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Proof

$$\lambda_{i-1} - (\beta+1)(\lambda_i - \lambda_{i+1}) - \lambda_{i+2} = \alpha_i, \quad 0 \leq i \leq d-2, \quad \lambda_{-1} = \lambda_d = 0$$

$$\alpha_i = (\beta+1) \left(\theta_i \theta_i^* - \theta_{i+2} \theta_{i+2}^* + (\theta_{i+1} \theta_{i+2}^* + \theta_{i+2} \theta_{i+1}^*) - (\theta_i \theta_{i+1}^* + \theta_{i+1} \theta_i^*) \right)$$

$$(A, A^*; \{V_{d-i}\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$$

$$\text{data } \left(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\tilde{\lambda}_i\}_{i=0}^{d-1} \right)$$

$\overbrace{\theta_i}^{\parallel}, \quad \overbrace{\theta_i^*}^{\parallel}$

$$\tilde{\alpha}_i = (\beta+1) \left(\tilde{\theta}_i \tilde{\theta}_i^* - \tilde{\theta}_{i+2} \tilde{\theta}_{i+2}^* + (\tilde{\theta}_{i+1} \tilde{\theta}_{i+2}^* + \tilde{\theta}_{i+2} \tilde{\theta}_{i+1}^*) - (\tilde{\theta}_i \tilde{\theta}_{i+1}^* + \tilde{\theta}_{i+1} \tilde{\theta}_i^*) \right)$$

$$= -\alpha_{d-i-2}$$

$$= \underbrace{\lambda_{d-i}}_{\overbrace{\tilde{\lambda}_{i-1}}^{\parallel}} - (\beta+1) \left(\underbrace{\lambda_{d-i-1}}_{\overbrace{\tilde{\lambda}_i}^{\parallel}} - \underbrace{\lambda_{d-i-2}}_{\overbrace{\tilde{\lambda}_{i+1}}^{\parallel}} \right) - \underbrace{\lambda_{d-i-3}}_{\overbrace{\tilde{\lambda}_{i+2}}^{\parallel}}$$



$$(A^*, A; \{V_i^*\}_{i=0}^d, \{V_i\}_{i=0}^d)$$

$$\text{data } \left(\{\theta_i^*\}_{i=0}^d, \{\theta_i\}_{i=0}^d, \{\lambda_i\}_{i=0}^d \right)$$

Proof α_i is symmetric w.r.t. $\{\theta_i\}_{i=0}^d \leftrightarrow \{\theta_i^*\}_{i=0}^d$



6th lecture

§6 Classification of dual systems of orth. polys.

§6.1 Standard basis and the associated tridiagonal matrices

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

$$V_i \ni w_i \neq 0, 0 \leq i \leq d$$

w_0, w_1, \dots, w_d basis of V

the matrices of A, A^*

$$D = \begin{pmatrix} \delta_0 & & & \\ & \ddots & & \\ & & \delta_1 & \\ 0 & & & \ddots & \delta_d \end{pmatrix}$$

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & & 0 \\ c_0^* & a_1^* & b_1^* & & & \\ & \ddots & \ddots & \ddots & & \\ 0 & & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ & & & & c_d^* & a_d^* \end{pmatrix} \in M$$

the class of matrices
that are

triangular

irreducible

diagonalizable

w_0, w_1, \dots, w_d standard if $w_i \in V_i, 0 \leq i \leq d$,
and

$B^* \in M(\delta_0^*)$ the subclass of M

that has row sum δ_0^*

$$c_i^* + a_i^* + b_i^* = \delta_0^*, 1 \leq i \leq d-1$$

$$a_0^* + b_0^* = c_d^* + a_d^* = \delta_0^*$$

$$V_i^* \ni w_i \neq 0, \quad 0 \leq i \leq d$$

w_0, w_1, \dots, w_d basis of V

the matrices of A, A^*

$$B = \begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & \ddots & \ddots & \ddots & & \\ 0 & & c_{d-1} & a_{d-1} & b_{d-1} & \\ & & & c_d & a_d & \end{pmatrix} \in M$$

