Construction of Rational Spline Motions of Low Degree

Karla Počkaj

University of Primorska
Andrej Marušič Institute

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1. Preliminaries
   - Motions of a rigid body
   - Geometric continuity for motions

2. Construction of G^1 Hermite rational spline motion
   - Geometric Hermite interpolation problem
   - Spherical motion
   - Translational part

Motivation

- Rational spline motions, prove to be very useful in many industrial applications.
- An important task is to construct a rational spline motion that matches a given sequence of positions.
- The solution of the interpolation problem is required in Computer Graphics in order to animate objects, as well as in Robotics, e.g. for path planning of robot manipulators.

Example

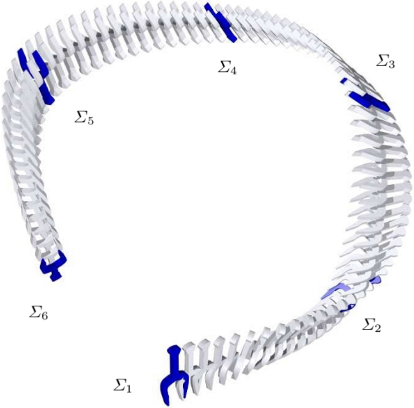

Figure : Some intermediate positions (silver) of a rigid body motion of a robot gripper arm interpolating six given input positions Σ_i (blue).

Motions of a rigid body

- Consider two coordinate systems in Euclidian 3-space:
  - the fixed coordinate system E^3
  - the moving coordinate system $\widehat{E}^3$
- Points can be described in either coordinate system: $\boldsymbol{p}$ or $\widehat{\boldsymbol{p}}$.

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- Coordinate transformation $\widehat{E}^3 \to E^3$?
- It can be represented in Cartesian coordinates:
  - by 3×3 matrix $\mathcal{R}$
  - by vector $\boldsymbol{c}$
- $\mathcal{R}$ is a special orthogonal matrix: $\mathcal{R}\mathcal{R}^\top = I, \quad \det(\mathcal{R}) = 1.$

- If $\mathcal{R}$ and $\boldsymbol{c}$ depend on time t, we speak of a rigid body motion.
- A trajectory of an arbitrary point $\widehat{\boldsymbol{p}}$ of a rigid body:

$$(\widehat{\boldsymbol{p}}, t) \mapsto \boldsymbol{p}(t) = \boldsymbol{c}(t) + \mathcal{R}(t)\,\widehat{\boldsymbol{p}}$$

- If $\boldsymbol{c} \equiv (0,0,0)^\top$, then $\boldsymbol{p}(t)$ of any point $\widehat{\boldsymbol{p}}$ lies on a sphere of radius $\|\widehat{\boldsymbol{p}}\|$, centered at the origin.
- $\mathcal{R}(t)$ describes motion of the unit sphere, we call it the spherical (rotational) part of the motion.
- $\boldsymbol{c}(t)$ defines the translational part.

- The motion is called *rational (spline) motion*, if the elements of $\boldsymbol{c}$ and $\mathcal{R}$ are rational (spline) functions.
- The *degree of the motion* is the maximal degree of the functions involved.
- Difficulty: $\mathcal{R} \in \mathrm{SO}_3$.

The kinematical mapping

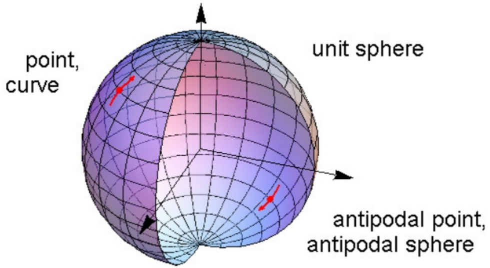
We define the kinematical mapping $\chi : \mathbb{H} \setminus \{\mathbf{0}\} \to \mathrm{SO}_3,$

$$\boldsymbol{Q} = (q_i)_{i=0}^3 \mapsto \chi(\boldsymbol{Q}) :=$$

$$\frac{1}{q_0^2+q_1^2+q_2^2+q_3^2}\begin{pmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{pmatrix}.$$

- $\mathcal{R} = \chi(\boldsymbol{Q})$
- $\chi(\lambda\boldsymbol{Q}) = \chi(\boldsymbol{Q}), \quad \lambda \in \mathbb{R} \setminus \{0\}$

- It defines a correspondence between the space of rotations and the unit quaternion sphere $\mathbb{S}^3$ with identified antipodal points.



