Resolving sets in finite geometries

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Overview

1. Resolving sets for graphs, metric dimension, some elementary observations.
2. Generalized polygons, examples, basic properties.
3. Blocking sets in finite planes.
4. The metrric dimensions of $\mathrm{PG}(2, q)$ and $\mathrm{AG}(2, q)$.
5. Resolving sets for biaffine planes and generalized quadrangles.

Definitions

Definition

Let $\Gamma = (V, E)$ *be a finite, simple, undirected graph. A vertex* $v \in V$ *is resolved by* $S = \{v_1, \ldots, v_n\} \subset V$ *if the list of distances* $(d(v, v_1), d(v, v_2), \ldots, d(v, v_n))$ *is unique.* S *is a resolving set for* Γ *if it resolves all the elements of* V*.*
The metric dimension of Γ*, denoted* $\mu(\Gamma)$*, is the smallest size of a resolving set for* Γ*.*

Motivation

The *base size of a permutation group* is the smallest number of points whose stabilizer is the identity. The study of base size dates back more than 40 years.
The *base size* of Γ, denoted $b(\Gamma)$, is the base size of its automorphism group.

A resolving set for Γ is a base for $\mathrm{Aut}(\Gamma)$, so the metric dimension of a graph gives an upper bound on its base size.

The *dimension jump* of Γ, denoted $\delta(\Gamma)$, is the difference of its metric dimension and its base size

$$\delta(\Gamma) = \mu(\Gamma) - b(\Gamma).$$

Elementary observations

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3. $\mu(K_n) = n - 1$.
4. $\mu(K_{m,n}) = m + n - 2$.
5. The metric dimension of the Petersen graph is 3.

Elementary observations

Lemma

If Γ has n vertices and its diameter is d, then

$$\mu(\Gamma) \leq n - d.$$

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Lemma

If Γ has n vertices, its diameter is d and $\mu(\Gamma) = k$ then

$$n \leq k + d^k.$$

The best known general bound

Theorem (Hernando et al., 2010)

If Γ *has* n *vertices, its diameter is* d *and* $\mu(\Gamma) = k$ *then*

$$n \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil d/3 \rceil} (2i-1)^{k-1}.$$

Distance regular graphs

A distance regular graph is *primitive* if each of its distance-i graphs (for $0 < i \leq d$) is connected.

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Theorem (Babai, 1981)

Suppose that Γ is a primitive distance regular graph with n vertices, valency $k \geq 3$ and diameter $d \geq 2$. Then

1. $\mu(\Gamma) < 4\sqrt{n} \log n$;
2. *if $d = 2$ (Γ is strongly regular) then*
   - $\mu(\Gamma) < 2\sqrt{n} \log n$, *and*
   - $\mu(\Gamma) < \frac{2n^2}{k(n-k)} \log n < \frac{4n}{k} \log n \quad (k \leq n/2)$.

Definitions

Generalized polygons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a connected, finite point-line incidence geometry.

$\mathcal{P}$ and $\mathcal{L}$ are two distinct sets, the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are called lines. $\mathrm{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric relation, called incidence.

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Chain of length h :

$$x_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, \ldots \,\mathrm{I}\, x_h$$

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Chain of length h :

$$x_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, \dots \,\mathrm{I}\, x_h$$

where $x_i \in \mathcal{P} \cup \mathcal{L}$.
The distance of two elements $d(x, y)$: length of the shortest chain joining them.

Generalized polygons

Definition

Let $n > 1$ *be a positive integer.* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *is called a generalized n-gon if it satisfies the following axioms.*

- **Gn1.** $d(x, y) \leq n \; \forall \; x, y \in \mathcal{P} \cup \mathcal{L}$.
- **Gn2.** *If* $d(x, y) = k < n$ *then* $\exists!$ *a chain of length* k *joining* x *and* y.
- **Gn3.** $\forall \; x \in \mathcal{P} \cup \mathcal{L} \; \exists \; y \in \mathcal{P} \cup \mathcal{L}$ *such that* $d(x, y) = n$.

