Graph Classes:
Interrelations, Structure, and Algorithmic Issues

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UP IAM and UP FAMNIT

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Outline

1. Algorithmic Graph Problems and Graph Classes
2. Perfect Graphs and Their Subclasses
3. Numerically Defined Graph Classes

ALGORITHMIC GRAPH PROBLEMS AND GRAPH CLASSES.

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Typically, every **search problem** of the above form also has a corresponding **decision problem**.

Examples

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Question: Does G contain a clique of size k?

clique: a set of pairwise adjacent vertices

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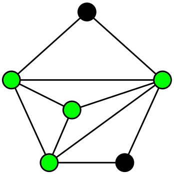

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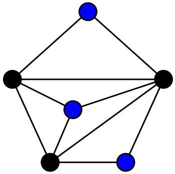

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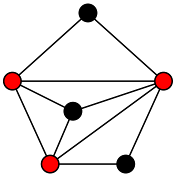

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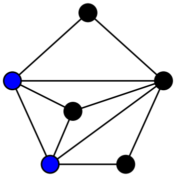

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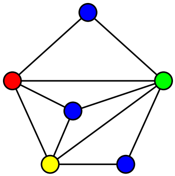

Complexity

All these problems

- Clique
- Stable Set
- Vertex Cover
- Dominating Set
- Colorability

(as well as hundreds of others) are NP-complete.

Coping With Intractability

There are several approaches to coping with the intractability of NP-hard problems:

- polynomial algorithms for particular input instances,
- approximation algorithms,
- heuristics, local optimization,
- "efficient" exponential algorithms (e.g., 1.5^n instead of 2^n),
- randomized algorithms,
- parameterized complexity
  (fixed-parameter tractable (FPT) algorithms $O(f(k)n^{O(1)})$),
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Graph Classes

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Examples:

- Planar graphs
- Connected graphs
- Trees
- Forests
- Bipartite graphs
- 3-colorable graphs
- Perfect graphs
- Cayley graphs
- vertex-transitive graphs

Main Questions

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- **Characterizations** of graphs in a given class.
- Computational complexity of **recognizing** graphs in a given class.

Recognition Problems

For a given graph class X we can define the following problem:

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Examples:

- If $X =$ the class of all 3-colorable graphs, the recognition problem is NP-complete.
- If $X =$ the class of planar graphs, the recognition problem is solvable in linear time.

www.graphclasses.org

Information System on Graph Classes and their Inclusions

- ISGCI home
- The Java application
- All classes
- References
- Smallgraphs
- About ISGCI
- Screenshots
- News
- FAQ
- Contact
- Impressum

Database contents

1437 classes
167680 inclusions
updated 2013-03-13

Latest news

2013-03-01 Smallgraphs available in graph6 format.

2012-09-08 The ISGCI database is included in Sage.

2012-01-14 'Find relation' in the Java application now gives a witness for proper inclusions.

Find class

What is ISGCI?

ISGCI is an encyclopaedia of graphclasses with an accompanying java application that helps you to research what's known about particular graph classes. You can:

- check the relation between graph classes and get a witness for the result
- draw clear inclusion diagrams
- colour these diagrams according to the complexity of selected problems
- find the P/NP boundary for a problem
- save your diagrams as Postscript, GraphML or SVG files
- find references on classes, inclusions and algorithms

Classic classes

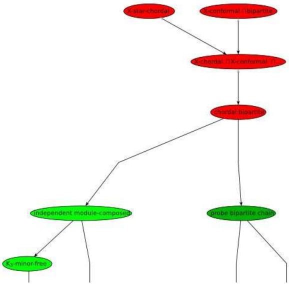
Meyniel
P_4-bipartite
P_4-reducible
P_4-sparse
bipartite
chordal
chordal bipartite
circle
circular arc
clique
cograph
comparability
distance-hereditary
even-hole-free
interval
median

