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Leonard pairs
and
the q-Racah polynomials
— Terwilliger's theory

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Leonard pairs and the q-Racah polynomials
— Terwilliger's theory

1. Background in algebraic combinatorics
2. L-pairs and dual system of orthogonal polynomials
3. TD-pairs: weight space decomposition
4. TD-pairs: TD-relations and Terwilliger's lemma
5. Classification of L-pairs
6. Classification of dual systems of orth. polynomials

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1st lecture Background in algebraic combinatorics

$\Gamma = (X, R)$ simple graph

X : point set

R : edge set

$R \subset X \times X$ no multiple edges

$R = {}^tR = \{(x, y) \mid (y, x) \in R\}$

undirected graph

$R \cap \Delta = \phi$

no loops

$\Delta = \{(x, x) \mid x \in X\}$

$|X| < \infty$ finite graph

$\partial(x, y)$ = the length of shortest paths joining x and y

Assume Γ is connected.

$d = \mathrm{Max}\{\partial(x, y) \mid x, y \in X\} < \infty$

the diameter

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§1.1

Examples of 'nice' graphs

(1) Johnson graph $J(v, k)$, $v \geq 2k$

V: finite set, $|V| = v$

$X = \binom{V}{k}$: the set of k-subsets of V

$R \ni (x, y) \iff |x \cap y| = k-1$

$d = k$ (diameter)

(2) q- Johnson graph $J_q(v, k)$, $v \geq 2k$

V: v-dim vector space $/ \mathbb{F}_q$

$X = \binom{V}{k}$: the set of k-dim subspaces of V

$R \ni (x, y) \iff \dim x \cap y = k-1$

$d = k$ (diameter)

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(3) Hamming graph $H(n, q)$

F : finite set, $|F| = q$

$X = F^n = F \times F \times \cdots \times F \ni x = (x_1, x_2, \cdots, x_n)$

$y = (y_1, y_2, \cdots, y_n)$

$R \ni (x, y) \iff \# \{ i \mid x_i \neq y_i \} = 1$

$d = n$ (diameter)

(4) Bilinear forms graph $Bil_{m \times n}(q)$

X : the set of $m \times n$ matrices $/ \mathbb{F}_q$

$R \ni (x, y) \iff \mathrm{rank}(x - y) = 1$

$d = \mathrm{Min}\{m, n\}$ (diameter)

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§1.2

How nice?

$\Gamma = (X, R)$: finite, connected simple graph
diameter d

A : the adjacency matrix of Γ

$$A(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

A_i : ith distance matrix

$$A_i(x,y) = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$A = A_1$$

1st nice property

$\exists\, v_i(x) \in \mathbb{R}[x]$ polynomial of degree i

s.t. $A_i = v_i(A)$

distance-regular, P-polynomial

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$$V = \mathbb{R}^X = \{ f : X \longrightarrow \mathbb{R} \} \simeq \mathbb{R}^n, \qquad |X| = n$$

$$M_X(\mathbb{R}) = \{ B : X \times X \longrightarrow \mathbb{R} \} \simeq M_n(\mathbb{R})$$

$M_X(\mathbb{R})$ acts on $V = \mathbb{R}^X$

by $$(Bf)(x) = \sum_{y \in X} B(x,y) f(y)$$

$$M_X(\mathbb{R}) \supset \mathfrak{A} = \langle \underset{\underset{I}{\parallel}}{A_0}, \underset{\underset{A}{\parallel}}{A_1}, A_2, \cdots, A_d \rangle$$

<u>Bose - Mesner algebra</u>

$\dim \mathfrak{A} = d + 1$

$V = V_0 + V_1 + \cdots + V_d$ eigenspace decomp. of A

θ_i : eigenvalue of A on V_i

$E_i : V \longrightarrow V_i$ projection

$$A = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d$$

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$$M_X(\mathbb{R}) \supset \mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle$$
$$= \langle E_0, E_1, \cdots, E_d \rangle$$

$$\begin{cases} I = E_0 + E_1 + \cdots + E_d \\ E_i E_j = \delta_{ij} E_i \end{cases}$$ primitive idempotents

$M_X(\mathbb{R})^\circ$: the algebra $M_X(\mathbb{R})$ w.r.t. the entry-wise product $\circ$
(Hadamard product, Schur-product)

$$(B_1 \circ B_2)(x, y) = B_1(x, y) B_2(x, y)$$

$$M_X(\mathbb{R})^\circ \supset \mathfrak{A}^\circ = \langle E_0, E_1, \cdots, E_d \rangle$$ dual BM-alg.
$$= \langle A_0, A_1, \cdots, A_d \rangle$$

$$\begin{cases} J = A_0 + A_1 + \cdots + A_d \\ \text{all one matrix (the identity of } M_X(\mathbb{R})^\circ) \\ A_i \circ A_j = \delta_{ij} A_i \end{cases}$$

2nd nice property

$\exists\ v_i^*(x) \in \mathbb{R}[x]$ polynomial of degree i

s.t. $n E_i = v_i^*(n E_1)$, $n = |X|$

Q-polynomial in $\mathfrak{A}^\circ$.