triangular
irreducible
diagonalizable

$$D^* = \begin{pmatrix} \delta_0^* & & 0 \\ & \delta_1^* & & \\ 0 & & \ddots & \\ & & & \delta_d^* \end{pmatrix}$$

w_0, w_1, \dots, w_d dual if $w_i \in V_i^*, 0 \leq i \leq d$,
and

$B \in M(\delta_0)$ the subclass of M
that has row sum δ_0

$$c_i + a_i + b_i = \delta_0, \quad 1 \leq i \leq d$$

$$a_0 + b_0 = c_d + a_d = \delta_0$$

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Proposition

$$(1) \quad V_0^* \ni v_0^* \neq 0, \quad w_i = E_i v_0^* \in V_i^*, \quad 0 \leq i \leq d$$

projection of v_0^* onto V_i

Then

w_0, w_1, \dots, w_d are a standard basis.

If w'_0, w'_1, \dots, w'_d are a standard basis

then $\exists \xi \in \mathbb{C}, \quad w'_i = \xi w_i, \quad 0 \leq i \leq d.$

$$(2) \quad V_0 \ni v_0 \neq 0, \quad w_i = E_i^* v_0 \in V_i^*, \quad 0 \leq i \leq d$$

projection of v_0 onto V_i^*

Then

w_0, w_1, \dots, w_d are a dual standard basis.

If w'_0, w'_1, \dots, w'_d are a dual standard basis,

then $\exists \xi \in \mathbb{C}, \quad w'_i = \xi w_i, \quad 0 \leq i \leq d.$

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Proof

$$(1) \quad V = \bigoplus_{i=0}^d V_i \quad \text{weight space decomposition}$$

$$V_i = (V_0^* + \dots + V_d^*) \cap (V_0 + \dots + V_d)$$

$$U_0 = V_0^* \ni v_0^* = u_0, \quad R^i u_0 = u_i \in U_i$$

$$E_i = \prod_{\nu \neq i} \frac{A - \theta_\nu}{\theta_i - \theta_\nu} = \prod_{\nu=i+1}^d \frac{A - \theta_\nu}{\theta_i - \theta_\nu} \cdot \prod_{\nu=0}^{i-1} \frac{A - \theta_\nu}{\theta_i - \theta_\nu}$$

$$\begin{aligned} w_i = E_i u_0 &= \left(\prod_{\nu=0}^{i-1} \frac{1}{\theta_i - \theta_\nu} \right) \underbrace{\prod_{\nu=i+1}^d \frac{A - \theta_\nu}{\theta_i - \theta_\nu}}_{\substack{u_i \mod U_{i+1} + \dots + U_d}} u_i \\ &= \frac{1}{(\theta_i - \theta_0) \dots (\theta_i - \theta_{i-1})} u_i \mod U_{i+1} + \dots + U_d \\ &= V_{i+1} + \dots + V_d \\ &\neq 0. \end{aligned}$$

$$So \quad 0 \neq w_i \in V_i$$

$$\text{and} \quad w_0 + w_1 + \dots + w_d = v_0^* \in V_0^*$$

$$A^*(w_0, w_1, \dots, w_d) = (w_0, w_1, \dots, w_d) B^*$$

$$A^* w_i = b_{i-1}^* w_{i-1} + a_i^* w_i + c_{i+1}^* w_{i+1}$$

$$A^* v_0^* = \theta_0^* v_0^* = \theta_0^* (w_0 + w_1 + \dots + w_d)$$

||

$$\sum_{i=0}^d A^* w_i$$

$$\text{So } b_i^* + a_i^* + c_i^* = \theta_0^* \quad \text{the coeff. of } w_i.$$

Suppose w'_0, w'_1, \dots, w'_d is a standard basis.

