No.
Date

2nd lecture — L-pairs and dual system of orth. polys.

§2.1 orthogonal polynomials / $\mathbb{C}$ with finite support

$u_0(x) = 1,\ u_1(x),\ \cdots,\ u_d(x) \in \mathbb{C}[x]$

$\deg u_i(x) = i, \quad 0 \le i \le d$

<u>orthogonal polynomials</u> with support $\theta_0, \theta_1, \cdots, \theta_d \in \mathbb{C}$ $(\theta_i \neq \theta_j)$

if
$$\sum_{\nu=0}^{d} u_i(\theta_\nu)\, u_j(\theta_\nu)\, \mu_\nu = \frac{1}{k_i}\delta_{ij}, \qquad \text{all } i, j$$

for some $\mu_0, \mu_1, \cdots, \mu_d \in \mathbb{C}$

$\mu_\nu \neq 0,\ 0 \le \nu \le d, \qquad \sum_{\nu=0}^{d} \mu_\nu = 1$

and $k_0, k_1, \cdots, k_d \in \mathbb{C}$

$k_i \neq 0,\ 0 \le i \le d, \qquad k_0 = 1.$

No.
Date

$\{u_i(x)\}_{i=0}^{d}$ orth. polys. with support $\{\theta_\nu\}_{\nu=0}^{d}$

$\Longrightarrow$

$$(*) \qquad x\,u_i(x) = c_i\,u_{i-1}(x) + a_i\,u_i(x) + b_i\,u_{i+1}(x), \qquad 0 \le i \le d$$

where $u_{-1}(x) = 0$, $b_d = 1$

$$u_{d+1}(x) = \frac{1}{b_0 b_1 \cdots b_{d-1}} (x-\theta_0)(x-\theta_1)\cdots(x-\theta_d)$$

$$B = \begin{pmatrix} a_0 & b_0 & & & 0 \\ c_1 & a_1 & b_1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & c_d & a_d \end{pmatrix}$$

<u>tridiagonal</u> matrix

<u>irreducible</u>

$c_i \neq 0$, $b_i \neq 0$ all i

<u>diagonalizable</u>

with eigenvalues $\{\theta_\nu\}_{\nu=0}^{d}$

No.
Date

Conversely

B : tridiagonal matrix
irreducible
diagonalizable with eigenvalues $\{\theta_\nu\}_{\nu=0}^{d}$

$\Longrightarrow$

$\{u_i(x)\}_{i=0}^{d}$ defined by $(*)$

orth. polys. with support $\{\theta_\nu\}_{\nu=0}^{d}$

$$k_0 = 1, \quad k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}, \quad 0 \le i \le d$$

$$\mu_\nu = \frac{1}{k_d\, u'_{d+1}(\theta_\nu)\, u_d(\theta_\nu)}, \qquad 0 \le \nu \le d$$

Christoffel number

Remark

B : tridiagonal, irreducible, diagonalizable

$\Longrightarrow$

Each eigenvalue of B has multiplicity 1.

No.
Date

$\mathcal{P} = \{$ orth. polys. with finite support $\}$

$\mathcal{M} = \{$ tridiagonal matrices that are irreducible and diagonalizable $\}$

$$\mathcal{P} \xleftrightarrow{1:1} \mathcal{M}$$

$$\mathcal{P} \supset \mathcal{P}(\theta_0) = \left\{ \{u_i(x)\} \in \mathcal{P} \;\middle|\; u_i(\theta_0) = 1 \quad \text{all } i \right\}$$

$$\mathcal{M} \supset \mathcal{M}(\theta_0) = \left\{ B \in \mathcal{M} \;\middle|\; \text{row sum of } B = \theta_0 \right\}$$

$$\mathcal{P}(\theta_0) \xleftrightarrow{1:1} \mathcal{M}(\theta_0)$$

No.
Date

§2.2 dual systems of orth. polys. and L-pairs

$\{u_i(x)\}_{i=0}^d$ orth. polys. with support $\{\theta_\nu\}_{\nu=0}^d$

$\{u_i^*(x)\}_{i=0}^d$ orth. polys. with support $\{\theta_\nu^*\}_{\nu=0}^d$

<u>dual</u>

if $\exists$ ordering $\theta_0, \theta_1, \cdots, \theta_d$
$\exists$ ordering $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$

$$u_j(\theta_i) = u_i^*(\theta_j^*) \quad \text{all } i, j$$

<u>Remark</u>

$$u_j(\theta_0) = u_0^*(\theta_j^*) = 1 \quad \text{all } j$$

$$u_i^*(\theta_0^*) = u_0(\theta_i) = 1 \quad \text{all } i$$

$\{u_i(x)\}_{i=0}^d \in \mathcal{P}(\theta_0)$, $\{u_i^*(x)\}_{i=0}^d \in \mathcal{P}(\theta_0^*)$

