No.
Date

5th lecture

§5 Classification of L-pairs

$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ L-system

$$V = \bigoplus_{i=0}^{d} V_i, \quad \dim V_i = 1, \quad 0 \le i \le d$$

eigenspace decomposition of A

θ_i : eigenvalue of A on V_i

$$V = \bigoplus_{i=0}^{d} V_i^*, \quad \dim V_i^* = 1, \quad 0 \le i \le d$$

eigenspace decomposition of A^*

θ_i^* : eigenvalue of A^* on V_i^*

(i) $A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$

$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$

(ii) V is irreducible as an $\langle A, A^* \rangle$-module

No.
Date

L - system

$\Longrightarrow$

pre L-system $(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$

$\{\theta_i\}_{i=0}^{d}$: (β, γ, δ) - sequence

$\{\theta_i^*\}_{i=0}^{d}$: $(\beta, \gamma^*, \delta^*)$ - sequence

$+$ $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$

Such pre L-systems are classified by Terwilliger's Lemma.

Moreover such a pre L-system satisfies

(i) $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$, $0 \le i \le d$. $V_{-1}^* = V_{d+1}^* = 0$

$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $0 \le i \le d$, $V_{-1} = V_{d+1} = 0$

by the converse theorem of TD-relations.

No.
Date

§ 5.1 Transition matrices

$A, A^* \in End(V)$ <u>pre L-pair</u>

with data $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1} \right)$

$$\begin{cases} \theta_i \neq \theta_j, & i \neq j \in \{0, 1, \cdots, d\} \\ \theta_i^* \neq \theta_j^*, & i \neq j \in \{0, 1, \cdots, d\} \\ \lambda_i \neq 0, & 0 \leq i \leq d-1 \end{cases}$$

Set $\lambda_{-1} = \lambda_d = 0.$

$$V = \bigoplus_{i=0}^{d} U_i, \quad \dim U_i = 1, \quad 0 \leq i \leq d$$

$F_i : V \longrightarrow U_i$ projection

$R, L \in End(V)$

$$\begin{cases} R U_i = U_{i+1}, & 0 \leq i \leq d, \quad U_{d+1} = 0 \\ L U_i = U_{i-1}, & 0 \leq i \leq d, \quad U_{-1} = 0 \end{cases}$$

$$A = R + \sum_{i=0}^{d} \theta_i F_i$$

$$A^* = L + \sum_{i=0}^{d} \theta_i^* F_i$$

No.

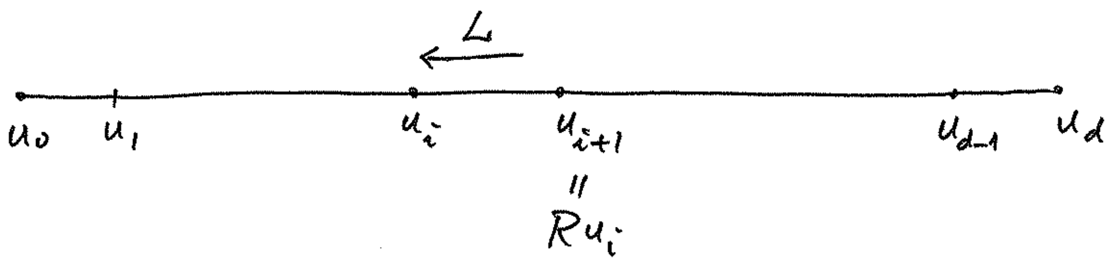
Date

$$\lambda_i = \operatorname{tr}\, LR|_{U_i} = \operatorname{tr}\, RL|_{U_{i+1}}, \qquad -1 \le i \le d$$

$$U_0 \ni u_0 \neq 0$$

$$u_i = R^i u_0 \in U_i$$

$$L u_{i+1} = \lambda_i u_i, \qquad -1 \le i \le d$$



No.

