Brandenburg
University of Technology
Cottbus - Senftenberg

# Linear structure of graphs and the knotting graph

Ekkehard Köhler

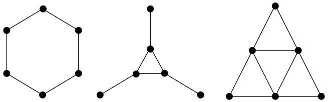
Rogla, June 2016

# 01

# Basic Properties

# AT-free Graphs — Definition

asteroidal triple: independent set of three vertices where each pair of vertices is joined by a path that avoids the neighborhood of the third vertex
$G$ AT-free: $G$ does not contain an asteroidal triple



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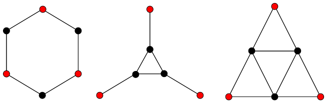

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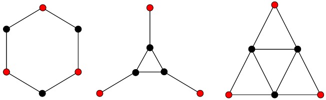
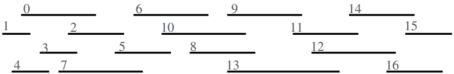
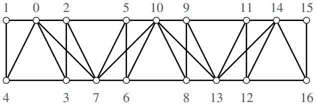


## Theorem (Boland, Lekkerkerker)

*$G$ interval graph if and only if $G$ chordal and AT-free*



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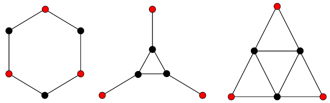

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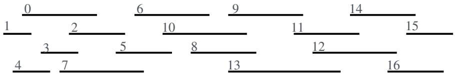
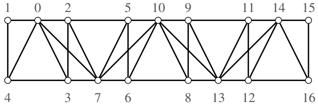
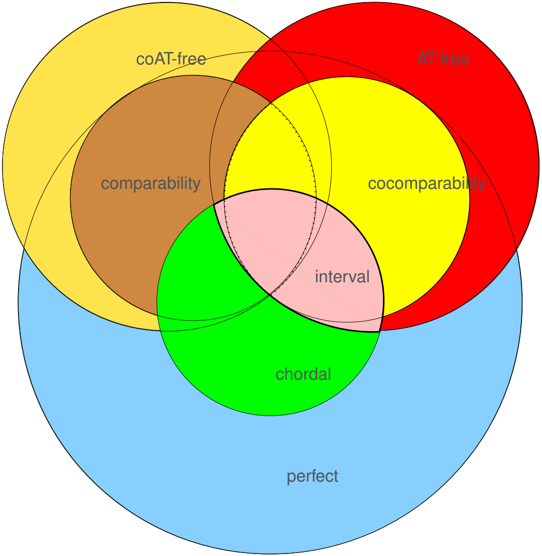



Claim: AT-free graphs have a linear structure.

# Relationship to other classes

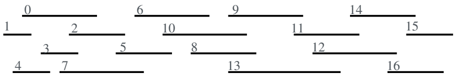


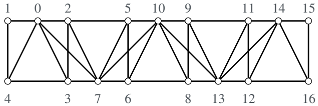
# Why is linear structure of any importance?

## Example:

maximum independent set in interval graphs

- Idea:
  - take interval model sorted by increasing right endpoint $\rightarrow$ scan from left to right
  - when interval $i$ is opened: update weight of $i$ plus weight of largest interval that has been closed before;
  - when interval $i$ is closed put its weight into (ordered) list of closed intervals
- linear time algorithm: $O(n+m)$
- what does it mean in complement?
  scan through partial order by iteratively visiting maximal elements and updating the weight function
- interval model imposes linear ordering $\Rightarrow$ linear structure does help!



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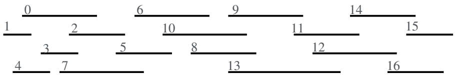


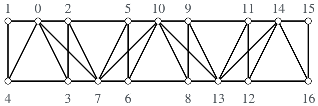


$\longrightarrow$ go to larger family: Cocomparability graphs

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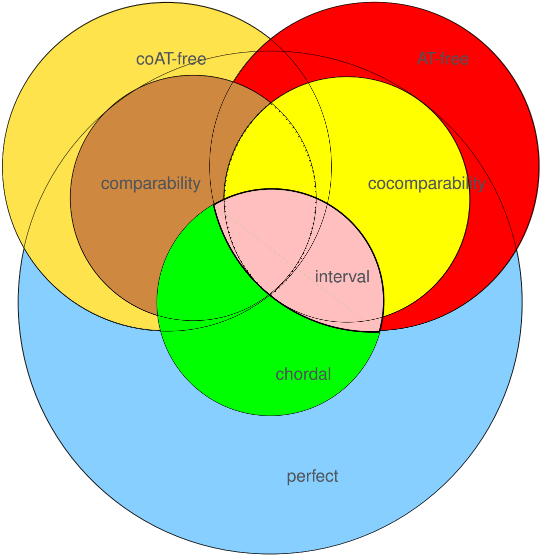

