Cayley Graphs On Abelian Groups

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Dedicated to Dragan Marušič
For His 60th Birthday Conference

University of Primorska
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Introduction

Happy Birthday, Dragan

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The complete graph K_2 is an exception but we dismiss it because it fails to have a cycle for the trivial reason that it has only two vertices.

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The striking common feature of these four graphs is that none of them is a Cayley graph. We now define Cayley graphs for completeness.

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- $S = S^{-1}$, that is, $s \in S$ if and only if $s^{-1} \in S$.

Then define the Cayley graph on the group G with connection set S, denoted $\mathrm{Cay}(G; S)$, to have its vertices labelled with the elements of G and x adjacent to y if and only if $y = xs$ for some $s \in S$.

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Conjecture. Connected Cayley graphs are hamiltonian.

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Many years ago I discussed the issue of the source of the Cayley graph conjecture with Laci Babai. His conclusion was that it arose in one of the informal discussions in Budapest while discussing Lovász's problem and it could not be attributed accurately to any individual. It should be regarded as a "folklore conjecture."

Cayley Graph Conjecture

In the first edition of Lovász's celebrated book *Combinatorial Problems and Exercises*, there is an exercise asking for a proof of the assertion that connected Cayley graphs on abelian groups are hamiltonian. So this fact was known already in 1979.

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However, something much stronger first appeared in 1979. This was the wonderful Chen-Quimpo Theorem which requires definitions before its statement.

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A family $\mathcal{F}$ of graphs is called H^*-connected if every bipartite graph in $\mathcal{F}$ is Hamilton-laceable and every non-bipartite graph in $\mathcal{F}$ is Hamilton-connected.

Chen-Quimpo Theorem

Theorem (Chen-Quimpo) The family of connected Cayley graphs of valency at least 3 on abelian groups is H^*-connected.

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I was at a conference in Geelong, Australia when C. C. Chen first presented this result.

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Over the next ten years, I received five manuscripts to referee with an affirmative answer to Boesch's question. Of course, it was easy to reject them and it meant five more people learned about the theorem.

Chen-Quimpo Theorem

The Chen-Quimpo Theorem has some nice applications, but I shall not explore that here. Let me simply say that the applications are based upon the fact that even though a vertex-transitive graph may not be a Cayley graph on an abelian group, it may have a vertex partition such that each part induces a Cayley graph on an abelian group

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The other lesson we learn from this theorem is that using Cayley graphs on abelian groups as a testing ground provides us with results we can attempt to extend to other families of Cayley graphs.

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Of course, graphs may fail to be pancyclic but still have cycles of many different lengths. For example, a graph may have large girth and have cycles of all possible lengths starting with the value of the girth.

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A bipartite graph cannot be pancyclic, but it can have cycles of all possible even lengths. In case the graph does, it is called even pancyclic.

Pancyclicity

Let X be a Cayley graph on an abelian group with connection set S satisfying $|S| \geq 3$. Let $s_1, s_2 \in S$ such that $s_1 \notin \{\pm s_2\}$.

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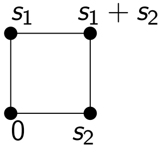
We then have the following 4-cycle in X.



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We see that a Cayley graph of valency at least 3 on an abelian group always has a 4-cycle. Thus, the girth is either 3 or 4. However, the situation is more complicated than this first glimpse indicates.

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The odd girth of a graph is the length of a shortest odd length cycle. If the graph is bipartite, the odd girth is undefined.

Pancyclicity

Consider the cartesian product $C_m \Box K_2$ of an odd length cycle C_m and K_2. First observe that it is a circulant graph, namely, $\mathrm{circ}(2m; \{2, m, 2m-2\}$. So it is a Cayley graph on an abelian group.

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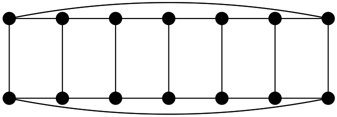
We now want to show that the odd girth of this graph is m. This demonstrates that the odd girth can be big relative to the order of the graph.

