Characterizing vertex-transitive graphs

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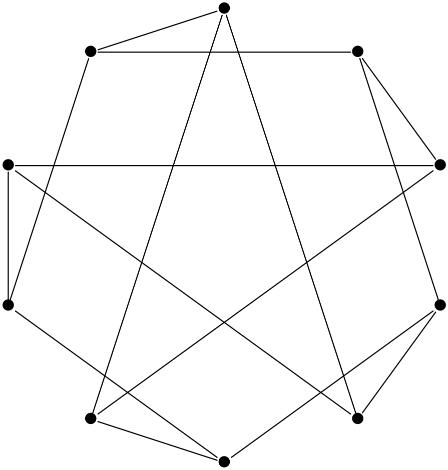

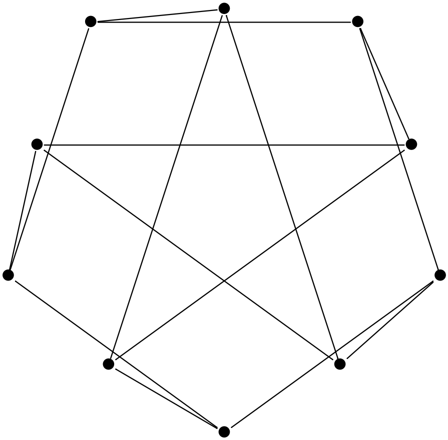

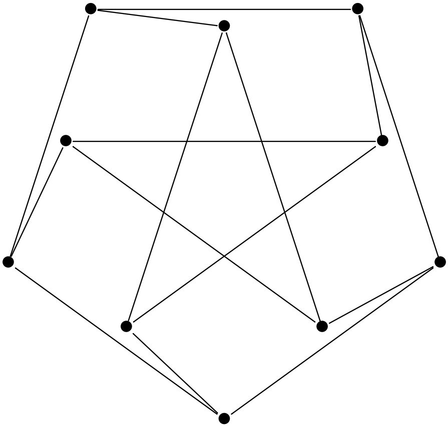

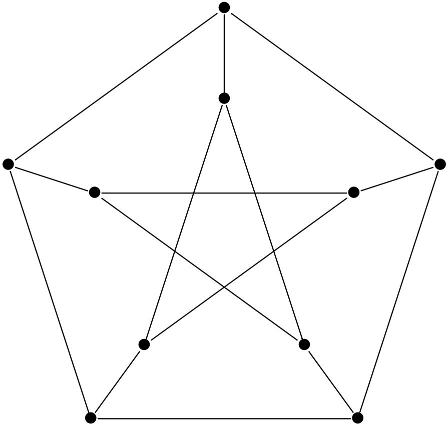

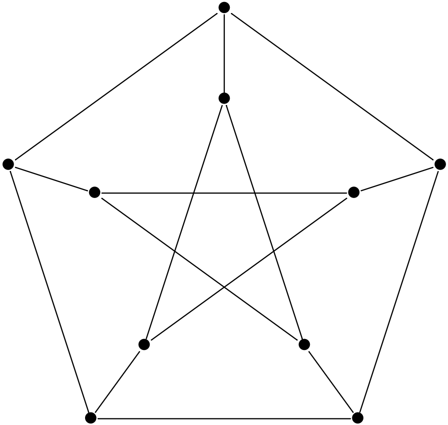

The Petersen graph is the unique graph of smallest order and smallest size that is not isomorphic to a Cayley graph.

Definition

Let $\alpha \in \mathbb{Z}_n^*$, and define $\rho, \tau : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $\rho(i,j) = (i, j+1)$ and $\tau(i,j) = (i+1, \alpha j)$. A graph Γ is an (m,n)-metacirculant graph if $V(\Gamma) = \mathbb{Z}_m \times \mathbb{Z}_n$ and $\langle \rho, \tau \rangle \leq \mathrm{Aut}(\Gamma)$.

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Problem

Characterize vertex-transitive graphs of order qp, and in particular, those which are not metacirculants.

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Theorem (Marušič and Scapellato, 1992)

A graph Γ *of order* qp *is isomorphic to a* (q,p)*-metacirculant graph if and only if* $\mathrm{Aut}(\Gamma)$ *contains a transitive subgroup* G *and* $H \triangleleft G$ *is nontrivial and intransitive.*

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> A nonsolvable permutation group of prime degree is by [15, Theorem 11.7] necessarily doubly transitive. On the other hand, it is easy to see that a vertex-transitive pq-graph whose automorphism group has a transitive subgroup with an intransitive normal subgroup must necessarily be a metacirculant. Note that $(k^n - 1)/(k - 1)$ is not a prime unless n is a prime and $(n, k - 1) = 1$. Hence Propositions 2.3 and 2.4 together imply the following result:

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Let Γ be a vertex-transitive graph of order pqr, p, q, and r distinct primes, whose automorphism group contains an imprimitive subgroup. Then one of the following is true:

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2. $\mathrm{Aut}(\Gamma)$ *is quasiprimitive, and is given in one of two tables in their paper.*

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A transitive group G of degree $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ and $m = \sum_{i=1}^{r} a_i$ is **genuinely** m**-step imprimitive** *if there exists normal subgroup $1 = N_0 \triangleleft N_1 \triangleleft \ldots \triangleleft N_{m-1} \triangleleft N_m = G$ and the orbits of N_i are properly contained in the orbits of N_{i+1}, $0 \leq i \leq m-1$.*

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Let n be such that $\gcd(n, \varphi(n)) = 1$. Then a vertex-transitive graph Γ of order n is isomorphic to a circulant graph of order n if and only if $\mathrm{Aut}(\Gamma)$ contains a genuinely m-step imprimitive subgroup.

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Let n be such that $\gcd(n, \varphi(n)) = q$ a prime. Then every vertex-transitive graph Γ of order n is isomorphic to a metacirculant graph of order n if and only if $\mathrm{Aut}(\Gamma)$ contains a genuinely m-step imprimitive subgroup.

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Let n be square-free such that there exists a unique prime p such that whenever r divides both n and $\varphi(n)$, then r divides $p-1$. Additionally, assume that if q divides n is prime, then $q^2 \not| (p-1)$. A vertex-transitive graph Γ of order n is isomorphic to a Cayley graph of order n if and only if $\mathrm{Aut}(\Gamma)$ contains an m-step imprimitive subgroup.

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Problem

Determine a minimal transitive subgroup of $\mathrm{Aut}(\Gamma)$ of order pqr such that $\mathrm{Aut}(\Gamma)$ is not quasiprimitive, primitive, and does not contain a genuinely 3-step imprimitive subgroup.

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A solution would lead to the classification of vertex-transitive graphs of some orders pqr. Some related work can be found in a paper by Iranmanesh and Praeger (2001).