$$\text{Then } \exists \zeta_i \in \mathbb{C}, \quad w'_i = \zeta_i w_i,$$

and the matrix of A^* w.r.t. w'_0, w'_1, \dots, w'_d is

$$B^{*\prime} = \begin{pmatrix} a_0^* & b_0^* \frac{\zeta_1}{\zeta_0} & & & 0 \\ c_1^* \frac{\zeta_0}{\zeta_1} & a_1^* & b_1^* \frac{\zeta_2}{\zeta_1} & & \\ & & & \ddots & \\ 0 & \ddots & \ddots & \ddots & \end{pmatrix}$$

$$\text{i-th row of } B^{*\prime} = 0, \dots, 0, c_i^* \frac{\zeta_{i-1}}{\zeta_i}, a_i^*, b_i^* \frac{\zeta_{i+1}}{\zeta_i}, 0, \dots, 0$$

$$\text{So } c_i^* \frac{\zeta_{i-1}}{\zeta_i} + a_i^* + b_i^* \frac{\zeta_{i+1}}{\zeta_i} = \theta_0^* = c_i^* + a_i^* + b_i^*.$$

Inductively $\zeta_0 = 1, \zeta_1 = 1, \zeta_2 = 1, \dots, \zeta_{i+1} = 1, \dots, \zeta_d = 1$



$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

data $(\{\alpha_i\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$

$(A, A^* ; \{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

↙ reversed

data $(\{\alpha_{d-i}\}_{i=0}^d, \{\alpha_i^*\}_{i=0}^d, \{\hat{\lambda}_i\}_{i=0}^d)$

dual standard basis w_0, w_1, \dots, w_d

$$A(w_0, w_1, \dots, w_d) = (w_0, w_1, \dots, w_d) B$$

$$B = \begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & \ddots & \ddots & \ddots & & \\ 0 & c_{d-1} & a_{d-1} & b_{d-1} & & \\ & c_d & a_d & & & \end{pmatrix} \in M(\mathcal{O})$$

standard basis w_0, w_1, \dots, w_d

$$A^*(w_0, w_1, \dots, w_d) = (w_0, w_1, \dots, w_d) B^*$$

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & & 0 \\ c_1^* & a_1^* & b_1^* & & & \\ & \ddots & \ddots & \ddots & & \\ 0 & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* & & \\ & c_d^* & a_d^* & & & \end{pmatrix} \in M(\mathcal{O}^*)$$

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Theorem

$$(1) \quad c_i = \hat{\lambda}_{i-1} \frac{(\vartheta_i^* - \vartheta_{i+1}^*) \cdots (\vartheta_i^* - \vartheta_d^*)}{(\vartheta_{i-1}^* - \vartheta_i^*) \cdots (\vartheta_{i-1}^* - \vartheta_d^*)}, \quad 1 \leq i \leq d$$

$$(2) \quad b_i = \lambda_i \frac{(\vartheta_i^* - \vartheta_{i-1}^*) \cdots (\vartheta_i^* - \vartheta_0^*)}{(\vartheta_{i+1}^* - \vartheta_i^*) \cdots (\vartheta_{i+1}^* - \vartheta_0^*)}, \quad 0 \leq i \leq d-1$$

$$(3) \quad c_i^* = \hat{\lambda}_{d-i} \frac{(\vartheta_i - \vartheta_{i+1}) \cdots (\vartheta_i - \vartheta_d)}{(\vartheta_{i-1} - \vartheta_i) \cdots (\vartheta_{i-1} - \vartheta_d)}, \quad 1 \leq i \leq d$$

$$(4) \quad b_i^* = \lambda_i \frac{(\vartheta_i - \vartheta_{i-1}) \cdots (\vartheta_i - \vartheta_0)}{(\vartheta_{i+1} - \vartheta_i) \cdots (\vartheta_{i+1} - \vartheta_0)}, \quad 0 \leq i \leq d-1$$

