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4th lecture

§4 TD-pairs : TD-relations and Terwilliger's Lemma

§4.1 TD-relations

Theorem (TD-relations)

$A,\ A^* \in \mathrm{End}(V)$ TD-pair
diameter d

(1) $\exists \beta, \gamma, \delta \in \mathbb{C}$

(TD) $$A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A).$$

Moreover if $d \geq 3$, β, γ, δ are uniquely determined,
if $d = 2$, β is arbitrary, γ, δ are uniquely det.
if $d = 1$, β, γ are arbitrary, δ is uniq. det.

(2) $\exists \beta^*, \gamma^*, \delta^* \in \mathbb{C}$

(TD)* $$A^{*3}A - (\beta^*+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*).$$

Moreover if $d \geq 3$, $\beta^*, \gamma^*, \delta^*$ are uniquely determined,
if $d = 2$, β^* is arbitrary, γ^*, δ^* are uniquely determined,
if $d = 1$, β^*, γ^* are arbitrary, δ^* is uniquely determined.

Remark $\beta = \beta^*$ holds.
(proof given later)

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<u>Proof</u> Notations as in §3.

(1) $\langle A \rangle \subseteq \mathrm{End}(V)$ subalg. generated by A

$$\mathcal{L} = \mathrm{Span}\{XA^*Y - YA^*X \mid X, Y \in \langle A \rangle\} \subseteq \mathrm{End}(V)$$

linear subspace

Then

$$\mathcal{L} = \mathrm{Span}\{E_j A^* E_i - E_i A^* E_j \mid 0 \le i, j \le d\}$$

$$= \mathrm{Span}\{E_{i+1} A^* E_i - E_i A^* E_{i+1} \mid 0 \le i \le d-1\}$$

by $E_j A^* E_i = 0$, $|j-i| > 1$.

In particular,

$$\dim \mathcal{L} \le d.$$

<u>claim</u> $\{A^i A^* - A^* A^i \mid 1 \le i \le d\}$ linearly independent

<u>proof of claim</u>

$$\sum_{i=0}^{r} c_i (A^i A^* - A^* A^i) = 0, \quad c_r \neq 0.$$

$$E_r^* (\quad\quad) E_0^* = \underbrace{c_r}_{\neq 0} \underbrace{(\theta_0^* - \theta_r^*)}_{\neq 0} \underbrace{E_r^* A^r E_0^*}_{\neq 0} \neq 0$$

#

So $\dim \mathcal{L} = d$

and $\{ A^i A^* - A^* A^i \mid 1 \le i \le d \}$ is a basis of $\mathcal{L}$.

If $d \le 2$, the theorem holds.

Assume $d \ge 3$.

$$\mathcal{L} \ni A^2 A^* A - A A^* A^2 = \sum_{i=1}^{r} c_i (A^i A^* - A^* A^i), \qquad c_r \ne 0.$$

Enough to show $r = 3$.

Suppose $r \ge 4$.

$$E_r^* (A^2 A^* A - A A^* A^2) E_0^* = 0 \qquad \text{since } A^* = \sum_{i=0}^{d} \theta_i^* E_i^*$$

$$E_r^* \Big(\sum_{i=1}^{r} c_i (A^i A^* - A^* A^i) \Big) E_0^* = c_r (\theta_0^* - \theta_r^*) E_r^* A^r E_0^* \ne 0$$

※

Suppose $r \le 2$.

$$E_3^* (A^2 A^* A - A A^* A^2) E_0^* = (\theta_1^* - \theta_2^*) E_3^* A^3 E_0^* \ne 0$$

$$E_3^* \Big(\sum_{i=1}^{r} c_i (A^i A^* - A^* A^i) \Big) E_0^* = 0$$

※

(2) the same argument is available. ▨

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§4.2 Eigenvalues

<u>Theorem</u>

$\left(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}\right)$ TD-system

θ_i : eigenvalue of A on V_i , $0 \le i \le d$
θ_i^* : eigenvalue of A^* on V_i^* , $0 \le i \le d$

$\beta, \gamma, \delta, \beta^*, \gamma^*, \delta^*$: the parameters of the TD-relations

