Symmetries of finite projective planes

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Basic Facts and Notation

A finite projective plane: $\left( P, L \right)$

$P$ := non empty set whose elements
$L$ := family of subsets of $P$ whose elements

Lines satisfying incident axioms:

- any two distinct points are contained in exactly one line;
- any two distinct lines have exactly one common point;
- there exists a non-degenerate quadrangle.

Basic facts:

(i) Each line consists of the same number of points
(ii) Each point is contained in the same number of lines
(iii) The numbers in (i) and (ii) coincide. This number is denoted by $n + 1$ where $n \geq 2$ is an integer the order of the projective plane
(iv) The total number of points, as well as of lines, is $n^2 + n + 1$. 
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Three equivalent definitions of $PG(2, q)$:

(i) $P := \{1\text{-dimensional subspaces of } V(3, q)\}$,

(ii) $L := \{2\text{-dimensional subspaces of } V(3, q)\}$,

(iii) $\mathbb{A}(2, q) := \text{affine plane over } \mathbb{F}_q$, $PG(2, q)$ its projective closure.

The order of $PG(2, q)$ is equal to $q^3$.

∃ many finite projective planes of order $q$ other than $PG(2, q)$.

Open problem: ∃ projective plane of order $n$ with $n \neq q$?
Classical plane: Projective plane $PG(2, q)$ over a finite field
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1. $V(3, q) := 3$-dimensional vector-space over the finite field $F_q$ with $q$ elements, $q := p^h$, $p$ prime, $h \geq 2$

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for $[u_1, u_2, u_3] \in \mathcal{U}$: $\ell := \{P(x_1, x_2, x_3) | : u_1x_1 + u_2x_2 + u_3x_3 = 0\}$. 

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Symmetries of a projective plane

Symmetry (automorphism, collineation) of $\Pi$: permutation on $P$ which takes lines to lines, i.e., $\sigma$ is an automorphism of $(P; L)$ $\iff \sigma(\ell) \in L$ for all $\ell \in L$.

$\text{Aut}(\Pi)$ is the set (group) of all automorphisms of $\Pi$.

For $\Pi = \text{PG}(2, q)$, $\text{Aut}(\Pi) \supseteq \text{PGL}(3, q)$ where $\text{PGL}(3, q) = \text{GL}(3, q) / Z(\text{GL}(3, q))$, $Z(\text{GL}(3, q)) = \{\lambda I_3 | \lambda \in \mathbb{F}_q^*\}$.

The study of $\text{Aut}(\Pi)$ is a difficult task: It combines three theories: Geometry, Group Theory, Combinatorics.

Typical problem: For some nice point-set $\Omega$ in $\Pi$, determine the subgroup $G$ of $\text{Aut}(\Pi)$ which leaves $\Omega$ invariant.

We shall see how to do this when $\Omega$ is an oval.
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Ovals of a projective plane

Oval is a combinatorial abstraction of a conic of the real projective plane:

\[ \Omega \subseteq \Pi \text{ s.t. no three points of } \Omega \text{ are collinear} \]

For every point \( P \in \Omega \), there exists a unique line \( \ell \) such that \( \Omega \cap \ell = \{ P \} \). The line \( \ell \) is the tangent of \( \Omega \) at \( P \).

\[ |\Omega| = n + 1 \] for any oval \( \Omega \) in a projective plane of order \( n \).

Conic in \( \operatorname{PG}(2, q) \) is the classical oval.

\[ \text{Conic } C := \text{set of all points } P(\mathbf{x}) = (x_1, x_2, x_3) \text{ in } \operatorname{PG}(2, q) \text{ s.t.} \]

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For \( q \) odd, conic \( C \) is the set of all self-conjugate points of an orthogonal polarity, the collineation group preserving \( C \) is \( \Gamma L(2, q) \) and hence contains \( \operatorname{PSL}(2, q) \).
Ovals of a projective plane

Oval is a combinatorial abstraction of a conic of the real projective plane:

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Geometry of an oval of a projective plane of odd order

Ω := oval in a projective plane Π of odd order,

Points of Π are partitioned in three types:

(i) Points on the oval
(ii) External points (through each there are exactly two tangents)
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<table>
<thead>
<tr>
<th>Ω</th>
<th>n+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(Ω)</td>
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<tr>
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Segre's Theorem (1955): In \(PG(2,q)\) with \(q\) odd, every oval is a conic.