Date

Then A, A^* are diagonalizable.

$$V = \bigoplus_{i=0}^{d} V_i \quad \text{eigenspace decomp. of } A$$

$$A|_{V_i} = \theta_i$$

$$E_i : V \longrightarrow V_i \quad \text{projection}$$

$$A = \sum_{i=0}^{d} \theta_i E_i$$

$$E_i = \prod_{\nu \neq i} \frac{A - \theta_\nu}{\theta_i - \theta_\nu}$$

$$V = \bigoplus_{i=0}^{d} V_i^* \quad \text{eigenspace decomp. of } A^*$$

$$A^*|_{V_i^*} = \theta_i^*$$

$$E_i^* : V \longrightarrow V_i^* \quad \text{projection}$$

$$A^* = \sum_{i=0}^{d} \theta_i^* E_i^*$$

$$E_i^* = \prod_{\nu \neq i} \frac{A^* - \theta_\nu^*}{\theta_i^* - \theta_\nu^*}$$

$$\begin{cases} U_0 + U_1 + \cdots + U_i = V_0^* + V_1^* + \cdots + V_i^* \\ U_i + U_{i+1} + \cdots + U_d = V_i + V_{i+1} + \cdots + V_d \end{cases}$$

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d)$$

No.

Date

$U \ni u_0 \neq 0$. $\quad u_i = R^i u_0 \in U_i$

$u_0, u_1, \cdots, u_d$: basis of V

$$U_i \simeq U_i + \cdots + U_d / U_{i+1} + \cdots + U_d$$
$$= V_i + \cdots + V_d / V_{i+1} + \cdots + V_d \simeq V_i$$

$\exists\, v_0, v_1, \cdots, v_d$: basis of V

$v_i \in V_i$, $\quad F_i v_i = u_i$

$E_i u_i = v_i$

<u>Lemma</u>

(1) $(u_0, u_1, \cdots, u_d) = (v_0, v_1, \cdots, v_d)\, C$

$$C = (c_{ij}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$c_{ij} = \prod_{\nu=j}^{i-1} (\theta_i - \theta_\nu)^{-1}, \quad 0 \leq j \leq i \leq d$$

(2) $(v_0, v_1, \cdots, v_d) = (u_0, u_1, \cdots, u_d)\, C^{-1}$

$$C^{-1} = (\tilde{c}_{ij}) = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{pmatrix}$$

$$\tilde{c}_{ij} = \prod_{\nu=j+1}^{i} (\theta_j - \theta_\nu)^{-1}, \quad 0 \leq j \leq i \leq d$$

No.

Date

Proof

(1) $$u_j = \sum_{i=j}^{d} c_{ij} v_i$$

$$A u_j = \theta_j u_j + u_{j+1} = \sum_{i=j}^{d} (\theta_j c_{ij} + c_{i\,j+1}) v_i$$

$$A \sum_{i=j}^{d} c_{ij} v_i = \sum_{i=j}^{d} \theta_i c_{ij} v_i$$

$$\theta_j c_{ij} + c_{i\,j+1} = \theta_i c_{ij}$$

$$c_{ij} = (\theta_i - \theta_j)^{-1} c_{i\,j+1} = \cdots = (\theta_i - \theta_j)^{-1} \cdots (\theta_i - \theta_{i-1})^{-1} c_{ii}$$

(2) $$v_j = \sum_{i=j}^{d} \tilde{c}_{ij} u_i$$

$$A v_j = \theta_j v_j = \sum_{i=j}^{d} \theta_j \tilde{c}_{ij} u_i$$

$$A \sum_{i=j}^{d} \tilde{c}_{ij} u_i = \sum_{i=j}^{d} (\theta_i \tilde{c}_{ij} + \tilde{c}_{i-1\,j}) u_i$$

$$\theta_j \tilde{c}_{ij} = \theta_i \tilde{c}_{ij} + \tilde{c}_{i-1\,j}$$

$$\tilde{c}_{ij} = (\theta_j - \theta_i)^{-1} \tilde{c}_{i-1\,j} = \cdots = (\theta_j - \theta_i)^{-1} \cdots (\theta_j - \theta_{j+1})^{-1} \tilde{c}_{jj}$$

$\blacksquare$

No.
Date

$$U_i \simeq U_0 + \cdots + U_i / U_0 + \cdots + U_{i-1}$$
$$= V_0^* + \cdots + V_i^* / V_0^* + \cdots + V_{i-1}^*$$
$$\simeq V_i^*$$