Then

(i)
$$\beta = \beta^* = \frac{\theta_{i+1} - \theta_i + \theta_{i-1} - \theta_{i-2}}{\theta_i - \theta_{i-1}} = \frac{\theta^*_{i+1} - \theta^*_i + \theta^*_{i-1} - \theta^*_{i-2}}{\theta^*_i - \theta^*_{i-1}}, \qquad 2 \le i \le d-1.$$

(ii)
$$\gamma = \theta_{i+1} - \beta\theta_i + \theta_{i-1}, \qquad 1 \le i \le d-1,$$
$$\gamma^* = \theta^*_{i+1} - \beta^*\theta^*_i + \theta^*_{i-1}, \qquad 1 \le i \le d-1.$$

(iii)
$$\delta = \theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 - \gamma(\theta_{i+1} + \theta_i), \qquad 0 \le i \le d-1,$$
$$\delta^* = \theta^{*2}_{i+1} - \beta^*\theta^*_{i+1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i+1} + \theta^*_i), \qquad 0 \le i \le d-1.$$

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<u>Remark</u>

(1) If $d \leq 2$, β, β^* are arbitrary. So we set $\beta = \beta^*$.

(2) $\{x_i\}_{i\in\mathbb{Z}}$: <u>β-sequence</u>

if $(x_i - x_{i-1})\beta = x_{i+1} - x_i + x_{i-1} - x_{i-2}$, $i \in \mathbb{Z}$,

<u>(β, γ)-sequence</u>

if $\gamma = x_{i+1} - \beta x_i + x_{i-1}$, $i \in \mathbb{Z}$,

<u>(β, γ, δ)-sequence</u>

if $\delta = x_{i+1}^2 - \beta x_{i+1} x_i + x_i^2 - \gamma(x_{i+1} + x_i)$, $i \in \mathbb{Z}$.

β-seq. $\iff$ (β, γ)-seq.

$$\begin{array}{rl} & \gamma = x_{i+1} - \beta x_i + x_{i-1} \\ -) & \gamma = x_i - \beta x_{i-1} + x_{i-2} \\ \hline & 0 = x_{i+1} - x_i - \beta(x_i - x_{i-1}) + x_{i-1} - x_{i-2} \end{array}$$

(β, γ)-seq. $\Longrightarrow$ (β, γ, δ)-seq.

$\Longleftarrow$ if $x_{i+1} \neq x_{i-1}$, $i \in \mathbb{Z}$

$$\begin{array}{rl} & \delta = x_{i+1}^2 - \beta x_{i+1} x_i + x_i^2 - \gamma(x_{i+1} + x_i) \\ -) & \delta = x_i^2 - \beta x_i x_{i-1} + x_{i-1}^2 - \gamma(x_i + x_{i-1}) \\ \hline & 0 = x_{i+1}^2 - x_{i-1}^2 - \beta x_i(x_{i+1} - x_{i-1}) - \gamma(x_{i+1} - x_{i-1}) \end{array}$$

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(3) $\{\theta_i\}_{i=0}^{d}$ is extended to $\{\theta_i\}_{i\in\mathbb{Z}}$ as a β-seq.

$\{\theta_i^*\}_{i=0}^{d}$ is extended to $\{\theta_i^*\}_{i\in\mathbb{Z}}$ as a β-seq.

(4) $\{x_i\}_{i\in\mathbb{Z}}$: β-sequence

$$\beta = q + q^{-1}$$

type I $\quad q \neq \pm 1$

$$x_i = a + b q^{i} + c q^{-i}, \qquad i \in \mathbb{Z}$$

type II $\quad q = 1$

$$x_i = a + b i + c i^2, \qquad i \in \mathbb{Z}$$

type III $\quad q = -1$

$$x_i = a + b(-1)^i + c(-1)^i i, \qquad i \in \mathbb{Z}$$

<u>Proof of Theorem</u> Notations as in §3.

