Plane-Width of Graphs

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Definition

Given a graph $G = (V, E)$ its **realization** is a function:

- mapping each vertex from V to a point in $\mathbb{R}^2$,
- such that the distance between two endpoints of an edge is at least 1.

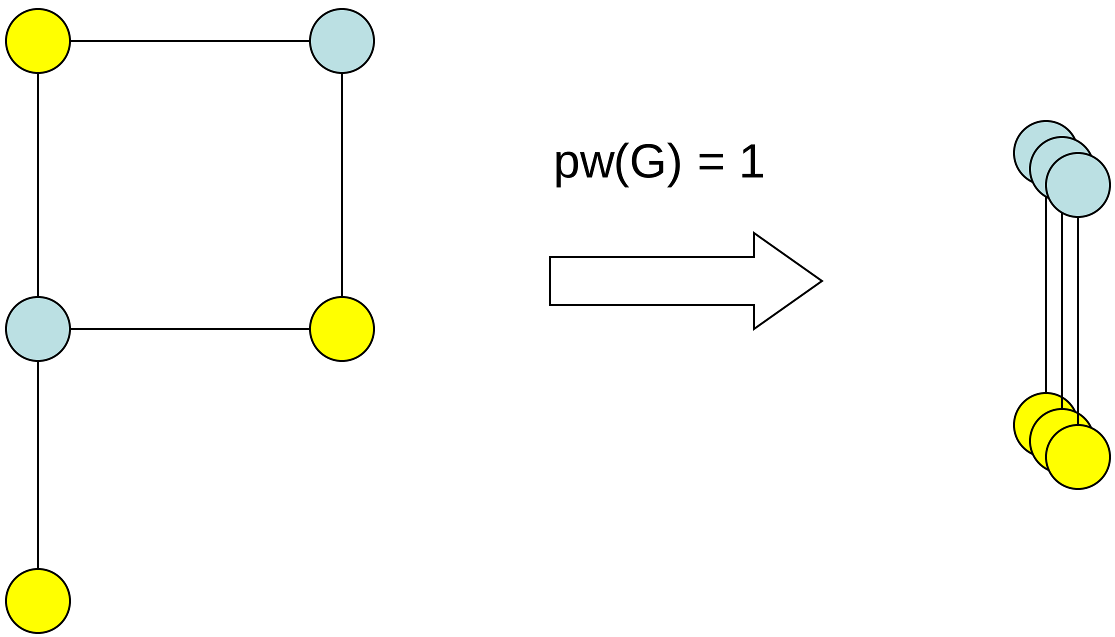
The **width** of a realization is the maximum distance between the images of any two vertices.

The infimum of widths of all valid realizations of graph G is the **plane-width** of G ($\mathrm{pwd}(G)$).

Outline

- Simple examples
- Relation to chromatic number
- Plane-width for complete graphs
- Relation to circular chromatic number
- Plane-width under graph operations
- Open problems