Proof

(1) $(A, A^* ; \{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

data $(\{\delta_{d-i}\}_{i=0}^d, \{\delta_i^*\}_{i=0}^d, \{\hat{\lambda}_i\}_{i=0}^{d-1})$

$$V = \bigoplus_{i=0}^d \hat{U}_i \quad \text{w.r.t. decomp.}$$

$$\hat{U}_i = (V_0^* + \dots + V_i^*) \cap (V_{d-i} + \dots + V_d)$$

$$\hat{F}_i : V \longrightarrow \hat{U}_i \quad \text{projection}$$

$$A = \hat{R} + \sum_{i=0}^d \delta_{d-i} \hat{F}_i, \quad \hat{R} \hat{U}_i = \hat{U}_{i+1}.$$

$$A^* = \hat{L} + \sum_{i=0}^d \delta_i^* \hat{F}_i, \quad \hat{L} \hat{U}_i = \hat{U}_{i-1}$$

dual standard basis w_0, w_1, \dots, w_d

$$V_0 = \hat{U}_d \ni \hat{u}_d \neq 0, \quad \hat{u}_i = \hat{R}^i \hat{u}_0, \quad 0 \neq \hat{u}_0 \in \hat{U}_0$$

$$w_i = E_i^* \hat{u}_d = \left(\prod_{v=0}^{i-1} \frac{A^* - \delta_v^*}{\delta_i^* - \delta_v^*} \right) \underbrace{\left(\prod_{v=i+1}^d \frac{A^* - \delta_v^*}{\delta_i^* - \delta_v^*} \right)}_{\hat{\lambda}_i \cdots \hat{\lambda}_{d-1}} \hat{u}_d$$

$$= \frac{\hat{\lambda}_i \cdots \hat{\lambda}_{d-1}}{(\delta_i^* - \delta_{i+1}^*) \cdots (\delta_i^* - \delta_d^*)} \hat{u}_i \mod \frac{\hat{U}_0 + \cdots + \hat{U}_{i-1}}{(\delta_i^* - \delta_{i+1}^*) \cdots (\delta_i^* - \delta_d^*)} V_0^* + \cdots + V_{i-1}^*$$

$$A w_i = b_{i-1} w_{i-1} + a_i w_i + c_{i+1} w_{i+1}$$

$$\text{mod } \widehat{U}_0 + \dots + \widehat{U}_i$$

$$RHS = c_{i+1} \frac{\widehat{\lambda}_{i+1} \cdots \widehat{\lambda}_{d-1}}{(\vartheta_{i+1}^* - \vartheta_{i+2}^*) \cdots (\vartheta_{i+1}^* - \vartheta_d^*)} \widehat{u}_{i+1}$$

LHS

$$A = \widehat{R} + \sum_{i=0}^d \vartheta_{d-i} \widehat{F}_i$$

$$A \widehat{u}_i = \widehat{u}_{i+1} \text{ mod } \widehat{U}_i$$

$$A(\widehat{U}_0 + \dots + \widehat{U}_{i-1}) \leq \widehat{U}_0 + \dots + \widehat{U}_i$$

$$= \frac{\widehat{\lambda}_i \cdots \widehat{\lambda}_{d-1}}{(\vartheta_i^* - \vartheta_{i+1}^*) \cdots (\vartheta_i^* - \vartheta_d^*)} \widehat{u}_{i+1}$$

$$\text{So } c_{i+1} = \frac{(\vartheta_{i+1}^* - \vartheta_{i+2}^*) \cdots (\vartheta_{i+1}^* - \vartheta_d^*)}{(\vartheta_i^* - \vartheta_{i+1}^*) \cdots (\vartheta_i^* - \vartheta_d^*)}, \quad 0 \leq i \leq d-1$$