# Cocomparability Graphs — Definition

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$G = (V, E)$ is a cocomparability graph if $\overline{G}$ admits a transitive orientation of its edges:

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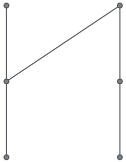
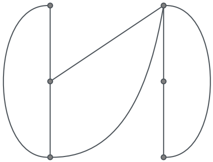


Hasse diagram of poset $P$



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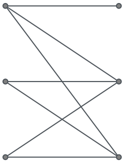

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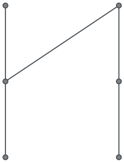

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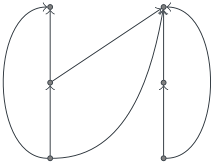

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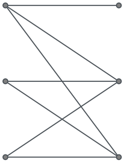

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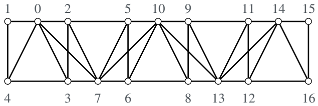
# Why is linear structure of any importance?

## Example (McConnell/Spinrad; Mouatadid/K.):

Maximum (weight) independent set in cocomparability graphs

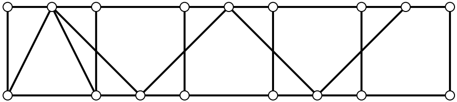
- Idea: work in $\overline{G}$
  - take linear extension of a corr. partial order of $\overline{G}$
  - create layering of the poset by iteratively removing the sets of maximal elements
  - when element $i$ is removed: for each direct predecessor of $i$ the weight function of $j$ is updated: $w'(j) = max\{w'(j), w(j) + w'(i)\}$
- $O(n+m) \rightarrow$ can do this algorithm in $\overline{G}$ in time linear in the size of $G$
- linear structure of $G$ used through linear structure of $\overline{G}$
- Can such an approach be generalized to AT-free graphs?



# How to generalize "linearity" for AT-free graphs?

## 1. Idea [Corneil/Olariu/Stewart]:

$x$,$y$ dominating pair iff all $x, y$-paths dominate $G$



## Theorem (Corneil/Olariu/Stewart)

*Every AT-free graph $G$ and each connected induced subgraph has a dominating pair.*

- dominating pair vertices define left and right "ends" of graph
- not uniquely determined
- no characterization of AT-free graphs

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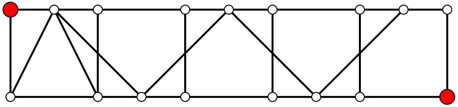

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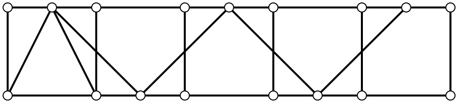
spine: elimination sequence of consecutive adjacent dominating pair vertices
$G$ has spine property iff $\forall$ nonadj. dom. pairs of $G$ form end-points of spine of $G$



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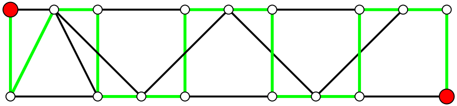

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minimal triangulation: inclusion minimal chordal completion

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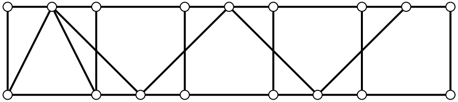
*$G$ AT-free iff each minimal triangulation is an interval graph.*



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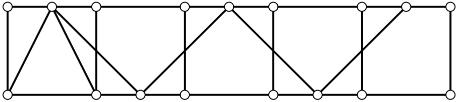

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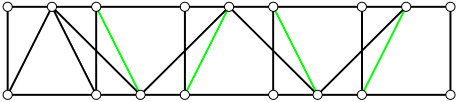

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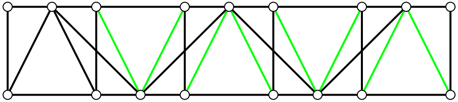

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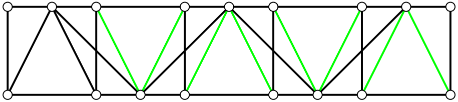

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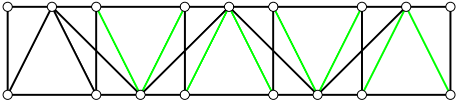

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Problem: Nice idea but how to use this algorithmically???