Some elementary observations

- For the graph-theorists: the point-line incidence graph of a generalized n-gon is a connected bipartite graph of diameter n and girth $2n$.
- The simplest example: n-gon in the euclidean plane.

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- The dual of a generalized n-gon is also a generalized n-gon.
- The distance of two points or two lines is even. The distance of a point and a line is odd.
- If $n = 2$ then any two points are collinear, any two lines intersect each other, hence generalized 2-gons are trivial structures (their Levi graphs are the complete bipartite graphs).

Almost trivial structures

$n = 3$

The distance of two distinct points is 2, hence the points are collinear. Because of Gn2 the line joining them is unique.
The distance of two distinct lines is 2, hence the lines intersect each other. Because of Gn2 the point of intersection is unique.

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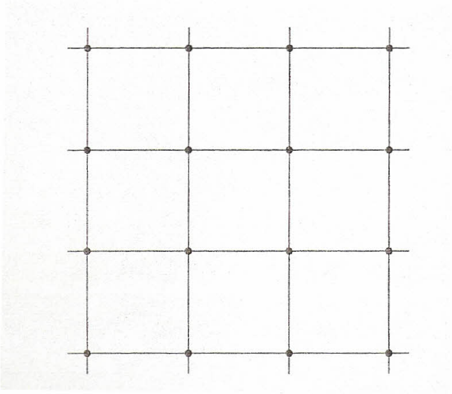
There are trivial structures,

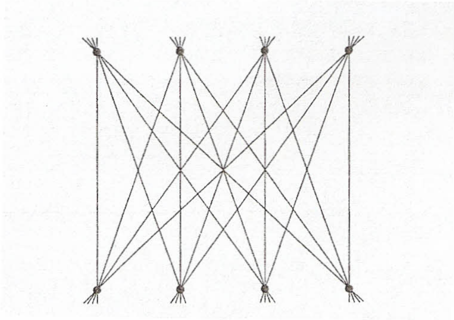
Almost trivial structures

$n = 4$

Grid



Its dual, bipartite graph.



Points: vertices
Lines: edges

Nontrivial structures

Definition

A generalized polygon is called thick, if it satisfies the following.
Gn4. *Each line is incident with at least three points and each point is incident with at least three lines.*

Nontrivial structures

Definition

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Gn4. *Each line is incident with at least three points and each point is incident with at least three lines.*

Theorem

In a thick finite generalized polygon each line is incident with the same number of points and each point is incident with the same number of lines.

Definition

The polygon is called of order (s, t) if these numbers are $s + 1$ and $t + 1$, respectively.

Existence

Theorem (Feit, Higman, 1964)

Finite thick generalized n-gons exist if and only if $n = 2, 3, 4, 6$ *and* 8.

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$n = 3$: projective planes

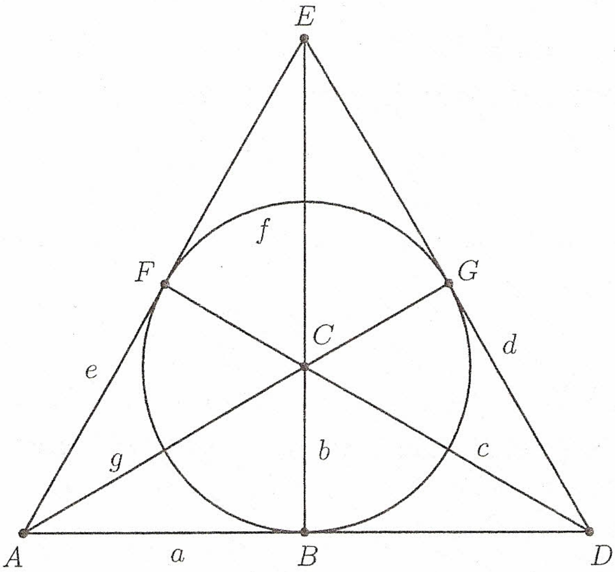
$n = 4$: generalized quadrangles

Definition

$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *is called a projective plane if it satisfies the following axioms.*

- **P1.** *For any two distinct points there is a unique line joining them.*
- **P2.** *For any two distinct lines there is a unique point of intersection.*
- **P3.** *Each line is incident with at least three points and each point is incident with at least three lines.*

The Fano plane



Basic combinatorial properties

Theorem

Let Π be a projective plane. If Π has a line which is incident with exatly $q+1$ points, then

1. *each line is incident with $q+1$ points,*
2. *each point is incident with $q+1$ lines,*
3. *the plane contains q^2+q+1 points,*
4. *the plane contains q^2+q+1 lines.*

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The number q is called the order of the plane.

