)^{ab}$, and there is an exact sequence of G-modules

$$0 \to \pi(X)^{ab} \to \mathbb{Z}[EX] \xrightarrow{\partial} \mathbb{Z}[VX] \to \mathbb{Z} \to 0. \tag{8}$$

Proof. Let $v \in VY$. As in the proof of Theorem 6.1 (see the claim), the paths of length 1 in X starting at v are the sequences of the form

$$v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^{\varepsilon}, gt_e^{\varepsilon}w,$$

where v, e^{ε}, w is a path in Y and $g \in G_v$. Hence, again as in the proof of Theorem 6.1, the natural projection homomorphism $\pi : P \to G$ is surjective (observe that the connectedness of X is fundamental). Let N be the kernel of π. Thus $G = P/N$.

For each $y \in Y$, the composition

$$G(y) \xrightarrow{inclusion} P \xrightarrow{\pi} G$$

is the natural embedding of $G(y) = G_y$. Therefore $N \cap G(y) = 1$, that is, N does not meet any vertex groups. Thus T is N-free and $N \cong \pi(T/N)$ by Corollary 6.2.

The group $P/N = G$ acts on the graph T/N; moreover, Y is a fundamental G-transversal, the t_e ($e \in EY$) are connecting elements, and the resulting graph of groups agrees with $(G(-), Y)$. As before, the path of length 1 in T/N starting at v are the sequences of the form

$$v, gt_e^{\frac{1}{2}(\varepsilon-1)}e^{\varepsilon}, gt_e^{\varepsilon}w,$$

where v, e^{ε}, w is a path in Y and $g \in G_v$. From this we deduce that $X \cong T/N$ as G-graphs, therefore $N \cong \pi(X)$. We treat this isomorphism as an identification, however for the rest of the proof we need to make this more precise.

For every element $c \in N$, the path in T from v_0 to cv_0 maps to a closed path in X at v_0 which corresponds to the element of $\pi(X)$ which we identify with c.

We have P acting on $\pi(X)$ by left conjugation, so by definition in the induced action on $\pi(X)^{ab}$, the group $\pi(X)$ acts trivially. Thus $\pi(X)^{ab}$ has the natural structure of a module over $P/N = G$. We need to analyse this action. The action under $g \in G$ sends an element of $\pi(X)^{ab}$ represented by a closed path q in X at v_0 to the element of $\pi(X)^{ab}$ represented by any g-conjugate of q, that is, a closed path p, gq, p^{-1} where p is a path in X from v_0 to gv_0. This action is independent of all choices.

The function which associates to a path $e_1^{\varepsilon_1}, \ldots, e_n^{\varepsilon_n}$ in X the element $\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n \in \mathbb{Z}EX$ induces a natural map $\pi(X)^{ab} \to \mathbb{Z}EX$ which is easily seen to be G-linear, and hence we have a complex as in (8).

Since X is connected, (8) is an exact complex at $\mathbb{Z}VX$.

Choose a maximal subtree X_0 of X. The group $\pi(X)$ is free on $EX \setminus EX_0$, therefore $\pi(X)^{ab} \cong \mathbb{Z}[EX \setminus EX_0]$, and the natural map to $\mathbb{Z}EX$ takes the form $\mathbb{Z}[EX \setminus EX_0] \to \mathbb{Z}EX$, $e \mapsto e + X_0[\tau e, \iota e]$, where $X_0[-,-]$ is as in Definition **??**. This is clearly injective, since composing with the projection onto $\mathbb{Z}[EX \setminus EX_0]$ gives the identity.

It remains to prove the exactness at $\mathbb{Z}EX$. Suppose that

$$\partial\left(\sum_{e \in EX} n_e e\right) = 0.$$

We know that

$$\partial\left(\sum_{e \in EX} n_e (e + X_0[\tau e, \iota e])\right) = 0$$

and hence

$$\partial\left(\sum_{e\in EX} n_e X_0[\tau e, \iota e]\right) = 0.$$

However ∂ restricted to $\mathbb{Z}EX_0$ is injective and hence

$$\sum_{e\in EX} n_e X_0[\tau e, \iota e] = 0.$$

Therefore

$$\sum_{e\in EX} n_e e = \sum_{e\in EX} n_e (e + X_0[\tau e, \iota e]),$$

which is the image of $\pi(X)^{ab}$, as desired. □

Corollary 11.2. *If X is a connected G-free G-graph, then there is an extension of groups $1 \to \pi(X) \to \pi(X/G) \to G \to 1$.*