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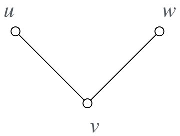


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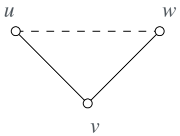


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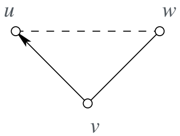


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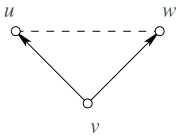


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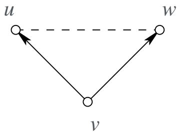


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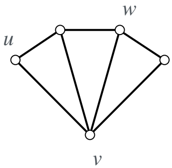

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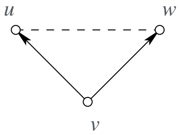


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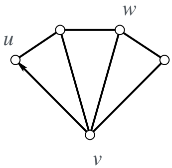

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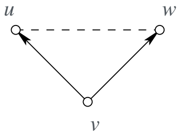


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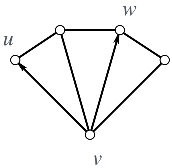
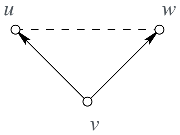
  $\rightarrow$ 'knot' edges $uv$ and $vw$ at $v$



# Linear structure of cocomparability graphs

- cocomp. graphs defined via comparability graphs and posets
- structure of cocomp. graphs mainly studied via complement
- comparability graphs: have transitive orientation
- early characterization by Gallai via knotting graph




## forcing of orientation

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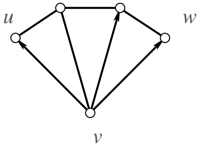

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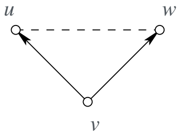


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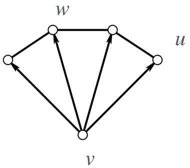

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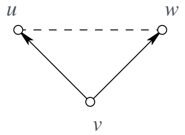


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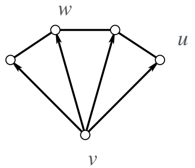
- $\rightarrow$ knotting graph



# 02
# Knotting Graph

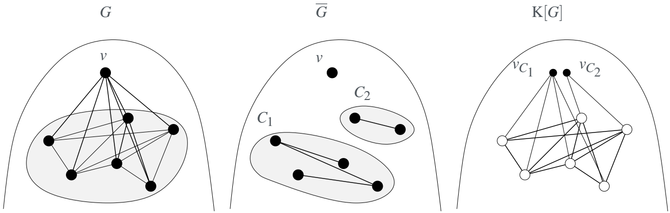
# The knotting graph

**Definition (Gallai):**

$G = (V, E)$ graph $\longrightarrow$ knotting graph of $G$: $\mathrm{K}[G] = (V_\mathrm{K}, E_\mathrm{K})$

$v \in V,\ C_1, C_2, \ldots, C_{i_v}$ connected components of $\overline{G}[N(v)] \Rightarrow v_{C_1}, v_{C_2}, \ldots, v_{C_{i_v}} \in V_\mathrm{K}$

$vw \in E,\ w \in C_i$ of $\overline{G}[N(v)]$ and $v \in C_j$ of $\overline{G}[N(w)] \Rightarrow v_{C_i} w_{C_j} \in E_K$



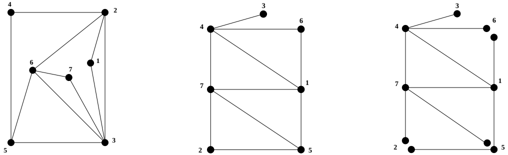

- forced edges $uv$, $vw$ are knotted at their copy of $v$

# Knotting graph — characterization of comparability graphs

- know: two edges that force eachother at $v \longrightarrow$ are knotted at $v$
- can show (Gallai): forcing sufficient to characterize comparability graphs: no odd cyles in $K[G]$

## Theorem (Gallai)

$G$ *comparability graph* $\Leftrightarrow K[G]$ *bipartite*
$G$ *cocomparability graph* $\Leftrightarrow K[\overline{G}]$ *bipartite*




- linear structure of cocomparability graphs imposed by transitive orientation of the non-edges
- Can this be generalized to AT-free graphs?

