A classification of tetravalent half-arc-transitive weak metacirculants of girth 4

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This is joint work with Primož Šparl.

Definition

Let X be a graph (without multiple edges, loops or semi-edges) with vertex set $V(X)$, edge set $E(X)$ and arc set $A(X)$. X is said to be vertex-transitive (VT), edge-transitive (ET) and arc-transitive (AT) if the automorphism group $Aut(X)$ is transitive on $V(X)$, $E(X)$ and $A(X)$ respectively.

Definition

A graph X is said to be half-arc-transitive (HAT) if it is VT and ET but not AT.

Weak metacirculants

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Let $m \geq 1$ and $n \geq 2$ be integers. An automorphism of a graph is called (m, n)-*semiregular* if it has m orbits of size n and no other orbits on the vertices of the graph.

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Definition

We call a graph X *weak* (m, n)-*metacirculant* if

1. there exists a (m, n)-semiregular automorphism ρ of X;
2. $\exists \sigma \in Aut(X) : \sigma^{-1}\rho\sigma = \rho^r$ for some $r \in \mathbb{Z}_n^*$ which cyclically permutes all of the the orbits of ρ.

Definition

We say that X is a *weak metacirculant* if it is a weak (m, n)-metacirculant for some positive integers m and n.

Weak metacirculant

Example

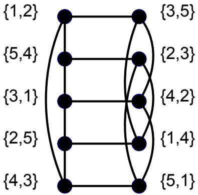

Figure: Petersen graph as a weak (2, 5)-metacirculant

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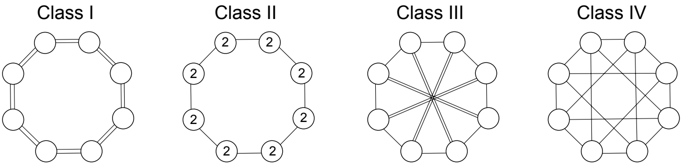
Quartic Weak metacirculants - classes

Proposition (Marušič, Šparl, 2008)

Let X be a connected HAT weak metacirculant relative to (ρ, σ). Then X belongs to one (or more) of the following four classes of graphs according to its quotient (multi)graph relative to ρ:



Weak metacirculants - $\mathcal{X}_o(m, n; r)$

Example

For each $m \geq 3$, $n \geq 3$ odd, $r \in \mathbb{Z}_n^*$, where $r^m = \pm 1$ let $\mathcal{X}_o(m, n; r)$ be the graph with vertex set $V = \left\{ u_i^j \mid i \in \mathbb{Z}_m, \, j \in \mathbb{Z}_n \right\}$ and edges defined by the following adjacencies

$$u_i^j \sim u_{i+1}^{j \pm r^i} \quad i \in \mathbb{Z}_n, \, j \in \mathbb{Z}_m. \tag{1}$$

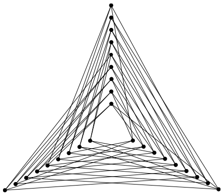


Figure: Doyle-Holt graph as a metacirculant $\mathcal{X}_o(3, 9; 2)$

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Weak metacirculants - $\mathcal{X}_e(m, n; r, t)$

Example

For integers m, $n \geq 4$, n even, $r \in \mathbb{Z}_n^*$, $t \in \mathbb{Z}_n$ satisfying $r^m = 1$, $t(r-1) = 0$ let $\mathcal{X}_e(m, n; r, t)$ be the graph with vertex set $V = \left\{ u_i^j \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n \right\}$ and edges defined by adjacencies

$$u_i^j \sim \begin{cases} u_{i+1}^j, u_{i+1}^{j+r^i} & ; i \in \mathbb{Z}_m \setminus \{m-1\}, j \in \mathbb{Z}_n \\ u_0^{j+t}, u_0^{j+r^{m-1}+t} & ; i = m-1, j \in \mathbb{Z}_n. \end{cases} \tag{2}$$