TD-relation

$$(\mathrm{TD})\quad A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$$

$$\begin{cases} E_{i+1}(\text{LHS of TD})E_i = \left(\theta_{i+1}^3 - (\beta+1)(\theta_{i+1}^2\theta_i - \theta_{i+1}\theta_i^2) - \theta_i^3\right)E_{i+1}A^*E_i \\ E_{i+1}(\text{RHS of TD})E_i = \left(\gamma(\theta_{i+1}^2 - \theta_i^2) + \delta(\theta_{i+1} - \theta_i)\right)E_{i+1}A^*E_i \end{cases}$$

$$\theta_{i+1} - \theta_i \neq 0, \quad E_{i+1}A^*E_i \neq 0$$

So

$$\theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 = \gamma(\theta_{i+1} + \theta_i) + \delta$$

$$\underset{(iii)}{(\beta, \gamma, \delta)\text{-seq.}} \Longrightarrow \underset{(ii)}{(\beta, \gamma)\text{-seq.}} \Longrightarrow \underset{(i)}{\beta\text{-seq.}}$$

$$\begin{cases} E_{i+3}^*(\text{LHS of TD})E_i^* = \left(\theta_i^* - (\beta+1)(\theta_{i+1}^* - \theta_{i+2}^*) - \theta_{i+3}^*\right)E_{i+3}^*A^3E_i^* \\ E_{i+3}^*(\text{RHS of TD})E_i^* = 0 \end{cases}$$

$$E_{i+3}^*A^3E_i^* = 0$$

So $\quad \theta_i^* - (\beta+1)(\theta_{i+1}^* - \theta_{i+2}^*) - \theta_{i+3}^* = 0$

$\{\theta_i^*\}_{i\in\mathbb{Z}}$ is a β-sequence. $\quad \beta = \beta^*$

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§ 4.3 TD-relations revisited

pre TD-pair: $A, A^* \in \mathrm{End}(V)$

$$V = \bigoplus_{i=0}^{d} U_i$$

$$F_i : V \longrightarrow U_i \quad \text{projection}, \quad 0 \le i \le d$$

$$R,\ L \in \mathrm{End}(V)$$

$$R U_i \subseteq U_{i+1}, \quad 0 \le i \le d, \qquad U_{d+1} = 0$$

$$L U_i \subseteq U_{i-1}, \quad 0 \le i \le d, \qquad U \ = 0$$

$$\theta_0, \theta_1, \cdots, \theta_d \in \mathbb{C}, \quad \theta_i \ne \theta_j$$

$$\theta_0^*, \theta_1^*, \cdots, \theta_d^* \in \mathbb{C}, \quad \theta_i^* \ne \theta_j^*$$

$$\begin{cases} A = R + \sum_{i=0}^{d} \theta_i F_i \\ A^* = L + \sum_{i=0}^{d} \theta_i^* F_i \end{cases} \qquad \underline{\text{pre TD-pair}}$$

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Then a pre TD-pair $A, A^* \in \mathrm{End}(V)$
are diagonalizable.

$$V = \bigoplus_{i=0}^{d} V_i \quad : \quad \text{eigenspace decomposition of } A$$

$$A|_{V_i} = \theta_i \; : \; \text{eigenvalue on } V_i$$

$$V = \bigoplus_{i=0}^{d} V_i^* \quad : \quad \text{eigenspace decomposition of } A^*$$

$$A^*|_{V_i^*} = \theta_i^* \; : \; \text{eigenvalue on } V_i^*$$

and

$$U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \quad 0 \le i \le d$$

$$U_i + \cdots + U_d = V_i + \cdots + V_d, \quad 0 \le i \le d$$

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \quad 0 \le i \le d$$

$(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ <u>pre TD-system</u>

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Theorem

$(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ pre TD-system

$$A = R + \sum_{i=0}^{d} \theta_i F_i,$$

$$A^* = L + \sum_{i=0}^{d} \theta_i^* F_i.$$

Assume $\{\theta_i\}_{i\in\mathbb{Z}}$ is a (β, γ, δ)-sequence,

$\{\theta_i^*\}_{i\in\mathbb{Z}}$ is a $(\beta, \gamma^*, \delta^*)$-sequence.

Set
$$\alpha_i = (\beta+1)\left(\theta_i\theta_i^* - \theta_{i+2}\theta_{i+2}^* + (\theta_{i+1}\theta_{i+2}^* + \theta_{i+2}\theta_{i+1}^*) - (\theta_i\theta_{i+1}^* + \theta_{i+1}\theta_i^*)\right)$$

Then

(1) (TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$\Longleftrightarrow$ $R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2$ on U_i, $0 \le i \le d-2$

(2) $(\mathrm{TD})^*$ $A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*$

$\Longleftrightarrow$ $L^3R - (\beta+1)(L^2RL - LRL^2) - LR^3 = -\alpha_i L^2$ on U_{i+2}, $0 \le i \le d-2$