# Knotting graph — characterization of AT-free graphs

- consider coAT-free graphs
- let vertices $u, v, w$ form AT in $\overline{G}$
- then $u, v, w$ form triangle in $G$ and
  - $\exists$ path of non-edges between $u$ and $w$ in $N(v)$
  - $\exists$ path of non-edges between $u$ and $v$ in $N(w)$
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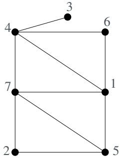
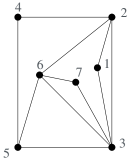
- $\Rightarrow$ edges are knotted at $u, v, w$

$G$:




$\overline{G}$:



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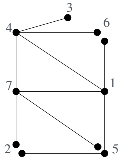
$K[G]$:

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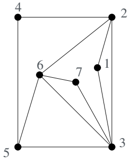

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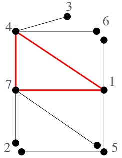


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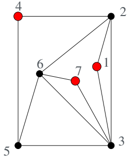

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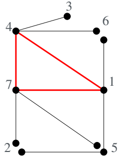
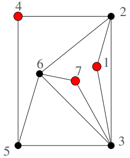



## Theorem

*$G$ coAT-free graph $\Leftrightarrow K[G]$ triangle-free*
*$G$ AT-free graph $\Leftrightarrow K[\overline{G}]$ triangle-free*

$\overline{G}$:




## Theorem

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## Theorem

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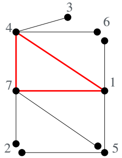
- recognition algorithm for AT-free graphs: construct $K[\overline{G}]$, check for triangles
- Does this imply linear structure?
- know not enough about the knotting graph

$K[G]$:

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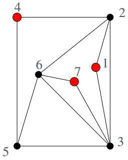
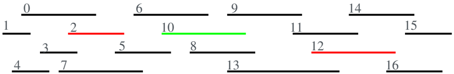


# Properties of the knotting graph — intervals in graphs

- Idea: linearity implies "betweeness"
- for two non-adjacent vertices $x, y$ there are vertices strictly between $x$ and $y$ in the linear order.
- example: interval model of interval graph: 10 between 2 and 12
  (other example: function diagram for cocomparability graphs)



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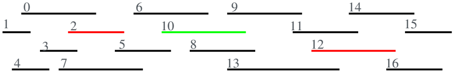


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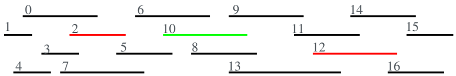


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$s$ between $x$ and $y$ if

- $y$ and $s$ in same component of $G \setminus N[x]$ and
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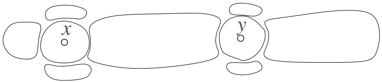
$C^x(y)$ component of $G \setminus N[x]$ that contains $y$



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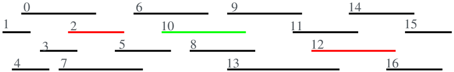


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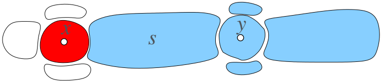

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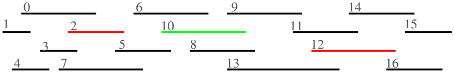


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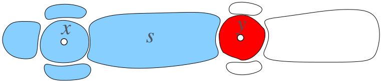

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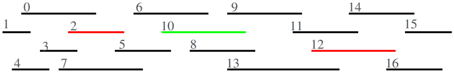


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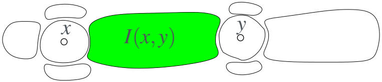
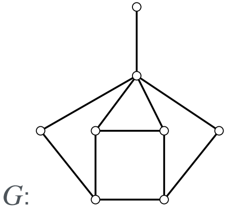



- $I(x,y) = C^x(y) \cap C^y(x)$ interval of $x$ and $y$ or $I(x,y)$ is the interval between $x$ and $y$

# Properties of the knotting graph — intervals in graphs II

- What is "betweeness" and "interval" in knotting graph?
- $I(x,y)$ set of vertices $u$ such that both
  - $ux$ knotted to $xy$ at $x$ and
  - $uy$ knotted to $xy$ at $y$
- intervals of $G$ correspond (somehow) to edges in $K[\overline{G}]$
- have "linear" property:




