

Block designs constructed from groups

Dean Crnković

Department of Mathematics
University of Rijeka
Croatia

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A $t - (v, k, \lambda)$ **design** is a finite incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ satisfying the following requirements:

- 1 $|\mathcal{P}| = v$,
- 2 every element of \mathcal{B} is incident with exactly k elements of \mathcal{P} ,
- 3 every t elements of \mathcal{P} are incident with exactly λ elements of \mathcal{B} .

Every element of \mathcal{P} is incident with exactly $r = \frac{\lambda(v-1)}{k-1}$ elements of \mathcal{B} . The number of blocks is denoted by b . If $b = v$ (or equivalently $k = r$) then the design is called **symmetric**.

If \mathcal{D} is a t -design, then it is also a s -design, for $1 \leq s \leq t - 1$.

Theorem 1 [J. D. Key, J. Moorj]

Let G be a **finite primitive permutation group** acting on the set Ω of size n . Further, let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α . If

$$\mathcal{B} = \{\Delta g : g \in G\}$$

and, given $\delta \in \Delta$,

$$\mathcal{E} = \{\{\alpha, \delta\}g : g \in G\},$$

then $\mathcal{D} = (\Omega, \mathcal{B})$ is a **symmetric** $1 - (n, |\Delta|, |\Delta|)$ **design**. Further, if Δ is a **self-paired orbit** of G_α then $\Gamma(\Omega, \mathcal{E})$ is a **regular connected graph** of valency $|\Delta|$, \mathcal{D} is **self-dual**, and G acts as an **automorphism group** on each of these structures, **primitive** on vertices of the graph, and on points and blocks of the design.

Instead of taking a single G_α -orbit, we can take Δ to be any **union of G_α -orbits**. We will still get a symmetric 1-design with the group G acting primitively on points and blocks of the design.

Theorem 2 [DC, V. Mikulić]

Let G be a finite permutation group **acting primitively on the sets Ω_1 and Ω_2 of size m and n , respectively**. Let $\alpha \in \Omega_1$, $\delta \in \Omega_2$, and let $\Delta_2 = \delta G_\alpha$ be the G_α -orbit of $\delta \in \Omega_2$ and $\Delta_1 = \alpha G_\delta$ be the G_δ -orbit of $\alpha \in \Omega_1$.

If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, |\Delta_1|)$ **design** with m blocks, and G **acts** on the design $\mathcal{D}(G, \alpha, \delta)$, primitively on points and blocks.

In the construction of the design described in Theorem 2, instead of taking a single G_α -orbit, we can take Δ_2 to be any **union of G_α -orbits**.

Corollary 1

Let G be a finite permutation group acting primitively on the sets Ω_1 and Ω_2 of size m and n , respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^s \delta_i G_\alpha$, where $\delta_1, \dots, \delta_s \in \Omega_2$ are representatives of distinct G_α -orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \dots, \delta_s) = (\Omega_2, \mathcal{B})$ is a 1-design $1 - (n, |\Delta_2|, \sum_{i=1}^s |\alpha G_{\delta_i}|)$ with m blocks, and G acts on the design $\mathcal{D}(G, \alpha, \delta_1, \dots, \delta_s)$, primitively on points and blocks.

In fact, this construction gives us **all 1-designs on which the group G acts primitively on points and blocks.**

Corollary 2

If a group G acts primitively on the points and the blocks of a 1-design \mathcal{D} , then \mathcal{D} can be obtained as described in Corollary 1, *i.e.*, such that Δ_2 is a union of G_α -orbits.

We can interpret the design (Ω_2, \mathcal{B}) from Corollary 1 in the following way:

- the point set is Ω_2 ,
- the block set is $\Omega_1 = \alpha G$,
- the block $\alpha g'$ is incident with the set of points $\{\delta_i g : g \in G_\alpha g', i = 1, \dots, s\}$.