Homogeneous coordinates, PG$(2, \mathbf{K})$

Let V_3 be a 3-dimensional vector space over the field $\mathbf{K}$. The projective plane PG$(2, \mathbf{K})$ is the following.

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points:	1-dim subspaces of V_3	$\mathbf{0} \neq \mathbf{v} = (v_0, v_1, v_2)$
lines:	2-dim subspaces of V_3 $\Leftrightarrow$	
	1-codim subspaces of V_3	$\mathbf{0} \neq \mathbf{u} = (u_0, u_1, u_2)$
incidence:	inclusion	$\sum_{i=0}^{2} u_i v_i = 0$

Homogeneous coordinates, $\mathrm{PG}(2, \mathbf{K})$

The relation $\sim$

$$\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists\, 0 \neq \lambda \in \mathbf{K} \,:\, \mathbf{x} = \lambda \mathbf{y}$$

is an equivalence relation. The equivalence class of the vector $\mathbf{v} \in V_3$ is denoted by $[\mathbf{v}]$.

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Homogeneous coordinates

– of the point represented by the class of vectors $[\mathbf{v}]$: $(v_0 : v_1 : v_2)$,
– of the line represented by the class of vectors $[\mathbf{u}]$: $[u_0 : u_1 : u_2]$.

Affine planes

Definition

$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called an affine plane if it satisfies the following axioms.

- **A1.** *For any two distinct points there is a unique line joining them.*
- **A2.** *For any non-incident point-line pair (P, e) $\exists!$ a line f such that $P \, \mathrm{I} \, f$ and $e \cap f = \emptyset$.*
- **A3.** $\exists$ *three non-collinear points.*

AG(2, **K**)

The affine plane AG(2, **K**) is the following.

Points: (a, b) $\qquad a, b \in \mathbf{K}$

Lines: $[c]$, $[m, k]$ $\qquad c, m, k \in \mathbf{K}$

Incidence:

$$(a, b)\,\mathrm{I}\,[c] \iff a = c,$$

$$(a, b)\,\mathrm{I}\,[m, k] \iff b = ma + k.$$

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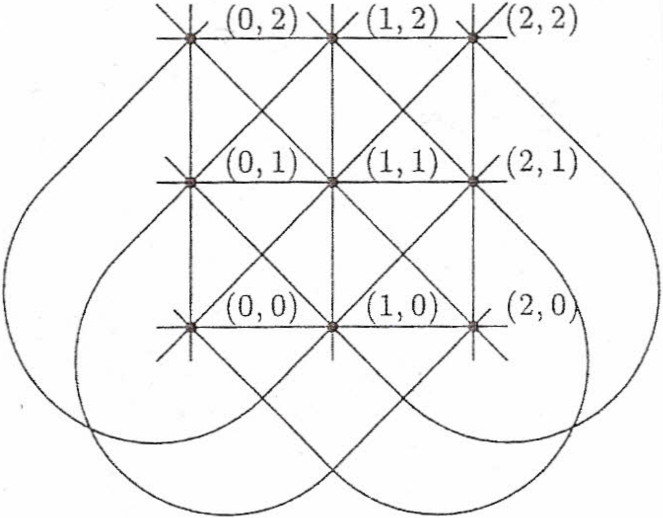
If $\mathbf{K} = \mathrm{GF}(q)$ then AG(2, q) denotes the finite affine plane.