## Lemma (BKKM)

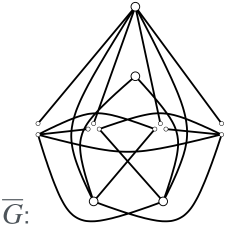
*$G$ AT-free and $u \in I(x,y)$ then $ux$, $uy$ not knotted at $u$*

## Theorem

*$G$ AT-free iff $\forall u \in I(x,y)$ $ux$, $uy$ not knotted at $u$ for all intervals*

## Lemma (BKKM)

*$G$ AT-free and $\forall s \in I(x,y)$: $I(x,s) \cap I(s,y) = \emptyset$*



# Properties of knotting graph — intervals in graphs III

- $G$ AT-free, $s \in I(x,y)$ then $s$ separates $x$ and $y$ in $G$: $x$ and $y$ in different conn. components of $N_G(s)$
- $\rightarrow$if $r$ has edge to same copy of $s$ as $x$ in $K[\overline{G}]$ then $r$ adj. to same copy of $y$ as $x$
- this implies

## Lemma (BKKM)

*if $G$ AT-free then $\forall\ s \in I(x,y)$: $I(x,s) \subset I(x,y)$ and $I(s,y) \subset I(x,y)$*

Using knotting graph, characterization of AT-free graphs:

## Theorem

*$G$ AT-free $\Leftrightarrow \forall I(x,y)$ and $\forall z \in I(x,y)$: $I(x,z) \subset I(x,y)$ and $I(z,y) \subset I(x,y)$*

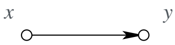
# What are intervals in (co-)comparability graphs

## Lemma

*$G$ comparability graph, $x, y \in V$, $z \in I(x,y)$ in $\overline{G}$*
*then in any transitive orientation of $G$ vertex $z$ is between $x$ and $y$ ($x < z < y$ or $y < z < x$).*

## Proof.

- wlog $x < y$
- suppose $z < x$
- $z \in I(x,y) \Rightarrow \exists$ path $P = v_1, \ldots, v_k$ between $z = v_1$ and $y = v_k$ in $\overline{G} \setminus N_{\overline{G}}[x]$
- all vertices of $P$ are knotted at $x$
- $\Rightarrow$ since $z$ predecessor of $x$, all vertices of $P$ predecessors of $x$, contradicting $x < y$

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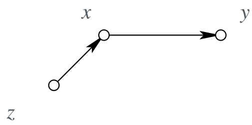


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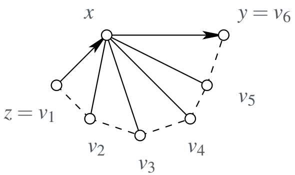


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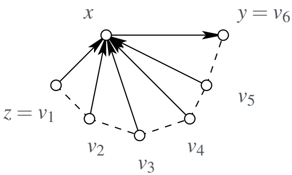
## What are intervals in (co-)comparability graphs

### Lemma

*$G$ comparability graph, $x, y \in V$, $z \in I(x,y)$ in $\overline{G}$*
*then in any transitive orientation of $G$ vertex $z$ is between $x$ and $y$ ($x < z < y$ or $y < z < x$).*

### Proof.

- wlog $x < y$
- suppose $z < x$
- $z \in I(x,y) \Rightarrow \exists$ path $P = v_1, \ldots, v_k$ between $z = v_1$ and $y = v_k$ in $\overline{G} \setminus N_{\overline{G}}[x]$
- all vertices of $P$ are knotted at $x$
- $\Rightarrow$ since $z$ predecessor of $x$, all vertices of $P$ predecessors of $x$, contradicting $x < y$

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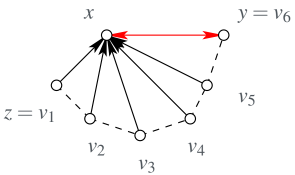


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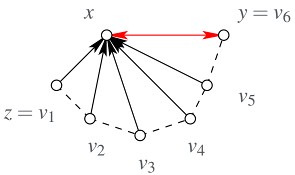


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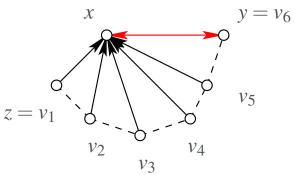


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Q: Does that extend to AT-free graphs?

A: No, not directly!