Let G be a **finite simple group** and let H_1 and H_2 be **maximal subgroups** of G . Then $N_G(H_1) = H_1$ and $N_G(H_2) = H_2$. G **acts primitively** on $ccl_G(H_1)$ and $ccl_G(H_2)$ by conjugation. We can construct a **primitive 1–design** such that:

- the point set of the design is $ccl_G(H_2)$,
- the block set is $ccl_G(H_1)$,
- the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i$, $i = 1, \dots, k$, where $\{G_1, \dots, G_k\} \subset \{H_2^x \cap H_1^y \mid x, y \in G\}$.

We denote a 1–design constructed in this way by $\mathcal{D}(G, H_2, H_1; G_1, \dots, G_k)$.

From the conjugacy class of a **maximal subgroup** H of a simple group G one can construct a **regular graph**, denoted by $\mathcal{G}(G, H; G_1, \dots, G_k)$, in the following way:

- the vertex set of the graph is $ccl_G(H)$,
- the vertex H^{g_i} is adjacent to the vertex H^{g_j} if and only if $H^{g_i} \cap H^{g_j} \cong G_i$, $i = 1, \dots, k$, where $\{G_1, \dots, G_k\} \subset \{H^x \cap H^y \mid x, y \in G\}$.

G acts **primitively** on the set of vertices of $\mathcal{G}(G, H; G_1, \dots, G_k)$.

Theorem 3 [DC, V. Mikulić, A. Švob]

Let G be a finite permutation group **acting transitively** on the sets Ω_1 and Ω_2 of size m and n , respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^s \delta_i G_\alpha$, where $\delta_1, \dots, \delta_s \in \Omega_2$ are representatives of distinct G_α -orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then the incidence structure $\mathcal{D}(G, \alpha, \delta_1, \dots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \frac{|G_\alpha|}{|G_{\Delta_2}} \sum_{i=1}^s |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}}|$ blocks. Then the group G acts on (Ω_2, \mathcal{B}) , **transitively on points and blocks** of the design.

Corollary 3

If a group G acts transitively on the points and the blocks of a 1-design \mathcal{D} , then \mathcal{D} can be obtained as described in Theorem 3.

Let M be a **finite group** and $H_1, H_2, G \leq M$. G acts transitively on the conjugacy classes $ccl_G(H_i)$, $i = 1, 2$, by conjugation. We can construct a 1–design such that:

- the point set of the design is $ccl_G(H_2)$,
- the block set is $ccl_G(H_1)$,
- the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i$, $i = 1, \dots, k$, where $\{G_1, \dots, G_k\} \subset \{H_2^x \cap H_1^y \mid x, y \in G\}$.

The group $G / \bigcap_{K \in ccl_G(H_2) \cup ccl_G(H_1)} N_G(K)$ acts as an automorphism group of the constructed design, **transitive on points and blocks**.

Using the described approach we have constructed a number of 2-designs and strongly regular graphs from the groups $U(3, 3)$, $U(3, 4)$, $U(3, 5)$, $U(3, 7)$, $U(4, 2)$, $U(4, 3)$, $U(5, 2)$, $L(2, 32)$, $L(2, 49)$, $L(3, 5)$, $L(4, 3)$, $S(6, 2)$ and He .

Let \mathbf{F}_q be the finite field of order q . A **linear code** of **length** n is a subspace of the vector space \mathbf{F}_q^n . A k -dimensional subspace of \mathbf{F}_q^n is called a linear $[n, k]$ code over \mathbf{F}_q .

For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbf{F}_q^n$ the number $d(x, y) = |\{i \mid 1 \leq i \leq n, x_i \neq y_i\}|$ is called a Hamming distance.

The **minimum distance** of a code C is

$$d = \min\{d(x, y) \mid x, y \in C, x \neq y\}.$$

A linear $[n, k, d]$ code is a linear $[n, k]$ code with the minimum distance d .

An $[n, k, d]$ linear code can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors.

The **dual** code C^\perp is the orthogonal complement under the standard inner product $(,)$. A code C is **self-orthogonal** if $C \subseteq C^\perp$ and **self-dual** if $C = C^\perp$.

Codes constructed from block designs have been extensively studied.