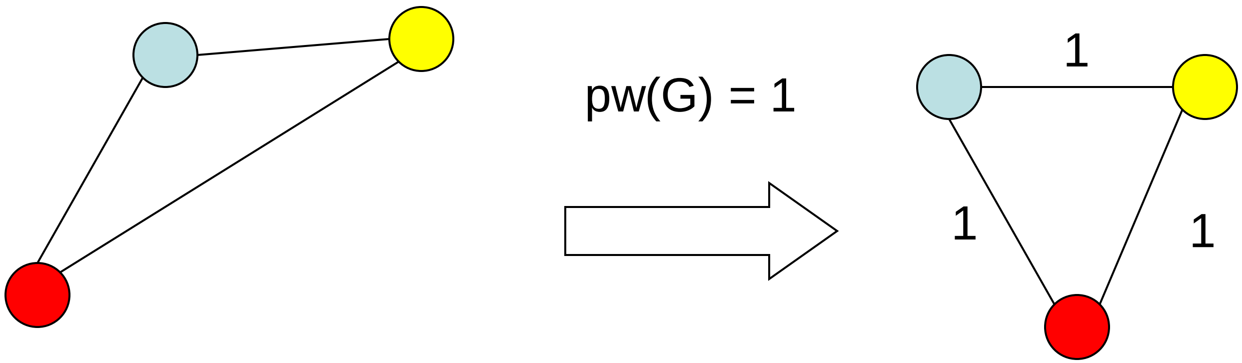
Warm-Up Examples



G Bipartite $\rightarrow$ pw(G) = 1

G Bipartite ?$\leftarrow$ pw(G) = 1

Warm-Up Examples



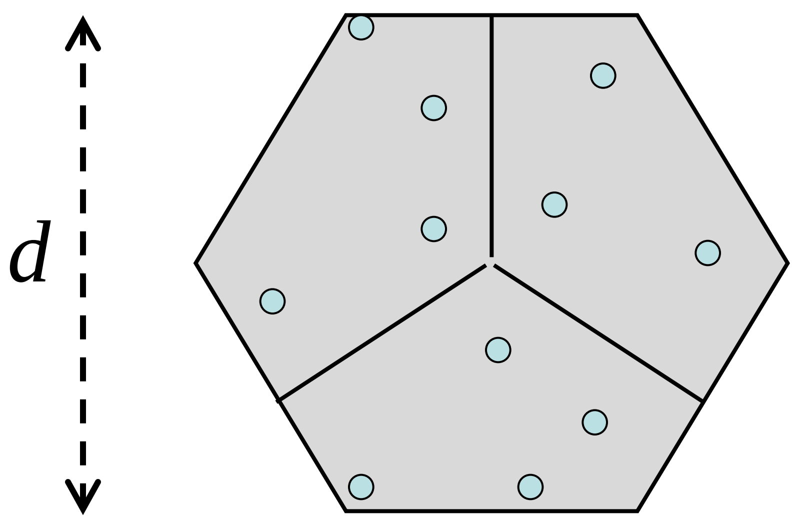
$$\mathrm{pw}(G) \leq \mathrm{pw}(K_{\chi(G)})$$

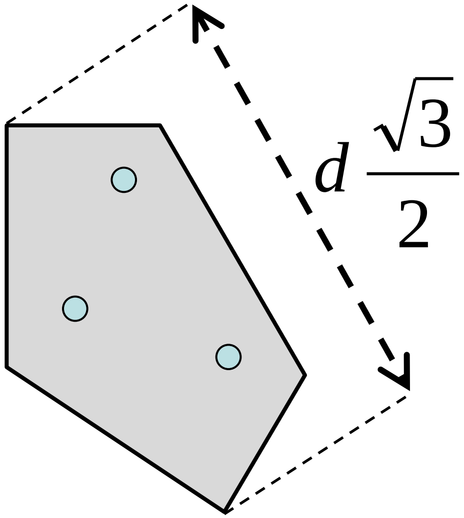
G 3-colourable → pw(G) = 1

G 3-colourable ?← pw(G) = 1

$\mathbf{pw}(G) \leq 2/\sqrt{3} \rightarrow G$ is 3-colourable

- Suppose you have a realization of width d
- Enclose it in a regular hexagon (J. Pál, 1920)
- Partition
- All the points in a partition must form an independent set if $d \leq 2/\sqrt{3}$





