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Graph Symmetries

Lectures by

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Summary

- Introduction to symmetries of graphs
- Vertex-transitive and arc-transitive graphs
- s-arc-transitivity (including theorems of Tutte and Weiss)
- Proof of Tutte's theorem on symmetric cubic graphs
- Use of amalgams and covers to analyse and construct examples
- Some recent developments

1 Introduction to symmetries of graphs

Generally, an object is said to have **symmetry** if it can be *transformed in way that leaves it looking the same as it did originally.*

Automorphisms: An *automorphism* (or *symmetry*) of a simple graph $X = (V, E)$ is a permutation of the vertices of X which preserves the relation of adjacency; that is, a bijection $\pi\colon V \to V$ such that $\{v^\pi, w^\pi\} \in E$ if and only if $\{v, w\} \in E$.

Under composition, the automorphisms form a group, called the *automorphism group* (or *symmetry group*) of X, and this is denoted by $\mathrm{Aut}(X)$, or $\mathrm{Aut}\, X$.

Examples

(a) Complete graphs and null graphs: $\mathrm{Aut}\, K_n \cong \mathrm{Aut}\, N_n \cong S_n$ for all n

(b) Simple cycles: $\mathrm{Aut}\, C_n \cong D_n$ (dihedral group of order $2n$) for all $n \geq 3$

(c) Simple paths: $\mathrm{Aut}\, P_n \cong S_2$ for all $n \geq 3$

(d) Complete bipartite graphs: $\mathrm{Aut}\, K_{m,n} \cong S_m \times S_n$ when $m \neq n$,
while $\mathrm{Aut}\, K_{n,n} \cong S_n \wr S_2 \cong (S_n \times S_n) \rtimes S_2$ (when $m = n$)

(e) Star graphs – see above: $\mathrm{Aut}\, K_{1,n} \cong S_n$ for all $n > 1$

(f) Wheel graphs (cycle C_{n-1} plus nth vertex joined to all): $\mathrm{Aut}\, W_n \cong D_{n-1}$ for all $n \geq 5$

(g) Petersen graph: $\mathrm{Aut}\, P \cong S_5$.

Exercise 1: How many automorphisms has the underlying graph (1-skeleton) of each of the five Platonic solids: the regular tetrahedron, cube, octahedron, dodecahedron and icosahedron?

Exercise 2: Find a simple graph on 6 vertices that has exactly one automorphism.

Exercise 3: Find a simple graph that has exactly three automorphisms. What is the *smallest* such graph?

Exercise 4: For large n, do 'most' graphs of order n have a large automorphism group? or just the identity automorphism?

One amazing fact about graphs and groups is **Frucht's theorem**: in 1939, Robert(o) Frucht proved that given any finite group G, there exist infinitely many connected graphs X such that $\mathrm{Aut}\, X$ is isomorphic to G. And then later, in 1949, he proved that X may be chosen to be 3-valent. There are several variants and generalisations of this such as regular representations for graphs and digraphs (GRRs and DRRs).

The simple graphs of order n with the largest number of automorphisms are the null graph N_n and the complete graph K_n, each with automorphism group S_n (the symmetric group on n symbols). For non-null, incomplete graphs of bounded valency (vertex degree), the situation is more interesting.

In this course of lectures, we will devote quote a lot of attention to the case of regular graphs of valency 3, which are often called *cubic graphs.*

Exercise 5: For each $n \in \{4, 6, 8, 10, 12, 14, 16\}$, which 3-valent connected graph on n vertices has the largest number of automorphisms?

For fixed valency, some of the graphs with the largest number of automorphisms do not look particularly nice, or do not have other good properties (e.g. strength/stability, or suitability for broadcast networks). The 'best' graphs possess special kinds of symmetry.

Transitivity: A graph $X = (V, E)$ is said to be

- *vertex-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the vertex-set V
- *edge-transitive* if $\mathrm{Aut}\,X$ has a single orbit on the edge-set E
- *arc-transitive* (or *symmetric*) if $\mathrm{Aut}\,X$ has a single orbit on the arc-set
  (that is, the set $A = \{(v, w) \mid \{v, w\} \in E\}$ of all ordered pairs of adjacent vertices)
- *distance-transitive* if $\mathrm{Aut}\,X$ has a single orbit on each of the sets $\{(v, w) \mid d(v, w) = k\}$
  for $k = 0, 1, 2, \ldots$
- *semi-symmetric* if X is edge-transitive but not vertex-transitive
- *half-arc-transitive* if X is vertex-transitive and edge-transitive but not arc-transitive.

Examples

(a) Complete graphs: K_n is vertex-transitive, edge-transitive, arc-transitive and distance-transitive

(b) Simple cycles: C_n is vertex-transitive, edge-transitive, arc-transitive and distance-transitive

(c) Complete bipartite graphs: $K_{m,n}$ is edge-transitive, but is vertex-transitive (and arc-transitive and distance-transitive) only when $m = n$

(d) Wheel graphs: W_n is neither vertex-transitive nor edge-transitive (for $n \geq 5$)

(e) The Petersen graph is vertex-transitive, edge-transitive, arc-transitive and distance-transitive.

Note that **every vertex-transitive graph is regular** (in the sense of having all vertices of the same degree/valency), since for any two vertices v and w, there is an automorphism θ taking v to w, and then θ takes the edges incident with v to the edges incident with w.

On the other hand, not every edge-transitive graph is regular: counter-examples include all $K_{m,n}$ for $m \neq n$. But there exist graphs that are edge-transitive and regular but not vertex-transitive. One example is the smallest semi-symmetric cubic graph, called the *Gray graph* (discovered by Gray and re-discovered later by Bouwer), on 54 vertices. The smallest semi-symmetric regular graph is the *Folkman graph*, which is 4-valent on 20 vertices.

Also note that every arc-transitive connected graph without isolated vertices is both vertex-transitive and edge-transitive, but the converse does not hold. Counter-examples are half-arc-transitive. The smallest half-arc-transitive graph is the *Holt graph*, which is a 4-valent graph on 27 vertices. There are infinitely many larger examples.

Exercise 6: Let X be a k-valent graph, where k is odd (say $k = 3$). Show that if X is both vertex-transitive and edge-transitive, then also X is arc-transitive. [Harder question: does the same thing always happen when k is even?]

Exercise 7: Prove that every semi-symmetric graph is bipartite.

Exercise 8: Every distance-transitive graph is arc-transitive. Can you find an arc-transitive graph that is not distance-transitive?

***s*-arcs**: An *s-arc* in a graph $X = (V, E)$ is a sequence $(v_0, v_1, \ldots, v_s)$ of vertices of X in which any two consecutive vertices are adjacent and any three consecutive vertices are distinct, that is, $\{v_{i-1}, v_i\} \in E$ for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. The graph $X = (V, E)$ is called *s-arc-transitive* if $\mathrm{Aut}\, X$ is transitive on the set of all s-arcs in X.

Examples

(a) Simple cycles: C_n is s-arc-transitive for all $s \ge 0$, whenever $n \ge 3$

(b) Complete graphs: K_n is 2-arc-transitive, but not 3-arc-transitive, for all $n \ge 3$

(c) The complete bipartite graph $K_{n,n}$ is 3-arc- but not 4-arc-transitive, for all $n \ge 2$

(d) The Petersen graph is 3-arc-transitive, but not 4-arc-transitive

(e) The Heawood graph (the incidence graph of the projective plane of order 2) is 4-arc-transitive, but not 5-arc-transitive.

