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3rd lecture

§3 TD-pairs : weight space decomposition

V: finite dim vector space $/\mathbb{C}$

$A, A^* \in \mathrm{End}(V)$ diagonalizable

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomp. of A

$V = \bigoplus_{i=0}^{d^*} V_i^*$ eigenspace decomp. of A^*

A, A^* : <u>TD-pair</u> (tridiagonal pair)

if

(i) $\exists$ <u>standard</u> ordering $V_0, V_1, \cdots, V_d$

$\exists$ <u>standard</u> ordering $V_0^*, V_1^*, \cdots, V_{d^*}^*$

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d^*, \quad V_{-1}^* = V_{d^*+1}^* = 0$$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$$

and

(ii) V is <u>irreducible</u> as an $\langle A, A^* \rangle$-module

$V \supseteq W$ A-inv., A^*-inv. subspace

$\Longrightarrow$ $W = V$ or 0.

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Remark

(1) $d = d^*$ holds. <u>diameter</u>

<u>trivial</u> TD-pair if $d=0$.

We assume $d \geq 1$ unless otherwise stated.

(2) $A, A^* \in \mathrm{End}(V)$: <u>L-pair</u> (Leonard pair)

if $\dim V_i = \dim V_i^* = 0$, $0 \leq i \leq d$.

(3) $V_0, V_1, \cdots, V_d$: standard

$\Longrightarrow$

$V_d, V_{d-1}, \cdots, V_0$: standard

and no other standard orderings for the eigenspaces of A.

The same holds for the eigenspaces of A^*.

$$(A, A^* ; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}) \quad \underline{\text{TD-system}}$$

standard ordering

3 more TD-systems

$$(\{V_{d-i}\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$$

$$(\{V_i\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$$

$$(\{V_{d-i}\}_{i=0}^{d}, \{V_{d-i}^*\}_{i=0}^{d})$$

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In what follows, we fix a TD-system
$$\left(A, A^*; \{V_i\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d}\right).$$

Set
$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \qquad 0 \le i \le d$$

<u>weight space</u>

$$U_0 = V_0^*, \qquad U_d = V_d$$

θ_i: eigenvalue of A on V_i

θ_i^*: eigenvalue of A^* on V_i^*

<u>Lemma</u>

$$(A - \theta_i) U_i \subseteq U_{i+1}, \qquad 0 \le i \le d, \qquad U_{d+1} = 0$$
$$(A^* - \theta_i^*) U_i \subseteq U_{i-1}, \qquad 0 \le i \le d, \qquad U_{-1} = 0$$

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<u>Proposition</u>

(1) $V = \bigoplus_{i=0}^{d} U_i$ <u>weight space decomposition</u>
(split decomposition)

(2) $U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*, \quad 0 \le i \le d$

$U_i + \cdots + U_d = V_i + \cdots + V_d, \quad 0 \le i \le d$

(3) $\dim U_i = \dim V_i = \dim V_i^*, \quad 0 \le i \le d$

(4) $\dim U_i = \dim U_{d-i}, \quad 0 \le i \le d$

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<u>Proof</u>

(1) $W = U_0 + U_1 + \cdots + U_d$ is $\langle A, A^* \rangle$-inv. by Lemma

$\supseteq U_0 = V_0^* \neq 0$

So $W = V$ by the irreducibility of V

$(U_0 + \cdots + U_i) \cap (U_{i+1} + \cdots + U_d)$

$\subseteq (V_0^* + \cdots + V_i^*) \cap (V_{i+1} + \cdots + V_d)$

$=: U_i'$

$W' = U_0' + U_1' + \cdots + U_{d-1}'$ is $\langle A, A^* \rangle$-inv.

$\subseteq V_0^* + \cdots + V_{d-1}^* \neq V$

$(A - \theta_{i+1}) U_i' \subseteq U_{i+1}'$, $(A - \theta_d) U_{d-1}' = 0$

$(A^* - \theta_i^*) U_i' \subseteq U_{i-1}'$, $(A^* - \theta_0^*) U_0' = 0$

So $W' = 0$

$(U_0 + \cdots + U_i) \cap (U_{i+1} + \cdots + U_d) = 0$

(2) $U_0 + \cdots + U_i \subseteq V_0^* + \cdots + V_i^*$

$$V_0^* + \cdots + V_i^* = (A^* - \theta_{i+1}^*) \cdots (A^* - \theta_d^*) V$$
$$\subseteq U_0 + \cdots + U_i \quad \text{by Lemma}$$
$$+ \quad V = \bigoplus_{j=0}^{d} U_j$$

$U_i + \cdots + U_d \subseteq V_i + \cdots + V_d$

$$V_i + \cdots + V_d = (A - \theta_{i-1}) \cdots (A - \theta_0) V$$
$$\subseteq U_i + \cdots + U_d \quad \text{by Lemma}$$
$$+ \quad V = \bigoplus_{j=0}^{d} U_j$$

(3) $U_i \simeq U_0 + \cdots + U_i / U_0 + \cdots + U_{i-1} = V_0^* + \cdots + V_i^* / V_0^* + \cdots + V_{i-1}^* \simeq V_i^*$