Corollary 11.3. *If F is a free group on a set S, N is a normal subgroup of F and $G = F/N$, then there is an exact sequence of G-modules*

$$0 \to N^{ab} \to \mathbb{Z}[G \times S] \to \mathbb{Z}G \to \mathbb{Z} \to 0.$$

Proof. Let $X = \mathrm{Cay}(G, S)$ be the Cayley graph for G with connection set S. Thus we have a group extension $1 \to \pi(X) \to \pi(X/G) \to G \to 1$ and an exact sequence of G modules $0 \to \pi(X)^{ab} \to \mathbb{Z}[G \times S] \to \mathbb{Z}G \to \mathbb{Z} \to 0$. Finally, we can identify $F = \pi(X/G)$ so that $N = \pi(X)$. □

To have a complete structure theorem for a group acting on a connected graph, we want an explicit description of $\pi(X)$ as a subgroup of P. This is also useful in computations. An element of $\pi(X)$ corresponds to a unique closed path in X at v_0 and this can be expressed in the form

$$\begin{aligned} &v_0, g_0 t_{e_1}^{\frac{1}{2}(\varepsilon_1 - 1)} e_1^{\varepsilon_1}, g_0 t_{e_1}^{\varepsilon_1} v_1 \\ &g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} v_n = v_0, \end{aligned}$$

where $v_0, e_1^{\varepsilon_1}, v_1, e_2^{\varepsilon_2}, v_2, \ldots, v_{n-1}, e_n^{\varepsilon_n}, v_n = v_0$ is a closed path in Y and $g_i \in G_{v_i}$ for all $i \in \{0, \ldots, n\}$. Then $g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n}$ is an element of G_{v_0}, and denoting it by g_n^{-1}, we get an expression

$$g_0 t_{e_1}^{\varepsilon_1} g_1 t_{e_2}^{\varepsilon_2} g_2 \cdots g_{n-1} t_{e_n}^{\varepsilon_n} g_n$$

representing the desired element of P.

For the purpose of presenting G, one wants a set of elements which generate $\pi(X)$ as a normal subgroup of P; geometrically this amounts to a set of closed paths at v_0 in X whose G-translates generate all of $\pi(X)$.

Exercise 11.4. Let G be the group generated by the four elements x_1, x_2, x_3, x_4 and by the four relations

$$x_2 x_1 x_2^{-1} = x_1^2,\ x_3 x_2 x_3^{-1} = x_2^2,\ x_4 x_3 x_4^{-1} = x_3^2,\ x_1 x_4 x_1^{-1} = x_4^2.$$

- **(i):** The group G has a unique subgroup of finite index, namely G itself.
- **(ii):** The group G is infinite.

Let H be a subgroup of finite index in G. Now, $N_G(H)$ has also finite index in G and hence H has a finite number of conjugates in G. This shows that $N = \bigcap_{g\in G} H^g$ has finite index in G. Therefore it suffices to show that $N = G$. Let $\bar{G} = G/N$ ans assume that $\bar{G} \neq 1$. Let n_i be the order of $\bar{x}_i$ in $\bar{G}$, for $i \in \{1, 2, 3, 4\}$. Since $\bar{G}_i \neq 1$, there exists $i \in \{1, 2, 3, 4\}$ with $n_i \neq 1$. Among all prime divisors p of n_1, or n_2, or n_3, or n_4, choose p as small as possible. By the symmetry on the relations defining G, we may assume that p divides n_1. Now

$$\bar{x}_1^{2^{n_2}} = \bar{x}_2^{n_2}\bar{x}_1\bar{x}_2^{-n_2} = \bar{x}_1$$

and hence $2^{n_2} \equiv 1 \pmod{n}_1$. Therefore $2^{n_2} \equiv 1 \pmod p$. Clearly, this gives $p \neq 2$. Let N be the order of 2 in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^*$ of the field $\mathbb{Z}/p\mathbb{Z}$. As $2 \not\equiv 1 \pmod p$, we have $1 < N \leq p-1$. Since $2^{n_2} \equiv 1 \pmod p$, we deduce $n_2 \equiv 0 \pmod N$. If p' is a prime factor of N, we deduce $n_2 \equiv 0 \pmod p'$ and $p' \leq N \leq p-1$, which contradicts the minimality of p.

