Symmetric graphs with 2-arc transitive quotients

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dedicated to Dragan on his 60th birthday

motivation

- A G-symmetric graph Γ, which is not necessarily $(G, 2)$-arc transitive, may admit a natural $(G, 2)$-arc transitive quotient with respect to a G-invariant partition.

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- When does this happen? (Iranmanesh, Praeger and Z, 2005)

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- When does this happen? (Iranmanesh, Praeger and Z, 2005)
- If there is such a quotient, what information does it give us about the original graph? (Iranmanesh, Praeger and Z, 2005)

Observation

If Γ admits a $(G, 2)$-arc transitive quotient, then a natural 2-point transitive and block transitive design $\mathcal{D}^*(B)$ arises and plays a significant role in understanding the structure of Γ.

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- Li, Praeger and Zhou (2010): $k = v - 2$ and a natural auxiliary graph is a cycle
- Lu and Zhou (2007): constructions were given when $\mathcal{D}^*(B)$ or its complement is degenerate

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- When $v - k = 3$ or 5, these necessary conditions are essentially sufficient.
- At the end of this talk, I will mention briefly a result about Hamiltonicity of vertex-transitive graphs.

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- $\Gamma_{\mathcal{B}}(B)$: neighbourhood of B in $\Gamma_{\mathcal{B}}$
- $\mathcal{D}(B)$: incidence structure with point set B and block set $\Gamma_{\mathcal{B}}(B)$, in which $\alpha \in B$ and $C \in \Gamma_{\mathcal{B}}(B)$ are incident if and only if $\alpha \in \Gamma(C)$

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- $\mathcal{D}(B)$ is a 1-(v, k, r) design with b blocks (Gardiner and Praeger 1995)

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- $\mathcal{D}^*(B)$: dual of $\mathcal{D}(B)$ (swap 'points' and 'blocks')
- $\overline{\mathcal{D}^*}(B)$: complementary of $\mathcal{D}^*(B)$ (swap 'flags' and 'antiflags')
- If $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive, then in general, for some λ,

  $\mathcal{D}^*(B)$ is a 2-(b, r, λ) design with v blocks, and

  $\overline{\mathcal{D}^*}(B)$ is a 2-$(b, b-r, \overline{\lambda})$ design, where $\overline{\lambda} = v - 2k + \lambda$,

  except in some 'degenerate cases'.
- $\mathcal{D}^*(B)$ and $\overline{\mathcal{D}^*}(B)$ admit G_B as a group of automorphisms acting 2-transitively on points and transitively on blocks.

$v - k = p$ an odd prime: necessary conditions

Theorem

[Xu and Zhou, 2011-12]
Suppose $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive and $v - k = p \geq 3$ is a prime. Then one of the following occurs:

Case	$\overline{\mathcal{D}^*}(B)$	(v, b, r, λ)	Conditions
(a)		$(p+1, p+1, 1, 0)$	
(b)		$(2p, 2, 1, 0)$	
(c)	$\mathrm{PG}_{n-1}(n, q)$	$\left(\frac{q^{n+1}-1}{q-1}, \frac{q^{n+1}-1}{q-1}, q^n, q^n - q^{n-1}\right)$	$p = \frac{q^n-1}{q-1}$, $n \geq 2$ q is a prime power $\frac{q^n-1}{q-1}$ is a prime
(d)	2-$(11, 5, 2)$	$(11, 11, 6, 3)$	$p = 5$
(e)		$(pa, a, a-1, p(a-2))$	$a \geq 3$
(f)		$\left(pa, ps+1, \frac{(ps+1)(a-1)}{a}, p(a-2) + \frac{ps-a+1}{as}\right)$	$a \geq 2$, $s \geq 1$ a is a divisor of $ps+1$ s is a divisor of $\frac{ps-a+1}{a}$ $\frac{a-1}{p-a} \leq s \leq a-1 \leq p-2$

Theorem
(cont'd)

Moreover, the following hold in each case:

(a) $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$, and any connected $(p+1)$-valent $(G, 2)$-arc transitive graph can occur as $\Gamma_{\mathcal{B}}$ in (a).

(b) $\Gamma \cong n \cdot \Gamma[B, C]$ and $\Gamma_{\mathcal{B}} \cong C_n$.

(c) $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \leq \mathrm{P\Gamma L}(n+1, q)$ (2-transitive subgroup).

(d) $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(2, 11)$.