$\exists\ v_0^*, v_1^*, \cdots, v_d^*$: basis of V

$v_i^* \in V_i^*$, $\quad F_i v_i^* = u_i$

$E_i^* u_i = v_i^*$

Lemma

(1) $(u_0, u_1, \cdots, u_d) = (v_0^*, v_1^*, \cdots, v_d^*)\, C^*$

$$C^* = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (c_{ij}^*)$$

$$c_{ij}^* = \prod_{\nu=i+1}^{j} \lambda_{\nu-1} (\theta_i^* - \theta_\nu^*)^{-1}, \qquad 0 \le i \le j \le d$$

(2) $(v_0^*, v_1^*, \cdots, v_d^*) = (u_0, u_1, \cdots, u_d)\, C^{*-1}$

$$C^{*-1} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (\tilde{c}_{ij}^*)$$

$$\tilde{c}_{ij}^* = \prod_{\nu=i}^{j-1} \lambda_\nu (\theta_j^* - \theta_\nu^*)^{-1}, \qquad 0 \le i \le j \le d$$

No.
Date

§5.2 Equations derived from Terwilliger's Lemma

$(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre L-system

$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$ data

<u>Proposition</u>

Set $\lambda_i' = \lambda_i - (\theta_i - \theta_d)(\theta_{i+1}^* - \theta_0^*)$. $0 \le i \le d-1$

$\lambda_{-1}' = \lambda_d' = 0$.

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\Longleftrightarrow$

$$\lambda_{i-1}' = \frac{\theta_i^* - \theta_1^*}{\theta_{i+1}^* - \theta_0^*} \lambda_i' + \lambda_0', \quad 0 \le i \le d, \quad \frac{\theta_d^* - \theta_1^*}{\theta_{d+1}^* - \theta_0^*} \lambda_d' \underset{\text{def}}{=} 0$$

(2) $A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$

$\Longleftrightarrow$

$$\lambda_i' = \frac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d} \lambda_{i-1}' + \lambda_{d-1}', \quad 0 \le i \le d, \quad \frac{\theta_0 - \theta_{d-1}}{\theta_{-1} - \theta_d} \lambda_{-1}' \underset{\text{def}}{=} 0$$

No.
Date

Proof

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\Leftrightarrow$

$E_0^*(A - AE_0^*)(A^* - \theta_1^*) = 0$ by Terwilliger's Lemma

$\Leftrightarrow$

$E_0^*(A - a_0)(A^* - \theta_1^*) = 0$

$a_0 = \theta_0 + \dfrac{\lambda_0}{\theta_0^* - \theta_1^*}$ proof $E_0^* A E_0^* = a_0 \in \mathbb{C}$ $V_0^* = U_0$

$E_0^* A E_0^* v_0^* = E_0^* A v_0^*$ $(v_0^* = u_0)$

$= E_0^*(\theta_0 u_0 + u_1)$

$= \theta_0 v_0^* + \varsigma_{01}^* v_0^*$

$= \left(\theta_0 + \dfrac{\lambda_0}{\theta_0^* - \theta_1^*}\right) v_0^*$

by Lemma on page 5-8

$V = \bigoplus_{i=0}^{d} U_i$, $U_i = \mathbb{C} u_i$

$u_i = R^i u_0$

$$E_0^*(A - a_0)\underbrace{(A^* - \theta_1^*) u_i}_{\substack{\| \\ (\theta_i^* - \theta_1^*) u_i + \lambda_{i-1} u_{i-1}}}$$

$$= E_0^*\Big((\theta_i^* - \theta_1^*) u_{i+1} + \big((\theta_i - a_0)(\theta_i^* - \theta_1^*) + \lambda_{i-1}\big) u_i + \lambda_{i-1}(\theta_{i-1} - a_0) u_{i-1}\Big) = 0$$

So $(\theta_i^* - \theta_1^*)\varsigma_{0\,i+1}^* + \big((\theta_i - a_0)(\theta_i^* - \theta_1^*) + \lambda_{i-1}\big)\varsigma_{0i}^* + \lambda_{i-1}(\theta_{i-1} - a_0)\varsigma_{0\,i-1}^*$

$$= \frac{\lambda_{i-1} \cdots \lambda_0}{(\theta_0^* - \theta_i^*) \cdots (\theta_0^* - \theta_1^*)}\left(\frac{\theta_i^* - \theta_1^*}{\theta_0^* - \theta_{i+1}^*}\lambda_i + \big((\theta_i - a_0)(\theta_i^* - \theta_1^*) + \lambda_{i-1}\big) + (\theta_{i-1} - a_0)(\theta_0^* - \theta_i^*)\right)$$