Exercise 9: For each of the five Platonic solids, what is the largest value of s such that the underlying graph (1-skeleton) is s-arc-transitive?

Exercise 10: Let X be an s-arc-transitive d-valent connected simple graph. Find a lower bound on the order of the stabiliser in $\mathrm{Aut}\, X$ of a vertex $v \in V(X)$, in terms of s and d.

Sharp transitivity ('regularity'): A graph $X = (V, E)$ is said to be

- *vertex-regular* if the action of $\mathrm{Aut}\, X$ on the vertex-set V is regular (that is, for every ordered pair (v, w) of vertices, there is a *unique* automorphism taking v to w)
- *edge-regular* if the action of $\mathrm{Aut}\, X$ on the edge-set E is regular
- *arc-regular* if the action of $\mathrm{Aut}\, X$ on the arc-set A is regular
- *s-arc-regular* if the action of $\mathrm{Aut}\, X$ on the set of s-arcs of X is regular.

The same terminology applies to actions of a subgroup of $\mathrm{Aut}\, X$ on X. For example, *Cayley graphs* (which will be encountered soon) are precisely the graphs that admit a vertex-regular group of automorphisms ... and possibly other automorphisms as well.

Important note: The term 'distance regular' means something quite different — a graph X is called *distance regular* if for all j and k, it has the property that for any two vertices v and w at distance j from each other, the number of vertices adjacent to w and at distance k from v is a constant (depending only on j and k, and not on v and w).

2 Vertex-transitive and arc-transitive graphs

Let X be a vertex-transitive graph, with automorphism group G, and let H be the stabiliser of any vertex v, that is, the subgroup $H = G_v = \{g \in G \mid v^g = v\}$. Let us also assume that X is not null, and hence that every vertex of X has the same positive valency.

Since G is transitive on $V = V(X)$, we may label the vertices with the right cosets of H in G such that each automorphism $g \in G$ takes the vertex labelled H to the vertex labelled Hg — that is, the action of G on $V(X)$ is given by right multiplication on the coset space $(G : H) = \{Hg : g \in G\}$.

Next, **define** $D = \{g \in G \mid v^g \text{ is adjacent to } v\} = \{g \in G \mid \{v, v^g\} \in E(X)\}$. Then:

Lemma 2.1:
(a) *D is a union of double cosets HaH of H in G*
(b) *D is closed under taking inverses*
(c) *v^x is adjacent to v^y in X if and only if $xy^{-1} \in D$*
(d) *The valency of X is the number of right cosets Hg contained in D*
(e) *X is connected if and only if D generates G.*

Proof.
(a) If $a \in D$ then for all $h, h' \in H$ we have $v^{hah'} = v^{ah'} = (v^a)^{h'}$, which is the image of a neighbour of v under an automorphism fixing v, and hence a neighbour of v, so $hah' \in D$. Thus $HaH \subseteq D$ whenever $a \in D$, and so D is the union of all such double cosets of H.
(b) If $a \in D$ then $\{v^a, v\} \in E(X)$, and hence $\{v, v^{a^{-1}}\} = \{v^a, v\}^{a^{-1}} \in E(X)$, so $a^{-1} \in D$.
(c) $\{v^y, v^x\} \in E(X) \;\Leftrightarrow\; \{v, v^{xy^{-1}}\}^y \in E(X) \;\Leftrightarrow\; \{v, v^{xy^{-1}}\} \in E(X) \;\Leftrightarrow\; xy^{-1} \in D$.
(d) By vertex-transitivity, the valency of X is the number of neighbours of v. These neighbours are all of the form v^a for $a \in D$, and if $v^a = v^{a'}$ for $a, a' \in D$ then $v^{a'a^{-1}} = v$ so $a'a^{-1} \in G_V = H$, or equivalently, $a' \in Ha$, and conversely, if $a' = ha$ where $h \in H$, then $v^{a'} = v^{ha} = v^a$. Hence this valency equals the number of right cosets of H contained in D.
(e) Neighbours of v are of the form v^a where $a \in D$, and their neighbours are of the form $v^{a'a}$ where $a, a' \in D$. By induction, vertices at distance at most k from v are of the form $v^{a_k a_{k-1} \dots a_2 a_1}$ where $a_i \in D$ for $1 \le i \le k$. It follows that X is connected if and only if every vertex can be written in this form (for some k), or equivalently, if and only if every element of G can be written as a product of elements of D. □

Lemma 2.2: *X is arc-transitive if and only if the stabiliser H of a vertex v of X is transitive on the neighbours of v.*

Proof. If X is arc-transitive, then for any two neighbours w and w' of v, there exists an automorphism $g \in G$ taking (v, w) to (v, w'). Any such g lies in G_v, and takes w to w', and it follows that G_v is transitive on the set $X(v)$ of all neighbours of v. Conversely, suppose that $H = G_v$ is transitive on $X(v)$. Then for any arcs (v, w) and (v', w'), some $g \in G$ takes

v to v', and if the pre-image of w' under g is w'', then also some $h \in G_v$ takes w to w''. From these it follows that $(v,w)^{hg} = (v,w'')^g = (v',w')$. Thus X is arc-transitive. □

Lemma 2.3: *X is arc-transitive if and only if $D = HaH$ for some $a \in G \setminus H$, indeed if and only if $D = HaH$ for some $a \in G$ such that $a \notin H$ but $a^2 \in H$.*

Proof. By Lemma 2.1, we know that X is arc-transitive if and only if $H = G_v$ is transitive on the neighbours of v, which occurs if and only if every neighbour of v is of the form w^h for some $w \in X(v)$ and some $h \in G_v = H$. By taking $v^a = w$, we find the equivalent condition that $D = HaH$ for some $a \in G \setminus H$.

For the second part, note that $a^{-1} \in D = HaH$, so $a^{-1} = hah'$ for some $h, h' \in H$. But then $aha = (h')^{-1}$, so $(ah)^2 = (h')^{-1}h \in H$, and also $H(ah)H = HahH = HaH = D$, so we can replace a by ah and then assume that $a^2 \in H$ (and still $a \notin H$). □

Constructions: The observations in the preceding lemmas can be turned around to produce *constructions* for vertex-transitive and arc-transitive graphs, as follows.

Let G be any group, H any subgroup of G, and D any union of double cosets of G such that $H \cap D = \emptyset$, and D is closed under taking inverses. [Note: there is also a construction for vertex-transitive digraphs that does not assume D is inverse-closed.]

Now **define** a graph $X = X(G,H,D)$ by taking $V = V(X)$ to be the right coset space $(G:H) = \{Hg : g \in G\}$, and $E = E(X)$ to be the set of all pairs of the form $\{Hx, Hax\}$ where $a \in D$ and $x \in G$. [This construction is due to Sabidussi (1964)]

The adjacency relation is symmetric, since $Hx = Ha(a^{-1}x)$, and so this is an undirected simple graph. Also right multiplication gives an action of G on X, with $g \in G$ taking a vertex Hx to the vertex Hxg, and an edge $\{Hx, Hax\}$ to the edge $\{Hxg, Haxg\}$. This action is transitive on vertices, since g takes H to Hg for any $g \in G$. The stabiliser of the vertex H is $\{g \in G \mid Hg = H\}$, which is the subgroup H itself (since $Hg = H$ if and only if $g \in H$). Valency and connectedness are as in Lemma 2.1.