No.
Date

$$\mathcal{P}(\theta_0) \ni \{u_i(x)\}_{i=0}^{d} \longleftrightarrow B \in \mathcal{M}(\theta_0)$$

$$\mathcal{P}(\theta_0^*) \ni \{u_i^*(x)\}_{i=0}^{d} \longleftrightarrow B^* \in \mathcal{M}(\theta_0^*)$$

Proposition

$\{u_i(x)\}_{i=0}^{d}$, $\{u_i^*(x)\}_{i=0}^{d}$ are dual

$\Longleftrightarrow$

$\exists\, S$ non singular matrix

$$S^{-1} B S = \begin{pmatrix} \theta_0 & & 0 \\ & \ddots & \\ 0 & & \theta_d \end{pmatrix} = D \quad \text{diagonal matrix}$$

$$S B^* S^{-1} = \begin{pmatrix} \theta_0^* & & 0 \\ & \ddots & \\ 0 & & \theta_d^* \end{pmatrix} = D^* \quad \text{diagonal matrix}$$

$B,\ D^* \in M_{d+1}(\mathbb{C}) \simeq \mathrm{End}(V), \quad V = \mathbb{C}^{d+1}$

eigenspace decomposition of B

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_d, \qquad \dim V_i = 1, \quad 0 \le i \le d$$

$$V_i = \langle \mathbb{S}_i \rangle, \qquad B\mathbb{S}_i = \theta_i \mathbb{S}_i$$

$$S = (\mathbb{S}_0, \mathbb{S}_1, \cdots, \mathbb{S}_d)$$

eigenspace decomposition of D^*

$$V = V_0^* \oplus V_1^* \oplus \cdots \oplus V_d^*, \qquad \dim V_i^* = 1, \quad 0 \le i \le d$$

$$V_i^* = \langle e_i \rangle, \qquad D^* e_i = \theta_i^* e_i$$

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} (i$$

(i) $B V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0$

B : tridiagonal

$D^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0$

$B^* = S^{-1} D^* S$: tridiagonal

(ii) V : irreducible as a $\langle B, D^* \rangle$-module

B : irreducible tridiagonal

No.
Date

$A, A^* \in \mathrm{End}(V)$ diagonalizable, $V \simeq \mathbb{C}^{d+1}$

$$V = \bigoplus_{i=0}^{d} V_i, \quad \dim V_i = 1, \quad 0 \le i \le d$$

$$A|_{V_i} = \theta_i \in \mathbb{C}, \quad \theta_i \neq \theta_j$$

eigenspace decomposition of A

$$V = \bigoplus_{i=0}^{d} V_i^*, \quad \dim V_i^* = 1, \quad 0 \le i \le d$$

$$A^*|_{V_i^*} = \theta_i^* \in \mathbb{C}, \quad \theta_i^* \neq \theta_j^*$$

eigenspace decomposition of A^*

<u>L-pair</u> (Leonard pair)

if $\exists$ ordering $\theta_0, \theta_1, \cdots, \theta_d$

$\exists$ ordering $\theta_0^*, \theta_1^*, \cdots, \theta_d^*$

$$\begin{cases} A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \quad 0 \le i \le d, \quad V_{-1}^* = V_{d+1}^* = 0 \\ A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \quad 0 \le i \le d, \quad V_{-1} = V_{d+1} = 0 \end{cases}$$

and

V is irreducible as an $\langle A, A^* \rangle$-module.

$A_1,\ A_1^* \in \mathrm{End}(V)$ L-pair

$A_2,\ A_2^* \in \mathrm{End}(V)$ L-pair

$(A_1, A_1^*) \simeq (A_2, A_2^*)$ isomorphic

If $\exists\, \phi : V \longrightarrow V$ Isomorphism as vector spaces

$$\begin{tikzcd} V \arrow[r, "{A_1, A_1^*}"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\phi"] \\ V \arrow[r, "{A_2, A_2^*}"'] & V \end{tikzcd}$$

$$A_2 = \phi A_1 \phi^{-1}$$
$$A_2^* = \phi A_2 \phi^{-1}$$

No.
Date

$$\mathcal{P}(\theta_0) \ni \{u_i(x)\}_{i=0}^{d} \longleftrightarrow B \in \mathcal{M}(\theta_0)$$

$$\mathcal{P}(\theta_0^*) \ni \{u_i^*(x)\}_{i=0}^{d} \longleftrightarrow B^* \in \mathcal{M}(\theta_0^*)$$

