Reachability relations, transitive digraphs and groups

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Introduction

P. J. Cameron, C. E. Praeger, N. C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs, Combinatorica 13 (4) (1993), 377–396.

Reachability relation on edges: e is *reachable* from f if there exists an alternating walk containing e and f.

Reachability relation on vertices:
$W = (v_0, \epsilon_1, v_1, \epsilon_2, v_2, \ldots, \epsilon_n, v_n)$ from v_0 to v_n is a sequence of $n+1$ vertices and n indicators $\epsilon_1, \ldots, \epsilon_n$ such that

$$\begin{aligned} \varepsilon_j = 1 \quad &\Rightarrow \quad (v_{j-1}, v_j) \in E(W), \\ \varepsilon_j = -1 \quad &\Rightarrow \quad (v_j, v_{j-1}) \in E(W). \end{aligned}$$

Weight $\omega(W) = \Sigma_{i=1}^{i=n} \epsilon_i$

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$$uR_k^+ v$$

if there exists a walk W from u to v with $\omega(W) = 0$ and $\omega({}_0W_j) \in [0, k]$ for every $0 \leq j \leq |W|$. Analogously $uR_k^- v$.

$$R_k^+(v) = \{u \in V(D) | vR_k^+ u\}$$
$$R_k^-(v) = \{u \in V(D) | vR_k^- u\}$$

$R_k^+ \subseteq R_{k+1}^+$, $R_k^- \subseteq R_{k+1}^-$

$$R^+ = \bigcup_{k \in \mathbb{Z}^+} R_k^+, \qquad R^- = \bigcup_{k \in \mathbb{Z}^+} R_k^-$$

$(R_k^+)_{k \in \mathbb{Z}^+}, (R_k^-)_{k \in \mathbb{Z}^+}$

exponent $\exp^+(D)$ is the smallest nonnegative integer k such that $R_k^+ = R^+$. Analogously $\exp^-(D)$.

Introduction

D . . . connected, vertex-transitive, infinite, locally finite

Structure of D/R^+:

- a finite cycle
- directed infinite line
- regular tree with indegree 1 and outdegree > 1

A. Malnič, D. Marušič, N.S., P. Šparl, B. Zgrablič, Reachability relations in digraphs, European J. Combin. 29 (2008), 1566 - 1581.

Are there connections between R_k^+ (R_k^-) and the end structure of D?

D has property **Z** if there exists a homomorphism from D onto the directed infinite line.

Introduction

- If D has infinitely many ends, then it has property **Z** if and only if at least one of the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ is infinite.
- If D has property **Z** and the sequences $(R_k^+)_{k\in\mathbb{Z}^+}$ and $(R_k^-)_{k\in\mathbb{Z}^+}$ are both finite and there exists an integer $k \geq 1$ such that R_k^+ (and hence R_k^-) has infinite equivalence classes, then D has one end.
- If D has two ends, then it has property **Z** if and only if for each integer $k \geq 1$ at least one (and hence both) of the relations R_k^+ and R_k^- have finite equivalence classes.

Introduction

Connections between R_k^+ (R_k^-) and growth properties?

$$f_D(v,n) = |\{u \in V(D) | dist_D(v,u) \leq n\}|$$

- polynomial growth: $f_D(n) \leq cn^d$ for all $n \geq 1$
- exponential growth: $f_D \geq c^n$ for all $n \geq 1$
- intermediate growth: E. g. $2^{\sqrt{n}} < f_D(n) < 2^{n^{log_{32}31}}$

If at least one of the sequences $(R_k^+)_{k \in \mathbb{Z}^+}$ and $(R_k^-)_{k \in \mathbb{Z}^+}$ is infinite, then D has exponential growth.

R_k^+ (R_k^-) and growth of groups

Both sequences finite $\Rightarrow$ polynomial or intermediate growth

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- Is it possible to find conditions for R_k^+ (R_k^-) which imply polynomial or intermediate growth?
- Do there exist bounds for $\exp^+(D)$ ($\exp^-(D)$) in the case of polynomial growth?

R_k^+ (R_k^-) and growth of groups

- If an abelian group acts transitively on D, then $\exp^+(D) = \exp^-(D) = 1$.