Note, however, that the action of G on $X(G,H,D)$ need not be faithful: the kernel K of this action is the *core* of H (the intersection of all conjugates $g^{-1}Hg$ of H) in G. Similarly, the group G/K induced on $X(G,H,D)$ need not be the full automorphism group $\mathrm{Aut}\,X$; it is often possible that the graph admits additional automorphisms.

Cayley graphs: Given a group G and a set D of elements of G, the *Cayley graph* $\mathrm{Cay}(G,D)$ is the graph with vertex-set G, and edge set $\{\{x, ax\} : x \in G, a \in D\}$. Note that this is a special case of the above, with $H = \{1\}$.

In particular, $\mathrm{Cay}(G,D)$ is vertex-transitive, and the group G acts faithfully and regularly on the vertex-set, but is not necessarily the full automorphism group. For example, a *circulant* (which is a Cayley graph for a cyclic group) can often have more than just simple rotations. Similarly, the n-dimensional hypercube Q_n is the Cayley graph $\mathrm{Cay}(\mathbb{Z}_2^n, B)$ where B is the standard basis (of elementary vectors) for $\mathbb{Z}_2^n$, but $\mathrm{Aut}\,Q_n \cong \mathbb{Z}_2 \wr S_n \cong \mathbb{Z}_2^n \rtimes S_n$.

3 s-arc-transitivity (and theorems of Tutte and Weiss)

As defined earlier, an *s-arc* in a graph X is a sequence $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices of X in which any 2 consecutive vertices are adjacent and any 3 consecutive vertices are distinct. The graph X is called *s-arc-transitive* if $\mathrm{Aut}\,X$ is transitive on the s-arcs in X.

Lemma 3.1: *Let X be a vertex-transitive graph of valency $k > 2$, and let $G = \mathrm{Aut}\,X$. Then X is 2-arc-transitive if and only if the stabiliser G_v of a vertex v is 2-transitive on the k neighbours of v.*

Proof. If X is 2-arc-transitive, then for any two ordered pairs (u_1, w_1) and (u_2, w_2) of neighbours of v, some automorphism $g \in G$ takes the 2-arc (u_1, v, w_1) to the 2-arc (u_2, v, w_2), in which case g fixes v and g takes (u_1, w_1) to (u_2, w_2); hence G_v is 2-transitive on the neighbourhood $X(v)$. Conversely, suppose G_v is 2-transitive on $X(v)$, and let (u, v, w) and (u', v', w') be any two 2-arcs in X. Then by vertex-transitivity, some $g \in G$ takes v to v', and then if g takes u' to u'' and w' to w'', say, then some $h \in G_v$ takes (u, v) to (u'', v''), in which case $(u, v, w)^{hg^{-1}} = (u'', v, w'')^{g^{-1}} = (u', v', w')$; hence X is 2-arc-transitive. □

Exercise 11: For a vertex-transitive graph X of valency 3, what are the possibilities for the permutation group induced on $X(v)$ by the stabiliser G_v in $G = \mathrm{Aut}\,X$ of a vertex v? Which of these correspond to arc-transitive actions?

Exercise 12: For an arc-transitive graph X of valency 4, what are the possibilities for the permutation group induced on $X(v)$ by the stabiliser G_v in $G = \mathrm{Aut}\,X$ of a vertex v?

Lemma 3.2: *Let X be a vertex-transitive graph of valency $k > 2$, and let $G = \mathrm{Aut}\,X$. Then X is $(s+1)$-arc-transitive if and only if X is s-arc-transitive and the stabiliser G_σ of an s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ is transitive on $X(v_s) \setminus \{v_{s-1}\}$ (the set of $k-1$ neighbours of v_s other than v_{s-1}).*

Proof. If X is $(s+1)$-arc-transitive, then for any s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ and any vertices w and w' in $X(v_s) \setminus \{v_{s-1}\}$, some automorphism $g \in G$ takes the $(s+1)$-arc $(v_0, v_1, \ldots, v_s, w)$ to the $(s+1)$-arc $(v_0, v_1, \ldots, v_s, w')$, in which case g fixes σ, and g takes w to w'; hence G_σ is transitive on $X(v_s) \setminus \{v_{s-1}\}$. Conversely, suppose G_σ is transitive on $X(v_s) \setminus \{v_{s-1}\}$, and let $(v_0, v_1, \ldots, v_s, v_{s+1})$ and $(w_0, w_1, \ldots, w_s, w_{s+1})$ be any two $(s+1)$-arcs in X. Then by s-arc-transitivity, some $g \in G$ takes $(v_0, v_1, \ldots, v_s)$ to $(w_0, w_1, \ldots, w_s)$, and then if g takes w_{s+1} to w', say, then some $h \in G_\sigma$ takes v_{s+1} to w', in which case $(v_0, v_1, \ldots, v_s, v_{s+1})^{hg^{-1}} = (v_0, v_1, \ldots, v_s, w')^{g^{-1}} = (w_0, w_1, \ldots, w_s, w_{s+1})$; hence X is $(s+1)$-arc-transitive. □

The simple cycle C_n (which has valency 2) is s-arc-transitive for all $s \geq 0$, as is the union of more than one copy of C_n. This case is somewhat exceptional. For $k > 2$, there is an upper bound on values of s for which there exists a finite s-arc-transitive graph of valency k, as shown by the theorems of Tutte and Weiss below.

The first theorem, due to W.T. Tutte, is for valency 3, and will be proved in Section 4. On the other hand, the second theorem, due to Richard Weiss, is for arbitrary valency $k \geq 3$, but its proof is much more difficult, and is beyond the scope of this course.

Theorem 3.3 [Tutte, 1959]: *Let X be a finite connected arc-transitive graph of valency 3. Then X is s-arc-regular (and so $|\mathrm{Aut}\, X| = 3 \cdot 2^{s-1} \cdot |V(X)|$) for some $s \leq 5$. Hence in particular, there are no finite 6-arc-transitive cubic graphs.*

The upper bound on s in Tutte's theorem is sharp; in fact, it is attained by infinitely many graphs, although these graphs are somewhat rare. The smallest example is given below.

Tutte's 8-cage: This is the smallest 3-valent graph of girth 8, and has 30 vertices. It can be constructed in many different ways. One way is as follows:

In the symmetric group S_6, there are $\binom{6}{2} = 15$ transpositions (2-cycles), and $5 \cdot 3 \cdot 1 = 15$ triple transpositions (sometimes called *synthemes*). Define a graph T by taking these 30 permutations as the vertices, and joining each triple transposition $(a, b)(c, d)(e, f)$ by an edge to each of its three transpositions (a, b), (c, d) and (d, e).

The resulting graph T is *Tutte's 8-cage.* It is 3-valent, bipartite and connected, and the group S_6 induces a group of automorphisms of T by conjugation of the elements.

Exercise 13: Write down the form of a typical 5-arc $(v_0, v_1, \ldots, v_5)$ in Tutte's cage T with initial vertex v_0 being a transposition (a, b). Use this to prove that (a) the group S_6 is transitive on all such 5-arcs, and (b) T is not 6-arc-transitive.

Exercise 14: Prove that the girth (the length of the smallest cycle) of Tutte's 8-cage is 8.