(e) $V(\Gamma)$ admits a G-invariant partition $\mathcal{P}$ with block size p that is a refinement of $\mathcal{B}$, such that $\Gamma_{\mathcal{P}}$ can be 'constructed' from $\Gamma_{\mathcal{B}}$ by the 3-arc graph construction.

(f) if $s = 1, 2$, then all possibilities are given in the next two tables, respectively.

$G_B^{\Gamma_{\mathcal{B}}(B)}$	$\mathcal{D}^*(B)$	(v, b, r, λ)	Conditions
A_{p+1}	$\overline{\mathcal{D}^*}(B) \cong K_{p+1}$		$a = \frac{p+1}{2}$
$\leq \mathrm{AGL}(n, 2)$		$\begin{pmatrix} 2^m(2^n - 1) \\ 2^n \\ 2^n - 2^{n-m} \\ (2^m - 1)(2^n - 2^{n-m} - 1) \end{pmatrix}$	$1 \leq m \leq n - 1$ $p = 2^n - 1$ a Mersenne prime $r^* = (2^n - 1)(2^m - 1)$
$\leq \mathrm{PGL}(2, p)$			$a - 1$ a divisor of $p - 1$
$\mathrm{Sp}_4(2)$	2-(6, 3, 2)		$p = 5$
M_{11}	2-(12, 6, 5)		$p = 11$ $\mathcal{D}^*(B)$ is a Hadamard 3-subdesign of the Witt design W_{12} (3-(12, 6, 2) design)

Table: Possibilities when $s = 1$ in case (f).

$G_B^{\Gamma_{\mathcal{B}}(B)}$	$\mathcal{D}^*(B)$	(v, b, r, λ)	Conditions
$\leq \mathrm{AGL}(n, 3)$		$\begin{pmatrix} \frac{(3^n-1)3^j}{2} \\ 3^n \\ 3^{n-j}(3^j-1) \\ \frac{(3^n-1)(3^j-2)}{2} + \frac{3^{n-j}-1}{2} \end{pmatrix}$	$n \geq 3$ odd $p = \frac{3^n-1}{2}$ $1 \leq j \leq n-1$
$\leq \mathrm{PGL}(n, 2)$		$\begin{pmatrix} a(2^{n-1}-1) \\ 2^n - 1 \\ \frac{(2^n-1)(a-1)}{a} \\ (2^{n-1}-1)(a-2) + \frac{2^n-1-a}{2a} \end{pmatrix}$	a an odd divisor of $2p+1$ $3 \leq a \leq \frac{2p+1}{3}$ $p = 2^{n-1} - 1$ a Mersenne prime ($n - 1 \geq 3$ a prime)
A_7	$\overline{\mathcal{D}^*}(B) \cong \mathrm{PG}(3, 2)$	$(35, 15, 12, 22)$	

Table: Possibilities when $s = 2$ in case (f).

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2. The condition $(v, b, r, \lambda) = (pa, a, a-1, p(a-2))$ in (e) is sufficient for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc transitive.

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3. We have examples for the third row of the second table (due to Yuqing Chen).
4. For general p, we do not know whether these necessary conditions are sufficient.

3-arc graph

Given a graph Γ, the *3-arc graph* of Γ, $X(\Gamma)$, is defined to have the set of arcs of Γ as its vertex set, such that two arcs uv and xy are adjacent if and only if (v, u, x, y) is a 3-arc of Γ.

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4. v is not a multiple of $p \Rightarrow \mathcal{D}^*(B)$ is a 2-transitive symmetric 2-$\left(pa+1, p(a-1)+1, p(a-2)+\frac{p+a-1}{a}\right)$ design $\Rightarrow \mathcal{D}^*(B)$ or $\overline{\mathcal{D}^*}(B)$ is known (due to Kantor) $\Rightarrow$ case (c) or (d)

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5. $v = pa$ is a multiple of $p \Rightarrow$ case (e) or (f)

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5. $v = pa$ is a multiple of $p \Rightarrow$ case (e) or (f)
6. $s = 1$ or 2 in case (f): classification of 2-transitive symmetric designs

$p = 3$

Theorem

[Xu and Zhou, 2011-12]

Suppose that $v - k = 3$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive iff one of the following holds:

(a) $(v, b, r, \lambda) = (4, 4, 1, 0)$, $G_B^B \cong A_4$ or S_4;

(b) $(v, b, r, \lambda) = (6, 2, 1, 0)$, $\Gamma_{\mathcal{B}} \cong C_n$;

(c) $(v, b, r, \lambda) = (7, 7, 4, 2)$, $G_B^B \cong \mathrm{PSL}(3, 2)$;

(d) $(v, b, r, \lambda) = (3a, a, a - 1, 3a - 6)$ for some $a \geq 3$;

(e) $(v, b, r, \lambda) = (6, 4, 2, 1)$, $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_4$ or S_4.

