Finite Geometries

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Generalized polygons

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a connected, finite point-line incidence geometry.

$\mathcal{P}$ and $\mathcal{L}$ are two distinct sets, the elements of $\mathcal{P}$ are called points, the elements of $\mathcal{L}$ are called lines. $\mathrm{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ is a symmetric relation, called incidence.

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$$x_0 \,\mathrm{I}\, x_1 \,\mathrm{I}\, \dots \,\mathrm{I}\, x_h$$

where $x_i \in \mathcal{P} \cup \mathcal{L}$.

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Chain of length h :

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where $x_i \in \mathcal{P} \cup \mathcal{L}$.
The distance of two elements $d(x, y)$: length of the shortest chain joining them.

Generalized polygons

Definition

Let $n > 1$ be a positive integer. $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a generalized n-gon if it satisfies the following axioms.

- **Gn1.** $d(x, y) \leq n \;\forall\; x, y \in \mathcal{P} \cup \mathcal{L}$.
- **Gn2.** *If* $d(x, y) = k < n$ *then* $\exists!$ *a chain of length* k *joining* x *and* y.
- **Gn3.** $\forall\; x \in \mathcal{P} \cup \mathcal{L} \;\exists\; y \in \mathcal{P} \cup \mathcal{L}$ *such that* $d(x, y) = n$.

Some elementary observations

- (For the graph-theorists:) A generalized n-gon is a connected bipartite graph of diameter n and girth $2n$.
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- The dual of a generalized n-gon is also a generalized n-gon.
- The distance of two points or two lines is even. The distance of a point and a line is odd.
- If $n = 2$ then any two points are collinear, any two lines intersect each other, hence generalized 2-gons are trivial structures (their Levi graphs are the complete bipartite graphs).

Almost trivial structures

$n = 3$

The distance of two distinct points is 2, hence the points are collinear. Because of Gn2 the line joining them is unique.
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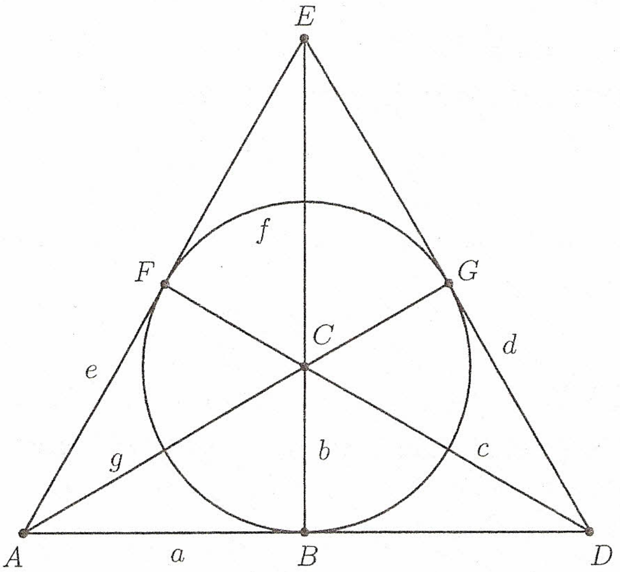
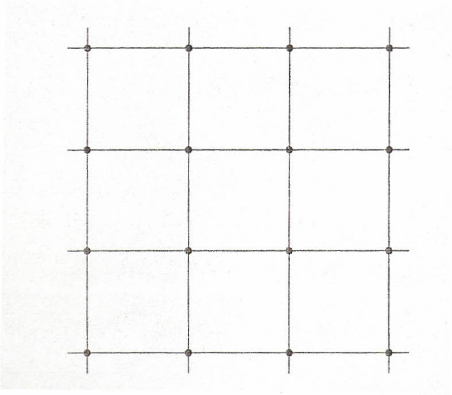
There are trivial structures,
and there are non-trivial ones:

The Fano plane

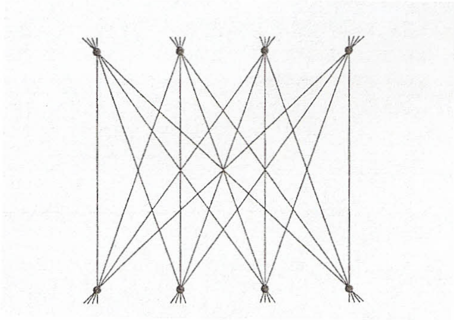


$n = 4$

Grid



Its dual, bipartite graph.