- Applying mapping χ to a polynomial (spline) curve of degree n gives a spherical rational (spline) motion of degree $2n$.

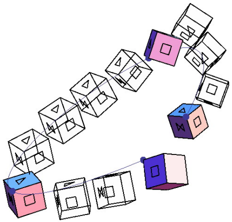
Translational part $\boldsymbol{c} = (c_i)_{i=1}^3$: the functions c_i should be choosen as

$$c_i = \frac{w_i}{r}, \quad r = \sum_{j=0}^{3} q_j^2, \quad i = 1, 2, 3,$$

where $\boldsymbol{w} := (w_i)_{i=1}^3$ is a polynomial (spline) curve of degree $\leq 2n$.

Interpolation problem

Consider $n + 1$ positions of a moving object. The positions are described by the corresponding Euclidian spatial displacements.
How to interpolate a certain set of positions?



Disadvantages of standard interpolation techniques:

- The resulting motion heavily depends on a particular parametrization which has to be chosen in advance.
- They lead to rational motions of relatively high polynomial degree.
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- a higher asymptotic approximation order,
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- a higher asymptotic approximation order,
- we obtain rational motions of lower degree.

The difficulty: the uniqueness and the existence analysis may be quite hard.

Literature

- H.-P. Schröcker, B. Jüttler, Motion Interpolation with Bennett Biarcs, Proc. Computational Kinematics, Springer (2009), 101-108.
- B. Jüttler, M. Krajnc, E. Žagar, Geometric interpolation by quartic rational spline motions, Advances in Robot Kinematics: Motion in Man and Machine, Springer (2010), 377-384.
- G. Jaklič, B. Jüttler, M. Krajnc, V. Vitrih, E. Žagar, Hermite interpolation by rational G^k motions of low degree, Journal of Computational and Applied Mathematics 240 (2013), 20-30.
- M. Krajnc, K. Počkaj, V. Vitrih, Construction of a low degree rational Lagrange motions, submitted

Geometric continuity for motions

The trajectories

$$\begin{aligned} \boldsymbol{p}_t(t) &= \boldsymbol{c}_t(t) + \mathcal{R}_t(t)\,\widehat{\boldsymbol{p}}, \quad t \in [t_0, t_1],\\ \boldsymbol{p}_s(s) &= \boldsymbol{c}_s(s) + \mathcal{R}_s(s)\,\widehat{\boldsymbol{p}}, \quad s \in [s_0, s_1]. \end{aligned}$$

of an arbitrary point $\widehat{\boldsymbol{p}}$ join with G^1 continuity at the common point $\boldsymbol{p}_t(\tau) = \boldsymbol{p}_s(\sigma)$ iff there exists a regular reparametrization $\varphi\colon [t_0, t_1] \to [s_0, s_1]$, such that $\varphi' > 0$, $\varphi(\tau) = \sigma$ and

$$\left.\begin{aligned} \frac{d^j \boldsymbol{q}_t(t)}{dt^j}\Big|_{t=\tau} &= \frac{d^j}{dt^j}\big(\lambda(t)\boldsymbol{q}_s(\varphi(t))\big)\Big|_{t=\tau},\\ \frac{d^j \boldsymbol{c}_t(t)}{dt^j}\Big|_{t=\tau} &= \frac{d^j(\boldsymbol{c}_s \circ \varphi)(t)}{dt^j}\Big|_{t=\tau}, \end{aligned}\right. \qquad j = 0, 1,$$

where $\boldsymbol{q}_t$, $\boldsymbol{q}_s$ represent the rotations $\mathcal{R}_t$, $\mathcal{R}_s$ and $\lambda\colon [t_0, t_1] \to \mathbb{R}$ is a zero free scalar function.

