On regular Cayley maps over dihedral groups

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joint work with Dragan Marušič and István Kovács

Dihedrants and circulants

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Problem (Marušič)

Classify connected arc-transitive (arc-regular) dihedrants.

Known results

2-arc-transitive dihedrants were classified by S.F. Du, A. Malnič and D. Marušič (2008).

Arc-transitive dihedrants of degrees $4, 6$ were studied by Y.H Kwak, Y.M Oh, C.Q. Wang, M. Xu and Z.Y. Zhou (2006) and I. Kovacs, B. Kuzman, A. Malnič.

D. Kim, Y.S. Kwon and J. Lee proved that an arc-regular dihedrants of prime degree are normal

Y.H. Kwak, Y.S. Kwon and Y.-M. Oh constructed arc-regular dihedrants of any prescribed valency (2008).

I. Kovács classified arc-transitive dihedrants of order $2p^e, p$ an odd prime (2012).

Dihedrants and circulants. Doubling of circulants

Let $\Gamma = (V, E)$ be a digraph. It's *canonical double cover* (CDC) $\widetilde{\Gamma}$ is the undirected bipartite graph with vertex set $V \times \mathbb{Z}_2$ where two vertices $(x, 0), (y, 1)$ are connected by an edge whenever (x, y) is an arc of Γ.
A CDC of a circulant $\mathrm{Cay}(\mathbb{Z}_n, S)$ produces a bipartite graph with vertex set $\mathbb{Z}_n \times \{0, 1\}$ where the vertices $(x, 0)$ and $(y, 1)$ are connected iff $x - y \in S$.
Another presentation of the same graph may be obtained if we connect two vertices $(x, 0), (y, 1)$ whenever $x + y \in S$. An isomorphism between two graphs is given by $(x, i) \mapsto ((-1)^i x, i)$.

Doubling of circulants

A permutation $s : (x, i) \mapsto (-x, i+1)$ is an order two automorphism of a CDC of $\mathrm{Cay}(\mathbb{Z}_n, S)$. Together with the permutation $c : (x, i) \mapsto (x+1, i)$ it generates a dihedral group of order $2n$ which acts regularly on the point set $\mathbb{Z}_n \times \mathbb{Z}_2$. Notice that every dihedrant $\mathrm{Cay}(D_{2n}, S)$ where S is a set of reflections may be obtained in this way.

Proposition

If Γ is a CDC of a G-arc-transitive circulant, then $\widetilde{\Gamma}$ is a $\langle \widetilde{G}, s \rangle$-arc-transitive dihedrant.

Dihedrants and circulants. Lexicographic product.

Definition

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two graphs. Their *lexicographic* product has a vertex set $V \times V'$, two vertices (v, v') and (w, w') are connected in $G[G']$ iff

$$vv' \in E \text{ or } v = v' \wedge ww' \in E'$$

Proposition

If $\Gamma = (V, E)$ is an arc-transitive circulant, then $\Gamma[\overline{K_2}]$ is an arc-transitive dihedrant.

Dihedrants as Cayley graphs over $\mathbb{Z}_n \times \mathbb{Z}_2$

Let $S \subseteq \mathbb{Z}_n \times \mathbb{Z}_2$ be a symmetric subset. Then the map

$$s : (x, y) \mapsto (-x, 1 - y)$$

is an automorphism of the Cayley graph $\Gamma := \mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S)$. Together with the automorphism $c : (x, y) \mapsto (x + 1, y)$ it generates a dihedral regular group of automorphisms of Γ.

Proposition

If Γ is a G-arc-transitive/regular and $s \in G$, then Γ is a G-arc-transitive/regular dihedrant. In particular, if Γ is arc-transitive/regular, then Γ is an arc-transitive/regular dihedrant.

Arc-regular dihedrant with a trivial cyclic core

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Theorem

Let Γ be a G-arc-regular connected dihedrant of order $2n$. Let $D \leq G$ be a regular dihedral subgroup and C a cyclic subgroup of D of index 2. If the core of C in G is trivial, then

- $n = 1, \Gamma \cong K_2, G = S_2$;
- $n = 2, \Gamma \cong K_4, G = A_4$;
- $n = 3, \Gamma \cong K_{2,2,2}, G = S_4$;
- $n = 4, \Gamma \cong Q_3, G = S_4$;
- $n = 2m, m$ odd, $\Gamma \cong K_{n,n}, G = (D_n \times D_n) \rtimes S_2$

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If H is a cyclic subgroup of G of index n, then

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In the case of equality n is a prime power and $G \cong AGL_1(n)$.

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Theorem

Let H be a dihedral subgroup of G of index n, then

$$[G : \mathrm{Core}_G(H)] \leq 2n^2.$$

In the case of equality $G \cong (D_n \times D_n) \rtimes S_2$, $n = 2m$, m is odd.