(There exist AT-free graphs not having linear order that respects intervals [Corneil, Olariu, K., Stewart]).

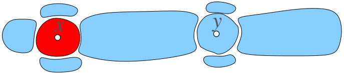
# 03

# Independent Sets

# Independent sets in AT-free graphs

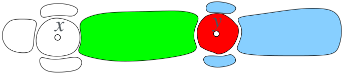
- Broersma/Kloks/Kratsch/Müller: $O(n^4)$ algorithm
- show here: improvement to $O(n\overline{m})$
  (or $O(nm)$ algorithm for maximum weighted clique in coAT-free graphs)
- main idea: dynamic programming using linear structure

$$\alpha(G) = 1 + \max_{x \in V} \left( \sum_{i=1}^{r(x)} \alpha(C_i^x) \right)$$



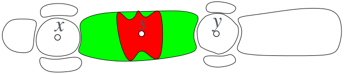

$C_1^x, C_2^x, \ldots, C_{r(x)}^x$: connected components of $G - N[x]$.

$$\alpha(C^x) = 1 + \max_{y \in C^x} \left( \alpha(I(x,y)) + \sum_i \alpha(D_i^y) \right)$$




$D_i^y$: component of $G - N[y]$ contained in $C^x$.

$$\alpha(I(x,y)) = 1 + \max_{s \in I(x,y)} \left( \alpha(I(x,s)) + \alpha(I(s,y)) + \sum_i \alpha(C_i^s) \right)$$



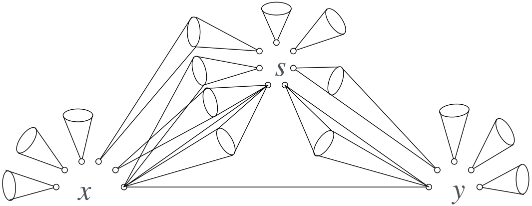

$C_i^s$: component of $G - N[s]$ contained in $I(x,y)$
Problem: computation of $\alpha(I(x,y))$ takes $O(n^4)$

# Independent sets in AT-free graphs II

## Theorem (BKKM)

$s \in I(x,y) \Rightarrow \exists$ *comp.* $C^s_1, \ldots, C^s_t$ *of* $G \setminus N[s]$ *such that* $I(x,y) \setminus N[s] = I(x,s) \cup I(s,y) \cup \bigcup_{i=1}^t C^s_i$.

in knotting graph:




new idea: can *characterize* connected components contained in an interval

## Theorem

$s \in I(x,y) \Rightarrow$ *and* $C^s_1, \ldots, C^s_t$ *be the components of* $\mathscr{C}^s \setminus (\mathscr{C}^x \cup \mathscr{C}^y \cup \{C^s(x), C^s(y)\})$
*then* $I(x,y) \setminus N[s] = I(x,s) \cup I(s,y) \cup \bigcup_{i=1}^t C^s_i$.

$\mathscr{C}^x$: set of components of $G \setminus N[x]$

# Independent sets in AT-free graphs III

## Theorem

*$s \in I(x,y) \Rightarrow$ and $C^s_1, \ldots, C^s_t$ be the components of $\mathscr{C}^s \setminus (\mathscr{C}^x \cup \mathscr{C}^y \cup \{C^s(x), C^s(y)\})$ then $I(x,y) \setminus N[s] = I(x,s) \cup I(s,y) \cup \bigcup_{i=1}^{t} C^s_i$.*

$\Rightarrow$ each of the components of $\mathscr{C}^z \setminus \{C^s(x), C^s(y)\}$ not contained in $I(x,y)$ is in $\mathscr{C}^x$ or in $\mathscr{C}^y$.

Another betweenness property:

## Lemma

*$s \in I(x,y)$ and $C$ component of both $G - N[x]$ and $G - N[y]$. Then $C$ is component of $G - N[s]$.*

## Theorem

*There is an $O(n\overline{m})$ algorithm to compute the independence number of a given AT-free graph.*

# 04

# Conclusion

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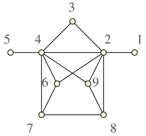
- many ways to see linear structure in AT-free graphs
- complementary graph helps to see structural properties that generalize partial orders
- algorithms can profit from this structure
- Open problem: Can linear structure be used for coloring or Hamilton path/cycle in AT-free graphs?



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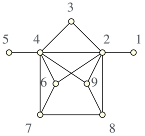


Thank you!