Geometric Hermite interpolation problem

- **Data:** $2N+1$ positions Pos_ℓ, described by a center position $\boldsymbol{C}_\ell \in \mathbb{R}^3$ and by a unit quaternion $\boldsymbol{Q}_\ell \in \mathbb{H}$. Additionally, every position is supplemented with unit tangent vector $\boldsymbol{d}_\ell$ and velocity quaternion $\boldsymbol{U}_\ell$.
- $\boldsymbol{Q}_\ell \cdot \boldsymbol{Q}_{\ell+1} \geq 0, \quad \ell = 1, 2, \ldots, 2N.$
- The task is to construct a spline motion of degree six, which interpolates these positions and the corresponding derivative data.

- $\boldsymbol{q}_S : [0, N] \to \mathbb{H}, \quad \boldsymbol{c}_S : [0, N] \to \mathbb{R}^3$?
- They have integer knots and consist of segments $\boldsymbol{q}^k$ and $\boldsymbol{c}^k$,

$$\boldsymbol{q}^k(t^k) := \boldsymbol{q}_S(u)\big|_{[k-1,k]}, \quad \boldsymbol{c}^k(t^k) := \boldsymbol{c}_S(u)\big|_{[k-1,k]},$$

$$t^k := u - k + 1 \in [0, 1], \quad k = 1, 2, \ldots, N.$$

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$$t^k := u - k + 1 \in [0, 1], \quad k = 1, 2, \ldots, N.$$

- Curve $\boldsymbol{q}^k$ must satisfy:

$$\begin{aligned} \boldsymbol{q}^k(t_i^k) &= \lambda_i^k \boldsymbol{Q}_{2k-2+i} \\ (\boldsymbol{q}^k)'(t_i^k) &= \mu_i^k \boldsymbol{Q}_{2k-2+i} + \lambda_i^k \varphi_i^k \boldsymbol{U}_{2k-2+i} \end{aligned} \qquad i = 1, 2, 3 \qquad (1)$$

Definition

The solution of the nonlinear system (1) is admissible if the following relations are satisfied,

$$0 < t_2^k < 1, \quad \lambda_2^k > 0, \quad \varphi_i^k > 0, \quad i = 1, 2, 3.$$

- If a given unit tangent vector $\boldsymbol{d}_\ell$, $\ell = 1, 2, \ldots, 2N+1$, is multiplied by an arbitrary positive constant, the tangent direction of the trajectory $\boldsymbol{c}^k$, $k = 1, 2, \ldots N$, does not change.
- Hence we obtain free parameters $\beta^k = (\beta_i^k)_{i=1}^3$ that influence the lengths of the tangents and therefore the shape of the trajectory $\boldsymbol{c}^k$.
- Each spline segment $\boldsymbol{c}^k = \boldsymbol{w}^k / r^k$, where $r^k = \sum_{i=0}^3 (q_i^k)^2$, must satisfy:

$$\begin{aligned} \boldsymbol{w}^k(t_i^k) &= r^k(t_i^k)\, \boldsymbol{C}_{2k-2+i} \\ (\boldsymbol{w}^k)'(t_i^k) &= \varphi_i^k\, r^k(t_i^k)\, \beta_i^k \boldsymbol{d}_{2k-2+i} + (r^k)'(t_i^k)\, \boldsymbol{C}_{2k-2+i} \end{aligned} \qquad i = 1, 2, 3, \qquad (2)$$

Some notation, which will be used further on:

- $A^k := (\boldsymbol{U}_{2k-1}, \quad \boldsymbol{Q}_{2k-1}, \quad \boldsymbol{Q}_{2k}, \quad \boldsymbol{Q}_{2k+1}, \quad \boldsymbol{U}_{2k}, \quad \boldsymbol{U}_{2k+1}) \in \mathbb{R}^{4\times 6}$,
- $(A^k)^{[i,j]}$ denotes a matrix A^k with columns i and j omitted,
- $\alpha_{i,j}^k := \det(A^k)^{[6-5i,j+1]}$, $i = 0, 1$, $j = i, i+1, \ldots, i+4$.

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The presented G^1 Hermite spline motion is **constructed entirely locally**, so it is enough to analyse the case $k = 1$ only.