We need to prove now that G is infinite, clearly by the previous part it suffices to show that $G \neq 1$. Write

$$\begin{aligned}
G_{12} &= \langle x_1, x_2 \mid x_2x_1x_2^{-1} = x_1^2\rangle\\
G_{23} &= \langle x_2, x_3 \mid x_3x_2x_3^{-1} = x_2^2\rangle\\
G_{34} &= \langle x_3, x_4 \mid x_4x_3x_4^{-1} = x_3^2\rangle\\
G_{41} &= \langle x_4, x_1 \mid x_1x_4x_1^{-1} = x_4^2\rangle.
\end{aligned}$$

We claim that each of these groups is infinite. It suffices to show that G_{12} is infinite.

Now denote by G_1, G_2, G_3, G_4 the subgroups (in each of the previous groups) generated by x_1, x_2, x_3, x_4, that is, $G_i = \langle x_i\rangle$, for $i \in \{1, 2, 3, 4\}$. Let $\mathbb{Z}[1/2] = \{a/2^n \mid a, n \in \mathbb{Z}\}$. It is not hard to see that G_{12} is isomorphic to $\mathbb{Z}[1/2] \rtimes \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\mathbb{Z}[1/2]$ is by multiplication by 2, that is, $a/2^n \mapsto 2a/2^n = a/2^{n-1}$.

Now define

$$\begin{aligned}
G_{123} &= G_{12} *_{G_2} G_{23}\\
G_{234} &= G_{23} *_{G_3} G_{34}\\
G_{341} &= G_{34} *_{G_4} G_{41}.
\end{aligned}$$

Clealy, each of G_{123}, G_{234} and G_{341} is infinite by the theory developed so far.

Inside G_{123} the group generated by G_1 and G_3 is free, thus we set $F = \langle G_1, G_3\rangle = G_1 * G_2$. We do the same for G_{341}. Now a moment's thought gives that G is actually $G_{123} *_F G_{341}$ and hence G is infinite because it is obtained by successive amalgamations and it contains the free group on two generators F.

Exercise 11.5. Let G be the group generated by the three elements x_1, x_2, x_3 and by the four relations

$$x_2x_1x_2^{-1} = x_1^2,\ x_3x_2x_3^{-1} = x_2^2,\ x_1x_3x_1^{-1} = x_3^2.$$

Prove that $G = 1$.

12. The amalgam method

In this lecture we introduce a method (invented by Goldschmidt) that can be used to determine the order (and possibly classify the structure) of vertex stabilisers in arc-transitive graphs. We present this method through an example proving a celebrated theorem of Tutte establishing that the vertex stabiliser in a finite cubic arc-transitive graph has order dividing 48.

Let Γ be a (not necessarily finite) cubic graph and let G be a subgroup of $\mathrm{Aut}(\Gamma)$ acting transitively on the arcs of Γ. Let α,β be two adjacent vertices, $A=G_\alpha$, $B=G_{\{\alpha,\beta\}}$ and $C=G_{\alpha,\beta}$. Consider the graph Y having vertices A, B and with an edge C pointing from A to C. Consider the graph of groups $G(-)$ with $G(A)=A$, $G(B)=B$ and $G(C)=C$. Replacing G by $\pi(G(-),Y,Y_0)$ and Γ by $T(G(-),Y,Y_0)$ we may think that $G=A*_C B$ and that Γ is a tree. Since $|A:C|=3$ and $|B:C|=2$, we have two classes of vertices in Γ, vertices of valency 2 and vertices of valency 3. Thus Γ is the subdivision of the 3-regular tree.

For simplifying the notation we think of Γ as a 3-regular tree. Let α,β be two adjacent vertices of Γ. We let G_α be the stabiliser in G of a vertex α. In particular, $\alpha: G_\alpha\cap G_\beta|=3$ and $|G_{\{\alpha,\beta\}}:G_{\alpha\beta}|=2$. We denote by Q_α the kernel of the action of G_α on its neighbourhood $\Gamma(\alpha)$. If $Q_\alpha=1$, then $|G_\alpha|\leq 3!=6$, and hence there is nothing to prove. Therefore we assume that $Q_\alpha\neq 1$. Observe that the permutation group induced by G_α on $\Gamma(\alpha)$ is $G_\alpha^{\Gamma(\alpha)}=G_\alpha/Q_\alpha\cong\mathrm{Sym}(3)$.