Now the group S_6 is somewhat special among symmetric groups in that $\mathrm{Aut}\, S_6$ is twice as large as S_6. In fact, S_6 admits an *outer automorphism* that interchanges the 15 transpositions with the 15 triple transpositions, and interchanges the 40 3-cycles with the 40 double 3-cycles $(a, b, c)(d, e, f)$. Any such outer automorphism reverses a 5-arc in Tutte's 8-cage, and it follows that Tutte's 8-cage is 5-arc-transitive.

Note that Tutte's theorem actually puts a bound on the order of the stabiliser of a vertex (in the automorphism group of a finite symmetric 3-valent graph). The same thing does not hold for 4-valent symmetric graphs, as shown by the following.

Necklace/wreath graphs: Take a simple cycle C_n, where $n \geq 3$, with vertices labelled $0, 1, 2, \ldots, n-1$ in cyclic order, and then replace every vertex j by a pair of vertices u_j and v_j, and join every such u_j and every such v_j by an edge to each of the four vertices u_{j-1}, v_{j-1}, u_{j+1} and v_{j+1}, with addition and subtraction of subscripts taken modulo n. The resulting 4-valent graph (called a 'necklace' or 'wreath' graph) has $2n$ vertices, and is arc-transitive, with automorphism group isomorphic to the wreath product $S_2 \wr D_n \cong (S_2)^n \rtimes D_n$. In particular, the stabiliser of any vertex has order 2^n, which is unbounded.

Exercise 15: What is the largest value of s for which the above graph (on $2n$ vertices) is s-arc-transitive?

It is also worth noting here that vertex-stabilisers are bounded for the automorphism groups of maps. A *map* is an embedding of a connected graph or multigraph on a surface, dividing the surface into simply-connected regions, called the *faces* of the map. By definition, an automorphism of a map M preserves incidence between vertices, edges and faces of M, and it follows that if a vertex v has degree k, then the stabiliser of v in $\mathrm{Aut}\, M$ is a subgroup of the dihedral group D_k. The most highly symmetric maps are called *rotary*, or *regular*.

Theorem 3.4 [Weiss, 1981]: *Let X be a finite connected s-arc-transitive graph of valency $k \geq 3$. Then $s \leq 7$, and if $s = 7$ then $k = 3^t + 1$ for some t. Hence in particular, there are no finite 8-arc-transitive graphs of valency k whenever $k > 2$.*

As with Tutte's theorem, the upper bound on s in Weiss's theorem is sharp. In fact, for every $t > 0$, the incidence graph of a generalised hexagon over $\mathrm{GF}(3^t)$ is a 7-arc-transitive graph of valency $3^t + 1$.

The proof of Weiss's theorem uses the fact that if X is s-arc-transitive for some $s \geq 2$, then X is 2-arc-transitive (by Lemma 3.2), and so the stabiliser in $G = \mathrm{Aut}\, X$ of a vertex v of X is 2-transitive on the neighbourhood $X(v)$ of v (by Lemma 3.1). It then uses the classification of finite 2-transitive groups, obtained by Peter Cameron in 1981 using the classification of the finite simple groups (CFSG).

Finally in this section, we give something that is useful in proving Tutte's theorem (and in other contexts as well):

Lemma 3.5 (The 'even distance' lemma): *For any connected arc-transitive graph X, let $G^+ = \langle G_v, G_w \rangle$ be the subgroup of $G = \mathrm{Aut}\, X$ generated by the stabilisers G_v and G_w of any two adjacent vertices v and w. Then*

(a) *the orbit of v under G^+ contains all vertices at even distance from v,*

(b) *G^+ contains the stabiliser of every vertex of X,*

(c) *G^+ has index 1 or 2 in $G = \mathrm{Aut}\, X$, and*

(d) *$|G : G^+| = 2$ if and only if X is bipartite.*

Proof. Let Ω and $\mho$ be the orbits of v and w under G^+. Then $\mho$ contains w^{G_v}, so contains all neighbours of v. Similarly, Ω contains all neighbours of w, so contains all vertices at distance 2 from v. Also G^+ contains their stabilisers; for example, if $h \in G^+$ takes v to z, then G^+ contains $h^{-1} G_v h = G_z$. Parts (a) and (b) now follow from these observations, by induction. By the same token, the orbit $\mho = w^{G^+}$ contains all vertices at even distance from w. Hence in particular, every vertex of X lies in $\Omega \cup \mho$. Also by the orbit-stabiliser theorem, $|G_v||\Omega| = |G^+| = |G_w||\mho|$, and then since $|G_v| = |G_w|$, this implies $|\Omega| = |\mho|$, and it follows that $|\Omega| = |\mho| = |V(X)|$ or $|V(X)|/2$. In the latter case, Ω and $\mho$ are disjoint, which happens if and only if X is bipartite (with parts Ω and $\mho$), and then also $|G| = |G_v||V(X)| = 2|G_v||\Omega| = 2|G^+|$, so G^+ has index 2 in G. On the other hand, in the former case, $\Omega = \mho = V(X)$ and $|G| = |G_v||V(X)| = |G_v||\Omega| = |G^+|$, and then $G^+ = G$. This proves parts (c) and (d). $\square$

4 Proof of Tutte's theorem on symmetric cubic graphs

Theorem [Tutte, 1959]: *Let X be a finite connected arc-transitive graph of valency 3. Then X is s-arc-regular (and so $|\mathrm{Aut}\,X| = 3 \cdot 2^{s-1} \cdot |V(X)|$) for some $s \le 5$. Hence in particular, there are no finite 6-arc-transitive cubic graphs.*

We will prove this in several stages, using only elementary theory of groups and graphs.

First, we let s be the largest positive integer t for which the graph X is t-arc-transitive, and let $G = \mathrm{Aut}\,X$. Then we let $\sigma = (v_0, v_1, \ldots, v_s)$ be any s-arc in X, and consider the stabilisers in G of the 0-arc (v_0), the 1-arc (v_0, v_1), the 2-arc (v_0, v_1, v_2), and so on. We use properties of these to show that X is s-arc-regular, and then by considering the smallest k for which the stabiliser in G of the k-arc $(v_0, v_1, \ldots, v_k)$ is abelian, we prove that $s \le 5$.

Lemma 4.1: *X is s-arc-regular.*

Proof. We have assumed X is s-arc-transitive, so all we have to do is show that the stabiliser of an s-arc is trivial. So assume the contrary. Then every s-arc σ is preserved by some non-trivial automorphism f, and by conjugating by a 'shunt' if necessary, we can choose $\sigma = (v_0, v_1, \ldots, v_s)$ such that f moves one of the neighbours of v_s, say w. Then since f fixes v_s and its neighbour v_{s-1}, it must interchange w with the third neighbour w' of v_s. It follows that the stabiliser of the s-arc $\sigma = (v_0, v_1, \ldots, v_s)$ is transitive on the set of two $(s+1)$-arcs extending σ, namely $(v_0, v_1, \ldots, v_s, w)$ and $(v_0, v_1, \ldots, v_s, w')$. Hence X is $(s+1)$-arc-transitive, contradiction. □

Stabilisers

Let $\sigma = (v_0, v_1, \ldots, v_s)$ be any s-arc of X, and let $G = \mathrm{Aut}\,X$, and now define

$$
\begin{array}{ccccc}
H_s & = & G^{(0)} & = & G_{(v_0)} \\
H_{s-1} & = & G^{(1)} & = & G_{(v_0,v_1)} \\
\vdots & & \vdots & & \vdots \\
H_{s-k} & = & G^{(k)} & = & G_{(v_0,v_1,\ldots,v_k)} \\
\vdots & & \vdots & & \vdots \\
H_0 & = & G^{(s)} & = & G_{(v_0,v_1,\ldots,v_{s-1},v_s)} = \{1\}.
\end{array}
$$

Then working backwards, we find that $|H_j| = |G^{(s-j)}| = 2^j$ for $0 \le j < s$, while also $|H_s| = |G^{(0)}| = 3 \cdot 2^{s-1}$ and $|G| = 3 \cdot 2^{s-1} \cdot |V(X)|$.