Theorem

Theorem *For all graphs G,*

$$(a)\ \ \mathrm{pw}(G) = 1 \text{ if and only if } \chi(G) \le 3\,,$$

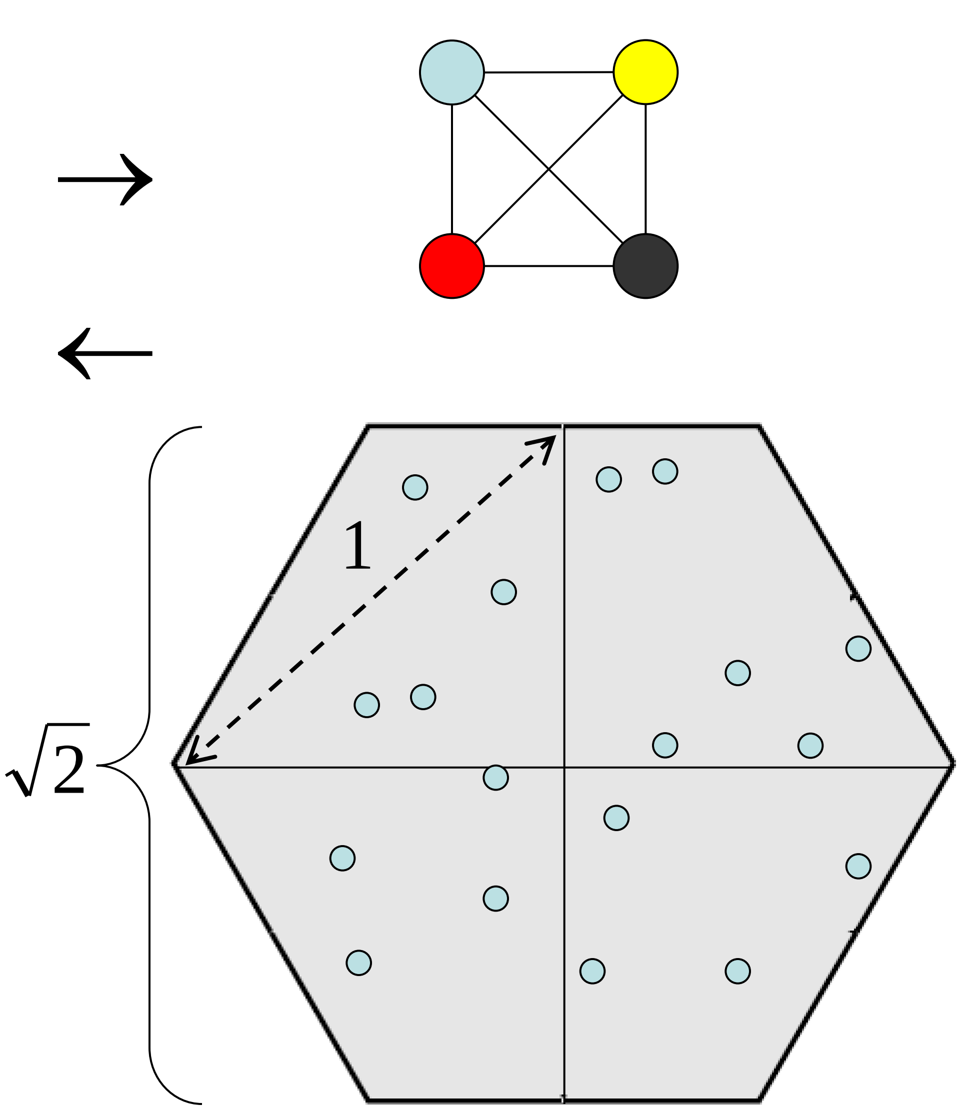
$$(b)\ \ \mathrm{pw}(G) \notin (1\,,2/\sqrt{3}]\,,$$

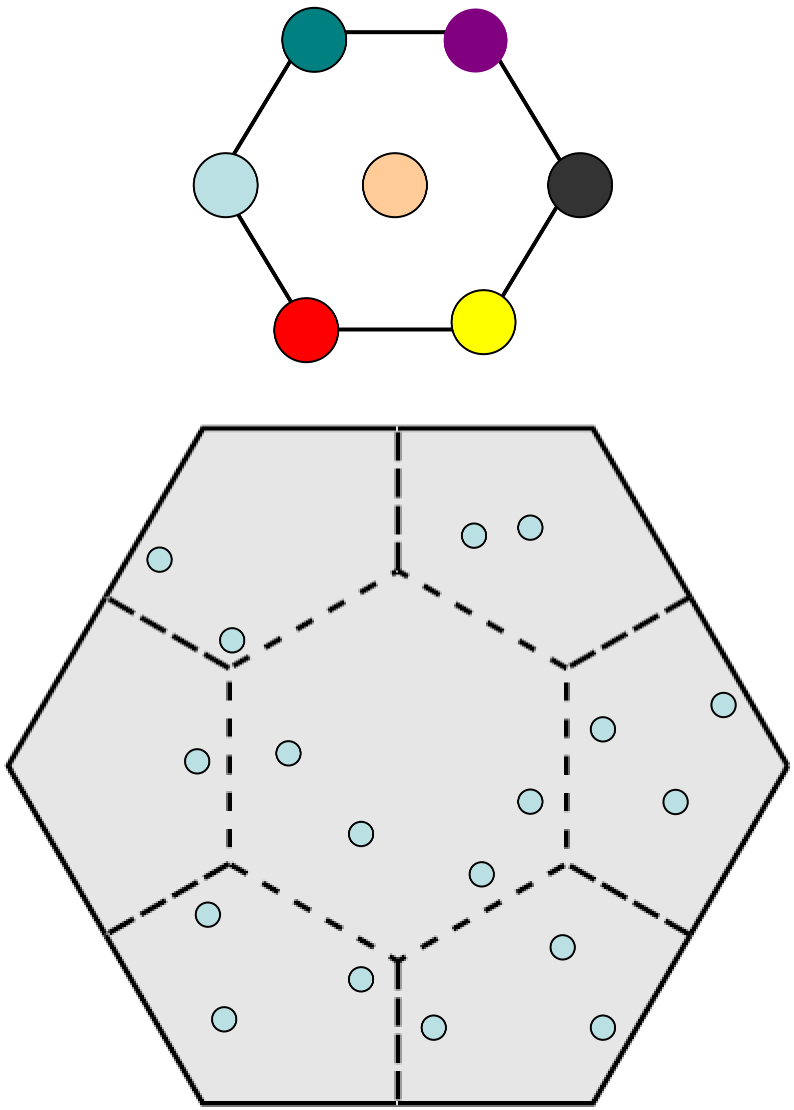
$$(c)\ \ \mathrm{pw}(G) \in (2/\sqrt{3}\,,\sqrt{2}] \text{ if and only if } \chi(G) = 4\,,$$