Spherical motion

Spherical motion

$$t_2^k = \frac{u^k}{1+u^k}, \quad u^k := \sqrt[3]{-\frac{\alpha_{0,3}^k}{\alpha_{0,4}^k}\,\frac{\alpha_{1,4}^k}{\alpha_{1,1}^k}}, \quad \lambda_2^k = -(1-t_2^k)^3\,\frac{\alpha_{1,2}^k}{\alpha_{1,1}^k} - (t_2^k)^3\,\frac{\alpha_{0,2}^k}{\alpha_{0,3}^k}, \tag{3}$$

$$\varphi_1^k = \left(\frac{t_2^k}{1-t_2^k}\right)^2 \frac{\alpha_{0,0}^k}{\alpha_{0,3}^k}, \quad \varphi_2^k = \frac{1}{\lambda_2^k}\,\frac{(t_2^k)^2}{1-t_2^k}\,\frac{\alpha_{0,4}^k}{\alpha_{0,3}^k}, \quad \varphi_3^k = \left(\frac{t_2^k}{1-t_2^k}\right)^{-2} \frac{\alpha_{0,0}^k}{\alpha_{1,1}^k}, \tag{4}$$

$$\mu_1^k = -\left(\frac{t_2^k}{1-t_2^k}\right)^2 \frac{\alpha_{0,1}^k}{\alpha_{0,3}^k} - \frac{2+t_2^k}{t_2^k}, \quad \mu_2^k = \frac{(t_2^k)^2}{1-t_2^k}\,\frac{\alpha_{0,2}^k}{\alpha_{0,3}^k} + \lambda_2^k\,\frac{2-3t_2^k}{t_2^k(1-t_2^k)},$$
$$\mu_3^k = \left(\frac{t_2^k}{1-t_2^k}\right)^{-2} \frac{\alpha_{1,3}^k}{\alpha_{1,1}^k} + \frac{3-t_2^k}{1-t_2^k}, \tag{5}$$

Theorem

Let $\boldsymbol{Q}_{2k-2+i} \in \mathbb{H}$ *be a sequence of normalized quaternions, and let* $\boldsymbol{U}_{2k-2+i}$ *be given velocity quaternions for* $i = 1, 2, 3$, *such that*

$$\begin{aligned} &\det(\boldsymbol{Q}_{2k-1}, \boldsymbol{Q}_{2k}, \boldsymbol{Q}_{2k+1}, \boldsymbol{U}_{2k-3+2j}) \neq 0, \\ &\det(\boldsymbol{Q}_{2k-2+j}, \boldsymbol{Q}_{2k-1+j}, \boldsymbol{U}_{2k-2+j}, \boldsymbol{U}_{2k-1+j}) \neq 0, \end{aligned} \qquad j = 1, 2,$$

for every $k = 1, 2, \ldots, N$. *Then there exists a unique cubic interpolating spline curve* $\boldsymbol{q}_S$, *satisfying* (1) *with* $\lambda_1^k = \lambda_3^k = 1$, *where* t_2^k, λ_2^k, $(\varphi_i^k)_{i=1}^3$ *and* $(\mu_i^k)_{i=1}^3$ *are determined by* (3), (4) *and* (5).

Theorem

Suppose that the assumptions of Theorem 2 hold. Then the interpolating quaternion spline $\boldsymbol{q}_S$ *is admissible iff*

$$\frac{\alpha_{0,0}^k}{\alpha_{0,3}^k}, \frac{\alpha_{0,0}^k}{\alpha_{1,1}^k}, \frac{\alpha_{0,4}^k}{\alpha_{0,3}^k} > 0, \quad \frac{\alpha_{1,4}^k}{\alpha_{1,1}^k} < 0, \quad \frac{\alpha_{0,2}^k}{\alpha_{0,4}^k} < \frac{\alpha_{1,2}^k}{\alpha_{1,4}^k}, \quad k = 1, 2, \ldots, N. \tag{6}$$

Translational part

$\boldsymbol{w}^k$, $\deg(\boldsymbol{w}^k) \leq 6$?