Particular automorphisms

As before, let w and w' be the other two neighbours of v_s. Also let h and h' be the two automorphisms that take $\sigma = (v_0, v_1, \ldots, v_{s-1}, v_s)$ to $(v_1, v_2, \ldots, v_s, w)$ and $(v_1, v_2, \ldots, v_s, w')$ respectively, and define $x_0 = h'h^{-1}$ and $x_j = h^j x_0 h^{-j}$ for $1 \le j \le s$. Note that $x_0 = h'h^{-1}$ preserves $(v_0, v_1, \ldots, v_{s-1})$ and is non-trivial, so x_0 must swap v_s with the third neighbour of v_{s-1}; hence x_0 has order 2. It follows that every x_j has order 2.

Moreover, x_{j-1} preserves $(v_0, v_1, \ldots, v_{s-j})$ and swaps v_{s-j+1} with the third neighbour of v_{s-j}, for $1 \leq j < s$. Hence in particular, $x_{j-1} \in H_j \setminus H_{j-1}$. Then since $|H_j| = 2|H_{j-1}|$, we find that H_j is generated by $\{x_0, x_1, \ldots, x_{j-1}\}$ for $1 \leq j < s$. Similarly, x_{s-1} fixes v_0 but moves v_1, so $x_{s-1} \in H_s \setminus H_{s-1}$. Then since $|H_s : H_{s-1}| = 3$ (a prime), H_{s-1} is a maximal subgroup of H_s, so H_s is generated by $\{x_0, x_1, \ldots, x_{s-1}\}$.

Next, consider the subgroup G^* generated by $\{x_0, x_1, \ldots, x_s\}$. This contains $H_s = G_{v_0}$ and $\langle x_1, .., x_{s-1}, x_s \rangle = G_u$ where $u^h = v_0$, and so by the 'even distance' lemma, G^* is a subgroup of index 1 or 2 in G (the one we called G^+ earlier). Hence in particular, $|G^*| = 3 \cdot 2^{s-1} \cdot |V(X)|$ or half of that. Finally, since h moves v_0 to v_1 (which is at distance 1 from v_0), we find that $G = \langle h, G^* \rangle = \langle h, x_0, x_1, \ldots, x_s \rangle = \langle h, x_0 \rangle$.

We can summarise this in the following lemma.

Lemma 4.2:

(a) $H_j = \langle x_0, x_1, \ldots, x_{j-1} \rangle$ *for* $1 \leq j \leq s$,

(b) $G^+ = \langle x_0, x_1, \ldots, x_s \rangle$, *and*

(c) $G = \langle h, x_0, x_1, \ldots, x_s \rangle = \langle h, x_0 \rangle$.

Note that $H_1 = \langle x_0 \rangle$ and $H_2 = \langle x_0, x_1 \rangle$ are abelian, with orders 2 and 4 respectively.

Define λ ***to be the largest value of*** j ***for which*** H_j ***is abelian.*** We will show that $\frac{2}{3}(s-1) \leq \lambda < \frac{1}{2}(s+2)$ whenever $s \geq 4$, and hence that $s \leq 5$ or $s = 7$, and then we will eliminate the possibility that $s = 7$.

Lemma 4.3: *If* $s \geq 4$, *then* $2 \leq \lambda < \frac{1}{2}(s+2)$.

Proof. Assume the contrary. We know that $\lambda \geq 2$, so the assumption implies $2\lambda \geq s+2$, and hence $\lambda - 1 \geq s - \lambda + 1$. Now $H_\lambda = \langle x_0, x_1, \ldots, x_{\lambda-1} \rangle$ is abelian, and therefore so is its conjugate $h^{s-\lambda+1} H_\lambda h^{-(s-\lambda+1)} = \langle x_{s-\lambda+1}, x_{s-\lambda+2}, \ldots, x_s \rangle$. Then since $\lambda - 1 \geq s - \lambda + 1$, both of these contain $x_{\lambda-1}$, and also together they generate $\langle x_0, x_1, \ldots, x_s \rangle = G^+$. It follows that $x_{\lambda-1}$ commutes with every element of G^+. In particular, $x_{\lambda-1}$ commutes with h^2 (which lies in G^+ since $|G : G^+| \leq 2$). But that implies $x_{\lambda-1} = h^2 x_{\lambda-1} h^{-2} = x_{\lambda+1}$, and then conjugating by $h^{\lambda-1}$ gives $x_0 = x_2$, contradiction. $\square$

Lemma 4.4: *The centre of* $H_j = \langle x_0, x_1, \ldots, x_{j-1} \rangle$ *is generated by* $\{x_{j-\lambda}, \ldots, x_{\lambda-1}\}$, *for* $\lambda \leq j < 2\lambda$.

Proof. Every element x of H_j can be written uniquely in the form $x = x_{i_1} x_{i_2} \ldots x_{i_r}$ with $0 \leq i_1 < i_2 < \cdots < i_r \leq j-1$. Now $[x_{i_1}, x_{i_1+\lambda}] \neq 1$ since otherwise $[x_0, x_\lambda] = 1$ and then x_λ commutes with $x_0, x_1, \ldots, x_{\lambda-1}$, so $H_{\lambda+1} = \langle x_0, x_1, \ldots, x_\lambda \rangle$ is abelian, contradiction. Thus $x_{i_1+\lambda} \notin Z(H_j)$ when $i_1 + \lambda < j$. Similarly $[x_{i_r}, x_{i_r-\lambda}] \neq 1$ when $i_r - \lambda \geq 0$. Hence if $x \in Z(H_j)$ then $x = x_{i_1} x_{i_2} \ldots x_{i_r}$ where $i_1 \geq j - \lambda$ and $i_r < \lambda$.

Conversely, if $0 \leq j - \lambda \leq i < \lambda \leq j$ then x_i commutes with all of $x_0, x_1, \ldots, x_{\lambda-1}$, because $H_\lambda = \langle x_0, x_1, \ldots, x_{\lambda-1} \rangle$ is abelian, and with all of $x_\lambda, \ldots, x_{j-1}$, since $h^\lambda H_\lambda h^{-\lambda} = \langle x_\lambda, \ldots, x_{2\lambda-1} \rangle$ is abelian (and $\lambda \leq j < 2\lambda$). Thus every such element $x_{i_1} x_{i_2} \ldots x_{i_r}$ is central in H_j. □

Lemma 4.5: *The derived subgroup of* $H_{j+1} = \langle x_0, x_1, \ldots, x_j \rangle$ *is a subgroup of* $\langle x_1, \ldots, x_{j-1} \rangle$, *for* $1 \leq j \leq s-2$.