AG(2, 3)



$\mathrm{BG}(2, q)$

We denote by $\mathrm{BG}(2, q)$ the Desarguesian biaffine plane; that is, the point-line incidence structure obtained from $\mathrm{AG}(2, q)$ by removing the lines of an arbitrary parallel class.
Without loss of generality, we may assume that the class in question is the set of vertical lines.

Basic combinatorial properties of GQ-s

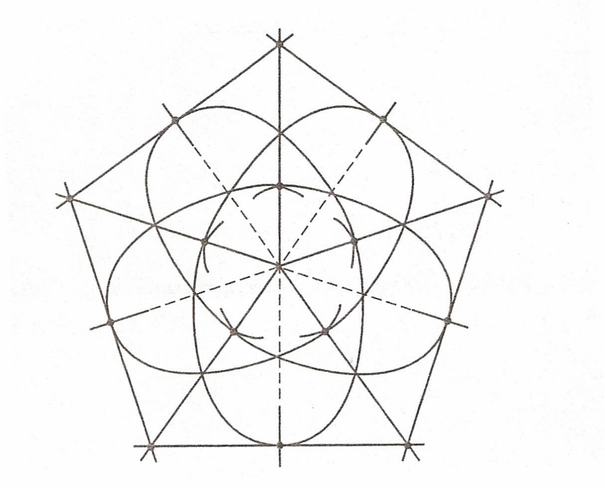
Theorem

Let $\mathcal{Q}$ be a finite, thick generalized quadrangle of order (s,t). Then

1. *For each non-incident point-line pair (P,e) $\exists!$ a point-line pair (R,f) such that $P \,\mathrm{I}\, f \,\mathrm{I}\, R \,\mathrm{I}\, e$,*
2. *$\mathcal{Q}$ contains $(s+1)(st+1)$ points,*
3. *$\mathcal{Q}$ contains $(t+1)(st+1)$ lines.*

The smallest nontrivial example

$s = t = 2$, 15 points, 15 lines



A GQ of order (q, q)

Let A be a 4×4 nonsingular, antisymmetric matrix over $\mathrm{GF}(q)$. Then A defines a null polarity π of the projective space $\mathrm{PG}(3, q)$.

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$$\mathbf{r}A\mathbf{s}^{\mathrm{T}} = 0 \iff R \in S^{\pi}.$$

Hence the self-conjugate lines through a point P are the elements of the pencil of lines in P^{π} having carrier P.

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Theorem

The points of $\mathrm{PG}(3, q)$ and the self-conjugate lines of a null polarity with the inherited incidence form a generalized quadrangle of order (q, q).

A GQ of order (q, q)

- Each line contains $q+1$ points.
- There are $q+1$ lines through each point.
- If (P, ℓ) is a non-incident point-line pair, then $\ell \not\subset P^{\pi}$. Hence $\exists\,!\ Q = \ell \cap P^{\pi}$. The line PQ is self-conjugate, contains P and meets ℓ.

Some notations

Let $S = \mathcal{P}_S \cup \mathcal{L}_S$, with $\mathcal{P}_S$ points and $\mathcal{L}_S$ lines, be a resolving set in a point-line incidence geometry. Then

- *inner points* and *inner lines* indicate points or lines in S, whereas *outer points* or *outer lines* denote points and lines not in S;
- a line ℓ is *skew* or *tangent* to S if $[\ell] \cap \mathcal{P}_S$ is empty or just one point respectively;
- an outer point is *covered* if it lies on at least one line of $\mathcal{L}_S$, is 1-covered if it lies on exactly one line of $\mathcal{L}_S$.

Lemma

A line ℓ which intersects $\mathcal{P}_S$ in at least two points is resolved by S. If a point lies in at least two inner lines then it is resolved by S.

Blocking sets

Definition

A point-set $\mathcal{B}$ of a projective or an affine plane is called *blocking set* if $\mathcal{B}$ meets every line of the plane.

Lemma

Let $\mathcal{B}$ be a blocking set in a projective plane of order q. Then $|\mathcal{B}| \geq q + 1$. If equality holds then $\mathcal{B}$ is a line.