Points: vertices
Lines: edges

Nontrivial structures

Definition

A generalized polygon is called thick, if it satisfies the following.
Gn4. *Each line is incident with at least three points and each point is incident with at least three lines.*

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Theorem

In a thick finite generalized polygon each line is incident with the same number of points and each point is incident with the same number of lines.

Definition

The polygon is called of order (s, t) if these numbers are $s + 1$ and $t + 1$, respectively.

Existence

Theorem (Feit-Higman)

Finite thick generalized n-gons exist if and only if $n = 2, 3, 4, 6$ *and* 8.

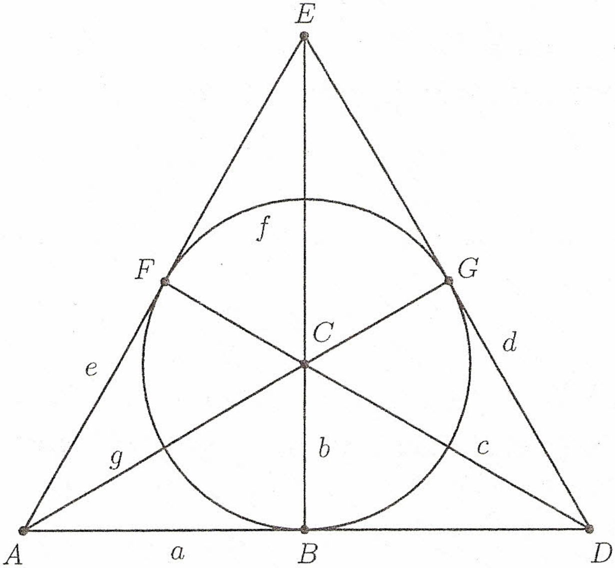
$n = 3$, projective planes

Definition

$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is called a projective plane if it satisfies the following axioms.

- **P1.** *For any two distinct points there is a unique line joining them.*
- **P2.** *For any two distinct lines there is a unique point of intersection.*
- **P3.** *Each line is incident with at least three points and each point is incident with at least three lines.*

The Fano plane



Theorem

Let Π be a projective plane. If Π has a line which is incident with exatly $n+1$ points, then

1. *each line is incident with $n+1$ points,*
2. *each point is incident with $n+1$ lines,*
3. *the plane contains n^2+n+1 points,*
4. *the plane contains n^2+n+1 lines.*

Basic combinatorial properties

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3. *the plane contains n^2+n+1 points,*
4. *the plane contains n^2+n+1 lines.*

The number n is called the order of the plane.

Proof

If (P, ℓ) is a non-incident point-line pair, then there is a bijection between the set of lines through P and the set of points on ℓ.

$$F_i \mathrm{I} \ell \Longleftrightarrow PF_i$$

Proof

The total number of points of the plane. Let H be any point of the plane. By (2) there are $n+1$ lines through H. Since any two points of the plane are joined by a unique line, every point of the plane except H is on exactly one of these $n+1$ lines. By (1) each of these lines contains n points distinct from H. Thus the total number of points is $1+(n+1)n = n^2+n+1$.

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Dually, the total number of lines of the plane. Let h be any line of the plane. By (1) there are $n+1$ points on h. Since any two liness of the plane intersect in a unique point, every line of the plane except h is on exactly one of these $n+1$ points. By (2) each of these points is incident with n liness distinct from h. Thus the total number of lines is $1+(n+1)n = n^2+n+1$.

The cyclic model

The plane of order 3 have $3^2 + 3 + 1 = 13$ points and 13 lines. Take the vertices of a regular 13-gon $P_1 P_2 \ldots P_{13}$. The chords obtained by joining distinct vertices of the polygon have 6 $(= 3(3+1)/2)$ different lengths. Choose 4 $(= 3+1)$ vertices of the regular 13–gon so that all the chords obtained by joining pairs of these points have different lengths. Four vertices define $4 \times 3/2 = 6$ chords. For example the vertices P_1, P_2, P_5 and P_7 form a good subpolygon. Let us denote this quadrangle by Λ_0.

The cyclic model

The points of the plane are the vertices of the regular 13-gon.