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(2) Reverse the ordering $V_0^*, V_1^*, \dots, V_d^*$

$$(A, A^*; \{V_i\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$$

standard basis $w_0, w_1, \dots, w_d - w_i \in V_i^*$

$$A(w_d, \dots, w_1, w_0) = (w_d, \dots, w_1, w_0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a_d c_d & 0 \\ b_{d-1} a_{d-1} c_{d-1} & \ddots \\ 0 & b_1 a_1 c_1 \\ & b_0 a_0 \end{pmatrix}$$

$$\begin{pmatrix} a'_0 b'_0 & 0 \\ c'_1 a'_1 b'_1 & \ddots \\ 0 & c'_{d-1} a'_{d-1} b'_{d-1} \\ & c'_d a'_d \end{pmatrix}$$

$$b_{d-i} = c'_i = \lambda_{d-i} \frac{(a_{d-i}^* - a_{d-i-1}^*) \cdots (a_{d-i}^* - a_0^*)}{(a_{d-i+1}^* - a_{d-i}^*) \cdots (a_{d-i+1}^* - a_0^*)}$$

by (1)

$$(A, A^*; \{V_{d-i}\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$$

$$\text{data } (\{a_{d-i}\}_{i=0}^d, \{a_{d-i}^*\}_{i=0}^d, \{\lambda_{d-i-1}\}_{i=0}^{d-1})$$

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(3) Use the L-system

$$(A^*, A ; \{V_i^*\}_{i=0}^d, \{V_i\}_{i=0}^d)$$

If the ordering of $V_0^*, V_1^*, \dots, V_d^*$ is reversed,

$$(A^*, A ; \{V_{d-i}^*\}_{i=0}^d, \{V_i\}_{i=0}^d) \text{ has data}$$

$$(\{\theta_{d-i}^*\}_{i=0}^d, \{\theta_i\}_{i=0}^d, \{\hat{\lambda}_{d-i-1}\}_{i=0}^d).$$

The claim for c_i^* follows from (1).

(4) Use the L-system

$$(A^*, A ; \{V_i^*\}_{i=0}^d, \{V_i\}_{i=0}^d).$$

The claim for b_i^* follows from (2),

since $(A^*, A ; \{V_i^*\}_{i=0}^d, \{V_i\}_{i=0}^d)$ has

$$\text{data } (\{\theta_i^*\}_{i=0}^d, \{\theta_i\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1}).$$



§ 6.3 Classification of dual systems of orth. polys.

$$(A, A^* ; \{v_i\}_{i=0}^d, \{v_i^*\}_{i=0}^d)$$

L-system on $V \cong \mathbb{C}^{d+1}$



$D, B^* \in M(\mathcal{O}_d^*)$ via standard basis
diag.

L-pair on \mathbb{C}^{d+1}

\cong

$B \in M(\mathcal{O}_d), D^*$ via dual standard basis
diag.

L-pair on \mathbb{C}^{d+1}

$\exists S$ nonsingular matrix

$$S^{-1} B S = D$$

$$S B^* S^{-1} = D^*$$

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$$M(\delta_0) \Rightarrow B = \begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_{d-1} & a_{d-1} & b_{d-1} & \\ 0 & & & c_d & a_d & \end{pmatrix}$$

tridiagonal
irreducible
diagonalizable
row sum δ_0



$$P(\delta_0) \Rightarrow \{P_i(x)\}_{i=0}^d$$

orthogonal polynomials
support $\{\delta_0, \delta_1, \dots, \delta_d\}$
 $P_i(\delta_0) = 1, \quad 0 \leq i \leq d$

$$x P_i(x) = c_i P_{i-1}(x) + a_i P_i(x) + b_i P_{i+1}(x), \quad 0 \leq i \leq d$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1, \quad b_d = 1$$

$$P_{d+1}(x) = \frac{1}{b_0 b_1 \cdots b_{d-1}} (x - \delta_0)(x - \delta_1) \cdots (x - \delta_d)$$

min. poly. of B

$$M(\theta_0^*) \rightarrow B^* = \begin{pmatrix} a_0^* & b_0^* & & & \\ c_1^* & a_1^* & b_1^* & & \\ & \ddots & \ddots & \ddots & \\ & & c_{d-1}^* & a_{d-1}^* & b_{d-1}^* \\ 0 & & & c_d^* & b_d^* \end{pmatrix}$$