Theorem
(cont'd)
Moreover, in each case we have the following:

(a) $\Gamma \cong (|V(\Gamma)|/2) \cdot K_2$, any connected 4-valent 2-arc transitive graph can occur as $\Gamma_{\mathcal{B}}$.
(b) $\Gamma \cong 3n \cdot K_2, n \cdot C_6$ or $n \cdot K_{3,3}$.
(c) $\overline{\mathcal{D}}(B) \cong \mathrm{PG}(2,2)$, $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(3,2)$, and $\Gamma[B,C] \cong 4 \cdot K_2$, $K_{4,4} - 4 \cdot K_2$ or $K_{4,4}$; in the first case Γ is $(G,2)$-arc transitive.
(d) $V(\Gamma)$ admits a G-invariant partition $\mathcal{P}$ with block size 3 that is a refinement of $\mathcal{B}$, such that $\Gamma_{\mathcal{P}}$ can be 'constructed' from $\Gamma_{\mathcal{B}}$ by the 3-arc graph construction.
(e) Γ can be constructed from $\Gamma_{\mathcal{B}}$ as a '2-path graph', and every connected 4-valent $(G,2)$-arc transitive graph can occur as $\Gamma_{\mathcal{B}}$ in (e).

$p = 5$

Theorem

[Xu and Zhou, 2011-12]

Suppose that $v - k = 5$. Then $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc transitive iff one of the following holds:

(a) $(v, b, r, \lambda) = (6, 6, 1, 0)$, $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_6$ or S_6;

(b) $(v, b, r, \lambda) = (10, 2, 1, 0)$, $\Gamma_{\mathcal{B}} \cong C_n$ and $G/G_{(\mathcal{B})} \leq D_{2n}$, where $n = |V(\Gamma)|/10$;

(c) $(v, b, r, \lambda) = (21, 21, 16, 12)$, $\overline{\mathcal{D}^*}(B) \cong \mathrm{PG}(2, 4)$, $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a 2-transitive subgroup of $\mathrm{P\Gamma L}(3, 4)$, and G is faithful on $\mathcal{B}$;

(d) $(v, b, r, \lambda) = (11, 11, 6, 3)$, $\overline{\mathcal{D}^*}(B)$ is isomorphic to the unique 2-$(11, 5, 2)$ design and $G_B^B \cong G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{PSL}(2, 11)$;

(e) $(v, b, r, \lambda) = (5a, a, a - 1, 5a - 10)$ for some $a \geq 3$;

Theorem

(cont'd)

(f) one of the following occurs:

1. $(v, b, r, \lambda) = (10, 6, 3, 2)$, $\mathcal{D}^*(B)$ is isomorphic to the unique 2-$(6, 3, 2)$ design, and $G_B^{\Gamma_{\mathcal{B}}(B)} \cong \mathrm{Sp}_4(2)$ or $\mathrm{PSL}(2, 5)$;
2. $(v, b, r, \lambda) = (15, 6, 4, 6)$, $\mathcal{D}^*(B)$ is isomorphic to the complementary design of K_6 and $G_B^{\Gamma_{\mathcal{B}}(B)} \cong A_6$;
3. $(v, b, r, \lambda) = (20, 16, 12, 11)$, $\overline{\mathcal{D}^*}(B) \cong \mathrm{AG}(2, 4)$ and $G_B^{\Gamma_{\mathcal{B}}(B)}$ is isomorphic to a 2-transitive subgroup of $\mathrm{A\Gamma L}(2, 4)$.

Hamiltonicity of 3-arc graphs

Theorem

[Xu and Zhou, 2011-12]
Let Γ be a graph without isolated vertices. The 3-arc graph $X(\Gamma)$ of Γ is hamiltonian if and only if

(a) $\delta(\Gamma) \geq 2$;

(b) no two degree-two vertices of Γ are adjacent; and

(c) the subgraph obtained from Γ by deleting all degree-two vertices is connected.

(Graphs and Combinatorics, to appear)

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Corollary

[Xu and Zhou, 2011-12]
If a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is hamiltonian.