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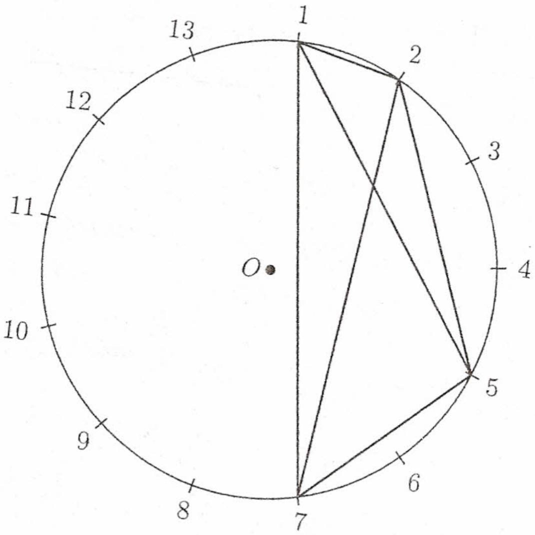
Cyvlic model



The cyclic model

The model obviously satisfies P3.

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If P_k and P_l are two distinct vertices of the regular 13-gon, then there uniquely exists a chord of the quadrangle Λ_0 which has a length equal to P_kP_l. Thus this chord is carried into the chord P_kP_l by a unique rotation with angle less than 2π. Thus the model satisfies P1.

If Λ_k and Λ_l are two distinct quadrangles, then $\exists!$ an angle $\phi < 2\pi$ which is the angle of the rotation carrying Λ_k into Λ_l. The quadrangle Λ_k has exactly one chord which corresponds to ϕ. The rotation by ϕ carries one endpoint of this chord into the other, but this second endpoint is also a vertex of Λ_l. So any two distinct line of the plane have at least one point in common.

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If any two lines have at least one point of intersection and P1 holds, then any two distinct lines meet in exactly one point. So the model satisfies P2.

The cyclic model

This proof works for an arbitrary $n \geq 2$, so we can construct a projective plane of order n, if we are able to choose $n + 1$ vertices of the regular $n^2 + n + 1$-gon in such a way that no two chords spanned by the choosen vertices have the same length.

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One can easily find such sets of vertices if n is equal to 2 or 3. The most elementary method, trial and error, leads to the solution if $n=4$ or $n=5$. But as n increases, the number of cases to be tested increases too rapidly, so the trial and error method does not work in practice.

The cyclic model

Some examples.

n	n^2+n+1	vertices of the subpolygon
2	7	1,2,4
3	13	1,2,5,7
4	21	1,2,5,15,17
5	31	1,2,4,9,13,19
6	43	???
7	57	1,2,4,14,33,37,44,53

Difference sets

Definition

Let G be an additive group. A subset $D = \{d_1, d_2, \ldots, d_k\}$ is called ***difference set****, if* $\forall\, 0 \neq g \in G \; \exists!\, d_i, d_j \in D$ *such that* $g = d_i - d_j$.

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Theorem

Let $n > 1$ be an integer and $v = n^2 + n + 1$. If the group $\mathbb{Z}_v$ contains a difference set then there exists a projective plane of order n.

Incidence matrix

	A	B	C	D	E	F	G
a	1	1	0	1	0	0	0
b	0	1	1	0	1	0	0
c	0	0	1	1	0	1	0
d	0	0	0	1	1	0	1
e	1	0	0	0	1	1	0
f	0	1	0	0	0	1	1
g	1	0	1	0	0	0	1

Incidence matrix

Let $n > 1$, $v = n^2 + n + 1$, A a $v \times v$ $0 - 1$ matrix,
$\mathbf{r}_i$ the i^{th} row vector of A, $\mathbf{c}_j$ the j^{th} column vector of A.

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Incidence matrix

$\mathbf{r}_i \cdot \mathbf{r}_j$: number of common points of the i-th and j-th lines.

Theorem

A is the incidence matrix of a projective plane if and only if

- $\mathbf{c}_i \cdot \mathbf{c}_j = 1$ *for all* $i \neq j$ ($\Longleftrightarrow$ **P1**),
- $\mathbf{r}_i \cdot \mathbf{r}_j = 1$ *for all* $i \neq j$ ($\Longleftrightarrow$ **P2**),
- $\mathbf{c}_i^2 = \mathbf{r}_i^2 = n + 1$ *for all* i ($\Longleftrightarrow$ **P3**).

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$$AA^{\mathrm{T}} = nI + J$$

Euclidean plane (as we know from high school):
Points: (a, b) $\qquad a, b \in \mathbb{R}$
Lines: $[c]$, $[m, k]$ $\qquad c, m, k \in \mathbb{R}$
Incidence:

$$(a, b) \,\mathrm{I}\, [c] \iff a = c,$$

$$(a, b) \,\mathrm{I}\, [m, k] \iff b = ma + k.$$

Euclidean plane

One can prove (by solving sets of linear equations) the following.