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Proof

$$A = R + F, \quad F = \sum_{i=0}^{d} \theta_i F_i$$
$$A^* = L + F^*, \quad F^* = \sum_{i=0}^{d} \theta_i^* F_i$$

(1) $$A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = X_3 + X_2 + X_1 + X_0 + X_{-1}$$

$$X_3 = R^3F^* - F^*R^3 - (\beta+1)(R^2F^*R - RF^*R^2)$$

$$\begin{aligned} X_2 = {} & R^3L - LR^3 + (R^2F + RFR + FR^2)F^* - F^*(R^2F + RFR + FR^2) \\ & - (\beta+1)(R^2LR - RLR^2) \\ & - (\beta+1)\{R^2F^*F + (RF+FR)F^*R - FF^*R^2 - RF^*(RF+FR)\} \end{aligned}$$

$$\begin{aligned} X_1 = {} & (R^2F + RFR + FR^2)L - L(R^2F + RFR + FR^2) \\ & + (RF^2 + FRF + F^2R)F^* - F^*(RF^2 + FRF + F^2R) \\ & - (\beta+1)\{R^2LF - FLR^2 + (RF+FR)LR - RL(RF+FR)\} \\ & - (\beta+1)\{(RF+FR)F^*F + F^2F^*R - FF^*(RF+FR) - RF^*F^2\} \end{aligned}$$

$$\begin{aligned} X_0 = {} & (RF^2 + FRF + F^2R)L - L(RF^2 + FRF + F^2R) + F^3F^* - F^*F^3 \\ & - (\beta+1)\{(RF+FR)LF + F^2LR - FL(RF+FR) - RLF^2\} \\ & - (\beta+1)(F^2F^*F - FF^*F^2) \end{aligned}$$

$$X_{-1} = F^3L - LF^3 - (\beta+1)(F^2LF - FLF^2)$$

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$$\gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A) = Y_2 + Y_1 + Y_0 + Y_{-1}$$

$$Y_2 = \gamma(R^2F^* - F^*R^2)$$

$$Y_1 = \gamma(R^2L - LR^2) + \gamma\{(RF+FR)F^* - F^*(RF+FR)\} + \delta(RF^* - F^*R)$$

$$Y_0 = \gamma\{(RF+FR)L - L(RF+FR)\} + \gamma(F^2F^* - F^*F^2) + \delta(RL - LR) + \delta(FF^* - F^*F)$$

$$Y_{-1} = \gamma(F^2L - LF^2) + \delta(FL - LF)$$

on U_i

$X_3 = 0$ ok by $\{\theta_i^*\}_{i\in\mathbb{Z}}$ β-sequence

$X_2 = Y_2 \iff R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2$

$X_1 = Y_1$ ok by $\{\theta_i\}_{i\in\mathbb{Z}}$ (β, γ)-seq.

$X_0 = Y_0$ ok by $\{\theta_i\}_{i\in\mathbb{Z}}$ (β, γ, δ)-seq.

$X_{-1} = Y_{-1}$ ok by $\{\theta_i\}_{i\in\mathbb{Z}}$ (β, γ, δ)-seq.

(2) by the same argument. $\blacksquare$

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Theorem

$(A, A^* ; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ pre TD-system

$A = R + \sum_{i=0}^d \theta_i F_i$,

$A^* = L + \sum_{i=0}^d \theta_i^* F_i$.

Assume $\{\theta_i\}_{i\in\mathbb{Z}}$ is a (β, γ, δ)-sequence,

$\{\theta_i^*\}_{i\in\mathbb{Z}}$ is a $(\beta, \gamma^*, \delta^*)$-sequence.

Set $\beta = q + q^{-1}$.

Then

(1) (TD) $A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)$

$\Rightarrow$ $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$, $0 \le i \le d$,

$V_{d+1} = 0$, $V_{-1} = \begin{cases} V_d & \text{if } q^{d+1} = 1,\ q \ne \pm 1 \\ 0 & \text{otherwise} \end{cases}$

(2) $(\mathrm{TD})^*$ $A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*)$

$\Rightarrow$ $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$, $0 \le i \le d$,

$V_{-1}^* = 0$, $V_{d+1}^* = \begin{cases} V_0^* & \text{if } q^{d+1} = 1,\ q \ne \pm 1 \\ 0 & \text{otherwise} \end{cases}$