Lemma 12.1. *If N is a subgroup of $G_{\alpha\beta}$ with $N_{G_\alpha}(N)$ transitive on $\Gamma(\alpha)$ and $N_{G_{\{\alpha,\beta\}}}(N)$ transitive on $\{\alpha,\beta\}$, then $N=1$.*

Lemma 12.2. *If $Q_\alpha\leq Q_\beta$, then $|G_\alpha|\leq 6$.*

Proof. If $Q_\alpha\leq Q_\beta$, then $Q_\alpha=Q_\beta$ because an automorphism of Γ interchanging α with β swaps Q_α with Q_β. Therefore Q_α is normal in G_α and in $G_{\{\alpha,\beta\}}$. Therefore $Q_\alpha=1$. □

From now on we may assume that $Q_\alpha\nleq Q_\beta$. In particular, $Q_\alpha Q_\beta=G_\alpha\cap G_\beta=G_{\alpha\beta}$ is a Sylow 2-subgroup of G_α and of G_β.

Lemma 12.3. *Either $C_{G_\alpha}(Q_\alpha)\leq Q_\alpha$ or $|G_\alpha|=12$.*

Proof. Set $C=C_\alpha(Q_\alpha)$ and assume that $C\nleq Q_\alpha$. As $C\trianglelefteq G_\alpha$, we see that C acts transitively on $\Gamma(\alpha)$. Now, consider $N=Q_{\alpha\beta}$. We have $N_{G_\alpha}(N)\geq C_{G_\alpha}(N)\geq C_{G_\alpha}(Q_\alpha)$ and hence $N_{G_\alpha}(N)$ is transitive on Γ. Clearly, $N_{G_{\{\alpha,\beta\}}}(N)=G_{\{\alpha,\beta\}}$ is transitive on $\{\alpha,\beta\}$. Therefore $N=1$ and $Q_{\alpha\beta}=1$. □

From the previous lemma we may assume that $C_{G_\alpha}(Q_\alpha)\leq Q_\alpha$. We write

$$Z_\alpha=\langle\Omega(Z(T))\mid T\in\mathrm{Syl}_2(G_\alpha)\rangle.$$

Lemma 12.4. **(i):** *$Z_\alpha\leq\Omega(Z(Q_\alpha))$;*
(ii): *$C_{G_\alpha}(Z_\alpha)=Q_\alpha$;*
(iii): *$Z_\alpha Z_\beta\neq Z_\alpha Z_\gamma$ for every $\gamma\in\Gamma(\alpha)\setminus\{\beta\}$.*

Proof. Part (i). Let $T\in\mathrm{Syl}_2(G_\alpha)$. Then $Q_\alpha\leq T$ and, as $C_{G_\alpha}(Q_\alpha)\leq Q_\alpha$, we obtain $\Omega(Z(T))\leq Z(Q_\alpha)$. Therefore $\Omega(Z(T))\leq\Omega(Z(Q_\alpha))$.

Part (ii). From (i), we have $Q_\alpha\leq C_{G_\alpha}(Z_\alpha)$. Set $C=C_{G_\alpha}(Z_\alpha)$. If $C\nleq Q_\alpha$, then C is transitive on $\Gamma(\alpha)$. Set $N=Z_\alpha$. We have that $N_{G_\alpha}(N)\geq C_{G_\alpha}(N)$ is transitive on $\Gamma(\alpha)$. Let $T=Q_\alpha Q_\beta$ be a Sylow 2-subgroup of G_α. Now, $\Omega(Z(T))$ is normalized by $G_{\{\alpha,\beta\}}$. Moreover, $C_{G_\alpha}(\Omega(Z/(T))\geq C_{G_\alpha}(Z_\alpha)$ and hence $C_{G_\alpha}(\Omega(Z(T)))$ is transitive on $\Gamma(\alpha)$. This gives $\Omega(Z(T))=1$ and $T=1$, a contradiction.

