Symmetries of finite projective planes

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No (analogous) results are known about case $n \equiv 3 \pmod 4$.

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Open problem: Does the case $H \cong PSU(3, q)$ actually occur (in some non-classical plane)?.

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