The translational part $\boldsymbol{c}_S$ which satisfies (2) is a G^1 continuous for an arbitrary choice of positive parameters

$$\beta^k \in \mathbb{R}^3, \; k = 1, 2, \ldots, N.$$

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What if we choose $\beta^k = (1, 1, 1)^\top$?

Example

$$\widetilde{\boldsymbol{q}} = \frac{\boldsymbol{q}}{\|\boldsymbol{q}\|}, \quad \boldsymbol{q}(s) = \left(s + \cos\left(\frac{\pi s}{4}\right), s^3, s + \sin\left(\frac{\pi s}{4}\right), \sqrt{s^2+1}\right)^\top$$

$$\widetilde{\boldsymbol{c}}(s) = 3\left(\log(s+1)\sin\left(\frac{\pi s}{3}\right), \log(s+1)\cos\left(\frac{\pi s}{3}\right)\sqrt{s^2+1}\right)^\top$$

The positions and the corresponding derivative data are sampled as

$$\boldsymbol{Q}_\ell = \widetilde{\boldsymbol{q}}(s_\ell), \;\; \boldsymbol{U}_\ell = \widetilde{\boldsymbol{q}}'(s_\ell), \;\; \boldsymbol{C}_\ell = \widetilde{\boldsymbol{c}}(s_\ell), \;\; \boldsymbol{d}_\ell = \frac{\widetilde{\boldsymbol{c}}'(s_\ell)}{\|\widetilde{\boldsymbol{c}}'(s_\ell)\|}, \tag{7}$$

where $s_\ell = \ell - 1$, $\ell = 1, 2, 3$. The polynomial $\boldsymbol{w}$ is of degree five and the parameters $(\beta_i)_{i=1}^3$ are set to one. Solution of the equations (3), (4):

$$\varphi_1 = 1.15, \;\; \varphi_2 = 2.17, \;\; \varphi_3 = 4.11, \;\; t_2 = 0.66, \;\; \lambda_2 = 0.77.$$

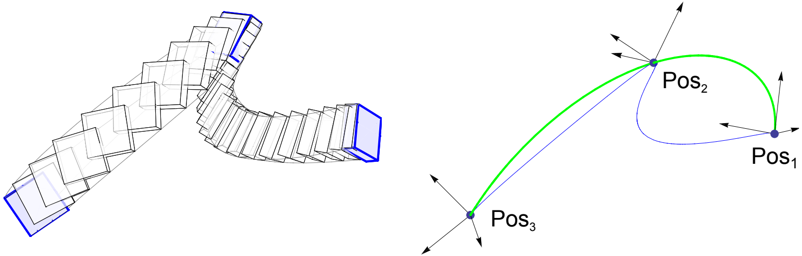

Figure : Three positions of a cuboid interpolated by rational motion of degree six (left) and two center trajectories (right), one of the original (bold curve) and one of the G^1 Hermite (thin curve) motion. The polynomial $\boldsymbol{w}$ is of degree five and the parameters $(\beta_i)_{i=1}^{3}$ are set to one.

- A common way in modeling smooth curves which satisfy some interpolation conditions is to minimize some functionals, such as

$$\int \kappa^j(t)\|\boldsymbol{c}'(t)\|dt, \quad j = 0, 2.$$

- The stretch ($j = 0$) and the bend energy ($j = 2$) can be approximated with the following functionals:

$$E_1 = \int_0^1 \|\boldsymbol{c}'(t)\|^2 dt, \quad E_2 = \int_0^1 \|\boldsymbol{c}''(t)\|^2 dt.$$

- We will minimize their convex combination,

$$E = \delta E_1 + (1-\delta) E_2, \tag{8}$$

where $\delta \in [0, 1]$ is a fixed weight given in advance.