$$\mathcal{P}(\theta_0^*) \rightarrow \{p_i^*(x)\}_{i=0}^d$$

orthogonal polynomials
support $\{\theta_0^*, \theta_1^*, \dots, \theta_d^*\}$

$$p_i^*(\theta_0^*) = 1, \quad 0 \leq i \leq d$$

$$\{p_i(x)\}_{i=0}^d, \quad \{p_i^*(x)\}_{i=0}^d : \quad \underline{\text{dual}}$$

$$p_i(\theta_j) = p_j^*(\theta_i), \quad \forall i, j \in \{0, 1, \dots, d\}$$

all dual systems of orthogonal polynomials
arise in this way from L-pairs.

$$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \quad L\text{-system}$$

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data $(\{\delta_i\}_{i=0}^d, \{\delta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$

Theorem

$$(1) \quad P_i(x) = \sum_{\nu=0}^i \frac{(\delta_i^* - \delta_\nu^*) \cdots (\delta_i^* - \delta_{\nu-1}^*)}{\lambda_0 \cdots \lambda_{\nu-1}} (x - \delta_0) \cdots (x - \delta_{\nu-1}), \quad 0 \leq i \leq d$$

$$(2) \quad P_i^*(x) = \sum_{\nu=0}^i \frac{(\delta_i - \delta_\nu) \cdots (\delta_i - \delta_{\nu-1})}{\lambda_0 \cdots \lambda_{\nu-1}} (x - \delta_0^*) \cdots (x - \delta_{\nu-1}^*), \quad 0 \leq i \leq d$$

Proof

(1) Set

$$P_i(x) = \sum_{\nu=0}^i t_{\nu i} (x - \delta_0) \cdots (x - \delta_{\nu-1}), \quad 0 \leq i \leq d.$$

$$V = \bigoplus_{i=0}^d U_i \quad \text{weight space decomposition}$$

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$$

$F_i : V \longrightarrow U_i$ projection

$$\begin{cases} A = R + \sum_{i=0}^d \delta_i F_i, & R U_i = U_{i+1}, \quad 0 \leq i \leq d, U_{d+1} = 0 \\ A^* = L + \sum_{i=0}^d \delta_i^* F_i, & L U_i = U_{i-1}, \quad 0 \leq i \leq d, U_{-1} = 0 \end{cases}$$

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$$U_0 \ni u_0 \neq 1,$$

$$U_i \ni u_i = R^i u_0 = (A - \delta_{i-1}) \cdots (A - \delta_0) u_0$$

$$P_i(A) u_0 = \sum_{\nu=0}^i t_{\nu i} u_\nu$$

Claim $P_i(A) u_0 \in V_i^*$

Proof

$$k_0 = 1, \quad k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} = \frac{b_{i-1}}{c_i} k_{i-1}$$

$$\text{Set } w_i = k_i P_i(A) u_0.$$

Then

$$A P_i(A) u_0 = c_i P_{i-1}(A) u_0 + a_i P_i(A) u_0 + b_i P_{i+1}(A) u_0$$

$$A \frac{w_i}{k_i} = c_i \frac{w_{i-1}}{k_{i-1}} + a_i \frac{w_i}{k_i} + b_i \frac{w_{i+1}}{k_{i+1}}$$

$$A w_i = b_{i-1} w_{i-1} + a_i w_i + c_{i+1} w_{i+1}$$

$$w_{-1} = 0, \quad w_0 = u_0 \in U_0$$

This means

 w_0, w_1, \dots, w_d are a dual standard basis.