$= 0$, $0 \le i \le d$, $\dfrac{\theta_d^* - \theta_1^*}{\theta_0^* - \theta_{i+1}^*}\lambda_d = 0$

No.
Date

(2) $A^*(V_0+\cdots+V_{d-2}) \subseteq V_0+\cdots+V_{d-1}$

$\Longleftrightarrow$

$E_d(A^*-A^*E_d)(A-\theta_{d-1})=0$ by Terwilliger's Lemma

$\Longleftrightarrow$

$E_d(A^*-a_d^*)(A-\theta_{d-1})=0$

$$a_d^* = \frac{\lambda_{d-1}}{\theta_d-\theta_{d-1}}+\theta_d^*$$

Proof $E_dA^*E_d = a_d^* \in \mathbb{C}$

$$E_dA^*E_d u_d = E_dA^*u_d \qquad (V_d = U_d \ni \underset{v_d}{u_d})$$
$$= E_d(\lambda_{d-1}u_{d-1}+\theta_d^*u_d)$$
$$= (\lambda_{d-1}c_{d\,d-1}+\theta_d^*)u_d$$
$$= \left(\frac{\lambda_{d-1}}{\theta_d-\theta_{d-1}}+\theta_d^*\right)u_d$$

by Lemma on page 5-6

$$E_d(A^*-a_d^*)(A-\theta_{d-1})u_i$$
$$= E_d(A^*-a_d^*)\left(u_{i+1}+(\theta_i-\theta_{d-1})u_i\right)$$
$$= E_d\Big((\theta_{i+1}^*-a_d^*)u_{i+1}+\big(\lambda_i+(\theta_i^*-a_d^*)(\theta_i-\theta_{d-1})\big)u_i+(\theta_i-\theta_{d-1})\lambda_{i-1}u_{i-1}\Big)$$
$$=0$$

So $(\theta_{i+1}^*-a_d^*)c_{d\,i+1}+\big(\lambda_i+(\theta_i^*-a_d^*)(\theta_i-\theta_{d-1})\big)c_{di}+(\theta_i-\theta_{d-1})\lambda_{i-1}c_{d\,i-1}$

$$= \frac{1}{(\theta_d-\theta_i)\cdots(\theta_d-\theta_{d-1})}\left((\theta_{i+1}^*-a_d^*)(\theta_d-\theta_i)+\big(\lambda_i+(\theta_i^*-a_d^*)(\theta_i-\theta_{d-1})\big)+\frac{\theta_i-\theta_{d-1}}{\theta_d-\theta_{i-1}}\lambda_{i-1}\right)$$

$=0$, $\quad 0\le i\le d$, $\quad \dfrac{\theta_0-\theta_{d-1}}{\theta_d-\theta_{-1}}\lambda_{-1} \underset{\text{def}}{=} 0$

$\boxed{//}$

No.
Date

Assume

$\{\theta_i\}_{i=0}^{d}$: (β, γ, δ)-sequence

$\{\theta_i^*\}_{i=0}^{d}$: $(\beta, \gamma^*, \delta^*)$-sequence

$\beta = q + q^{-1}$

type I $\qquad \theta_i = a + bq^i + cq^{-i}, \quad i \in \mathbb{Z}$

$(q \neq \pm 1) \qquad \theta_i^* = a^* + b^* q^i + c^* q^{-i}, \quad i \in \mathbb{Z}$

Askey-Wilson parametrization,

$$\theta_i = \theta_0 + h q^{-i}(1-q^i)(1-sq^{i+1})$$

$$\theta_i^* = \theta_0^* + h^* q^{-i}(1-q^i)(1-s^* q^{i+1})$$

case $s \neq 0, s^* \neq 0$ $\qquad$ q-Racah

type II $\qquad q \longrightarrow 1, \quad (1-q)^2 h \longrightarrow h', \quad (1-q)^2 h^* \longrightarrow h^{*\prime}$