- Let us assume that the degree of $\boldsymbol{w}^k$ is equal to m.
- First, consider the case $m = 4$:
- We obtain a linear system for unknown β

$$
\begin{aligned}
&\left(\varphi_1\, r(0)t_2(1-t_2)^3\boldsymbol{d}_1,\ \varphi_2\, r(t_2)t_2(1-t_2)\boldsymbol{d}_2,\ \varphi_3\, r(1)t_2^3(1-t_2)\boldsymbol{d}_3\right)\beta \\
=&-\left(2\, r(0)(1+t_2)(1-t_2)^3 + r'(0)t_2(1-t_2)^3\right)\boldsymbol{C}_1 \qquad (9)\\
&-\left(2\, r(t_2)(2t_2-1) + r'(t_2)t_2(1-t_2)\right)\boldsymbol{C}_2 \\
&-\left(2\, r(1)t_2^3(t_2-2) + r'(1)t_2^3(1-t_2)\right)\boldsymbol{C}_3.
\end{aligned}
$$

- Once $(\beta_i)_{i=1}^3$ are computed, the polynomial $\boldsymbol{w}$ can be determined by any standard interpolation scheme componentwise.

Example

- The polynomial $\boldsymbol{w}$ is of degree four and the solution of the system (9) is equal to

$$\boldsymbol{\beta} = (-4.50, 3.46, 1.70)^\top.$$

- The value of the functional (8) where $\delta = 1/2$, is for the obtained center trajectory equal to 916.32, while for the original center trajectory it is equal to 462.34.

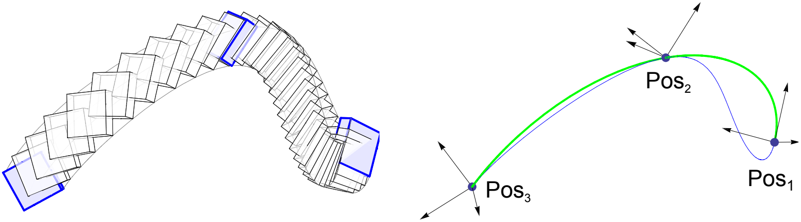

Figure : Three positions of a cuboid interpolated by rational motion of degree six (left) and two trajectories, one of the original (bold curve) and one of the G^1 Hermite (thin curve) motion (right). The polynomial $\boldsymbol{w}$ is of degree four.

- A more interesting case is $m = 5$, where three degrees of freedom are left for the construction.
- Let $(\beta_i)_{i=1}^3$ be taken as free parameters. They are used to minimize the shape functional (8).
- Using quintic Hermite basis polynomials $(h_{k,5})_{k=1}^6$ the polynomial $\boldsymbol{w}$ can be written as

$$\boldsymbol{w} = \sum_{i=1}^{3} r(t_i)\, \boldsymbol{C}_i\, h_{i,5} + (\varphi_i\, r(t_i)\, \beta_i \boldsymbol{d}_i + r'(t_i)\, \boldsymbol{C}_i)\, h_{i+3,5}.$$

- $\boldsymbol{w}$ can be expressed in a matrix form as $\boldsymbol{w} = A_6 \boldsymbol{v}$, where

$$A_6 := \left(\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \beta_1 \boldsymbol{d}_1, \beta_2 \boldsymbol{d}_2, \beta_3 \boldsymbol{d}_3\right) \in \mathbb{R}^{3\times 6},$$

$$\boldsymbol{v} := \begin{pmatrix} \left(r(t_i)\, h_{i,5} + r'(t_i)\, h_{i+3,5}\right)_{i=1}^3 \\ \left(\varphi_i\, r(t_i)\, h_{i+3,5}\right)_{i=1}^3 \end{pmatrix} \in \mathbb{R}^6[t].$$

- Let us define mappings

$$d_1 : \mathbb{R}^n[t] \to \mathbb{R}^n[t], \quad d_1 : \boldsymbol{v} \mapsto d_1(\boldsymbol{v}) := \frac{1}{r}\boldsymbol{v}' - \frac{r'}{r^2}\boldsymbol{v},$$

$$d_2 : \mathbb{R}^n[t] \to \mathbb{R}^n[t], \quad d_2 : \boldsymbol{v} \mapsto d_2(\boldsymbol{v}) := \frac{1}{r}\,\boldsymbol{v}'' - 2\frac{r'}{r^2}\,\boldsymbol{v}' + \frac{2(r')^2 - r''r}{r^3}\,\boldsymbol{v},$$

$$F : \mathbb{R}^n[t] \to \mathbb{R}^{n\times n}[t], \quad F(\boldsymbol{v}) := \delta\, d_1(\boldsymbol{v}) d_1(\boldsymbol{v})^\top + (1-\delta)\, d_2(\boldsymbol{v}) d_2(\boldsymbol{v})^\top.$$