Nilpotent groups?

$G^0 = G,\ G^{i+1} = [G^0, G^i], i \geq 0$

$$G = G^0 \triangleright G^1 \triangleright \ldots \triangleright G^k \triangleright G^{k+1} = 1$$

nilpotent of class k.

- Let G be a nilpotent group of class $k \geq 0$ acting transitively on D. Then $\exp^+(D) \leq k+1$ and $\exp^-(D) \leq k+1$.

R_k^+ (R_k^-) and growth of groups

This bound is tight! $\rightarrow D_8$

Infinite family of nilpotent groups:
G_n semidirect product of the elementary abelian group $\mathbb{Z}_2^n$ by the cyclic group $\mathbb{Z}_{2^{n-1}}$ generated by $G_n = \langle f, a_1, a_2, \ldots, a_n \rangle$. f cyclic of order 2^{n-1}, a_i involutions. $fa_if^{-1} = a_ia_{i+1}$, $1 \leq i \leq n-1$. $a_ia_j = a_ja_i$, $fa_n = a_nf$.

$S = \{f, fa_1\}$, $\langle S^{-i}S^i \rangle = \langle a_1, a_2, \ldots, a_i \rangle$, $1 \leq i \leq n$. $\Rightarrow$
$\exp^-(Cay(G_n, S)) = n$.

G_n is nilpotent of class $n-1$. $G^{(i)} = \langle a_{i+1}, a_{i+2}, \ldots, a_n \rangle$ holds for each i, $1 \leq i \leq n-1$. Also $G^{(n)} = 1$.

R_k^+ (R_k^-) and growth of groups

No bound for solvable groups! $\rightarrow$ lamplighter group. L is the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}$

$$L = \langle a, t | a^2, [t^m a t^{-m}, t^n a t^{-n}], m, n \in \mathbb{Z} \rangle.$$

$S = \{t, at\}$, $Cay(L, S)$ horocyclic product of two trees with indegree 1, outdegree 2.

R_k^+ (R_k^-) and growth of groups

G finitely generated with polynomial growth $\Rightarrow$ G contains a normal nilpotent subgroup N of finite index.

- Let the finitely generated group G act transitively on the connected digraph D such that a normal nilpotent subgroup N of G, where N is nilpotent of class $k \geq 0$, acts with m, $1 \leq m < \infty$, orbits on D. Then $\exp^+(D) \leq m(k+1) + m - 1$ and $\exp^-(D) \leq m(k+1) + m - 1$.

All examples we know satisfy $\exp^+(D) \leq m(k+1)$ and $\exp^-(D) \leq m(k+1)$.

- Let the finitely generated group G act transitively on the connected digraph D such that a normal abelian subgroup N of G acts with m, $1 \leq m < \infty$, orbits on D. Then $\exp^+(D) \leq m$ and $\exp^-(D) \leq m$.

R_k^+ (R_k^-) and growth of groups

- The orders of the finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ are bounded by some function $g(n)$ alone.
- Let G be a finitely generated torsion-free group with polynomial growth of degree d. Then G contains a normal nilpotent subgroup of class $< \sqrt{2d}$ and index at most $g(d)$, where $g(d)$ is the above function.
- Let G be a finitely generated torsion-free group with polynomial growth of degree d. Then for any Cayley graph D of G,
  $\exp^+(D) \leq g(d)(\sqrt{2d} + 1) + g(d) - 1$ and
  $\exp^-(D) \leq g(d)(\sqrt{2d} + 1) + g(d) - 1$.

R_k^+ (R_k^-) and growth of groups

Is it true that every finitely generated infinite simple group has exponential growth? (Grigorchuk)

- If a finitely generated infinite simple group G does not have exponential growth, then for every finite generating set S of G there is a finite integer $k_S \geq 1$, such that $R_{k_S}^+ = R_{k_S}^-$ is universal in $C(G, S)$.
- Let G be a finitely generated infinite simple group and let S denote a finite generating set. Furthermore, let $H \subseteq G$ denote the set of all those $h \in G$ which leave invariant at least one equivalence class of R_1^+ on $C(G, S)$. Then $\langle H \rangle = G$.