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§1.3

P- and Q- polynomial scheme

$\Gamma(X, R)$: finite, connected simple graph
diameter d, $|X| = n$

$$M_X(\mathbb{R}) \supset \mathfrak{A} = \langle \underset{\overset{\|}{I}}{A_0}, \underset{\overset{\|}{A}}{A_1}, \cdots, A_d \rangle \quad \text{BM-alg.}$$

$$= \langle E_0, E_1, \cdots, E_d \rangle \quad \text{primitive idempotents}$$

$$A = A_1 = \theta_0 E_0 + \theta_1 E_1 + \cdots + \theta_d E_d$$

$$M_X(\mathbb{R})^\circ \supset \mathfrak{A}^\circ = \langle \underset{\overset{\|}{J}}{nE_0}, \underset{\overset{\|}{nE}}{nE_1}, \cdots, nE_d \rangle \quad \text{dual BM-alg}$$

$$= \langle A_0, A_1, \cdots, A_d \rangle$$

$$nE = nE_1 = \theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d$$

$\exists\ v_i(x),\ v_i^*(x) \in \mathbb{R}[x]$ polynomials of degree i
$(0 \leq i \leq d)$

s.t.

$$A_i = v_i(A)$$

$$nE_i = v_i^*(nE)$$

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Fact

$\Gamma(X, R)$: P- and Q-polynomial

$\Longrightarrow$

$\{v_i(x)\}_{i=0}^{d}$: orthogonal polynomial with support $\{\theta_0, \theta_1, \cdots, \theta_d\}$

$\{v_i^*(x)\}_{i=0}^{d}$: orthogonal polynomial with support $\{\theta_0^*, \theta_1^*, \cdots, \theta_d^*\}$

and

$$\frac{v_i(\theta_j)}{v_i(\theta_0)} = \frac{v_j^*(\theta_i^*)}{v_j^*(\theta_0^*)} \qquad \text{all } i, j$$

dual system of orthogonal polynomials

Leonard Theorem

$\{v_i\}_{i=0}^{d}$, $\{v_i^*\}_{i=0}^{d}$: dual system of orth. polys.

$\Longrightarrow$

$$\left\{\frac{1}{v_i(\theta_0)} v_i(x)\right\}_{i=0}^{d}, \quad \left\{\frac{1}{v_i^*(\theta_0^*)} v_i^*(x)\right\}_{i=0}^{d}$$

the q-Racah polynomials
(Askey - Wilson polynomials)

or
their limits

Leonard 1982

Bannai - Ito 1984

Terwilliger 2001

in terms of Leonard pairs

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§1.4

Terwilliger algebra

$\Gamma = (X, R)$ finite connected simple graph

P- and Q-polynomial

$X \ni x_0$ base point fixed

$$X_i = X_i(x_0) = \{ x \in X \mid \partial(x_0 x) = i \}$$

$$V_i^* = V_i^*(x_0) = \{ f : X \longrightarrow \mathbb{C} \mid f(x) = 0 \quad \text{if} \quad x \notin X_i \}$$

Then

$$V = \mathbb{C}^X = V_0^* \oplus V_1^* \oplus \cdots \oplus V_d^*$$

$$E_i^* = E_i^*(x_0) : \quad V \longrightarrow V_i^* \qquad \text{projection}$$

$$\mathfrak{A}^* = \mathfrak{A}^*(x_0) = \langle E_0^*, E_1^*, \cdots, E_d^* \rangle \subset M_X(\mathbb{C}) \simeq End(V)$$

$$\begin{cases} I = E_0^* + E_1^* + \cdots\cdots + E_d^* \\ E_i^* E_j^* = \delta_{ij} E_i^* \end{cases} \qquad \text{primitive idempotents}$$