- E. F. Assmus Jnr, J. D. Key, Designs and their codes, Cambridge University Press, Cambridge, 1992.
- A. Baartmans, I. Landjev, V. D. Tonchev, On the binary codes of Steiner triple systems, Des. Codes Cryptogr. 8 (1996), 29–43.
- V. D. Tonchev, Quantum Codes from Finite Geometry and Combinatorial Designs, Finite Groups, Vertex Operator Algebras, and Combinatorics, Research Institute for Mathematical Sciences 1656, (2009) 44-54.

An automorphism of a code is any permutation of the coordinate positions that maps codewords to codewords.

The **code** $C_F(\mathcal{D})$ **of the design** \mathcal{D} over the finite field \mathbf{F} is the vector space spanned by the incidence vectors of the blocks over \mathbf{F} . It is known that $Aut(\mathcal{D}) \leq Aut(C_F(\mathcal{D}))$.

Any linear code is isomorphic to a code with generator matrix in so-called **standard form**, *i.e.* the form $[I_k|A]$; a check matrix then is given by $[-A^T|I_{n-k}]$. The first k coordinates are the **information symbols** and the last $n - k$ coordinates are the **check symbols**.

Permutation decoding was first developed by MacWilliams in 1964, and involves finding a set of automorphisms of a code called a **PD-set**.

Definition 1

If C is a t -error-correcting code with information set \mathcal{I} and check set \mathcal{C} , then a **PD-set** for C is a set S of automorphisms of C which is such that every t -set of coordinate positions is moved by at least one member of S into the check positions \mathcal{C} .

The property of having a PD-set will not, in general, be invariant under isomorphism of codes, *i.e.* it depends on the choice of information set.

If S is a PD-set for a t -error-correcting $[n, k, d]_q$ code C , and $r = n - k$, then

$$|S| \geq \left[\frac{n}{r} \left[\frac{n-1}{r-1} \left[\cdots \left[\frac{n-t+1}{r-t+1} \right] \cdots \right] \right] \right].$$

Good candidates for permutation decoding are linear codes with a large automorphism group and the large size of the check set (small dimension).

By the construction described in Teorem 3 we can construct designs admitting a large transitive automorphism group. Codes of these designs are good candidates for permutation decoding.

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be a $2 - (v, k, \lambda)$ design and $G \leq \text{Aut}(\mathcal{D})$. We denote the G -orbits of points by $\mathcal{P}_1, \dots, \mathcal{P}_n$, G -orbits of blocks by $\mathcal{B}_1, \dots, \mathcal{B}_m$, and put $|\mathcal{P}_r| = \omega_r$, $|\mathcal{B}_i| = \Omega_i$, $1 \leq r \leq n$, $1 \leq i \leq m$.

Denote by γ_{ij} the number of points of \mathcal{P}_j incident with a representative of the block orbit \mathcal{B}_i . For these numbers the following equalities hold:

$$\sum_{j=1}^n \gamma_{ij} = k, \quad (1)$$

$$\sum_{i=1}^m \frac{\Omega_i}{\omega_j} \gamma_{ij} \gamma_{is} = \lambda \omega_s + \delta_{js} \cdot (r - \lambda). \quad (2)$$

Definition 2

A $(m \times n)$ -matrix $M = (\gamma_{ij})$ with entries satisfying conditions (1) and (2) is called an **orbit matrix** for the parameters $2 - (v, k, \lambda)$ and orbit lengths distributions $(\omega_1, \dots, \omega_n)$, $(\Omega_1, \dots, \Omega_m)$.

Orbit matrices are often used in construction of designs with a presumed automorphism group.

The intersection of rows and columns of an orbit matrix M that correspond to non-fixed points and non-fixed blocks form a submatrix called the **non-fixed part of the orbit matrix** M .

Example

The incidence matrix of the symmetric $(7,3,1)$ design

$$\left[\begin{array}{c|ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

Corresponding orbit matrix for Z_3

$$\begin{array}{c|cc} & 1 & 3 & 3 \\ \hline 1 & 0 & 3 & 0 \\ 3 & 1 & 1 & 1 \\ 3 & 0 & 1 & 2 \end{array}$$

Theorem 4 [M. Harada, V. D. Tonchev]

Let \mathcal{D} be a 2 - (v, k, λ) design with a **fixed-point-free** and **fixed-block-free automorphism** ϕ of order q , where q is prime. Further, let M be the orbit matrix induced by the action of the group $G = \langle \phi \rangle$ on the design \mathcal{D} . If p is a prime dividing r and λ then the **orbit matrix** M generates a **self-orthogonal code** of length $b|q$ over \mathbf{F}_p .

