# Cayley graphs on abelian groups

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An automorphism of $\Gamma$ is a permutation of $\mathcal{V}$ which preserves the the relation $\mathcal{A}$.

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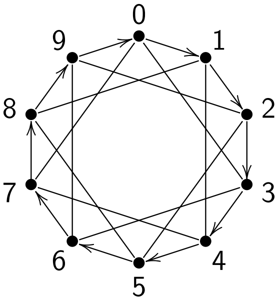
$\mathrm{Cay}(\mathbb{Z}_{10}, \{1, 3, 7\})$:



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Proof : Let $x \in G$ and let $\sigma_x : G \longrightarrow G$, $\sigma_x(g) = gx$. Then $gh^{-1} = gxx^{-1}h^{-1} = \sigma_x(g)\sigma_x(h)^{-1}$, hence $(g, h)$ is an arc of $\mathrm{Cay}(G, S)$ if and only if $(\sigma_x(g), \sigma_x(h))$ is. This shows that $\sigma_x$ is an automorphism of $\mathrm{Cay}(G, S)$. Clearly $\sigma_x\sigma_y = \sigma_{yx}$ hence $\{\sigma_x \mid x \in G\}$ is a group of automorphisms of $\mathrm{Cay}(G, S)$.

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Babai and Godsil proved the conjecture for nilpotent groups of odd order.

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Conclusion : if $A$ is an abelian group and $A \ncong (\mathbb{Z}_2)^n$, then no Cayley graph on $A$ is a GRR.

# Corresponding conjectures

**Conjecture (Babai, Godsil, Imrich, Lóvasz, 1982)**

*Let $G$ be a group of order $n$ which is neither generalized dicyclic nor abelian. The proportion of inverse-closed subsets $S$ of $G$ such that $\mathrm{Cay}(G, S)$ is a GRR goes to $1$ as $n \to \infty$.*

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This is now a theorem. (Dobson, Spiga, V.) We also proved the digraph conjecture for abelian groups.

# An important idea

**Lemma**

*Let $A$ be a group of order $n$. The number of subsets of $A$ which are fixed setwise by some element of $\mathrm{Aut}(A) \setminus \{1\}$ is at most $2^{3n/4+o(n)}$.*

**Proof.**

Note that $A$ is at most $\lfloor \log_2(n) \rfloor$-generated and hence $|\mathrm{Aut}(A)| \leq n^{\log_2(n)} \leq 2^{o(n)}$. We now count the number of subsets which are fixed setwise by a given $\varphi \in \mathrm{Aut}(A) \setminus \{1\}$. Let $\mathbf{C}_A(\varphi)$ denote the elements of $A$ that are fixed by $\varphi$. Note that $\varphi$ induces orbits of length 1 on $\mathbf{C}_A(\varphi)$ and of length at least 2 on $A \setminus \mathbf{C}_A(\varphi)$. Let $c = |\mathbf{C}_A(\varphi)|$. The number of subsets of $A$ which are fixed setwise by $\varphi$ is at most $2^{c+(n-c)/2} = 2^{n/2+c/2}$. Since $\mathbf{C}_A(\varphi)$ is a subgroup of $A$, we have $c \leq n/2$ and $n/2 + c/2 \leq 3n/4$. $\square$

# Outline of proof ideas

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What we actually do is study transitive permutation groups containing a self-normalizing regular abelian subgroup.

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In the graph case, there are some extra complications.

# Future work

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1. 2-groups,
2. nilpotent groups,
3. certain classes of solvable groups, etc..