Proof. Each of $\langle x_1, \ldots, x_j \rangle$ and $\langle x_0, \ldots, x_{j-1} \rangle$ has index 2 in $H_{j+1} = \langle x_0, x_1, \ldots, x_j \rangle$, and is therefore normal in H_{j+1}. Their intersection $\langle x_1, \ldots, x_{j-1} \rangle$ is a normal subgroup, of index 4, and (so) the quotient is abelian. Thus $\langle x_1, \ldots, x_{j-1} \rangle$ contains all commutators of elements of H_{j+1}, and hence contains the derived subgroup of H_{j+1}. □

Next, consider the element $[x_0, x_\lambda] = x_0^{-1} x_\lambda^{-1} x_0 x_\lambda = (x_0 x_\lambda)^2$. By Lemma 4.5, this lies in $\langle x_1, \ldots, x_{\lambda-1} \rangle$, so can be written in the form $x_{i_1} \ldots x_{i_r}$ with $1 \leq i_1 < \cdots < i_r \leq \lambda - 1$.

We will take $\mu = i_1$ and $\nu = i_r$, and show that $\mu + \lambda \geq s - 1$ and $2\lambda - \nu \geq s - 1$, and hence that $\frac{2}{3}(s-1) \leq \lambda$.

Lemma 4.6: *If* $[x_0, x_\lambda]$ *is written as* $x_{i_1} \ldots x_{i_r}$ *with* $0 < i_1 < \cdots < i_r < \lambda$, *then*
(a) $i_1 + \lambda \geq s - 1$, *and* (b) $2\lambda - i_r \geq s - 1$.

Proof. Take $\mu = i_1$ and $\nu = i_r$, so that $0 < \mu \leq \nu < \lambda$.

For (a), suppose that $\mu + \lambda \leq s - 2$. Then Lemma 4.5 implies that $[x_0, x_{\mu+\lambda}]$ lies in $\langle x_1, \ldots, x_{\mu+\lambda-1} \rangle$, the centre of which is $\langle x_\mu, \ldots, x_\lambda \rangle$. The latter contains x_λ and $x_\mu \ldots x_\nu = [x_0, x_\lambda]$, so both of these commute with $[x_0, x_{\mu+\lambda}]$. This gives

$$\begin{aligned}
&[x_0, x_\lambda]^{x_{\mu+\lambda}} \;=\; [x_0^{x_{\mu+\lambda}}, x_\lambda^{x_{\mu+\lambda}}] \;=\; [x_0^{x_{\mu+\lambda}}, x_\lambda] \\
&= [x_0[x_0, x_{\mu+\lambda}], x_\lambda] \;=\; [x_0, x_{\mu+\lambda}]^{-1} x_0^{-1} x_\lambda^{-1} x_0 [x_0, x_{\mu+\lambda}] x_\lambda \\
&= [x_0, x_{\mu+\lambda}]^{-1} [x_0, x_\lambda] x_\lambda^{-1} [x_0, x_{\mu+\lambda}] x_\lambda \\
&= [x_0, x_{\mu+\lambda}]^{-1} [x_0, x_\lambda][x_0, x_{\mu+\lambda}] \;=\; [x_0, x_\lambda],
\end{aligned}$$

and therefore $x_\mu \ldots x_\nu = [x_0, x_\lambda]$ commutes with $x_{\mu+\lambda}$, contradiction.

Similarly, if $2\lambda - \nu \leq s - 2$ then $[x_0, x_{2\lambda-\nu}] \in \langle x_1, \ldots, x_{2\lambda-\nu-1} \rangle$, the centre of which is $\langle x_{\lambda-\nu}, \ldots, x_\lambda \rangle$, so $[x_0, x_{2\lambda-\nu}]$ commutes with $x_{\lambda-\nu}$ and $x_{\mu+\lambda-\nu} \ldots x_\lambda = h^{\lambda-\nu}(x_\mu \ldots x_\nu) h^{\nu-\lambda} = h^{\lambda-\nu}[x_0, x_\lambda] h^{\nu-\lambda} = [x_{\lambda-\nu}, x_{2\lambda-\nu}]$. This gives

$$\begin{aligned}
&[x_{\lambda-\nu}, x_{2\lambda-\nu}]^{x_0} \;=\; [x_{\lambda-\nu}{}^{x_0}, x_{2\lambda-\nu}{}^{x_0}] \;=\; [x_{\lambda-\nu}, x_{2\lambda-\nu}{}^{x_0}] \\
&= [x_{\lambda-\nu}, [x_0, x_{2\lambda-\nu}] x_{2\lambda-\nu}] \\
&= x_{\lambda-\nu}\, x_{2\lambda-\nu} [x_0, x_{2\lambda-\nu}]^{-1} x_{\lambda-\nu} [x_0, x_{2\lambda-\nu}] x_{2\lambda-\nu} \\
&= x_{\lambda-\nu}\, x_{2\lambda-\nu}\, x_{\lambda-\nu}\, x_{2\lambda-\nu} \;=\; [x_{\lambda-\nu}, x_{2\lambda-\nu}],
\end{aligned}$$

so $x_{\mu+\lambda-\nu} \ldots x_\lambda = [x_{\lambda-\nu}, x_{2\lambda-\nu}]$ commutes with $x_{\mu+\lambda}$, contradiction. This proves part (b).

Lemma 4.7: *If* $s \geq 4$*, then* $\lambda \geq \frac{2}{3}(s-1)$*.*

Proof. Lemma 4.6 gives $s-1-\lambda \;\leq\; \mu \;\leq\; \nu \;\leq\; 2\lambda - s + 1$, and then forgetting μ and ν and rearranging gives $2s - 2 \leq 3\lambda$.

Lemma 4.8: *If* $s \geq 4$*, then* $s = 4, 5$ *or* 7*.*

Proof. By Lemmas 3 and 6 we have $\frac{2}{3}(s-1) \leq \lambda < \frac{1}{2}(s+2)$. Forgetting λ and rearranging gives $4s - 4 < 3s + 6$, so $s < 10$, but on the other hand, for $s \in \{6, 8, 9\}$ there is no integer solution for λ, so $s = 4, 5$ or 7.

Lemma 4.9: $s \neq 7$*.*

Proof. Assume that $s = 7$. Then $\lambda = 4$, and $2 \leq \mu \leq \nu \leq 2$ so $\mu = \nu = 2$, which gives $[x_0, x_4] = x_2$. Next we consider $[x_0, x_5]$. By Lemma 4.5, this lies in $\langle x_1, x_2, x_3, x_4 \rangle$.

Suppose that $(x_0 x_5)^2 = [x_0, x_5]$ lies in $\langle x_1, x_2, x_3 \rangle$. Then $x_5 x_0 x_5$ lies in $\langle x_0, x_1, x_2, x_3 \rangle = H_4$ and so fixes vertex v_3 of our original 7-arc $\sigma = (v_0, v_1, \ldots, v_7)$, and hence x_0 fixes ${v_3}^{x_5}$. Observe that x_5 fixes v_0 and v_1 but not v_2 or v_3, and so ${v_2}^{x_5}$ is the third neighbour of v_1, different from v_0 and v_2 but then also fixed by x_0. It follows that x_0 preserves the 7-arc $({v_3}^{x_5}, {v_2}^{x_5}, v_1, v_2, v_3, v_4, v_5, v_6)$, contradiction.

Thus $[x_0, x_5] = y x_4$ for some $y \in \langle x_1, x_2, x_3 \rangle$. In particular, y commutes with x_0 and x_4 (since $\lambda = 4$), and also $y^2 = 1$ since $\langle x_1, x_2, x_3 \rangle$ is abelian. But now it follows that

$$\begin{aligned} x_2 &= [x_0, x_4] = (x_0 x_4)^2 = (x_0 y x_4)^2 = (x_0 [x_0, x_5])^2 \\ &= (x_5 x_0 x_5)^2 = x_5 {x_0}^2 x_5 = 1, \text{ a final contradiction.} \end{aligned}$$

□

This completes the proof of Tutte's theorem.