- $G_6 := A_6^\top A_6 \in \mathbb{R}^{6\times 6}$
- $\|\boldsymbol{c}'\|^2 = (d_1(\boldsymbol{v}))^\top\, G_6\, d_1(\boldsymbol{v}), \quad \|\boldsymbol{c}''\|^2 = (d_2(\boldsymbol{v}))^\top\, G_6\, d_2(\boldsymbol{v})$

- The unknowns $(\beta_i)_{i=1}^3$ are computed as the solution of the equations

$$\int_0^1 \frac{d}{d\beta_i} \left(\delta \left(d_1(\boldsymbol{v})(t)\right)^\top G_6\, d_1(\boldsymbol{v})(t) + (1-\delta) \left(d_2(\boldsymbol{v})(t)\right)^\top G_6\, d_2(\boldsymbol{v})(t)\right) dt = 0.$$

- Written as a linear system:

$$D(\boldsymbol{v})\, \beta = \boldsymbol{g}(\boldsymbol{v}), \tag{10}$$

where

$$D(\boldsymbol{v}) := \left(\boldsymbol{d}_i^\top \boldsymbol{d}_j \int_0^1 (F(\boldsymbol{v})(t))_{i+3,j+3}\, dt\right)_{i,j=1}^3,$$

$$\boldsymbol{g}(\boldsymbol{v}) := -\left(\int_0^1 \boldsymbol{d}_k^\top \sum_{i=1}^3 (F(\boldsymbol{v})(t))_{i,k+3} \boldsymbol{C}_i\, dt\right)_{k=1}^3.$$

Theorem

Let $\boldsymbol{d}_\ell \in \mathbb{R}^3$, $\ell = 1, 2, \ldots, 2N+1$, *be given unit tangent vectors. If* $\det\left(D(\boldsymbol{v}^k)\right) \neq 0$, *then there exists a unique spline curve* $\boldsymbol{c}_S$ *determined by the polynomials* $\boldsymbol{w}^k$, $k = 1, 2, \ldots, N$, *of degree* 5, *which minimize the integral* (8) *and satisfy* (2). *The parameters* β^k *are defined by* (10).

Example

The polynomial $\boldsymbol{w}$ is of degree 5 and $\delta = 1/2$. The solution of (10) is equal to $\beta = (3.11, 3.90, 2.95)^\top$ and the value of the functional (8) is 443.83.

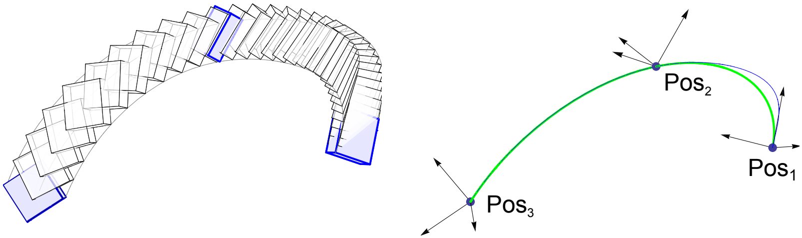

Figure : Three positions of a cuboid interpolated by rational motion of degree six (left) and the trajectories of the original (bold curve) and of the G^1 Hermite (thin curve) motion (right). The polynomial $\boldsymbol{w}$ is of degree five.

- Now assume that the degree of $\boldsymbol{w}$ is equal to 6.
- β and three additonal free parameters are left for the construction.
- $\boldsymbol{w}$ can be written as

$$\boldsymbol{w} = \sum_{i=1}^{3} (r(t_i)\, \boldsymbol{C}_i\, h_{i,6} + (\varphi_i\, r(t_i)\, \beta_i \boldsymbol{d}_i + r'(t_i)\, \boldsymbol{C}_i))\, h_{i+3,6} + \boldsymbol{e}\, h_{7,6}$$

- β_i and $\boldsymbol{e} := \boldsymbol{w}''(t_2)$ are unknown and will be computed by minimizing the integral (8)