Classes by definition

All classes
Chords & chordality
(De)composition
Forbidden subgraphs
(Forbidden) minors
Helly property
Hypergraphs
Intersection graphs
Matrix
Neighbourhood
Ordering
Partitionable
Perfection
Planarity
Posets
Probe graphs
Threshold
Tolerance

Problems

3-Colourability
Clique
Clique cover
Cliquewidth
Cliquewidth expression
Colourability
Cutwidth
Domination
Feedback vertex set
Hamiltonian cycle
Hamiltonian path
Independent set
Recognition
Treewidth
Weighted clique
Weighted feedback vertex set
Weighted independent set

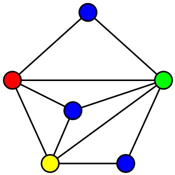


Source: `http://www.graphclasses.org/`

PERFECT GRAPHS AND THEIR SUBCLASSES.

χ and ω

$\chi(G)$: chromatic number of G = the smallest k such that G admits a k-coloring



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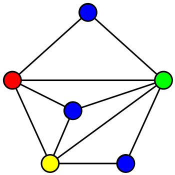
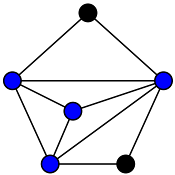


$\omega(G)$: clique number of G = the maximum size of a clique in G.



Perfect Graphs

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The class of perfect graphs is hereditary (closed under vertex deletions).

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Theorem (Lovász 1972, Perfect Graph Theorem)

A graph G is perfect if and only if its complement $\overline{G}$ is perfect.

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- their complements: $\overline{C_5}, \overline{C_7}, \overline{C_9}, \ldots$

Berge Graphs

Berge graph: a $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \ldots\}$-free graph.

Claude Berge, 1926–2002, a French mathematician

Source: `http://www.ecp6.jussieu.fr/GT04/`

The Strong Perfect Graph Theorem

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Total length of the proof $\approx$ 150 pages.

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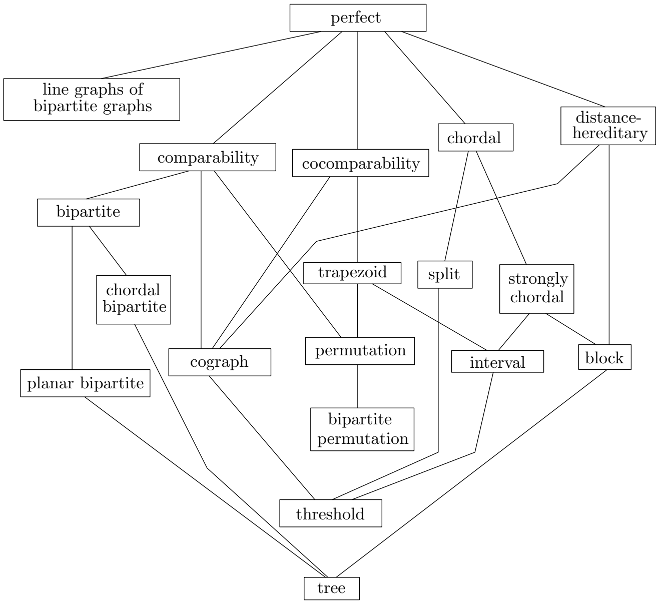
Existence of combinatorial algorithms for the COLORABILITY, STABLE SET and CLIQUE problems on perfect graphs is an open problem.

Recognizing Perfect Graphs

Theorem (Chudnovsky, Cornuéjols, Liu, Seymour, Vušković 2005)

There is a polynomial-time algorithm for recognizing Berge graphs.

Hasse Diagram of Some Classes of Perfect Graphs



CHORDAL GRAPHS.

Definition

A graph is chordal if every cycle on at least 4 vertices contains a chord.

chord: an edge connecting two non-consecutive vertices of the cycle.

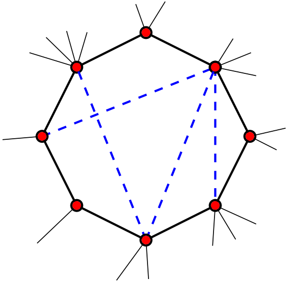
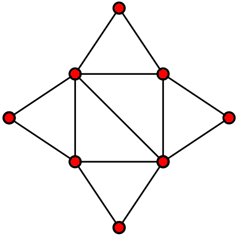


Figure: A cycle with four chords.