$$(d)\ \ \mathrm{pw}(G) \in (\sqrt{2}\,,2] \text{ if and only if } \chi(G) \in \{5,6,7\}\,.$$

G is 4-colourable $\leftrightarrow \mathrm{pw}(G) \leq \sqrt{2}$

G is 7-colourable $\leftrightarrow \mathrm{pw}(G) \leq 2$





Theorem *For all graphs G,*

(a) $\operatorname{pw}(G) = 1$ *if and only if* $\chi(G) \leq 3$,

(b) $\operatorname{pw}(G) \notin (1, 2/\sqrt{3}]$,

(c) $\operatorname{pw}(G) \in (2/\sqrt{3}, \sqrt{2}]$ *if and only if* $\chi(G) = 4$,

(d) $\operatorname{pw}(G) \in (\sqrt{2}, 2]$ *if and only if* $\chi(G) \in \{5, 6, 7\}$.

Further Questions:

1. How does this extend to graphs with large chromatic number?
2. Are these intervals "tight" ?
3. For non-bipartite graphs, can the graph's chromatic number be expressed as a function of its plane-width?

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Why Complete Graphs?

- **Theorem** *For all graphs G,*

$$\mathrm{pw}(K_{\omega(G)}) \leq \mathrm{pw}(G) \leq \mathrm{pw}(K_{\chi(G)})$$

- For some graphs, this bound is tight.
- Well-studied problem known as disc-packing

Small Complete Graphs.

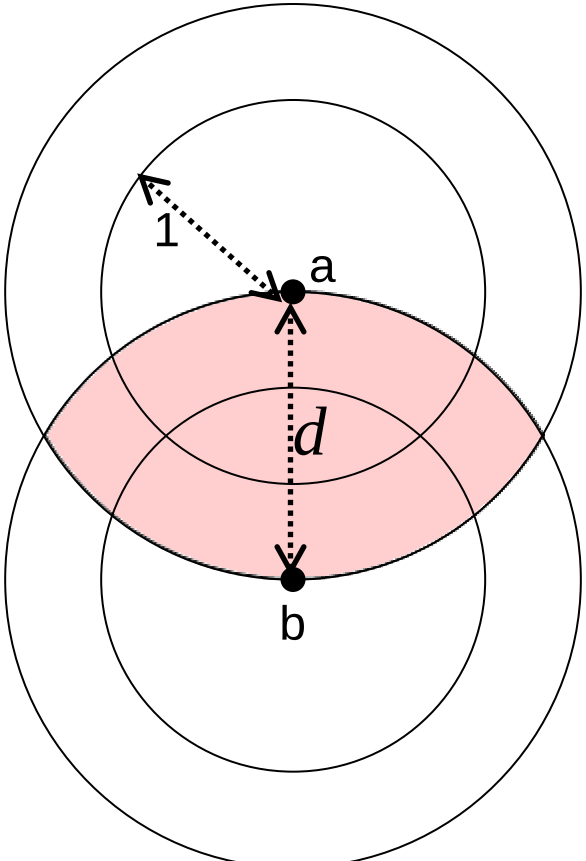
$\mathrm{pw}(K_3) = 1$

$\mathrm{pw}(K_4) = \sqrt{2}$

$\mathrm{pw}(K_5) = \frac{1+\sqrt{5}}{2}$

Proof of optimality for K_4 and K_5.