- The unknowns β and $\boldsymbol{e}$, which minimize the integral (8) are the solution of a system

$$\begin{pmatrix} D(\boldsymbol{u}), & \overbrace{H(\boldsymbol{u})^\top} \\ H(\boldsymbol{u}), & \int_0^1 (F(\boldsymbol{u})(t))_{7,7} dt\, I \end{pmatrix} \begin{pmatrix} \beta \\ \boldsymbol{e} \end{pmatrix} = \begin{pmatrix} \overbrace{-\int_0^1 \sum_{i=1}^3 (F(\boldsymbol{u})(t))_{i,7}\, \boldsymbol{C}_i \, dt\, \mathbf{1}}^{\boldsymbol{g}(\boldsymbol{u})} \end{pmatrix}, \tag{11}$$

where

$$H(\boldsymbol{u}) := \left(\boldsymbol{d}_1 \int_0^1 (F(\boldsymbol{u})(t))_{4,7}\, dt,\ \boldsymbol{d}_2 \int_0^1 (F(\boldsymbol{u})(t))_{5,7}\, dt,\ \boldsymbol{d}_3 \int_0^1 (F(\boldsymbol{u})(t))_{6,7}\, dt \right).$$

Theorem

Let $\boldsymbol{d}_\ell \in \mathbb{R}^3$, $\ell = 1, 2, \ldots, 2N+1$, *be given unit tangent vectors. If*

$$\det\left(D(\boldsymbol{u}^k)\int_0^1 (F(\boldsymbol{u}^k)(t))_{7,7}dt\, I - H(\boldsymbol{u}^k)^\top H(\boldsymbol{u}^k)\right) \neq 0,$$

then there exists a unique interpolating spline curve $\boldsymbol{c}_S$ *determined by polynomials* $\boldsymbol{w}^k$, *of degree 6, which minimize the integral* (8) *and satisfy* $(\boldsymbol{w}^k)''(t_2^k) = \boldsymbol{e}^k$ *and* (2). *The parameters* β^k *and* $\boldsymbol{e}^k$ *are defined by* (11).

Example

The polynomial $\boldsymbol{w}$ is of degree 6 and $\delta = 1/2$. The solution of the linear system (11) is equal to

$$\beta = (2.94, 3.93, 3.16)^\top, \quad \boldsymbol{e} = (7.20, -12.13, 22.42)^\top,$$

and the value of the functional (8) is 433.09.

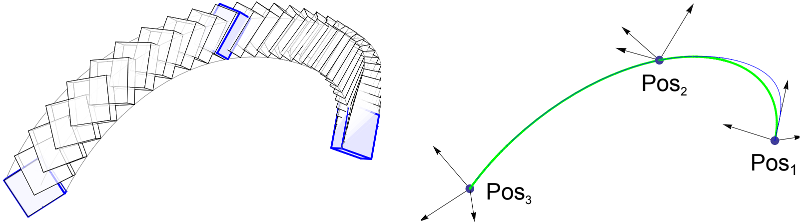

Figure : Three positions of a cuboid interpolated by rational motion of degree six (left) and two center trajectories (right), one of the original (bold curve) and one of the G^1 Hermite (thin curve) motion. The polynomial $\boldsymbol{w}$ is of degree six.

Example

Let us compare the values of the functional (8) for different weights δ and different methods to determine the parameters $(\beta_i)_{i=1}^3$.

The values of (8) *where:*	$\delta = \frac{1}{3}$	$\delta = \frac{1}{2}$	$\delta = \frac{2}{3}$
$\deg(\boldsymbol{w}) = 5$, $\beta = (1,1,1)^\top$	2910.04	2200.20	1490.37
$\deg(\boldsymbol{w}) = 4$, β is defined by (9)	1201.16	916.32	631.48
$\deg(\boldsymbol{w}) = 5$, β is defined by (10)	570.85	443.83	316.78
$\deg(\boldsymbol{w}) = 6$, β and $\boldsymbol{e}$ are defined by (11)	556.52	433.09	309.63
original curve	595.57	462.34	329.10

Table : The values of the functional (8) with $\delta \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$, for the original curve and for different choices of free parameters.

Thank you.