Pancyclicity



The above graph illustrates how the proof works. The number of edges between the two 7-cycles must be even and it is not hard to see that using none of them gives the shortest odd length cycle.

Pancyclicity

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I then received several manuscripts for refereeing that dealt with special subclasses of circulant graphs and ignored Bogdanowicz's work. This irritated me so I decided to take a look at extending his result in some kind of definitive manner.

Pancyclicity

Working with two honors students at Newcastle, we proved the following result.

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Theorem. (Alspach, Bendit and Maitland, to appear). If X is a connected Cayley graph of valency at least 3 on an abelian group, then X is even edge-pancyclic. If X is not bipartite, then X contains odd length cycles of all possible odd lengths starting with the odd girth. Furthermore, every edge lies in odd length cycles of all lengths except possibly the odd girth length.

Pancyclicity

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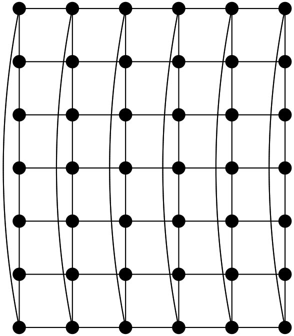
If the connection set contains an element of odd order, it is not hard to show it has a spanning subgraph like this.



Pancyclicity

One major part of the proof, as seen on the preceding slide, is to find a convenient odd length cycle of length ℓ, and then show that we can find cycles of all lengths $\ell + 2, \ell + 4$, and so on up through $|V(X)|$ or $|V(X)| - 1$.

Pancyclicity

Frequently, the convenient odd length cycle used for the preceding part of the proof has length much greater than the odd girth. For example, in the cartesian product shown earlier, the length of the column cycle could be very large, but the odd girth could be 3.

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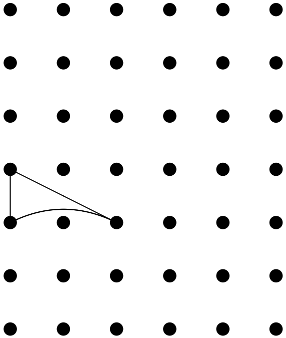
Frequently, the convenient odd length cycle used for the preceding part of the proof has length much greater than the odd girth. For example, in the cartesian product shown earlier, the length of the column cycle could be very large, but the odd girth could be 3.

So the second major part of the proof is to find cycles whose lengths fill in the gap from the odd girth through $\ell - 2$. This is the hardest part of the proof.

Pancyclicity



We see how to extend the length by 2 at a time until reaching ℓ.

Pancyclicity

We now are investigating pancyclicity for Cayley graphs on dihedral groups. The most interesting situation is when all the elements in the connection set are involutions because if other elements are in the connection set, we can use the results for Cayley graphs on abelian groups.

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We have uncovered some behavior that does not occur when dealing with Cayley graphs on abelian groups. We don't really know what is going on at this point in time.

Hamilton Decompositions

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A graph that admits a Hamilton decomposition is said to be Hamilton-decomposable.

Hamilton Decompositions

In 1982 I posed the following question: Does every connected Cayley graph on an abelian group admit a Hamilton decomposition?

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The first progress on this problem was by Bermond, Favaron and Maheo who proved that the answer is yes when the valency is 4.

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That is all that has been achieved in the direction of approaching the problem via valency. Another approach is to consider the structure of the connection set.

Hamilton Decompositions

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A connection set S for a Cayley graph $\mathrm{Cay}(G; S)$ is minimal if S generates the group G, but $S - \{s, s^{-1}\}$ generates a proper subgroup for every $s \in S$.

Theorem (Liu) Let G be an abelian group. If S is a minimal Cayley set for G, then $\mathrm{Cay}(G; S)$ is Hamilton-decomposable when $|G|$ is odd, or $|G|$ is even and $2s \notin S$ for $s \in S$.