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Proof

(1) $\quad V_i \ni v$

$$\left(A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3\right)v = \left(A^3 - (\beta+1)(\theta_i A^2 - \theta_i^2 A) - \theta_i^3\right)A^*v$$
$$= (A-\theta_i)(A^2 - \beta\theta_i A + \theta_i^2)A^*v$$
$$\left(\gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A)\right)v = \left(\gamma(A^2-\theta_i^2) + \delta(A-\theta_i)\right)A^*v$$
$$= (A-\theta_i)\left(\gamma(A+\theta_i)+\delta\right)A^*v$$

$-)$

$$0 = (A-\theta_i)\left(A^2 - (\beta\theta_i+\gamma)A + \theta_i^2 - \gamma\theta_i - \delta\right)A^*v$$

$$\beta\theta_i + \gamma = \beta\theta_i + (\theta_{i+1} - \beta\theta_i + \theta_{i-1}) = \theta_{i+1} + \theta_{i-1}$$
$$\theta_i^2 - \gamma\theta_i - \delta = \theta_i^2 - \gamma\theta_i - \left(\theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 - \gamma(\theta_{i+1}+\theta_i)\right)$$
$$= \theta_{i+1}(\gamma - \theta_{i+1} + \beta\theta_i) = \theta_{i+1}\theta_{i-1}$$

So $\quad 0 = (A-\theta_{i+1})(A-\theta_i)(A-\theta_{i-1})A^*v$

$1 \le i \le d-1, \quad A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$

$i = d, \quad A^*V_d \subseteq V_{d-1} + V_d \qquad$ as $\quad A^* = L + \sum_{i=0}^{d} \theta_i^* F_i$

$i = 0, \quad A^*V_0 \subseteq V_{-1} + V_0 + V_1$

where $\quad V_{-1} = \{ v \in V \mid Av = \theta_{-1} v \}$.

θ_{-1} is an eigenvalue of $A \iff q^{d+1} = 1, \quad q \neq \pm 1$

$\theta_{-1} = \theta_d$

type I $\quad \theta_i = a + bq^i + cq^{-i}$

type II $\quad \theta_i = a + bi + ci^2$

type III $\quad \theta_i = a + b(-1)^i + c(-1)^i i$

and $\theta_i \neq \theta_j$, $\; i \neq j \in \{0, 1, \cdots, d\}$

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§ 4.4 Terwilliger's Lemma

$A, A^* \in \mathrm{End}(V)$ diagonalizable

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomposition of A

$A|_{V_i} = \theta_i$ eigenvalue of A on V_i

$V = \bigoplus_{i=0}^{d} V_i^*$ eigenspace decomposition of A^*

$A^*|_{V_i^*} = \theta_i^*$ eigenvalue of A^* on V_i^*

$E_i : V = \bigoplus_{\nu=0}^{d} V_\nu \longrightarrow V_i$ projection

$E_i^* : V = \bigoplus_{\nu=0}^{d} V_\nu^* \longrightarrow V_i^*$ projection

If $(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ is a TD-system,

$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_{-1}^* = V_{d+1}^* = 0$

$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \leq i \leq d, \quad V_{-1} = V_{d+1} = 0.$

If $(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ is a pre TD-system,

$A V_i^* \subseteq V_0^* + \cdots + V_{i+1}^*, \quad 0 \leq i \leq d, \quad V_{d+1}^* = 0$

$A^* V_i \subseteq V_{i-1} + \cdots + V_d, \quad 0 \leq i \leq d, \quad V_{-1} = 0.$

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Terwilliger's Lemma

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\iff$

$E_0^*(A - AE_0^*)(A^* - \theta_1^*) = 0$

(2) $A^*(V_0 + \cdots + V_{d-2}) \subseteq V_0 + \cdots + V_{d-1}$

$\iff$

$E_d(A^* - A^*E_d)(A - \theta_{d-1}) = 0$

Proof

(1) $A(V_2^* + \cdots + V_d^*) \subseteq V_1^* + \cdots + V_d^*$

$\iff$

$E_0^* A V_i^* = 0, \quad 2 \le i \le d$

$\iff$

$E_0^* A E_i^* = 0, \quad 2 \le i \le d$

$\iff$

$E_0^*(A - AE_0^*)(A^* - \theta_1^*) = 0$

$\blacksquare$