$U_i \simeq U_i + \cdots + U_d / U_{i+1} + \cdots + U_d = V_i + \cdots + V_d / V_{i+1} + \cdots + V_d \simeq V_i$

(4) $(A, A^* ; \{V_{d-i}\}_{i=0}^{d}, \{V_i^*\}_{i=0}^{d})$ TD-system

$$\tilde{U}_i = (V_0^* + \cdots + V_i^*) \cap (V_{d-i} + \cdots + V_0)$$

Then by (3)

$$\begin{array}{ccccc} \dim \tilde{U}_i & = & \dim V_i^* & = & \dim V_{d-i} \\ & & \| & & \| \\ & & \dim U_i & & \dim U_{d-i} \end{array}$$

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$$V = \bigoplus_{i=0}^{d} U_i \qquad \text{w. s. decomp.}$$

$$F_i : V \longrightarrow U_i \qquad \text{projection}$$

$$R = A - \sum_{i=0}^{d} \theta_i F_i \qquad \underline{\text{raising map}}$$

θ_i : eigenvalue of A on V_i

$$L = A^* - \sum_{i=0}^{d} \theta_i^* F_i \qquad \underline{\text{lowering map}}$$

θ_i^* : eigenvalue of A^* on V_i^*

$$\begin{cases} R U_i \subseteq U_{i+1}, & 0 \le i \le d, \quad U_{d+1} = 0 \\ L U_i \subseteq U_{i-1}, & 0 \le i \le d, \quad U_{-1} = 0 \end{cases}$$

<u>Proof</u>

$$R|_{U_i} = A - \theta_i|_{U_i}$$

$$L|_{U_i} = A^* - \theta_i^*|_{U_i}$$

The result follows from Lemma. $\blacksquare$

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Proposition

(1) $R^{d-2i} : U_i \longrightarrow U_{d-i}$ bijection, $0 \le i \le \frac{d}{2}$

(2) $L^{d-2i} : U_{d-i} \longrightarrow U_i$ bijection, $0 \le i \le \frac{d}{2}$

Corollary

$\dim U_0 \le \dim U_1 \le \dim U_2 \le$ ----- $\ge \dim U_{d-1} \ge \dim U_d$

Theorem* (shape conjecture)

$\dim U_0 = 1$.

Proof of Proposition

(1)
$$R^{d-2i}\Big|_{U_i} = (A-\theta_{d-i-1})\cdots(A-\theta_{i+1})(A-\theta_i)\Big|_{U_i}$$

$$\ker R^{d-2i}\Big|_{U_i} = U_i \cap (V_{d-i-1}+\cdots+V_{i+1}+V_i)$$
$$\subseteq (V_0^*+\cdots+V_i^*)\cap(V_{d-i-1}+\cdots+V_0)$$
$$\underset{\text{claim}}{=} 0$$

Proof of claim

$$(A, A^*; \{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d) \quad \text{TD-system}$$

reversed ordering

$$V = \bigoplus_{i=0}^d \tilde{U}_i \qquad \text{w.s. decomp.}$$

where $\tilde{U}_i = (V_0^*+\cdots+V_i^*)\cap(V_{d-i}+\cdots+V_0)$

Then
$$V_0^*+\cdots+V_i^* = \tilde{U}_0+\cdots+\tilde{U}_i$$
$$V_{d-i}+\cdots+V_0 = \tilde{U}_i+\cdots+\tilde{U}_d,$$

$$(V_0^*+\cdots+V_i^*)\cap(V_{d-i-1}+\cdots+V_0)$$
$$= (\tilde{U}_0+\cdots+\tilde{U}_i)\cap(\tilde{U}_{i+1}+\cdots+\tilde{V}_d) = 0 \quad //$$

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So $\ker R^{d-2i}|_{U_i} = 0$.

The mapping is injective.

Since $\dim U_i = \dim U_{d-i}$, the surjectivity follows from the injectivity.

(2)

$$L^{d-2i}|_{U_{d-i}} = (A^* - \theta^*_{i+1}) \cdots (A^* - \theta^*_{d-i})|_{U_{d-i}}$$

$$\ker L^{d-2i}|_{U_{d-i}} = U_{d-i} \cap (V^*_{i+1} + \cdots + V^*_{d-i})$$

$$\subseteq (V_{d-i} + \cdots + V_d) \cap (V^*_{i+1} + \cdots + V^*_d)$$

$$= 0$$

claim

$(A, A^*; \{V_i\}_{i=0}^d, \{V^*_{d-i}\}_{i=0}^d)$ TD-system

reversed ordering

$$\tilde{U}_i = (V^*_d + \cdots + V^*_{d-i}) \cap (V_i + \cdots + V_d)$$

$$V = \bigoplus_{i=0}^d \tilde{U}_i$$

$$V^*_d + \cdots + V^*_{d-i} = \tilde{U}_0 + \cdots + \tilde{U}_i$$

$$V_i + \cdots + V_d = \tilde{U}_i + \cdots + \tilde{U}_d$$

$$(V_{d-i} + \cdots + V_d) \cap (V^*_{i+1} + \cdots + V^*_d)$$

$$= (\tilde{U}_{d-i} + \cdots + \tilde{U}_d) \cap (\tilde{U}_0 + \cdots + \tilde{U}_{d-i-1}) = 0$$