Hamilton Decompositions

Another direction that has been taken has been to look at the order of the group. For example, it is easy to see that every Cayley graph of prime order is Hamilton-decomposable.

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The following theorem has yet to appear.

Theorem (Alspach, Bryant and Kreher) Every connected Cayley graph of prime-squared order is Hamilton-decomposable.

Hamilton Decompositions

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Let's take a brief look at the proof of the theorem.

Hamilton Decompositions

The result is trivial for graphs of order 4 so that we let p be an odd prime. There are two groups of order p^2, namely, $Z_p \times Z_p$ and Z_{p^2}.

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Consider Cayley graphs on $Z_p \times Z_p$ first. Recall that the cartesian product $C_p \Box C_p$ of two p-cycles is Hamilton-decomposable.

Hamilton Decompositions

Use the language of linear algebra for Cayley graphs on $Z_p \times Z_p$. Let S be the connection set for a connected Cayley graph X on $Z_p \times Z_p$.

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Define an equivalence relation on S by letting two elements be related if they are scalar multiples of each other. There are at least two equivalence classes because X is connected. If we choose s_1 and s_2 from different equivalence classes, then $\{\pm s_1, \pm s_2\}$ generate a 4-valent subgraph of X that is isomorphic to $C_p \Box C_p$ and, thus, Hamilton-decomposable.

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Situation 1. We are left with three sets of the form $\{\pm s_1\}, \{\pm s_2\}, \{\pm s_3\}$. They generate a 6-valent subgraph which is Hamilton-decomposable via a theorem mentioned earlier that was proved by Kreher, Liu and Westland.

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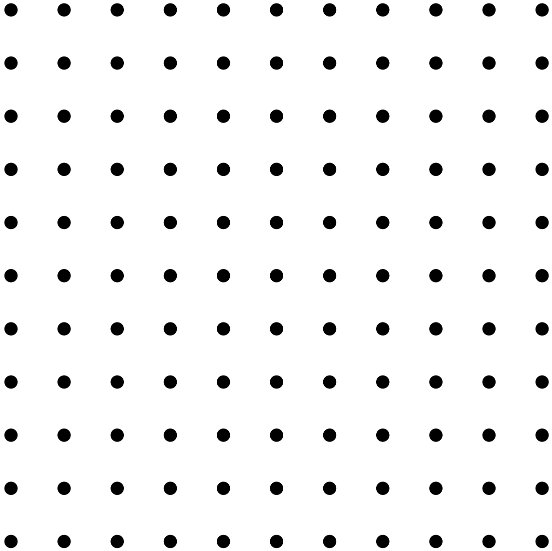
Situation 2. We are left with a set of the form $\{\pm s_1\}$ and a set with four or more elements. They generate a subgraph isomorphic to the cartesian product of a connected circulant graph of order p and C_p. This is the main portion of the proof.

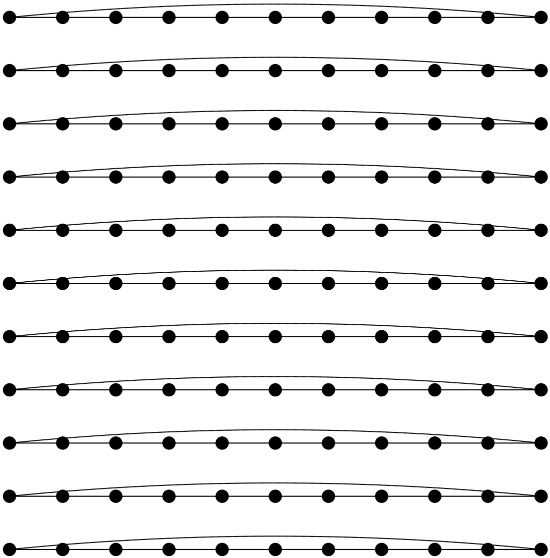
Hamilton Decompositions

We are going to do an example by picture to illustrate the idea. We'll use the prime 11 and let the connection set for the circulant graph be $\{\pm 1, \pm 3\}$.