5 Amalgams and covers

Recall (from Section 3) that if X is an arc-transitive graph, with automorphism group G, and H is the stabiliser of a vertex v, then X may be viewed as a double coset graph $X(G, H, D)$ where $D = HaH$ for some $a \in G$ (moving v to a neighbour), with $a^2 \in H$.

Lemma 5.1: *In the arc-transitive graph $X(G, H, HaH)$, the stabiliser in G of the arc (H, Ha) is the intersection $H \cap a^{-1}Ha$, and the stabiliser of the edge $\{H, Ha\}$ is the subgroup generated by $H \cap a^{-1}Ha$ and a. In particular, the valency of $X(G, H, HaH)$ is equal to the index $|H : H \cap a^{-1}Ha|$.*

Proof. Let v be the vertex H. Then since $a^2 \in H$, the element a interchanges v with its neighbour $w = v^a$, and hence reverses the arc (v, w). Also $a^{-1}Ha$ is the stabiliser of the vertex $v^a = w$, so $H \cap a^{-1}Ha$ is the stabiliser of the arc (v, w). The rest follows easily from this (and transitivity of H on the neighbours of v). □

Amalgams for symmetric graphs

For the next part of this section, we will abuse notation and use V, E and A respectively for the stabilisers of the vertex v, the edge $\{v, w\}$, and the arc (v, w), where w is the neighbour of v that is interchanged with v by the arc-reversing automorphism a.

The triple (V, E, A) may be called an *amalgam*. Note that $V \cap E = A$. Note also that if X is connected, then $G = \langle HaH \rangle = \langle H, a \rangle = \langle H, H \cap a^{-1}Ha, a \rangle = \langle V, E \rangle$.

This amalgam *specifies the kind of group action on X.* For example, if X is 3-valent and $V \cong S_3 \cong D_3$ and $E \cong V_4$ (the Klein 4-group), with $A = V \cap E \cong C_2$, then the action of G on X is 2-arc-regular, with the element a being an involution (reversing the arc (v, w)).

Conversely, from any such triple (V, E, A) we can form the amalgamated free product $\mathcal{U} = V *_A E$ (of the groups V and E with their intersection $A = V \cap E$ as amalgamated subgroup), which is a kind of *universal* group for such actions.

Specifically, G is an arc-transitive group of automorphisms of the symmetric graph X, acting in the way that is specified, if and only if G is a quotient of $\mathcal{U}$ via some homomorphism which preserves the amalgam (that is, preserves the orders of V, E and A). When that happens, the homomorphism takes V, E and A (faithfully) to the stabilisers of some vertex v, incident edge $\{v, w\}$ and arc (v, w) respectively,

This gives a way of classifying such graphs, or finding all of the examples of small order.

Exercise 16: What is the amalgam for the action of $S_3 \wr S_2$ on the graph $K_{3,3}$? Is is the same as the amalgam for the Petersen graph?

In 1980, Djoković and Miller determined all possible amalgams for an arc-transitive action of a group on a 3-valent graph with finite vertex-stabiliser. There are precisely seven such amalgams, which they called $1'$, $2'$, $2''$, $3'$, $4'$, $4''$ and $5'$. In each case, the given number is the value of s for which the group acts regularly on s-arcs, and $'$ indicates that the group contains arc-reversing elements that are involutions (of order 2), while $''$ indicates that every arc-reversing element has order greater than 2. (Note that we require $a^2 \in H$ but not necessarily $a^2 = 1$.) The first examples of finite 3-valent graphs with full automorphism groups of the types $2''$ and $4''$ were found by Conder and Lorimer (1989).

The universal groups for the seven Djoković-Miller amalgams are now customarily denoted by G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2 and G_5, with s being the subscript, and with G_s and G_s^1 corresponding to s', and G_s^2 corresponding to s''.

For example, the group G_1 is the modular group $\langle\, h, a \mid h^3 = a^2 = 1 \,\rangle$, which is the free product of $V = \langle h \rangle \cong C_3$ and $E = \langle a \rangle \cong C_2$ (with $A = V \cap E = \{1\}$ amalgamated). Quotients of this group (in which the orders of V and E are preserved) are groups that act regularly on the arcs of a connected 3-valent symmetric graph.

Similarly, G_2^1 is the extended modular group $\langle\, h, p, a \mid h^3 = p^2 = a^2 = (hp)^2 = (ap)^2 = 1 \,\rangle$, which is the amalgamated free product of $V = \langle h, p \rangle \cong S_3$ and $E = \langle p, a \rangle \cong V_4$ with amalgamated subgroup $A = V \cap E = \langle p \rangle \cong C_2$, while on the other hand, G_2^2 is the group $\langle\, h, p, a \mid h^3 = p^2 = a^4 = (hp)^2 = a^2 p = 1 \,\rangle$, which is the amalgamated free product of $V = \langle h, p \rangle \cong S_3$ and $E = \langle a \rangle \cong C_4$ with amalgamated subgroup $A = V \cap E = \langle p \rangle = \langle a^2 \rangle \cong C_2$.

The group G_5 has a presentation on five generators h, p, q, r, s and a, obtainable from the amalgam $5'$, with $V = \langle h, p, q, r, s \rangle \cong S_4 \times C_2$ (of order 48), and $E = \langle p, q, r, s, a \rangle \cong D_4 \rtimes V_4$, and $A = V \cap E = \langle p, q, r, s \rangle \cong D_4 \times C_2$ (of order 16). This was used by Conder (1988) to prove that for all but finitely many n, both A_n or S_n occur as the automorphism groups of 5-arc-transitive cubic graphs.

Relationships between these seven groups were investigated by Djoković and Miller (1980) and Conder and Lorimer (1989), and recently by Conder and Nedela (2009) to refine the Djoković-Miller classification of arc-transitive group actions on symmetric cubic graphs.

Exercise 17: Find an example of a symmetric cubic graph that admits actions of arc-transitive groups of types $1'$, $2'$, $2''$ and $3'$.

Exercise 18: Can you find an example of a symmetric cubic graph that admits an action by an arc-transitive group of type $3'$ but not one of type $1'$?

Also in 1987, Richard Weiss identified the amalgams for several different kinds of s-arc-transitive group actions on **graphs of valency greater than 3**.

For example, Weiss produced one that gives the universal group for 7-arc-transitive group actions on 4-valent graphs, with V a group of order 11664 (being an extension of a group of order 3^5 by $\mathrm{GL}(2,3)$), and E a group of order 5832, with $A = V \cap E$ having index 4 in V and index 2 in E. This was used by Conder and Walker (1998) to prove the existence of infinitely many 7-arc-transitive 4-valent graphs (indeed with automorphism group A_n or S_n, for all but finitely many n).

Amalgams for semi-symmetric graphs

The same kind of thing happens for semi-symmetric graphs. These can be analysed in terms of amalgams (H, K, L) consisting of the stabilisers $H = G_v$ and $K = G_w$ of adjacent vertices and their intersection $L = G_v \cap G_w$, which is the stabiliser of the edge $\{v, w\}$, since semi-symmetric graphs are edge- but not arc-transitive.