- Take an optimal realization of a complete graph *G* with $d < (1+\sqrt{5})/2$
- There exists two points *a* and *b* at a distance *d*
- Draw circles
- All other points must lie in shaded area
- Diameter of each area is < 1.
  Thus *G* contains at most 4 vertices.
- If $d < \sqrt{2}$, then the shortest distance between two areas is $> \sqrt{2}$
  Then *G* must contain at most 3 vertices.



Small Complete Graphs.

$\mathrm{pw}(K_3) = 1$

$\mathrm{pw}(K_4) = \sqrt{2}$

$\mathrm{pw}(K_5) = \frac{1+\sqrt{5}}{2}$

$\mathrm{pw}(K_6) = 2\sin 72^\circ$ [2]

$\mathrm{pw}(K_7) = 2$ [2]

$\mathrm{pw}(K_8) = (2\sin(\pi/14))^{-1}$ [3]

[2] P. Bateman and P. Erdős. Geometrical extrema suggested by a lemma of Besicovitch. *American Math. Monthly*, 58:306–314, 1951.

[3] A. Bezdek and F. Fodor. Minimal diameter of certain sets in the plane. *J. Comb. Theory, Ser. A*, 85:105–111, 1999.

Large Complete Graphs.

Theorem

$$\lim_{n\to\infty} \mathrm{pw}(K_n) = 2^{1/2}3^{1/4}\pi^{-1/2}\sqrt{n} \approx 1.05\sqrt{n}\,.$$

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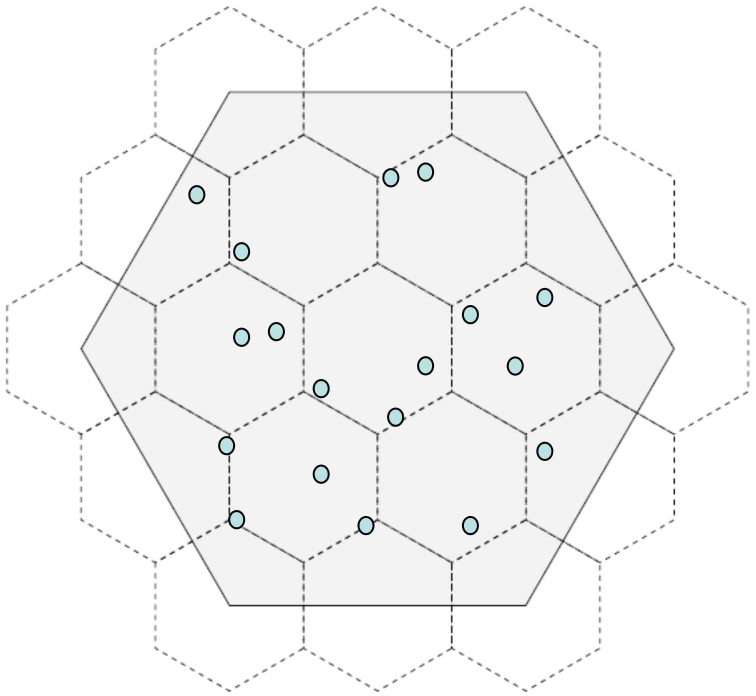
[10] P. Erdős. *Geometry and Differential Geometry, Lecture Notes in Mathematics 792*, Chapter: Some combinatorial problems in geometry, pages 46–53. New York: Springer-Verlag, 1980.

Graphs with Large Chromatic Number

Theorem *For every* $\varepsilon > 0$ *there exists a* $k \geq 3$ *such that for all graphs* G *of chromatic number at least* k,

$$0.86\sqrt{\chi(G)} - \varepsilon < \mathrm{pw}(G) < 1.05\sqrt{\chi(G)} + \varepsilon.$$



- Upper bound follows from

  $$\mathrm{pw}(G) \leq \mathrm{pw}(K_{\chi(G)})$$

  and

  $$\lim_{n\to\infty} \mathrm{pw}(K_n) = 1.05\sqrt{n}.$$

- Lower bound:

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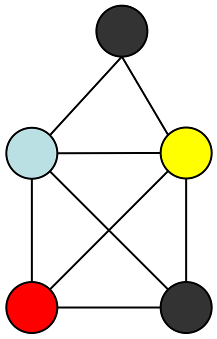
Definition