Hamilton Decompositions



Hamilton Decompositions



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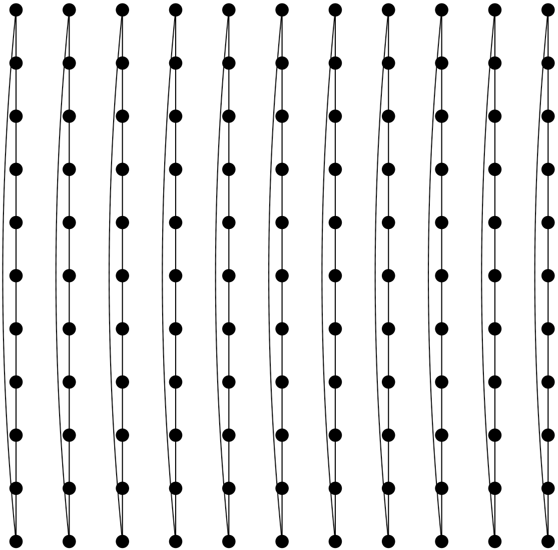
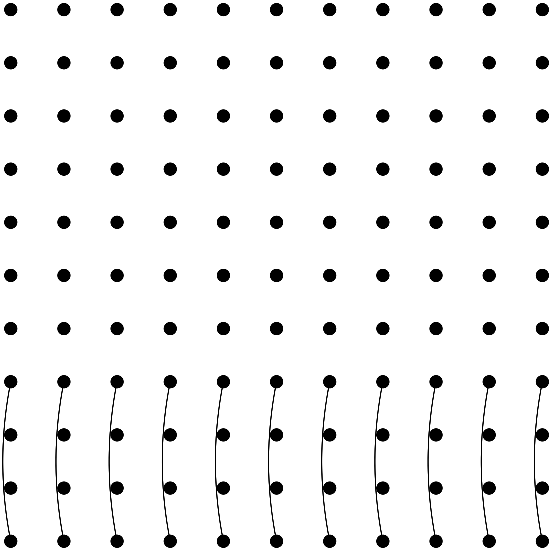