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$A, A^* \in \mathrm{End}(V)$ TD-pair
diameter d

$V = \bigoplus_{i=0}^{d} V_i$ eigenspace decomposition of A

$E_i : V \longrightarrow V_i$ projection

$$A = \sum_{i=0}^{d} \theta_i E_i , \qquad \theta_i : \text{eigenvalue of } A \text{ on } V_i$$

$$E_i = \prod_{\nu \neq i} \frac{A - \theta_\nu}{\theta_i - \theta_\nu}$$

$$I = E_0 + E_1 + \cdots + E_d , \qquad E_i E_j = \delta_{ij} E_i$$

$V = \bigoplus_{i=0}^{d} V_i^*$ eigenspace decomposition of A^*

$E_i^* : V \longrightarrow V_i^*$ projection

$$A^* = \sum_{i=0}^{d} \theta_i^* E_i^* , \qquad \theta_i^* : \text{eigenvalue of } A^* \text{ on } V_i^*$$

$$E_i^* = \prod_{\nu \neq i} \frac{A^* - \theta_\nu^*}{\theta_i^* - \theta_\nu^*}$$

$$I = E_0^* + E_1^* + \cdots + E_d^* , \qquad E_i^* E_j^* = \delta_{ij} E_i^*$$

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$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$$

$$E_j^* A E_i^* \begin{cases} = 0 & \text{if } |j-i| > 1 \\ \neq 0 & \text{if } |j-i| = 1 \end{cases}$$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$$

$$E_j A^* E_i \begin{cases} = 0 & \text{if } |j-i| > 1 \\ \neq 0 & \text{if } |j-i| = i \end{cases}$$

<u>Proposition</u> $\ell = |d - 2i|, \quad 0 \le i \le d$

(1) $E_{d-i}^* A^\ell E_i^* : V_i^* \longrightarrow V_{d-i}^*$ bijection

(2) $E_{d-i} A^{*\ell} E_i : V_i \longrightarrow V_{d-i}$ bijection

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Proof

(1) Case $d-i \geq i$

Set $W_i = V_0^* + \cdots + V_i^* = U_0 + \cdots + U_i$.

Then $V_i^* \simeq W_i / W_{i-1} \simeq U_i$

$$E_{i+1}^* A E_i^* : V_i^* \longrightarrow V_{i+1}^*$$
$$\simeq$$
$$A : W_i / W_{i-1} \longrightarrow W_{i+1} / W_i$$
$$\simeq$$
$$F_{i+1} A F_i : U_i \longrightarrow U_{i+1}$$
$$\|$$
$$F_{i+1} R F_i$$

Since $F_{d-i} A^\ell F_i : U_i \longrightarrow U_{d-i}$ is bijective,

so is $E_{d-i}^* A^\ell E_i^* : V_i^* \longrightarrow V_{d-i}^*$.

Case $d-i < i$

Apply the same argument to the TD-system

$$(A, A^* ; \{V_i\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$$

reversed ordering

(2) The same argument as in (1). $\square$

Corollary

(1) $E_j^* A^\ell E_i^* = 0 \quad \text{if } |j-i| > \ell$

$\qquad\qquad\quad \neq 0 \quad \text{if } |j-i| = \ell$

(2) $E_j A^{*\ell} E_i = 0 \quad \text{if } |j-i| > \ell$

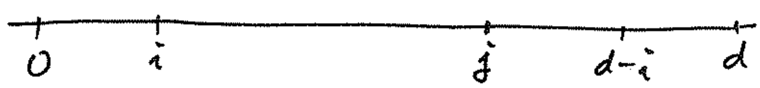
$\qquad\qquad\quad \neq 0 \quad \text{if } |j-i| = \ell$

Proof

(1) <u>case</u> $\ell = j-i$, $\quad d-i \geq j \geq i$

$$0 \neq E_{d-i}^* A^{d-2i} E_i^* = E_{d-i}^* A^{d-i-j} E_j^* A^\ell E_i^*$$

So $E_j^* A^\ell E_i^* \neq 0$.



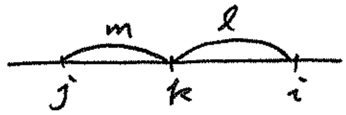
Other cases are proved in the same way.

∎

Lemma

(1) $$E_j^* A^m E_k^* A^\ell E_i^* = E_j^* A^{m+\ell} E_i^*$$

if $k = j - m = i + \ell$ or $k = j + m = i - \ell$.



(2) $$E_j A^{*m} E_k A^{*\ell} E_i = E_j A^{*m+\ell} E_i$$

if $k = j - m = i + \ell$ or $k = j + m = i - \ell$

Proof

(1) $$E_j^* A^{m+\ell} E_i^* = E_j^* A^m (E_0^* + E_1^* + \cdots + E_d^*) A^\ell E_i^*$$
$$= E_j^* A^m E_k^* A^\ell E_i^*$$

$\square$