Blocking sets

Definition

A blocking set is called *nontrivial* if it does not contain any line completely.
A blocking set is called *minimal* if no of its real subsets is a blocking set.

Lemma

Let $\mathcal{B}$ be a nontrivial blocking set in a projective plane Π_q of order q. Then each line of Π_q meets $\mathcal{B}$ in at most $|\mathcal{B}| - q$ points.

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Theorem (Bruen, Pelikán)

Let $\mathcal{B}$ be a nontrivial blocking set in the projective plane Π_q of order n. Then $\mathcal{B}$ contains at least $q + \sqrt{q} + 1$ points. Equality holds if and only if $\mathcal{B}$ is a Baer subplane.

Theorem (Blokhuis)

Let $\mathcal{B}$ be a nontrivial blocking set in $\mathrm{PG}(2, p)$, $p > 2$ prime. Then $\mathcal{B}$ contains at least $3(p+1)/2$ points.

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Theorem (Blokhuis)

Let $\mathcal{B}$ be a nontrivial blocking set in $\mathrm{PG}(2, q)$, where p^h), $h \geq 2$. Then

- $|\mathcal{B}| \geq q + \sqrt{q} + 1$ *if h is even,*
- $|\mathcal{B}| \geq q + \sqrt{pq} + 1$ *if h is odd.*

Blocking sets

Theorem (Blokhuis)

Let $\mathcal{B}$ be a t-fold blocking set in $\mathrm{AG}(2,q)$, $q = p^h$, and assume that $(t,q) = 1$. Then $|\mathcal{B}| \geq (t+1)q - 1$.
In particular if $t = 1$, then each blocking set in $\mathrm{AG}(2,q)$ contains at least $2q - 1$ points.

Blocking sets

Proposition

Let $\mathcal{B}$ be a t-fold blocking set in $\mathrm{PG}(2,q)$. If $\mathcal{B}$ contains a line ℓ then $|\mathcal{B}| \geq (t+1)q - t + 2$.

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Theorem (Ball)

Let $\mathcal{B}$ be a t-fold blocking set in $\mathrm{PG}(2,q)$, $q = p$ prime. If $t \leq (q+1)/2$ then

$$|\mathcal{B}| \geq tq + t + \frac{q+1}{2}.$$

Projective planes

Proposition

A set $S = \mathcal{P}_S \cup \mathcal{L}_S$ *is a resolving set in* Π_q *if and only if the following hold.*

1. *There is at most one outer line skew to* $\mathcal{P}_S$.
2. *There is at most one outer point not covered by* $\mathcal{L}_S$.
3. *There is at most one outer point on every line of* $\mathcal{L}_S$ *which is* 1*-covered.*
4. *Through every inner point there is at most one outer line tangent to* $\mathcal{P}_S$.
5. *On every inner line there is at most one outer point that is* 1*-covered by* $\mathcal{L}_S$.

Projective planes

Theorem

The metric dimension of the Fano plane is five.

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Theorem (Héger and Takáts, 2012)

If $q \geq 23$ then the metric dimension of a finite projective plane of order q is 4q-4.
There are 32 types of resolving sets of size 4q-4.

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There are 32 types of resolving sets of size 4q-4.

A construction: Let P, Q and R be three non-collinear points and let

- $\mathcal{P}_S = [PQ] \cup [PR] \setminus \{P, Q, R\}$,
- $\mathcal{L}_S = [P] \cup [R] \setminus \{PQ, RQ, PR\}$,

Projective planes

The base size of $\mathrm{PG}(2,q)$ is 4 (if q is a prime) or 5 (otherwise).
The incidence graph has $2(q^2+q+1)$ vertices.

$$\delta(\Gamma) \geq (4q-4)-5 = 4q-9.$$

Very large in terms of the order, roughly $2\sqrt{2n}$.

Affine planes

The main difference between projective and affine planes is the existence of parallel lines. The distance between two lines in an affine plane can be either 2 or 4, depending if they intersect or not.

Proposition

A set $S = \mathcal{P}_S \cup \mathcal{L}_S$ is a resolving set in $\mathcal{A}_q$ if and only if the following hold.