$(q=1) \qquad \dfrac{1-s}{1-q} \longrightarrow s', \quad \dfrac{1-s^*}{1-q} \longrightarrow s^{*\prime}$

type III $\qquad q \longrightarrow -1, \quad (1+q)h \longrightarrow h', \quad (1+q)h^* \longrightarrow h^{*\prime}$

$(q=-1) \qquad \dfrac{1-s}{1+q} \longrightarrow s', \quad \dfrac{1-s^*}{1+q} \longrightarrow s^{*\prime}$

No.
Date

type I

Lemma

$$\begin{cases} \lambda'_{i-1} = \dfrac{\theta^*_i - \theta^*_1}{\theta^*_{i+1} - \theta^*_0}\lambda'_i + \lambda'_0 , & 0 \le i \le d-1 \\ \lambda'_i = \dfrac{\theta_i - \theta_{d-1}}{\theta_{i-1} - \theta_d}\lambda'_{i-1} + \lambda'_0 , & 1 \le i \le d, \quad \lambda'_{d-1} = \lambda'_0 \end{cases}$$

$\Longleftrightarrow$

$$\lambda'_i = \frac{(1-q^{d-i})(1-q^{i+1})}{(1-q)(1-q^d)}\lambda'_0 , \qquad 0 \le i \le d-1$$

$\Longleftrightarrow$

$$\lambda_i = hh^* q^{-2i-1}(1-q^{i-d})(1-q^{i+1})(1-r_1 q^{i+1})(1-r_2 q^{i+1}), \qquad 0 \le i \le d-1$$

$$r_1 r_2 = ss^* q^{d+1}$$

type II $\quad \dfrac{1-r_1}{1-q} \longrightarrow r'_1 , \quad \dfrac{1-r_2}{1-q} \longrightarrow r'_2$

type III $\quad \dfrac{1+r_1}{1+q} \longrightarrow r'_1 , \quad \dfrac{1+r_2 q^{d+1}}{1+q} \longrightarrow r'_2 + d + 1$

No.
Date

§5.3 Equations derived from TD-relations

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre L-system

$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$ data

Assume

$\{\theta_i\}_{i=0}^d$: (β, γ, δ)-sequence

$\{\theta_i^*\}_{i=0}^d$: $(\beta, \gamma^*, \delta^*)$-sequence

Set

$$\alpha_i = (\beta+1)\Big(\theta_i\theta_i^* - \theta_{i+2}\theta_{i+2}^* + (\theta_{i+1}\theta_{i+2}^* + \theta_{i+2}\theta_{i+1}^*) - (\theta_i\theta_{i+1}^* + \theta_{i+1}\theta_i^*)\Big)$$

Proposition

(TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$\Longleftrightarrow$

$$\lambda_{i-1} - (\beta+1)(\lambda_i - \lambda_{i+1}) - \lambda_{i+2} = \alpha_i, \quad 0 \le i \le d-2, \quad \lambda_{-1} = \lambda_d = 0$$

$\Longleftrightarrow$

$(TD)^*$ $A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*)$

Proof

$$(TD) \iff R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2 \quad \text{on } U_i, \quad 0 \le i \le d-2$$

$$\iff \lambda_{i-1} - (\beta+1)(\lambda_i - \lambda_{i+1}) - \lambda_{i+2} = \alpha_i, \quad 0 \le i \le d-2$$

$$(TD)^* \iff L^3R - (\beta+1)(L^2RL - LRL^2) - RL^3 = -\alpha_i L^2 \quad \text{on } U_{i+2}, \quad 0 \le i \le d-2$$

$$\iff \lambda_{i+2}\lambda_{i+1}\lambda_i - (\beta+1)(\lambda_{i+1}\lambda_{i+1}\lambda_i - \lambda_{i+1}\lambda_i\lambda_i) - \lambda_{i+1}\lambda_i\lambda_{i-1} = -\alpha_i\lambda_{i+1}\lambda_i, \quad 0 \le i \le d-2$$

$$\iff \lambda_{i+2} - (\beta+1)(\lambda_{i+1} - \lambda_i) - \lambda_{i-1} = -\alpha_i, \quad 0 \le i \le d-2$$