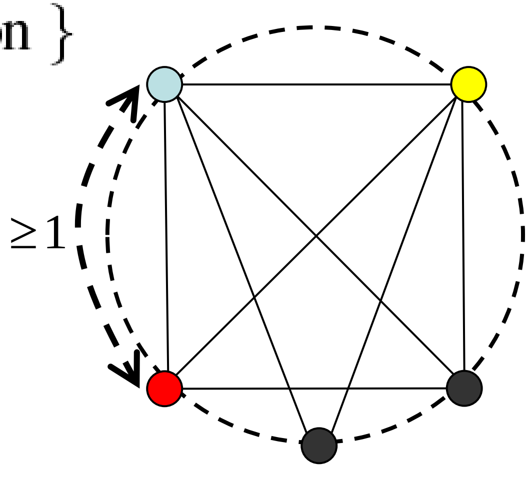
Given a graph $G = (V, E)$, an **r-circular realization** is

- A mapping which assigns each $v \in V$ to a point on a circle of radius r.
- Such that two adjacent vertices are at least a distance of 1 apart along the circumference.

The **circular chromatic number** is

$$\chi_c(G) = \inf \{ \; 2\pi c \; : \; G \text{ admits a } c\text{-circular realization} \; \}$$

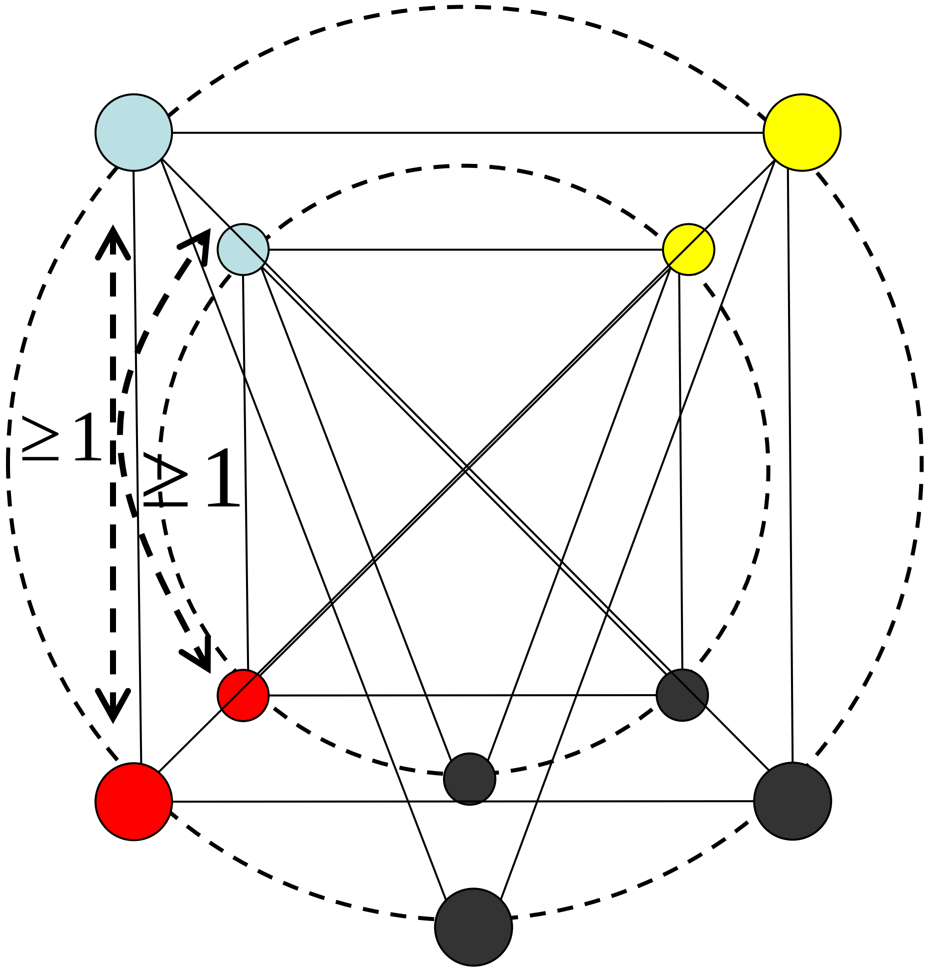




Theorem (Vince '88): For every rational q, there exists a graph with circular chromatic number q.