Chordal Graphs

Example:



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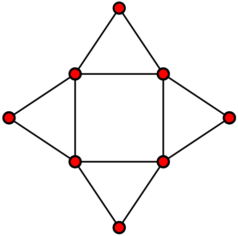


not chordal

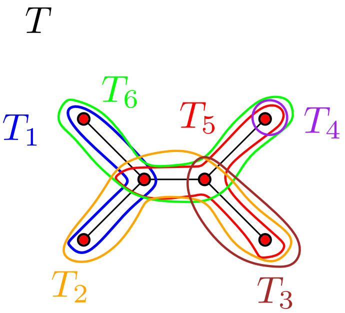
Properties of Chordal Graphs

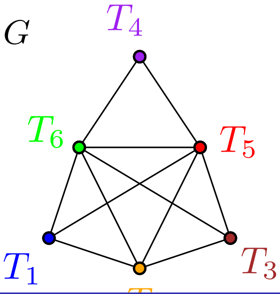
A graph is chordal if and only if it is $\{C_4, C_5, \ldots\}$-free.

Theorem (Gavril, 1974)

Chordal graphs are precisely the vertex-intersection graphs of subtrees in a tree.

Example:



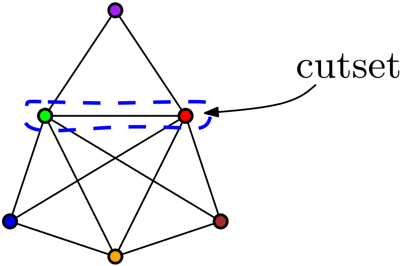


Chordal Graphs: Structural Properties

A cutset: a set of vertices $X \subseteq V$ such that the graph $G - X$ is disconnected.

Theorem (Dirac, 1961)

Every minimal cutset in a chordal graph is a clique.



Chordal Graphs: Algorithmic Aspects

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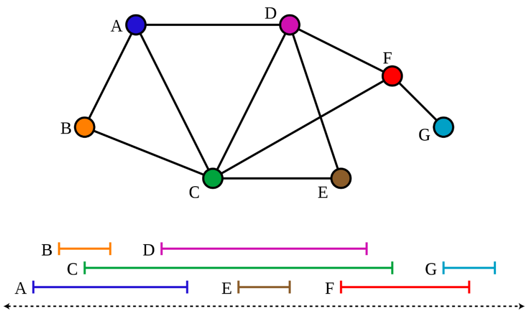
On the other hand, the DOMINATING SET problem is NP-complete on chordal graphs.

INTERVAL GRAPHS.

Definition

A graph is an interval graph if its vertices can be put into one-to-one correspondence with a set of intervals on the real line such that two vertices are connected by an edge if and only if their corresponding intervals have nonempty intersection.



Source: http://en.wikipedia.org/wiki/Interval_graph

Algorithmic Aspects

Theorem (Booth and Lueker 1976)

Interval graphs can be recognized in linear time.

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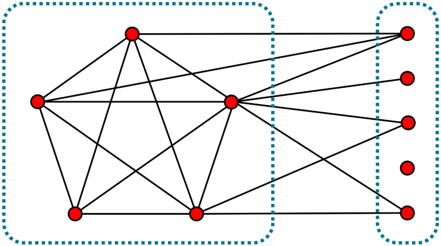
Other algorithmic problems on interval graphs:

- COLORABILITY: In P.
- CLIQUE: In P.
- STABLE SET: In P.
- DOMINATING SET: In P.

SPLIT GRAPHS.

Definition

A graph is split if there exists a partition of its vertex set into a clique and a stable set.