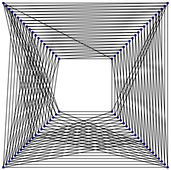

Figure: $\mathcal{X}_e(4, 20; 3, 10)$

Weak metacirculants - Class IV

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Let X be any connected HAT weak (m, n)-metacirculant of Class IV. Then X is completely determined by integers $m \geq 5$, $n \geq 3$, $r \in \mathbb{Z}_n^*$, $t \in \mathbb{Z}_n$, $p < q \in \mathbb{Z}_m \setminus \{0\}$, a, $b \in \mathbb{Z}_n$, thus we can denote $X = \mathcal{X}_{IV}(m, n; r, t, p, a, q, b)$.
The vertex set is $V = \{u_i^j;\ i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and the edges of X are defined by the following adjacency:

$$u_i^j \sim \begin{cases} u_{i+p}^{j+ar^i},\ u_{i+q}^{j+br^i} & 0 \leq i < m-q,\ j \in \mathbb{Z}_n \\ u_{i+p}^{j+ar^i},\ u_{i+q}^{j+br^i+t} & m-q \leq i < m-p,\ j \in \mathbb{Z}_n \\ u_{i+p}^{j+ar^i+t},\ u_{i+q}^{j+br^i+t} & m-p \leq i < m,\ j \in \mathbb{Z}_n \end{cases} \tag{3}$$

Weak metacirculants - $\mathcal{T}(m, n; r)$

Example

Let $m, \ n$ be such integers that $4|m$ and let $r \in \mathbb{Z}_n^*$ be such that $r^m = 1$. Then the graph $\mathcal{X}_{IV}(m, n; r, -1 - r^{\frac{m}{2}}, 1, 0, \frac{m}{2} - 1, 1)$ is denoted by $\mathcal{T}(m, n; r)$. Graph $\mathcal{T}(m, n; r)$ has girth 4.

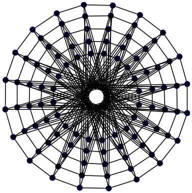

Figure: $\mathcal{T}(20; 5, 2)$

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HAT tetravalent weak metacirculants of girth 4

Theorem (A., Šparl; 201?)

Let $m \geq$, $n \geq 3$ and $r \in \mathbb{Z}_n^*$ be integers. A connected graph X is a half-arc-transitive weak (m, n)-metacirculant of valency 4 and girth 4 if and only if one of the following holds:

- $X \cong \mathcal{X}_o(4, n; r)$ for n odd, $r^4 = 1$, $r^2 \neq \pm 1$ and either $1 + r + r^2 + r^3 = 0$ or $1 - r + r^2 - r^3 = 0$.
- $X \cong \mathcal{X}_e(4, n; r, t)$ for n even, $r^4 = 1$, $r^2 \neq \pm 1$, $t(r-1) = 0$, $1 + r + r^2 + r^3 + 2t = 0$ and $t \in \{0, -1 - r^2\}$.
- $X \cong \mathcal{T}(m, n; r)$ for $m \geq 5$, $m \equiv 4 \pmod 8$, $r^4 = 1$, $r^2 \neq \pm 1$ and $1 - r + r^2 - r^3 = 0$.

HAT graphs of Class IV and girth 4

Theorem (A.,Šparl; 201?)

Let $m \geq 5$, $n \geq 3$ be integers. A connected quartic graph X of girth 4 is a half-arc-transitive weak (m, n)-metacirculant of Class IV if and only if $X \cong \mathcal{T}(m, n; r)$, where the following conditions are satisfied:

(i) $m \equiv 4 \, (mod\, 8)$,

(ii) $r^4 = 1$ and $r^2 \neq 1$,

(iii) $1 - r + r^2 - r^3 = 0$.