In particular, $w_i \in V_i^*$. //

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$$V_i^* \ni p_i(A) u_0 = \sum_{\nu=0}^i t_{\nu i} u_\nu$$

By Lemma on page 5-8

$$v_i^* = \sum_{\nu=0}^i \tilde{c}_{\nu i}^* u_\nu \in V_i^*$$

$$\tilde{c}_{\nu i}^* = \frac{\lambda_\nu \cdots \lambda_{i-1}}{(\theta_i^* - \theta_\nu^*) \cdots (\theta_i^* - \theta_{i-1}^*)}$$

On the other hand,

$$t_{0i} = p_i(\theta_0) = 1$$

So

$$t_{\nu i} = \frac{\tilde{c}_{\nu i}^*}{\tilde{c}_{0i}^*} = \frac{(\theta_i^* - \theta_\nu^*) \cdots (\theta_i^* - \theta_{i-1}^*)}{\lambda_0 \cdots \lambda_{i-1}}$$



Type I $q \neq \pm 1$

$$\phi_i = \phi_0 + h \frac{1}{q^i} (1-q^i)(1-sq^{i+1}), \quad 0 \leq i \leq d$$

$$\phi_i^* = \phi_0^* + h^* \frac{1}{q^i} (1-q^i)(1-s^*q^{i+1}), \quad 0 \leq i \leq d$$

$$\lambda_i = hh^* q^{-2i-1} (1-q^{i+1})(1-q^{i-d})(1-r_1 q^{i+1})(1-r_2 q^{i+1}), \quad 0 \leq i \leq d-1$$

$$r_1 r_2 = ss^* q^{d+1}$$

Theorem

$$(1) \quad p_i(x) = \sum_{4,3} \left(\begin{matrix} q^{-i}, s^*q^{i+1}, q^{-y}, sq^{y+1} \\ q^{-d}, r_1 q, r_2 q \end{matrix}; q, q \right), \quad 0 \leq i \leq d$$

$$x = \tilde{\gamma}(y) = \phi_0 + h \frac{1}{q^y} (1-q^y)(1-sq^{y+1})$$

$$b_i = \frac{h (1-q^{i-d})(1-s^*q^{i+1})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-s^*q^{2i+2})(1-s^*q^{2i+1})}, \quad 0 \leq i \leq d-1$$

$$c_i = \frac{hq}{s^*} \frac{(1-q^i)(q^{-d-1}-s^*q^i)(r_1 - s^*q^i)(r_2 - s^*q^i)}{(1-s^*q^{2i})(1-s^*q^{2i+1})}, \quad 1 \leq i \leq d$$

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$$(2) P_i^*(x) = {}_4\phi_3 \left(\begin{matrix} q^{-i}, sq^{i+1}, q^{-y}, s^*q^{y+1} \\ q^{-d}, r_1 q, r_2 q \end{matrix}; q, q \right), \quad 0 \leq i \leq d$$

$$x = \tilde{z}^*(y) = \theta_0^* + h^* \frac{1}{q^y} (1-q^y)(1-s^*q^{y+1})$$

$$b_i^* = \frac{h^*(1-q^{i-d})(1-sq^{i+1})(1-r_1 q^{i+1})(1-r_2 q^{i+1})}{(1-sq^{2i+2})(1-sq^{2i+1})}, \quad 0 \leq i \leq d-1$$

$$c_i^* = \frac{h^*q}{s} \frac{(1-q^i)(q^{-d-1}-sq^i)(r_1 - sq^i)(r_2 - sq^i)}{(1-sq^{2i})(1-sq^{2i+1})}, \quad 1 \leq i \leq d$$

Type II $q \rightarrow 1, \quad h(1-q)^2 \rightarrow h', \quad h^*(1-q)^2 \rightarrow h^{*'},$

$$\frac{1-s}{1-q} \rightarrow s', \quad \frac{1-s^*}{1-q} \rightarrow s^{*'}$$

$$\frac{1-r_1}{1-q} \rightarrow r_1', \quad \frac{1-r_2}{1-q} \rightarrow r_2'$$

Type III $q \rightarrow -1, \quad (1+q)h \rightarrow h', \quad (1+q)h^* \rightarrow h^{*'},$

$$\frac{1-s}{1+q} \rightarrow s', \quad \frac{1-s^*}{1+q} \rightarrow s^{*'}$$

$$\frac{1+r_1}{1+q} \rightarrow r_1', \quad \frac{1+r_2 q^{d+1}}{1+q} \rightarrow r_2' + d + 1$$

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