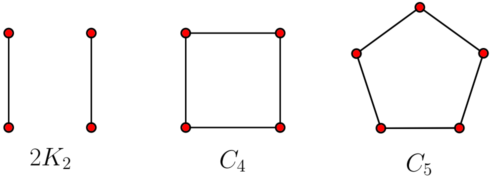


Source: http://en.wikipedia.org/wiki/Split_graph

Forbidden Induced Subgraphs

Theorem (Földes and Hammer, 1977)

A graph is split if and only if it is $\{2K_2, C_4, C_5\}$-free.



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Theorem

Let $d_1 \geq d_2 \geq \ldots \geq d_n$ be the degree sequence of a graph G. Also, let $m = \max\{i \ : \ d_i \geq i - 1\}$. Then, G is a split graph if and only if $\sum_{i=1}^{m} d_i = m(m-1) + \sum_{i=m+1}^{n} d_i$.

Algorithmic Aspects

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Other algorithmic problems on split graphs:

- COLORABILITY: In P.
- CLIQUE: In P.
- STABLE SET: In P.
- DOMINATING SET: NP-complete.

Threshold Graphs.

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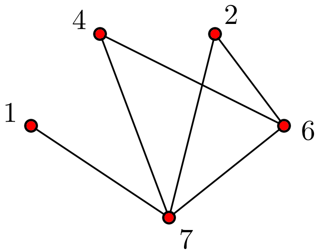
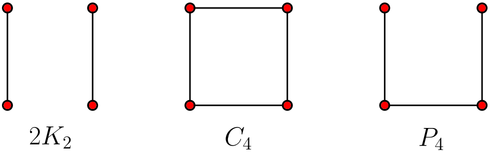


$t = 7$

Forbidden Induced Subgraphs

Theorem (Chvátal, Hammer 1977)

A graph is threshold if and only if it is $\{2K_2, C_4, P_4\}$*-free.*



Further Characterizations

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A graph G is threshold if and only if can be constructed from the one-vertex graph by repeated applications of the following two operations:

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Other algorithmic problems on threshold graphs:

- COLORABILITY: In P.
- CLIQUE: In P.
- STABLE SET: In P.
- DOMINATING SET: In P.

Numerically Defined Graph Classes.

A General Framework

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For example, $\mathcal{P}$ could be any of the following:

- a matching,
- a clique,
- a stable set,
- a dominating set,
- a total dominating set,
- etc.

Total Dominating Sets

total dominating set: a set of vertices such that every vertex has a neighbor in it

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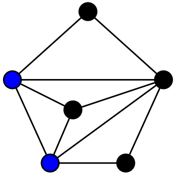

A General Framework

Problem

Given a graph G, does G admit
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positive integer weights $\mathbf{w}$ *on its vertices (or edges) and a set* T *such that*
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Several graph classes can be defined with a suitable choice of property $\mathcal{P}$ and restriction on the set T.

Example:
$\mathcal{P}$ = a stable set; T $=$ an interval unbounded from below:

A General Framework – Examples

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A General Framework – Examples

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total domishold graphs (Chiarelli-M. 2013)

$\mathcal{P}$ = a perfect matching; T = a single number:
all graphs

EQUISTABLE GRAPHS.

Equistable Graphs

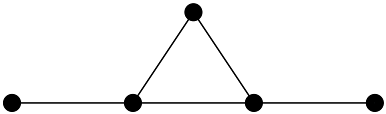
Definition

A graph $G = (V, E)$ is **equistable** if there exists a weight function $w : V \to \mathbb{N}$ and a positive integer $t \in \mathbb{N}$ such that for every $S \subseteq V$:

$$S \text{ is a maximal stable set in } G \quad \Leftrightarrow \quad w(S) = t\,.$$

Equistable graphs: example

The following graph is equistable:



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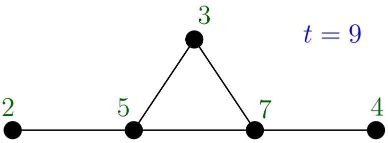

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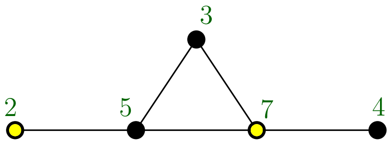