On a HAT graph of valency 4 and girth 4

Theorem (Marušič, Nedela; 2002)

Let X be a half-arc-transitive graph of valency 4 and girth 4. Then the set of 4-cycles of X decomposes the edge set $E(X)$ and furthermore, either

(i) every 4-cycle is alternating or

(ii) every 4-cycle is directed.

Moreover, in case (ii) the vertex stabilizer $(Aut(X))_v$, is isomorphic to $\mathbb{Z}_2$.

Idea of the proof

$\Rightarrow$

Let $X = \mathcal{X}_{IV}(m, n; r, t, p, a, q, b)$ be a HAT graph of Class IV.

1. X is a Cayley graph $\Rightarrow r \neq 1$.

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4. Some computation gives that $X \cong \mathcal{T}(m, n; r)$ with $r^4 = 1$ and $m \equiv\ 4\ (mod\ 8)$.

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5. If $r^2 = 1$ we can find an automorphism that interchanges two adjacent vertices $\Rightarrow r^2 \neq 1$.
6. Definition of $\mathcal{T}(m, n; r)$ with $rt = t$ gives $1 - r + r^2 - r^3 = 0$.

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Let $X = \mathcal{T}(m, n; r)$, where $m \equiv 4 \pmod 8$, $1 = r^4 \neq r^2$ and $1 - r + r^2 - r^3 = 0$.

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1. X is of girth 4.
2. $\langle \rho, \sigma \rangle \leq Aut(X)$.

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1. X is of girth 4.
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3. $$\tau(u_i^j) = \begin{cases} u_0^{-j} & ; j \in \mathbb{Z}_n \\ u_{iq}^{1+r^q+r^{2q}+\cdots+r^{(i-1)q}-j+\lceil \frac{i-2}{2} \rceil t} & ; i \leq \frac{m}{2},\, j \in \mathbb{Z}_n \\ u_{iq}^{1+r^q+r^{2q}+\cdots+r^{(i-1)q}-j+\lceil \frac{i-3}{2} \rceil t} & ; i > \frac{m}{2},\, j \in \mathbb{Z}_n. \end{cases}$$

   is an automorphism of X.

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   is an automorphism of X.
4. Inspecting the structure of X with emphesis on the 4-cycles it can be shown that any automorphism of X that fixes a 4-cycle setwise and fixes one of its vertices is the identity.

Isomorphism classes

$\mathcal{X}_o(m, n; r)$

Let $X = \mathcal{X}_o(m, n; r)$ and $X' = \mathcal{X}_o(m', n'; r')$ be HAT. Then $X \cong X'$ iff $m = m'$, $n = n'$ and $r' \in \{r, -r, r^{-1}, -r^{-1}\}$

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$\mathcal{X}_e(m, n; r, t)$

Let $X = \mathcal{X}_e(m, n; r)$ and $X' = \mathcal{X}_e(m', n'; r')$ be HAT. Then $X \cong X'$ iff $m = m'$, $n = n'$ and

- $r' = r$ and $t' = t$ or
- $r' = -r$ and $t' = t + r + r^3 + \cdots + r^{m-1}$ or
- $r' = r^{-1}$ and $t' = t$ or
- $r' = -r^{-1}$ and $t' = t + r + r^3 + \cdots + r^{m-1}$.

Isomorphism classes

Let $X = \mathcal{T}(m, n; r)$ be HAT. Then there exists such $X' = \mathcal{T}(M, N; r')$ that and $X' \cong X$ and M is the largest possible.

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Let $X = \mathcal{T}(m,\, n;\, r)$ and $X' = \mathcal{T}(m,\, n;\, r')$ be HAT. Then $X \cong X'$ if and only if $r' \in \left\{r,\, r^{-1}\right\}$.

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$\mathcal{T}(m, n; r)$

Two quartic HAT weak (m, n)-metacirculants of girth 4 of Class IV are isomorphic if and only if they are isomorphic to the same graph $\mathcal{T}(M, N; r)$.

Thank you!