$\{u_i(x)\}_{i=0}^{d}$, $\{u_i^*(x)\}_{i=0}^{d}$: dual

$\Longleftrightarrow$

B, D^* : L-pair

In this case,

$(B, D^*) \simeq (D, B^*)$ isomorphic as L-pairs

<u>Remark</u>

$A, A^* \in \mathrm{End}(V)$ L-pair

Then

$\exists$ a basis of V, the matrix B of A $\in \mathcal{M}(\theta_0)$
the matrix D^* of A^* is diagonal

and

$\exists$ a basis of V, the matrix D of A is diagonal
the matrix B^* of A^* $\in \mathcal{M}(\theta_0^*)$

No.
Date

§2.3 Leonard's theorem — classification of dual systems of orth. polys.

$$(a)_i = a(a+1)\cdots(a+i-1), \qquad i=1,2,\cdots$$
$$(a)_0 = 1$$

<u>shifted factorial</u>

$${}_{r+1}F_r\left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix} ; x\right) = \sum_{i=0}^{\infty} \frac{(a_1)_i \cdots\cdots (a_{r+1})_i}{(b_1)_i \cdots\cdots (b_r)_i} \frac{x^i}{i!}$$

<u>hypergeometric series</u>

$$(a;q)_i = (1-a)(1-aq)\cdots(1-aq^{i-1}), \qquad i=1,2,\cdots$$
$$(a;q)_0 = 1$$

<u>q-shifted factorial</u>

$${}_{r+1}\phi_r\left(\begin{matrix} a_1, \cdots, a_{r+1} \\ b_1, \cdots, b_r \end{matrix} ; q, x\right) = \sum_{i=0}^{\infty} \frac{(a_1;q)_i \cdots\cdots (a_{r+1};q)_i}{(b_1;q)_i \cdots\cdots (b_r;q)_i} \frac{x^i}{(q;q)_i}$$

<u>basic hypergeometric series</u>

(q-hypergeometric series)

$$q \longrightarrow 1, \qquad {}_{r+1}\phi_r \longrightarrow {}_{r+1}F_r$$

No.
Date

<u>Theorem</u> (Leonard 1982, Bannai - Ito 1984)

$\{u_i(x)\}_{i=0}^d$, $\{u_i^*(x)\}_{i=0}^d$: a dual system of orthogonal polynomials with support $\{\theta_\nu\}_{\nu=0}^d$, $\{\theta_\nu^*\}_{\nu=0}^d$, respectively

$\Longleftrightarrow$

$\{u_i(x)\}_{i=0}^d$, $\{u_i^*(x)\}_{i=0}^d$: a dual system of q-Racah polynomials or its limiting case

$$u_i(x) = {}_4\phi_3\left(\begin{matrix} q^{-i}, & s^* q^{i+1}, & q^{-y}, & s q^{y+1} \\ & q^{-d}, & r_1 q, & r_2 q \end{matrix} ; q, q\right)$$

$$x = \xi(y) = \theta_0 + h \frac{1}{q^y}(1-q^y)(1-sq^{y+1})$$

$$\theta_i = \xi(i), \quad 0 \le i \le d$$

$$u_i^*(x) = {}_4\phi_3\left(\begin{matrix} q^{-i}, & s q^{i+1}, & q^{-y}, & s^* q^{y+1} \\ & q^{-d}, & r_1 q, & r_2 q \end{matrix} ; q, q\right)$$

$$x = \xi^*(y) = \theta_0^* + h^* \frac{1}{q^y}(1-q^y)(1-s^* q^{y+1})$$

$$\theta_i^* = \xi^*(y), \quad 0 \le i \le d$$

(details in 6th lecture)

No.
Date

§ 2.4 Terwilliger's Theorem — Classification of L-pairs

start with

$A, A^* \in \mathrm{End}(V)$, $\quad V \simeq \mathbb{C}^{d+1}$

L-pair

$$V = \bigoplus_{i=0}^{d} V_i \qquad \text{e. s. decomp. of } A, \quad \dim V_i = 1$$

$$= \bigoplus_{i=0}^{d} V_i^* \qquad \text{e. s. decomp. of } A^*, \quad \dim V_i^* = 1$$

$$\begin{cases} A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, & V_{-1}^* = V_{d+1}^* = 0 \\ A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, & V_{-1} = V_{d+1} = 0 \end{cases}$$