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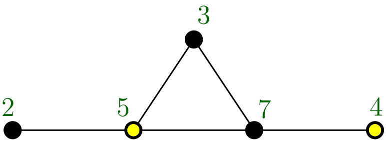

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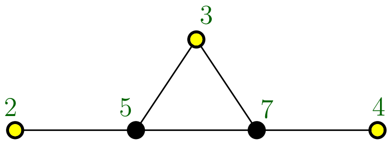

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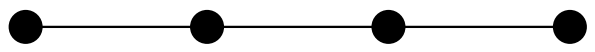
The following graph is not equistable:



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If

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$$w_2 + w_3 = t\,.$$

Complexity of algorithmic problems on equistable graphs:

Algorithmic Aspects of Equistable Graphs

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- CLIQUE: NP-complete.
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DOMISHOLD GRAPHS.

Domishold Graphs

Definition

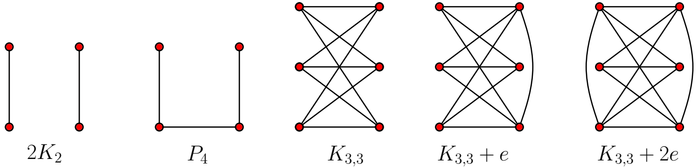
A graph $G = (V, E)$ is **domishold** if there exists a weight function $w : V \to \mathbb{N}$ and a positive integer $t \in \mathbb{N}$ such that for every $S \subseteq V$:

$$S \text{ is a dominating set in } G \quad \Leftrightarrow \quad w(S) \geq t\,.$$

Characterizations of Domishold Graphs

Theorem (Benzaken and Hammer 1978)

A graph G is domishold if and only if G is $\{2K_2, P_4, K_{3,3}, K_{3,3}+e, K_{3,3}+2e\}$-free.



Algorithmic Aspects of Domishold Graphs

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TOTAL DOMISHOLD GRAPHS.

Total Domishold Graphs

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Algorithmic Aspects of Total Domishold Graphs

Complexity of algorithmic problems on total domishold graphs (Chiarelli-M. 2013):

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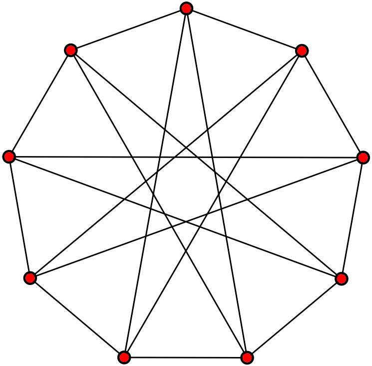
Circulant Graphs.

Circulants

A **circulant** is a Cayley graph over a cyclic group.



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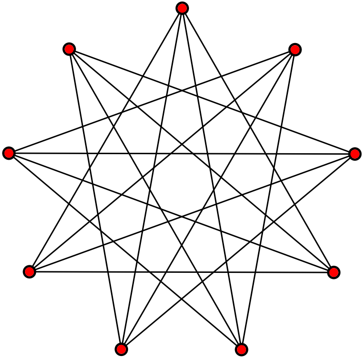

$C_9(\{3, 4, 5, 6\})$

Proposition

A circulant $G = C_n(D)$ *is*
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Some Graph-theoretic Properties of Circulants

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Proposition

A connected circulant $G = C_n(D)$ *with at least two vertices is*
bipartite
if and only if
n *is even, while every* $d \in D$ *is odd.*

Algorithmic Aspects

Theorem (Evdokimov and Ponomarenko 2004)

Circulant graphs can be recognized in polynomial time.

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CONCLUSION.

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Methods from different branches of mathematics and computer science apply to the study of graph classes:

1. algebraic and Boolean methods,
2. combinatorial methods,
3. mathematical programming (linear programming, polyhedral combinatorics, semidefinite programming),
4. algorithm design and computational complexity analysis,
5. etc.

Questions?

Thank you!