Quotients of arc-transitive dihedrants

Theorem (Wielandt)

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Corollary

If Γ is a connected non-complete G-arc-transitive dihedrant, then G is imprimitive.

If $\mathcal{B}$ is an imprimitivity system of G, then its block $\mathcal{B}(1)$ is a subgroup of D and all blocks of $\mathcal{B}$ are the right cosets of $\mathcal{B}(1)$.
In what follows C is an index two cyclic subgroup of D (it's unique if $|D| \neq 4$).

Quotients of arc-transitive dihedrants

Definition

An imprimitivity system $\mathcal{B}$ is called *cyclic* if $\mathcal{B}(1) \leq C$. Otherwise, it's called *dihedral*.

Proposition

If Γ is G-arc-transitive dihedrant and $\mathcal{B}$ is an imprimitivity system of G. Then the quotient graph $\Gamma/\mathcal{B}$ is dihedrant if $\mathcal{B}$ is cyclic and circulant otherwise. In both cases it is $G^{\mathcal{B}}$-arc-transitive.

Proposition

Let Γ be G-arc-transitive dihedrant. Then there exists a unique (possibly trivial) maximal cyclic imprimitivity system $\mathcal{C}$ of G. If $\mathrm{Core}_G(C)$ is non-trivial, then $\mathcal{C}$ is non-trivial too.

Quotients of arc-transitive dihedrants

One can always factor out by a maximal cyclic imprimitivity system and obtain an arc-transitive dihedrant with a trivial maximal cyclic imprimitivity system.

Problem

Classify G-arc-transitive dihedrants with trivial maximal cyclic imprimtivity system.

Quotients of arc-regular dihedrants

In general, a quotient of an arc-regular graph may be not arc-regular.

Proposition

Let Γ be a G-arc-regular vertex transitive graph. Assume that a point stabilizer of G is a Hamiltonian group. Then the quotiemt graph $\Gamma/\mathcal{B}$ is $G^{\mathcal{B}}$-arc-regular provided that $\mathcal{B}$ is a normal imprimtivity system of G.

In particular, this statement works when a point stabilizer is an abelian group.

Cayley maps

Definition

Let H be a finite group, $S \subseteq H$ an inverse closed subset s.t. $1 \notin H$, p a cyclic permutation on S. Then *Cayley map* $\mathrm{CM}(H, S, p)$ over H is a map with underlying graph $\mathrm{Cay}(H, S)$ with vertex rotation $(x, sx) \mapsto (x, p(s)x)$.

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Two Cayley maps $\mathrm{CM}(H, S, p)$ and $\mathrm{CM}(H', S', p')$ are *isomorphic* if there exists a graph isomorphism $\varphi : H \to H'$ which preserves the arc orientation at each vertex, that is

$$\{(sx, x, p(s)x) \,|\, x \in H, s \in S\}^{\varphi} = \{(s'y, y, p'(s')y) \,|\, y \in H, s' \in S'\}.$$

A map automorphism is defined in a natural way. The maps are called *Cayley* isomorphic if there exists a group isomorphism which induces a map isomorphism.

Regular Cayley maps

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A Cayley map $\mathcal{M} = \mathrm{CM}(H, S, p)$ is called *regular* if $\mathrm{Aut}(\mathcal{M})$ acts transitively on the arc set of the underlying Cayley graph.

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1. $\mathrm{Cay}(H, S)$ is a G-arc-regular graph;
2. a point stabilizer $Y = G_1$ is cyclic;
3. if $\mathcal{B}$ is normal imprimitivity system of G, then the quotient graph is G/N-arc-regular, where $N = G_{\mathcal{B}}$.

Regular Cayley maps

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Given a finite group H. Find all skew morphisms of H.

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$$c^\sigma = c^\ell, r^\sigma = r;$$

- $\langle D_*, \sigma \rangle$ is a subgroup of $\mathrm{Sym}(D)$ isomorphic to $D \rtimes \mathbb{Z}_m$.

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- $G := N\langle D_*, \sigma \rangle$ is a subgroup of $\mathrm{Sym}(D)$ which normalizes N

Regular non-balanced dihedral Cayley maps

Theorem

- The point stabilizer G_1 is a cyclic group of order $4m$ generated by $\mu_1 r_* \sigma$;
- The permutation $\mu_1 r_* \sigma$ is a skew-morphism of D whose action on the r^{G_1} has the following form:

$$(c,\ rc^{-\ell},\ rc^{\ell^2},\ c^{-\ell^3},\ \cdots\ c^{\ell^{4m-4}},\ rc^{-\ell^{4m-3}},\ rc^{\ell^{4m-2}},\ c^{-\ell^{4m-1}}).$$

- G is an automorphism group of a regular Cayley map $\mathrm{CM}(n,\ell)$ over D;
- Any non-balanced regular Cayley map over D is isomorphic to $\mathrm{CM}(n,\ell)$.