$\blacksquare$

No.
Date

Corollary

$$A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$$

$$A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$$

$\Longrightarrow$

(TD) + $(\mathrm{TD})^*$

In particular

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0.$$

Proof

Use the Proposition on page 5-9, the Lemma on page 5-13 and the Theorem on page 4-13. $\square$

No.
Date

§5.4 The irreducibility

L-system $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$

$\Longrightarrow$

pre L-system $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$

data $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d; \{\lambda_i\}_{i=0}^{d-1})$

$\{\theta_i\}_{i\in\mathbb{Z}}$: (β, γ, δ)-sequence

$\{\theta_i^*\}_{i\in\mathbb{Z}}$: $(\beta, \gamma^*, \delta^*)$-sequence

$\beta = q + q^{-1}$

type I $(q \neq \pm 1)$

$$\lambda_i = h h^* q^{-2i-1}(1-q^{i+1})(1-q^{i-d})(1-r_1 q^{i+1})(1-r_2 q^{i+1}), \quad 0 \le i \le d-1$$

$$r_1 r_2 = s s^* q^{d+1}$$

type II, III limiting cases

Call such pre L-system <u>special</u>.

No.
Date

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ special pre L-system

$\Longrightarrow$

$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_{-1}^* = V_{d+1}^* = 0$

$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d, \quad V_{-1} = V_{d+1} = 0$

by Prop. on page 5-9
Lemma on page 5-13
Cor on page 5-16

So L-system

$\Longleftrightarrow$
special pre L-system

\+
the irreducibility

No.
Date

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ special pre L-system

$E_i : V = \bigoplus_{\nu=0}^{d} V_\nu \longrightarrow V_i$ projection

$E_i^* : V = \bigoplus_{\nu=0}^{d} V_\nu^* \longrightarrow V_i^*$ projection

Proposition

V : irreducible as an $\langle A, A^* \rangle$-module

$\Longleftrightarrow$

$E_0^* V_0 \neq 0$

Proof

($\Longrightarrow$) Suppose $E_0^* V_0 = 0$.

Then $V_0 \subseteq V_1^* + \cdots + V_d^*$.

Set $W_i = (V_0 + \cdots + V_i) \cap (V_{i+1}^* + \cdots + V_d^*)$

$W = W_0 + W_1 + \cdots + W_{d-1}$.

Then $(A - \theta_i) W_i \subseteq W_{i-1}$, $(A - \theta_0) W_0 = 0$,
$(A^* - \theta_{i+1}^*) W_i \subseteq W_{i+1}$, $(A^* - \theta_d^*) W_{d-1} = 0$,

and so W is invariant under A, A^*.

$0 \neq V_0 = W_0 \subseteq W \subseteq V_1^* + \cdots + V_d^* \neq V$.

V is not irreducible. #

No.

Date

$(\Leftarrow)$

$V \supseteq W \neq 0 \quad : \quad \langle A, A^* \rangle$-submodule

$$W = \bigoplus_{i=0}^{d} W \cap V_i$$

$$i_0 = \mathrm{Min}\{ i \mid W \cap V_i \neq 0 \}$$

claim $\quad i_0 = 0$

$$V = \bigoplus_{i=0}^{d} U_i$$

$$U_i + \cdots + U_d = V_i + \cdots + V_d$$

$$U_{i_0} \simeq U_{i_0} + \cdots + U_d / U_{i_0+1} + \cdots + U_d$$

$$= V_{i_0} + \cdots + V_d / V_{i_0+1} + \cdots + V_d$$

$$\simeq V_{i_0}$$

$$(A^* - \theta^*_{i_0}) V_{i_0} = (A^* - \theta^*_{i_0}) U_{i_0} \quad \bmod U_{i_0+1} + \cdots + U_d$$

$$= L U_{i_0}$$

$$= U_{i_0 - 1}$$

$i_0 = 0$ by the minimality. //

Then

$W \supseteq V_0 \quad$ as $\dim V_0 = 1$.