Part (iii). Suppose that $Z_\alpha Z_\beta=Z_\alpha Z_\gamma$ for some $\gamma\in\Gamma(\alpha)\setminus\{\beta\}$. Then $Z_\alpha Z_\beta$ is normal in G_α. It is always normal in $G_{\{\alpha,\beta\}}$. Therefore $Z_\alpha Z_\beta=1$, contradicting $Z_\alpha\neq 1$. □

Lemma 12.5. *If $Z_\alpha \not\leq Q_\beta$, then $|G_\alpha| \leq 48$.*

Proof. □

In view of Lemma 12.5, we may assume that $Z_\alpha \leq Q_\beta$. We will prove that this yields a contradiction. For this part of the argument the following parameter b (introduced by Goldschmidt) plays a crucial role.

$$b = \min\{d(\mu, \lambda) \mid \mu, \lambda \in V\Gamma, Z_\mu \not\leq Q_\lambda\}.$$

A pair of vertices (μ, λ) such that $d(\mu, \lambda) = b$ and $Z_\mu \not\leq Q_\lambda$ is called a critical pair. Observe that $Z_\mu \not\leq Q_\lambda$ implie $Z_\mu \not\leq G_\lambda$ and hence Z_μ acts as a cyclic group of order 2 on $\Gamma(\mu)$.

As G acts transitively on the arcs of Γ, we see that $Z_\alpha \not\leq Q_\alpha$ and $Z_\beta \not\leq Q_\alpha$ and hence $b > 1$.

13. Computer computations

Suppose that Γ a finite connected directed graph with outvalency 2 and invalency 2. Suppose that there exists a group G acting transitively on the vertices of Γ and with G_v having two orbits on the neighbours (in- and out-) of v. Suppose that $G_v = \langle x, y, z \mid x^2 = y^2 = [x, y] = 1\rangle = A$ is the Klein 4 group. And that the stabiliser of an edge (v, w) is $G_{(v,w)} = \langle x \mid x^2 = 1\rangle = C$. Now the graph of groups of G consists of a single vertex v and of a single edge/loop. Let $t_e : C \to A$ be the embedding corresponding to the stabilizer G_w. This t_e sends x to y. Therefore the group G is a quotient of the group with presentation

$$P = \langle x, y, z, t \mid x^2 = y^2 = [x, y] = 1, t^{-1}xt = y\rangle,$$

that is, $G = P/N$ for some normal subgroup N of P. Moreover, the original graph Γ is the quotient T/N, where T is the fundamental free corresponding to P.

Now, the computer algebra system `magma` gives an invaluable tool for obtaining the small index normal subgroups of P, via the command

`LowIndexNormalSubgroups`.

The example has nothing special, and this procedure can be applied each time that we have complete information on the graph of groups of a certain group action on graphs.

For example, the famous census of connected cubic arc-transitive graphs of order at most 10 000 was obtained in this way, by applying this procedure to the possible graph of groups of finite connected cubic arc-transitive graphs. The same procedure was also applied more recently to obtain all connected cubic vertex-transitive graphs of order at most 1 280.

For this method to work it becomes relevant the description of the graph of groups of the family of graphs we like to generate. In particular, for vertex-transitive graphs this procedure requires a detailed knowledge of the vertex-stabilizers.

This is still a naive way to attack the problem of building a census of certain vertex transitive graphs, however it is the key idea and the key method.

The drawback of this method is that there are G-graphs with relatively small order, compared to the order of their vertex stabilisers. This makes the invaluable `LowIndexNormalSubgroups` command impractical to use. To make this method than to work one need to prove theorems like the following.

Theorem 13.1. *Let Γ be a connected 3-valent G-vertex-transitive graph. Then one of the following holds:*

(A): *Γ is in a well described family of examples;*
(B): *$|G_v| \leq 2^4 3^6$;*
(C): *$|V\Gamma| \geq 2|G_v| \log_2(|G_v|/2)$.*

Morally, this theorem is saying that, from a vertex stabiliser point of view, nature is rather meager: vertex stabilisers have size comparatively small compared to the number of vertices, or they are classified (these graphs are usually referred to as Praeger-Xu graphs, under the name of the people who first investigated rather symmetric cubic graphs). The method for proving this theorem is rather different from the theory developed so far, and pertain to the theory of finite groups and in particular it is an application of the Classification of the Finite Simple Groups.

A key ingredient for the theorem above is the following result (Gardiner, Praeger, Xu).

Theorem 13.2. *Let Γ be a connected finite cubic G-vertex-transitive graph, let A be an abelian normal subgroup of G. Then either A acts semiregular on $V\Gamma$ or Γ is a Praeger-Xu graph.*

This theorem exhibits a rather peculiar behaviour. A cubic vertex-transitive G-graph having a normal abelian subgroup, not acting freely (that is semiregularly) has a very restricted structure. Any theorem that can generalise this result to other classes of graphs might (in principle) open the door for classifying other classes of symmetric graphs. So far, no analogue (as powerful) of this result is known.

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