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$$\mathfrak{A} = \langle A_0, A_1, \cdots, A_d \rangle \subset M_X(\mathbb{C}), \qquad \text{BM-alg}$$
$$= \langle E_0, E_1, \cdots, E_d \rangle$$

$$\mathfrak{A}^\circ = \langle nE_0, nE_1, \cdots, nE_d \rangle \subset M_X(\mathbb{C})^\circ, \qquad \text{dual BM-alg}$$
$$= \langle A_0, A_1, \cdots, A_d \rangle \qquad n = |X|$$

$$\begin{cases} J = A_0 + A_1 + \cdots + A_d \\ A_i \circ A_j = \delta_{ij} A_i \end{cases}$$

$$nE = nE_1 = \theta_0^* A_0 + \theta_1^* A_1 + \cdots + \theta_d^* A_d$$

$$\begin{array}{ccc} M_X(\mathbb{C})^\circ & & M_X(\mathbb{C}) \\ \cup & & \cup \\ \mathfrak{A}^\circ & \xrightarrow{\sim} & \mathfrak{A}^* \\ A_i & \longmapsto & E_i^* \\ nE & \longmapsto & A^* = \theta_0^* E_0^* + \theta_1^* E_1^* + \cdots + \theta_d^* E_d^* \\ nE_i = v_i^*(nE) & \longmapsto & A_i^* = v_i^*(A^*) \end{array}$$

dual BM-alg

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$$T = T(x_0) = \langle \mathfrak{A}, \mathfrak{A}^* \rangle \subset M_X(\mathbb{C})$$

<u>Terwilliger algebra</u>

$$T = \langle A, A^* \rangle$$

$$A_i = v_i(A) \qquad (A = A_1)$$

$$A_i^* = v_i^*(A^*) \qquad (A^* = A_1^*)$$

E_i : $V = \bigoplus_{j=0}^{d} V_j \longrightarrow V_i$ projection

V_i: ith eigenspace of A

E_i^* : $V = \bigoplus_{j=0}^{d} V_j^* \longrightarrow V_i^*$ projection

V_i^*: ith eigenspace of A^*

<u>Proposition</u>

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*, \qquad 0 \le i \le d \quad (V_{-1}^* = V_{d+1}^* = 0)$$

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}, \qquad 0 \le i \le d \quad (V_{-1} = V_{d+1} = 0)$$

more precisely

$$E_j^* A E_i^* = 0 \quad \text{if } |i-j| > 1$$
$$\neq 0 \quad \text{if } |i-j| = 1$$

$$E_j A^* E_i = 0 \quad \text{if } |i-j| > 1$$
$$\neq 0 \quad \text{if } |i-j| = 1$$

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Proposition

$T = T(x_0)$ is a semi-simple algebra.

In particular

$V = \mathbb{C}^X$ is a direct sum of irreducible T-submodules

Corollary

$V = \mathbb{C}^X \supset W$: irreducible T-submodule

$\Longrightarrow$

$A|_W,\ A^*|_W \in \mathrm{End}(W)$

are a TD-pair.

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<u>TD-pair</u> (tridiagonal pair)

W : finite dim. vector space $/\mathbb{C}$

$A, A^* \in End(W)$: diagonalizable

$$W = \bigoplus_{i=0}^{d} W_i \quad \text{eigenspace decomp. of } A$$

$$W = \bigoplus_{i=0}^{d^*} W_i^* \quad \text{eigenspace decomp. of } A^*$$

The pair A, A^* is called a TD-pair if

(i) $A W_i^* \subseteq W_{i-1}^* + W_i^* + W_{i+1}^*$, $0 \leq i \leq d^*$

$(W_{-1}^* = W_{d^*+1}^* = 0)$

$A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$, $0 \leq i \leq d$

$(W_{-1} = W_{d+1} = 0)$

and

(ii) W is irreducible as an $\langle A, A^* \rangle$-module.

<u>Remark</u> $d = d^*$

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Def

$A, A^* \in \mathrm{End}(W)$: **L-pair** (Leonard pair)

$\Longleftrightarrow$

TD-pair

and

$\dim V_i = \dim V_i^* = 1, \quad 0 \le i \le d.$

Example

$T = T(x_0) = \langle A, A^* \rangle \subset \mathrm{End}(V), \quad V = \mathbb{C}^X$

Terwilliger alg. of a P- and Q-poly. scheme

Set

$W = \{ f \in V \mid f \text{ is constant on each } X_i \},$

where

$X_i = \{ x \in X \mid \partial(x_0, x) = i \}.$

Then W is an irreducible T-module

and

$\dim W \cap V_i = 1, \quad 0 \le i \le d$

$\dim W \cap V_i^* = 1, \quad 0 \le i \le d.$

W is called the principal T-module

(primary T-module)