Set

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d), \qquad 0 \le i \le d$$

<u>weight space</u>

Then

$$V = \bigoplus_{i=0}^{d} U_i \qquad \dim U_i = 1, \qquad 0 \le i \le d$$

<u>weight space decomposition</u>
(split decomposition)

$$F_i : V \longrightarrow U_i \qquad \text{projection}$$

$$R = A - \sum_{i=0}^{d} \theta_i F_i , \qquad \theta_i : \text{eigenvalue of } A \text{ on } V_i$$

$$L = A^* - \sum_{i=0}^{d} \theta_i^* F_i , \qquad \theta_i^* : \text{eigenvalue of } A^* \text{ on } V_i^*$$

Then

$$\begin{cases} R U_i = U_{i+1} , & 0 \le i \le d , \quad U_{d+1} = 0 \\ L U_i = U_{i-1} , & 0 \le i \le d . \quad U_{-1} = 0 \end{cases}$$

R <u>raises</u> $\{U_i\}_{i=0}^{d}$

L <u>lowers</u> $\{U_i\}_{i=0}^{d}$

No.
Date

Conversely, start with

$$V = \bigoplus_{i=0}^{d} U_i, \qquad \dim U_i = 1$$

$$F_i : V \longrightarrow U_i \qquad \text{projection}.$$

Given

$$\begin{cases} \theta_0, \theta_1, \cdots, \theta_d \in \mathbb{C}, & \theta_i \neq \theta_j \\ \theta_0^*, \theta_1^*, \cdots, \theta_d^* \in \mathbb{C}, & \theta_i^* \neq \theta_j^* \end{cases}$$

and

$R, L \in \mathrm{End}(V)$ raising, lowering maps

$$\begin{cases} R U_i = U_{i+1}, & 0 \le i \le d, \quad U_{d+1} = 0 \\ L U_i = U_{i-1}, & 0 \le i \le d, \quad U_{-1} = 0 \end{cases}$$

Set

$$\begin{cases} A = R + \sum_{i=0}^{d} \theta_i F_i \\ A^* = L + \sum_{i=0}^{d} \theta_i^* F_i . \end{cases}$$

Such $A, A^* \in \mathrm{End}(V)$ are called a <u>pre L-pair</u>.

No.
Date

Choose $u_0 \in U_0$, $u_0 \neq 0$

Set $u_i = R^i u_0$.

Then
$$U_i = \mathbb{C} u_i, \quad 0 \le i \le d$$

and
$$L u_{i+1} = \lambda_i u_i, \quad 0 \le i \le d-1$$

for some $\lambda_i \in \mathbb{C}^\times = \mathbb{C} - \{0\}$.

The <u>data</u> $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1} \right)$

determines the isomorphism class of a pre L-pair $A, A^* \in End(V)$.

If $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$, $i \neq j \in \{0, 1, \cdots, d\}$

and $\lambda_i \neq 0$, $0 \le i \le d-1$,

then $\left(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1} \right)$

is the data of some pre L-pair A, A^*.

No.
Date

<u>Theorem</u> (Terwilliger 2001)

$A, A^* \in \mathrm{End}(V)$ pre L-pair

with data $(\{\theta_i\}_{i=0}^{d}, \{\theta_i^*\}_{i=0}^{d}, \{\lambda_i\}_{i=0}^{d-1})$.

Then

A, A^* are an L-pair

$\Longleftrightarrow$

$$\theta_i = \theta_0 + h\frac{1}{q^i}(1-q^i)(1-sq^{i+1}), \quad 0 \le i \le d$$

$$\theta_i^* = \theta_0^* + h^*\frac{1}{q^i}(1-q^i)(1-s^*q^{i+1}), \quad 0 \le i \le d$$

$$\lambda_i = hh^* q^{-2i-1}(1-q^{i+1})(1-q^{i-d})(1-r_1q^{i+1})(1-r_2q^{i+1}), \quad 0 \le i \le d-1$$

for some $r_1, r_2, s, s^*, h, h^*, \theta_0, \theta_0^*$ and $q \in \mathbb{C}$

$r_1 r_2 = ss^* q^{d+1}$

$hh^* \neq 0$, $q^i \neq 1$, $1 \le i \le d$

$s, s^* \notin \{q^{-2}, q^{-3}, \cdots, q^{-2d}\}$

$r_1, r_2 \notin \{q^{-1}, q^{-2}, \cdots, q^{-d}\} \cup \{s^*q, s^*q^2, \cdots, s^*q^d\}$ if $s^* \neq 0$

$r_2 = 0$, $r_1 \notin \{q^{-1}, q^{-2}, \cdots, q^{-d}\} \cup \{sq, sq^2, \cdots, sq^d\}$ if $s^* = 0$

or the limiting case

(details in 5th lecture)