Segre's theorem fails for \(q\) even.

Open problem: Classification of ovals in \(PG(2,q)\) for \(q\) even.

Open problem: Do exist finite projective planes without any oval?
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\( \Omega = n + 1, \quad E(\Omega) = \frac{1}{2} (n + 1)^2, \quad I(\Omega) = \frac{1}{2} (n - 1)n \)

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Involutory collineations of an oval in an odd order plane

Let $\Pi$ be a projective plane of odd order $n$. A central-axial collineation $(C, \ell)$ is a collineation that fixes a line $\ell$ pointwise (axis) and each line through a point $C$ (center).

A homology is a central-axial collineation $(C, \ell)$ with $C \not\in \ell$.

An elation is a central-axial collineation $(C, \ell)$ with $C \in \ell$.

Let $\sigma \in \text{Aut}(\Pi)$. The fix set $\text{Fix}(\sigma)$ is the set of points $P$ such that $\sigma(P) = P$, $P \in \Pi$.

If $\sigma$ is involutory (i.e. $\sigma^2 = 1$, $\sigma \neq 1$) then $\sigma$ is either a homology $(C, \ell)$ with $C \not\in \Omega$, $\ell$ is not tangent, or $n = m^2$, and $\sigma$ is a Baer involution, i.e. $\text{Fix}(\sigma)$ is a subplane $\Pi_0$ of order $m$ and if $|\text{Fix}(\sigma) \cap \Omega| \geq 1$ then $\Omega_0 = \Pi_0 \cap \Omega$ is an oval in $\Pi_0$.

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Gábor Korchmáros
Symmetries of finite projective planes
Π:=projective plane of odd order $n$
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*Central-axial collineation* $(C, \ell) :=$ collineation fixing a line $\ell$ pointwise (axis) and each line through a point $C$ (center)

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Involutory collineations of an oval in an odd order plane

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2-groups of collineations of an oval in an odd order plane

$S_2$ := subgroup of $\text{Aut}(\Pi)$ of order $2^h \geq 1$ preserving an oval $\Omega$

Results on the structure of $S_2$:

- If $S_2$ contains no involutory homology then $S_2$ is cyclic;
- $r_2(S_2) \leq 3$, i.e. $S_2$ has no elementary abelian subgroup of order $> 8 = 2^3$;
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$H(S_2)$ := subgroup generated by all (involutory) homologies in $S_2$.

- $r(S_2) = 1$ $\Rightarrow$ $H(S_2) \cong C_2$,
- $r(S_2) = 2$ $\Rightarrow$ $H(S_2) \cong C_2 \times D_{2m} \times C_4 \times D_{2m} \times Q_2 \times D_{2m}$, $m, u \geq 3$;
- $r(S_2) = 3$ $\Rightarrow$ $H(S_2) \cong E_4 \times D_{2n}$.

Theorem: If $G$ is a (non-abelian) simple collineation group preserving $\Omega$ then $G \cong \text{PSL}(2, q)$ with $5 \leq q \leq n$.

Remark: The above results fail when $\Pi$ has even order: In the dual L"uneburg plane of order $2^h$, $h \geq 3$ odd, $\text{Sz}(2^h)$ preserves an oval.
$S_2 :=$ subgroup of $\text{Aut}(\Pi)$ of order $2^h$ $h \geq 1$ preserving an oval $\Omega$
2-groups of collineations of an oval in an odd order plane

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**Theorem**

If $G$ is a (non-abelian) simple collineation group preserving $\Omega$ then $G \simeq \text{PSL}(2, q)$ with $5 \leq q \leq n$.

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The above results fail when $\Pi$ has even order: In the dual L"uneburg plane of order $2^h$, $h \geq 3$ odd, $S_z(2^h)$ preserves an oval.
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2-groups of collineations of an oval in an odd order plane

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Gábor Korchmáros

Symmetries of finite projective planes
2-groups of collineations of an oval in an odd order plane

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**Theorem** If $G$ is a (non-abelian) simple collineation group preserving $\Omega$ then $G \cong PSL(2, q)$ with $5 \leq q \leq n$.