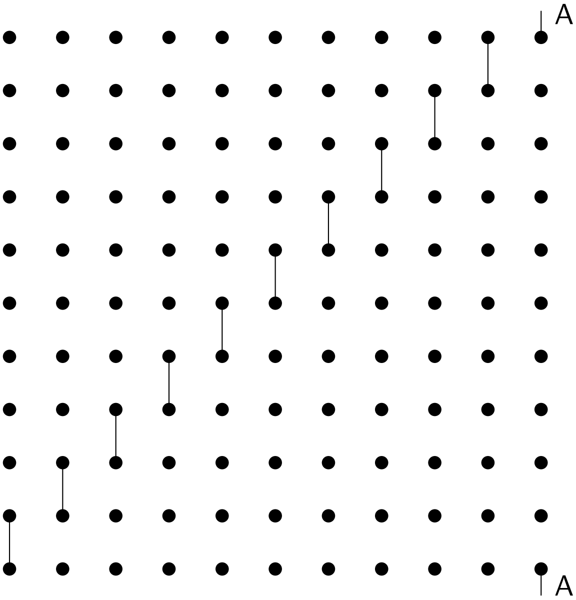
Hamilton Decompositions



Repeated vertically in a circulant fashion

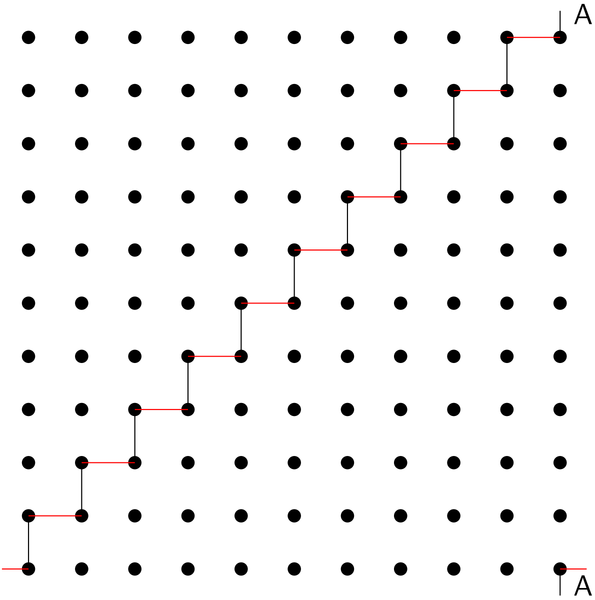
Hamilton Decompositions

Remove each of the edges shown from the vertical cycles made up of edges of length 1



Hamilton Decompositions

and replace
with the
red edges.



Hamilton Decompositions

It is clear that a Hamilton cycle is obtained and is formed by all the vertical edges of length 1, except for the one that was removed from each cycle, and the borrowed horizontal edges.

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We then do the same operation on the vertical edges of length 3, namely, we remove one from each column and cycle the choice one up and one over each time. We connect the spanning paths in each column by borrowing an edge from the horizontal edges as before.

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We then do the same operation on the vertical edges of length 3, namely, we remove one from each column and cycle the choice one up and one over each time. We connect the spanning paths in each column by borrowing an edge from the horizontal edges as before.

The key is to choose the deleted edges in the columns so that the remaining edges form a single cycle. It is not hard to make a choice using the canonical orthogonal matching with respect to a Hamilton decomposition of a prime order circulant graph.

Hamilton Decompositions

Switching to circulant graphs of order p^2, we know the connection set must contain a unit because the graph is connected. Hence, we may assume the connection set contains ± 1 by taking an appropriate isomorph.

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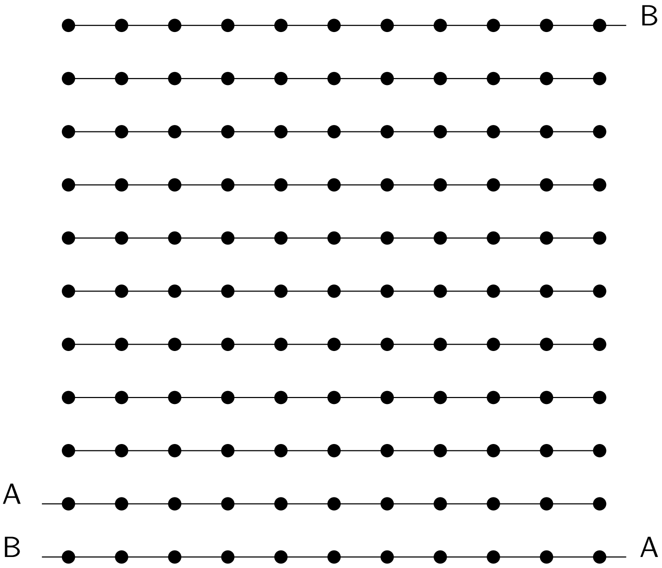
Any other unit in the connection set generates a Hamilton cycle itself. Thus, we may assume S contains ± 1 and multiples of p.

Hamilton Decompositions

If we form a $p \times p$ array again and successively label vertices so that vertices in the same column are congruent modulo p, then we get isomorphic circulant graphs in the columns and the edges from ± 1 being almost horizontal. It looks as follows.

Hamilton Decompositions



Hamilton Decompositions

The spanning subgraph on the preceding page is so close to being the same as the one used for $Z_p \times Z_p$, but the problems caused by the wrap-around edges jumping by 1 are very different from those caused in the previous case.

The essential difference is that over and up by one is a graph automorphism for $Z_p \times Z_p$ but not for circulant graphs. Nevertheless, we found a way around the problem which I shall not describe here.

Thank You