Lemma *For all graphs G,* $\mathrm{pw}(G) \leq \left[\sin\left(\frac{\pi}{\chi_c(G)}\right)\right]^{-1}$

Proof.



Theorem *For every $\varepsilon > 0$ there exists*

(a) A 4-chromatic graph G such that $\mathrm{pw}(G) < \boxed{2/\sqrt{3}} + \varepsilon$,

(b) A 5-chromatic graph G such that $\mathrm{pw}(G) < \boxed{\sqrt{2}} + \varepsilon$,

(c) An 8-chromatic graph G such that $\mathrm{pw}(G) < 2 + \varepsilon$.

Theorem *For all graphs G,*

(a) $\mathrm{pw}(G) = 1$ *if and only if* $\chi(G) \leq 3$,

(b) $\mathrm{pw}(G) \notin (1, 2/\sqrt{3}]$,

(c) $\mathrm{pw}(G) \in \boxed{(2/\sqrt{3}}, \sqrt{2}]$ *if and only if* $\chi(G) = 4$,

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Graph Operations

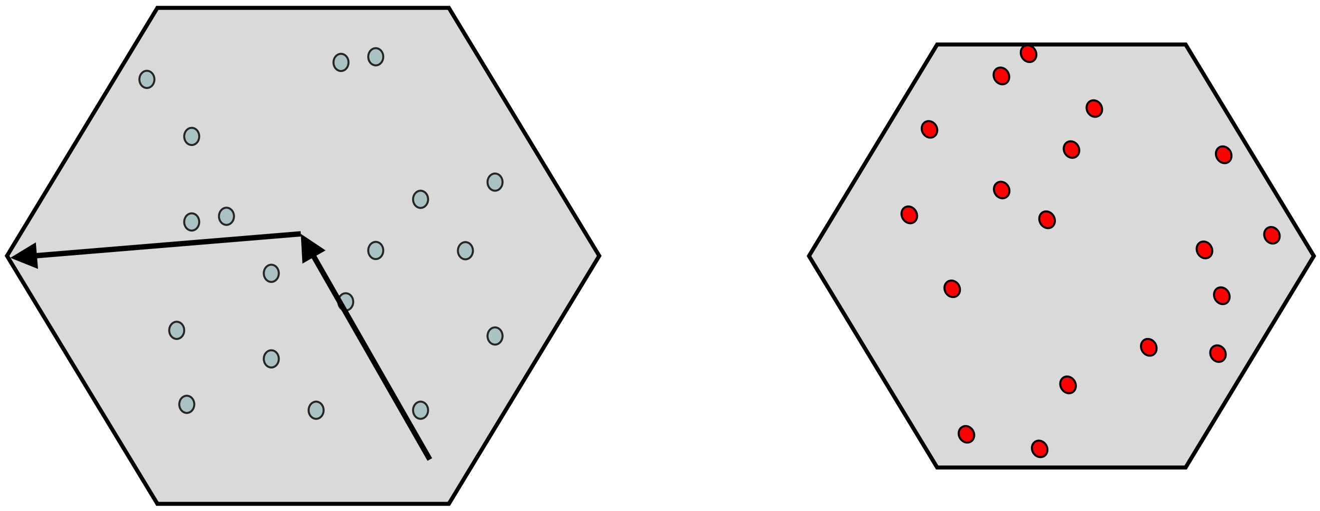
- Homomorphism
- Join
- Disjoint Union
- Cartestian Product
- Complement

Disjoint Union

Theorem *For every two graphs G and H, we have that*

$$\mathrm{pw}(G \uplus H) \leq \max\left(\mathrm{pw}(G), \mathrm{pw}(H), \frac{1}{\sqrt{3}}(\mathrm{pw}(G) + \mathrm{pw}(H))\right).$$

Proof.



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Open problems

- Let $\mathbb{P} = \{\mathrm{pw}(G) \ : \ G \textit{ is a graph}\}$. Determine whether there exists a monotone function $f : \mathbb{P} \to \mathbb{Z}$ such that for every non-bipartite graph G, $f(\mathrm{pw}(G)) = \chi(G)$.

- What is the value of $\inf\{\mathrm{pw}(G) : \chi(G) = k\}$?

- Is it true that, for n>2, $\mathrm{pw}(K_n) < \mathrm{pw}(K_{n+1})$

- Generalization to other norms
- Generalization to more dimensions
- Algorithmic aspects / Practical motivation