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2-groups of collineations of an oval in an odd order plane

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Action of collineation group of an oval in odd order plane

Theorem
$G$ is 2-transitive on $\Omega$ $\Rightarrow \Pi \cong PG(2, q)$ (and $\Omega = C$).

Theorem
$G$ is primitive on $\Omega$, $\Rightarrow$ either $G$ is 2-transitive on $\Omega$, or $\Pi \cong PG(2, 9)$, $\Omega = C$, and $PSL(2, 5) \leq G \leq PGL(2, 5)$.

Conjecture
$G$ is transitive on $\Omega$, $\Rightarrow \Pi \cong PG(2, q)$ (and $\Omega = C$).

Let $G$ be minimal transitive on $\Omega$.

For $n \equiv 1 \pmod{4}$, either $G$ is primitive on $\Omega$, or

(i) $|G| = 2^d$ $d$ odd;
(ii) $G \cong PSL(2, q)$, and

$n = q(q+1)−1$ or $n = q(q−1)+1$ according as $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$.

Remark: Case (ii) occurs in $PG(2, 29)$ with $G \cong PSL(2, 5)$ (and also in $PG(2, 5)$ with $G \cong PSL(2, 3)$). No (analogous) results are known about case $n \equiv 3 \pmod{4}$.
Action of collineation group of an oval in odd order plane

\[ G := \text{collineation group preserving an oval } \Omega \text{ in a projective plane } \Pi \text{ of odd order } n \]
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The action of \( G \) on \( \Omega \) is faithful

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Action of collineation group of an oval in odd order plane

$G :=$ collineation group preserving an oval $\Omega$ in a projective plane $\Pi$ of odd order $n$

The action of $G$ on $\Omega$ is faithful

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**Conjecture** $G$ is transitive on $\Omega \Rightarrow \Pi \cong PG(2, q)$ (and $\Omega = C$).
G := collineation group preserving an oval Ω in a projective plane Π of odd order n

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**Theorem** G is 2-transitive on Ω ⇒ Π ≅ PG(2, q) (and Ω = C).

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Action of collineation group of an oval in odd order plane

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Let $G$ be minimal transitive on $\Omega$. For $n \equiv 1 \pmod{4}$, either $G$ is primitive on $\Omega$, or
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Action of collineation group of an oval in odd order plane

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No (analogous) results are known about case $n \equiv 3 \pmod{4}$. 
Irreducible collineation groups in projective planes

Irreducible collineation group $G$ fixes no point, preserves no line and triangle in $\Pi$.

Strongly irreducible group $G$ is irreducible and preserves no subplane in $\Pi$.

Hering's classification of strongly irreducible groups containing central-axial collineations. (1970-1985)

Local version of irreducibility on an oval $\Omega$:

$G$ := collineation group preserving an oval $\Omega$ in a projective plane.

$G$ := irreducible on $\Omega$ if $G$ fixes no point of $\Omega$, preserves no chord or triangle of $\Omega$.

$G$ := strongly irreducible on $\Omega$ if $G$ is irreducible on $\Omega$, and preserves no suboval of $\Omega$.

Suboval $\Omega_0 = \Omega \cap \Pi_0$ with $\Pi_0$ a subplane and $\Omega_0$ an oval in $\Pi_0$.

Remark $G$ transitive on $\Omega$ $\Rightarrow$ $G$ strongly irreducible on $\Omega$.

Strongly irreducible oval $\Rightarrow$ $\exists$ strongly irreducible collineation group on the oval.
Irreducible collineation group $G := G$ fixes no point, preserves no line and triangle in $\Pi$. 

Remark $G$ transitive on $\Omega \Rightarrow G$ strongly irreducible on $\Omega$.

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Hering’s classification of strongly irreducible groups containing central-axial collineations. (1970-1985)

Local version of irreducibility on an oval.

\( G := \) collineation group preserving an oval \( \Omega \) in a projective plane.