$W \supseteq E^*_0 V_0 \neq 0 \quad$ as $\dim V_0^* = 1$

$\quad \quad \; \| $

$\quad \quad V_0^* = U_0$

$W \supseteq U_i = (A - \theta_{i-1}) \cdots (A - \theta_0) U_0 \quad \square$

$W = V$

No.
Date

Lemma

$E_0^* V_0 \neq 0$

$\Longleftrightarrow$

$$\sum_{i=0}^{d} \frac{\lambda_0 \lambda_1 \cdots\cdots \lambda_{i-1}}{(\theta_0-\theta_1)\cdots(\theta_0-\theta_i)(\theta_0^*-\theta_1^*)\cdots(\theta_0^*-\theta_i^*)} \neq 0$$

Proof

$$V_0 \ni v_0 = \sum_{i=0}^{d} \tilde{c}_{i0} u_i$$

$$\tilde{c}_{i0} = \frac{1}{(\theta_0-\theta_1)\cdots(\theta_0-\theta_i)}$$ by Lemma on page 5-6

$$U_i \ni u_i = c_{0i}^* v_0^* + c_{1i}^* v_1^* + \cdots$$

$$c_{0i}^* = \frac{\lambda_0 \lambda_1 \cdots \lambda_{i-1}}{(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_i^*)}$$ by Lemma on page 5-8

$$v_0 = \left(\sum_{i=0}^{d} \tilde{c}_{i0} c_{0i}^* \right) v_0^* + \cdots$$

$\blacksquare$

No.
Date

<u>q-Racah case</u> type I + $s \neq 0$, $s^* \neq 0$

Set

$$\hat{\lambda}_i = \frac{hh^*}{s^*} q^{-2i-1} (1-q^{i+1})(1-q^{i-d})(r_1 - s^* q^{i+1})(r_2 - s^* q^{i+1}),$$

$$0 \leq i \leq d-1$$

<u>Theorem</u>

$$\sum_{i=0}^{d} \frac{\lambda_0 \cdots\cdots \lambda_{i-1}}{(\theta_0-\theta_1)\cdots(\theta_0-\theta_i)(\theta_0^*-\theta_1^*)\cdots(\theta_0^*-\theta_i^*)}$$

$$= \frac{\hat{\lambda}_0 \cdots \hat{\lambda}_{d-1}}{(\theta_0-\theta_1)\cdots(\theta_0-\theta_d)(\theta_0^*-\theta_1^*)\cdots(\theta_0^*-\theta_d^*)}$$

<u>Cor</u>

V : irreducible as an $\langle A, A^* \rangle$-module

$\Longleftrightarrow$

$\hat{\lambda}_i \neq 0$, $0 \leq i \leq d-1$.

Proof of Theorem   out line

$$\mathrm{LHS} = {}_3\phi_2\left(\begin{matrix} r_1 q,\ r_2 q,\ q^{-d} \\ s q^2,\ s^* q^2 \end{matrix} ; q, q\right)$$

calculation

$$= \mathrm{RHS} \qquad \blacksquare$$

q- analogue of Pfaff-Saalschütz formula.

Pfaff - Saalschütz formula

$${}_3F_2\left(\begin{matrix} a,\ b,\ -n \\ c,\ 1+a+b-c-n \end{matrix} ; 1\right) = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n},$$

$$n = 0, 1, 2, \cdots$$

q- analogue

$${}_3\phi_2\left(\begin{matrix} a,\ b,\ q^{-n} \\ c,\ abc^{-1}q^{1-n} \end{matrix} ; q, q\right) = \frac{(\frac{c}{a};q)_n\,(\frac{c}{b};q)_n}{(c;q)_n\,(\frac{c}{ab};q)_n},$$

$$n = 0, 1, 2, \cdots$$

No.

Date

$A, A^* \in \mathrm{End}(V)$ TD-pair

$\Longrightarrow$

4 TD-systems	data
$(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$	$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\lambda_i\}_{i=0}^{d-1})$
$(\{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$	$(\{\theta_{d-i}\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\hat{\lambda}_i\}_{i=0}^{d-1})$
$(\{V_i\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$	$(\{\theta_i\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\hat{\lambda}_{d-1-i}\}_{i=0}^{d-1})$
$(\{V_{d-i}\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$	$(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\lambda_{d-1-i}\}_{i=0}^{d-1})$