For semi-symmetric 3-valent graphs, there are 15 different amalgams, determined by Goldschmidt (1980). These were used by Conder, Malnič, Marušič, Pisanski and Potočnik to find all semi-symmetric 3-valent graphs of small order (in 2001), and led to their discovery of the *Ljubljana graph*, which is a semi-symmetric 3-valent graph of order 112 with interesting properties.

Graph quotients and covers

If X and Y are graphs for which there exists a graph homomorphism (preserving adjacency) from Y onto X, then X is called a *quotient* of Y, and if the homomorphism preserves valency, then Y is called a *cover* of X.

Exercise 19: Show that K_4 is a quotient of the cube graph Q_3. [Hint: antipodes!]

There are various ways of constructing covers of a given connected graph X. Some involve *voltage graph* techniques, which can be roughly described as follows:

Choose a spanning tree for X, and a permutation group P on some set Ω, and assign elements of P to the co-tree edges (the edges not included in the spanning tree), with each such edge given a specific orientation, to make it an arc. Then take $|\Omega|$ identical copies of X, and for each co-tree arc (v, w), use the label π (from P) to define copies of the arc, from the vertex v in the jth copy of X to the vertex w in the (j^π)th copy of X, for all $j \in \Omega$. This gives a cover of X, with *voltage group* P.

Exercise 20: Construct Q_3 as a cover of K_4, using $P = S_2$.

In the 1970s, John Conway used a covering technique to produce infinitely covers of Tutte's 8-cage, and hence prove (for the first time) that there are infinitely many finite 5-arc-transitive cubic graphs. [This is described in Bigg's book *Algebraic Graph Theory.*]

Another way to construct covers of a symmetric graph X is to use the universal group $\mathcal{U} = V *_A E$ associated with the action of some arc-transitive group G of automorphisms of X. The group G is a quotient $\mathcal{U}/K$ for some normal subgroup K of $\mathcal{U}$, and then *for any normal subgroup L of $\mathcal{U}$ contained in K, the quotient $\mathcal{U}/L$ is an arc-transitive group of automorphisms of some cover of X.*

6 Some recent developments

This final section describes a number of recent developments on topics mentioned earlier.

Foster census:

In the 1930s, Ronald M. Foster (a mathematician/engineer working for Bell Labs) began compiling a list of all known connected symmetric 3-valent graphs of order up to 512. This 'census' was published in 1988, and was remarkably good, with only a few gaps.

The Foster census was extended by Conder and Dobcsányi (2002), with the help of some computational group theory and distributed computing. The extended census filled the gaps in Foster's list, and took it further, up to order 768. This also produced the smallest symmetric cubic graph of Djoković-Miller type $2''$, on 448 vertices. (The previously smallest known example had order 6652800.) In 2011/12, with the help of a new algorithm for finding finite quotients of finitely-presented groups, Conder extended this census, to find all connected symmetric 3-valent graphs of order up to 10000.

Other such lists:

Primož Potočnik, Pablo Spiga and Gabriel Verret have developed new methods for finding all *vertex-transitive* cubic graphs of small orders, and in 2012 used these to find all such graphs of order up to 1280, as well as all symmetric 4-valent graphs of order up to 640 (using a relationship between these kinds of graphs).

In 2013, Conder and Potočnik extended the list of all *semi-symmetric* 3-valent graphs, up to order 10000 as well.

There are similar lists of arc transitive graphs embedded on surfaces as *regular maps*, with large automorphism group. See `www.math.auckland.ac.nz/∼conder` for some of these.

Related open problems (concerning pathological examples):

(a) *What is the smallest symmetric cubic graph of Djoković-Miller type* $4''$*?*

Such a graph must be 4-arc-regular, but have no arc-reversing automorphisms of order 2. The smallest known example has order 5314410, with automorphism group an extension of $(C_3)^{11}$ by $\mathrm{PGL}(2,9)$. There is another nice (but larger) example of order 20401920, with automorphism group the simple Mathieu group M_{24}.

(b) *What is the smallest* 5*-arc-transitive cubic graph* X *with the property that its automorphism group is the only arc-transitive group of automorphisms of* X*?*

The smallest known example has order 2497430038118400, with automorphism group $M_{24} \wr S_2$. Examples are known with an alternating group A_n as automorphism group, but the smallest such n is 26.

(c) *What is the smallest half-arc-transitive graph* X *for which the stabiliser of a vertex in* $\mathrm{Aut}\,X$ *is neither abelian nor dihedral?*

Conder and Potočnik found one recently of order $90 \cdot 3^{10}$, with vertex-stabiliser $D_4 \times C_2$.

Covers:

Cheryl Praeger and some of her colleagues have done a lot of work on decomposing and constructing symmetric graphs via their quotients, and are using this to form a (loose) classification of all 2-arc-transitive finite graphs.

Group-theoretic covering methods have been applied to find all symmetric (or semisymmetric) regular covers of various small graphs, with abelian covering groups. For example:

- Cyclic symmetric coverings of Q_3 [Feng & Wang (2003)]
- Cyclic symmetric coverings of $K_{3,3}$ [Feng & Kwak (2004)]
- Elementary abelian symmetric covers of the Petersen graph [Malnič & Potočnik (2006)]
- Semisymmetric elementary abelian covers of the Möbius-Kantor graph [Malnič, Marušič, Miklavič & Potočnik (2007)]
- Elementary abelian symmetric covers of the Pappus graph [Oh (2009)]
- Elementary abelian symmetric covers of the octahedral graph [Kwak & Oh (2009)]
- Elementary abelian symmetric covers of K_5 [Kuzman (2010)].

In joint work with PhD student Jicheng Ma (2009–2012) we now have all symmetric abelian regular covers of K_4, $K_{3,3}$, Q_3, the Petersen graph and the Heawood graph.

Degree-diameter problem:

The *degree-diameter problem* involves finding the largest (regular) connected graph with given vertex-degree d and diameter D; for example, the Petersen graph is the largest for $(d, D) = (3, 2)$. In his PhD thesis project (2005-2008), Eyal Loz used voltage graphs to find covers of various small vertex- and/or arc-transitive graphs that are now the best known graphs in over half of the cases in the degree-diameter table. For more information, see: `moorebound.indstate.edu/wiki/The_Degree_Diameter_Problem_for_General_Graphs`.

Locally arc-transitive graphs:

A semi-symmetric graph is not vertex-transitive, but nevertheless can have a high degree of symmetry (subject to that constraint). A graph X is *locally s-arc-transitive* if the stabiliser in $\mathrm{Aut}\, X$ of a vertex v is transitive on all s-arcs in X starting at v.

An unpublished theorem of Stellmacher (1996) states that *If X is a finite locally s-arc-transitive graph, then $s \leq 9$.* Until recently, the only known examples for $s = 9$ came from classical generalised octagons and their covers. Such graphs are semi-symmetric (and hence bipartite) but not regular: vertices in different parts can have different valencies.

The smallest example for $s = 9$ has order 4680, with vertices of valency 3 in one part and 5 in the other. Its automorphism group is ${}^2F_4(2)$ (a Ree simple group), with vertex-stabilisers H and K of orders 12288 and 20480, and arc/edge-stabiliser $L = H \cap K$ of order 4096. In response to a comment by Michael Giudici at Rogla in 2011, Conder proved that the amalgamated free product $H *_L K$ has all but finitely many alternating groups A_n as quotients. Hence there exist *infinitely locally 9-arc-transitive bipartite graphs with vertices of valency 3 in one part and 5 in the other.*

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