1. *There is at most one outer point not covered by $\mathcal{L}_S$.*
2. *There is at most one outer point on every line of $\mathcal{L}_S$ which is 1-covered.*
3. *There is at most one outer skew line for each determined direction $p_1, \ldots, p_r$ and at most one skew line having not determined direction.*
4. *For each inner point there is at most one tangent line having not determined direction; in particular, all the tangent lines having directions $p_1, \ldots, p_r$ are resolved.*

Affine planes

Theorem (Bartoli, Héger, GyK and Takáts, 201x)

Let $q \geq 4$. Then the metric dimension of $\mathcal{A}_q$ is at most $3q - 4$.

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A construction of a resolving set in $\mathcal{A}_q$ of size $3q - 4$:
Consider a line ℓ and two points $P, Q \in \ell$. Let $\mathcal{P}_S := [\ell] \setminus \{P, Q\}$ and $\mathcal{L}_S := ([P] \setminus \{\ell\}) \cup \{\ell_1, \ell_2, \ldots, \ell_{q-2}\}$, where $\ell_1, \ell_2, \ldots, \ell_{q-2}$ are $q - 2$ distinct lines parallel to ℓ. Let $\bar{\ell}$ be the parallel line to ℓ not in $\{\ell_1, \ell_2, \ldots, \ell_{q-2}\}$.

- All the outer points are covered by $\mathcal{L}_S$.
- On a line $r \in ([P] \setminus \{\ell\})$ the unique point which is only 1-covered is $r \cap \bar{\ell}$. On a line $r \in \{\ell_1, \ell_2, \dots, \ell_{q-2}\}$ all the points are 2-covered.
- The skew outer lines are $\bar{\ell}$ an $[Q] \setminus \{\ell\}$. Each of these $q+1$ lines has different direction.
- All the directions are determined by $\mathcal{L}_S$.

The assumption $q \geq 4$ ensures that the line ℓ is not a tangent line to $\mathcal{P}_S$.

Affine planes

Theorem (Bartoli, Héger, GyK and Takáts, 201x)

Let $q \geq 11$. Then the metric dimension of $\mathrm{AG}(2, q)$ is $3q - 4$.

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Proposition

Let S be a resolving set in $\mathrm{AG}(2,q)$, suppose that $|S| \leq 3q-4$ and $q > 16$. Then $|\mathcal{P}_S| \geq q-2$ and therefore $|\mathcal{L}_S| \leq 2q-2$. Also, at least $q-2$ points are collinear.

Theorem (Weiner-Szőnyi, 2014)

Let B be a point set in $\mathrm{PG}(2,q)$. Pick a point P not from B and assume that through P there pass exactly r lines meeting B (that is containing at least 1 point of B). Then the total number of lines meeting B is at most

$$1 + rq + (|B| - r)(q + 1 - r).$$

Biaffine planes

Theorem (Bartoli, Héger, GyK and Takáts, 201x)

Let Γ_q *be the incidence graph of* $\mathrm{BG}(2, q)$. *Then*

$$8q/3 - 8 \leq \mu(\Gamma_q) \leq 3q - 6.$$

Theorem (Bartoli, Héger, GyK and Takáts, 201x)

Let Γ_q *be the incidence graph of* $\mathrm{BG}(2,q)$*. Then*

$$8q/3 - 8 \leq \mu(\Gamma_q) \leq 3q - 6.$$

The proof consists of several steps by showing a connection between small resolving sets of $\mathrm{BG}(2,q)$ and blocking sets of $\mathrm{PG}(2,q)$.

$W(q)$

Theorem (Bartoli, Héger, GyK and Takáts, 201x)

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$$\max\{6q-126, 4q-8\} \leq \mu(\Gamma_q) \leq \begin{cases} 8q & \text{if } q \text{ is even,} \\ 10q-5 & \text{if } q \text{ is odd.} \end{cases}$$

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The proof is based on the geometric properties of the null-polarity from which $W(q)$ arises.

Hvala za pozornost!