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Irreducible collineation group $G := G$ fixes no point, preserves no line and triangle in $\Pi$.  
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suboval: $= \Omega_0 = \Omega \cap \Pi_0$ with $\Pi_0$ a subplane and $\Omega_0$ an oval in $\Pi_0$.  

Irreducible collineation groups in projective planes

*Irreducible collineation group* $G := G$ fixes no point, preserves no line and triangle in $\Pi$.

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Hering’s classification of strongly irreducible groups containing central-axial collineations. (1970-1985)

*Local version of irreducibility on an oval.*

$G := $ collineation group preserving an oval $\Omega$ in a projective plane.

$G := $ irreducible on $\Omega$ if $G$ fixes no point of $\Omega$, preserves no chord or triangle of $\Omega$.

$G := $ strongly irreducible on $\Omega$ if $G$ is irreducible on $\Omega$, and preserves no suboval of $\Omega$.

*Suboval*: $\Omega_0 = \Omega \cap \Pi_0$ with $\Pi_0$ a subplane and $\Omega_0$ an oval in $\Pi_0$.

**Remark** $G$ transitive on $\Omega \Rightarrow G$ strongly irreducible on $\Omega$. 
Irreducible collineation group $G := G$ fixes no point, preserves no line and triangle in $\Pi$.

Strongly irreducible group $G := G$ is irreducible and preserves no subplane in $\Pi$.

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Suboval: $= \Omega_0 = \Omega \cap \Pi_0$ with $\Pi_0$ a subplane and $\Omega_0$ an oval in $\Pi_0$.

Remark $G$ transitive on $\Omega \Rightarrow G$ strongly irreducible on $\Omega$.

Strongly irreducible oval: $= \exists$ strongly irreducible collineation group on the oval.
Classification of strongly irreducible ovals I

\[ \Pi := \text{projective plane of order } n \]
\[ \Omega := \text{strongly irreducible oval in } \Pi \]
\[ G := \text{strongly irreducible collineation group of } \Omega \]

**Theorem**

If \( G \leq \text{Alt} \Omega \) then either

(i) \( G \) is isomorphic to a subgroup of \( P\Gamma L(2,q) \) containing \( PSL(2,q) \) for some odd prime power \( q \);

(ii) \( G \) fixes a point-line pair \( \{P, \ell\} \), where \( P \) is an internal point whereas \( \ell \) is an external line to \( \Omega \) and all involutions in \( G \) are homologies.

Gábor Korchmáros

Symmetries of finite projective planes
\Pi:=\text{projective plane of order } n \text{ with } n \equiv 1 \pmod{4}
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**Theorem** If \( G \leq \text{Alt}_\Omega \) then either

(i) \( G \) is isomorphic to a subgroup of \( \text{PGL}(2, q) \) containing \( \text{PSL}(2, q) \) for some odd prime power \( q \);

(ii) \( G \) fixes a point-line pair \( \{P, \ell\} \), where \( P \) is an internal point whereas \( \ell \) is an external line to \( \Omega \) and all involutions in \( G \) are homologies.
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(i) \( G \) is isomorphic to a subgroup of \( P\Gamma L(2, q) \) containing \( PSL(2, q) \) for some odd prime power \( q \); or
Π:=projective plane of order \( n \) with \( n \equiv 1 \pmod{4} \)

Ω:=strongly irreducible oval in Π

G:=strongly irreducible collineation group of Ω

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Classification of strongly irreducible ovals II

Let $\Pi$ be the projective plane of even order $n$, $\Omega$ be a strongly irreducible oval in $\Pi$, and $G$ be the strongly irreducible collineation group of $\Omega$.

Theorem

The subgroup $H$ of $G$ generated by all (involutory) elations is either

(i) $H = O(G) \rtimes C_2$; or

(ii) $H \cong PSL(2,q)$, $Sz(q)$, $PSU(3,q)$ with $q = 2^h$.

Open problem: Does the case $H \cong PSU(3,q)$ actually occur (in some non-classical plane)?
Π: = projective plane of even order $n$

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\[ (i) \quad H \cong \text{PSL}(2, q) \]
\[ \text{Sz}(q) \]
\[ \text{PSU}(3, q) \]

with \( q = 2^h \).

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