Using Theorem 4 Harada and Tonchev constructed a ternary $[63,20,21]$ code with a record breaking minimum weight from the symmetric 2 - $(189,48,12)$ design found by Janko.

Theorem 5 [V. D. Tonchev]

If G is a cyclic group of a prime order p that does not fix any point or block and $p|(r - \lambda)$, then the rows of the orbit matrix M generate a self-orthogonal code over \mathbf{F}_p .

Theorem 6 [DC, L. Simčić]

Let \mathcal{D} be a 2 - (v, k, λ) design with an automorphism group G which acts on \mathcal{D} with f fixed points, h fixed blocks, $\frac{v-f}{w}$ point orbits of length w and $\frac{b-h}{w}$ block orbits of length w . If a prime p divides w and $r - \lambda$, then the **columns** of the non-fixed part of the orbit matrix M for the automorphism group G generate a self-orthogonal code of length $\frac{b-h}{p}$ over \mathbf{F}_p .

Theorem 7

Let Ω be a finite non-empty set, $G \leq S(\Omega)$ and H a normal subgroup of G . Further, let x and y be elements of the same G -orbit. Then $|xH| = |yH|$.

Theorem 8

Let Ω be a finite non-empty set, $H \triangleleft G \leq S(\Omega)$ and $xG = \bigsqcup_{i=1}^h x_i H$,
for $x \in \Omega$. Then a group G/H acts transitively on the set $\{x_i H \mid i = 1, 2, \dots, h\}$.

Let \mathcal{D} be a 2 -(v, k, λ) design with an automorphism group G , and $H \triangleleft G$. Further, let H acts on \mathcal{D} with f fixed points, h fixed blocks, $\frac{v-f}{w}$ point orbits of length w and $\frac{b-h}{w}$ block orbits of length w . If a prime p divides w and $r - \lambda$, then the **columns** of the non-fixed part of the orbit matrix M for the automorphism group H generate a self-orthogonal code C of length $\frac{b-h}{p}$ over \mathbf{F}_p , and G/H acts as an automorphism group of C .

If G acts transitively on \mathcal{D} , then G/H acts transitively on C . Thus, we can construct codes admitting a large transitive automorphism group, which are good candidates for permutation decoding.

Definition 3

Let t be a positive integer, v an infinite cardinal, k and \bar{k} cardinals with $k + \bar{k} = v$, and Λ a $(t + 1) \times (t + 1)$ matrix with rows and columns indexed by $\{0, \dots, t\}$ with (i, j) entry a cardinal number if $i + j \leq t$ and blank otherwise. Then a **simple infinite** $t - (v, (k, \bar{k}), \Lambda)$ **design** consists of a set V of points and a set \mathcal{B} of subsets of V , having the properties

- $|B| = k$ and $V \setminus B = \bar{k}$, for all $B \in \mathcal{B}$.
- For $0 \leq i + j \leq t$, let $x_1, \dots, x_i, y_1, \dots, y_j$ be distinct points of V . Then the number of elements of \mathcal{B} containing all of x_1, \dots, x_i and none of y_1, \dots, y_j is precisely $\Lambda_{i,j}$.
- No block contains another block.

In a **nonsimple infinite design** repeated blocks are allowed and the last condition should be replaced by

- No block strictly contains another block.

$$\Lambda_{0,0} = b$$

$$\Lambda_{1,0} = r$$

Let G be an **infinite group acting transitively** on the **infinite sets** Ω_1 and Ω_2 . In a similar way as in Teorem 3 one constructs an infinite 1-design having an automorphism group isomorphic to $G/\bigcap_{x \in \Omega_2} G_x$ that acts transitively on points and blocks of the design.