- **E1.** For any two distinct points there is a unique line joining them.
- **E2.** For any non-incident point-line pair (P, e) $\exists!$ a line f such that $P \mathrm{I} f$ and $e \cap f = \emptyset$.

Affine planes

Definition

$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *is called an* *affine plane* *if it satisfies the following axioms.*

- **A1.** *For any two distinct points there is a unique line joining them.*
- **A2.** *For any non-incident point-line pair* (P, e) $\exists!$ *a line* f *such that* $P \,\mathrm{I}\, f$ *and* $e \cap f = \emptyset$.
- **A3.** $\exists$ *three non-collinear points.*

$\mathrm{AG}(2, \mathbf{K})$

Replace $\mathbb{R}$ by any field $\mathbf{K}$. The affine plane $\mathrm{AG}(2, \mathbf{K})$ is the following.

Points: (a, b) $\qquad a, b \in \mathbf{K}$

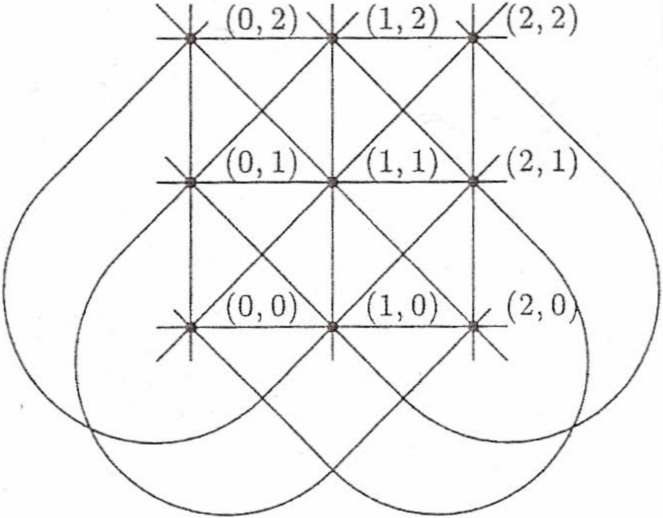
Lines: $[c]$, $[m, k]$ $\qquad c, m, k \in \mathbf{K}$

Incidence:

$$(a, b) \,\mathrm{I}\, [c] \iff a = c,$$

$$(a, b) \,\mathrm{I}\, [m, k] \iff b = ma + k.$$

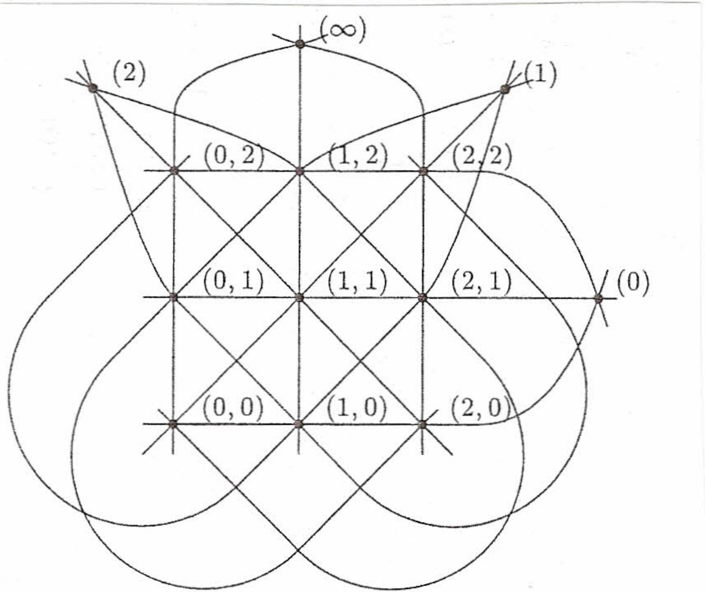
AG(2, 3)



The classical projective plane

The classical projective plane is an extension of the euclidean plane. It contains all points and lines of the euclidean plane and some extra points, called *points at infinity* and an extra line called the *line at infinity*. The points at infinity correspond to the classes of parallel lines of the euclidean plane. Each line of the euclidean plane is incident with exactly one point at infinity such a way that parallel lines have the same point at infinity, while the line at infinity contains all points at infinity and no euclidean point.

PG(2, 3)



Let V_3 be a 3-dimensional vector space over the field $\mathbf{K}$. The projective plane $\mathrm{PG}(2, \mathbf{K})$ is the following.

Homogeneous coordinates, $\mathrm{PG}(2, \mathbf{K})$

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points:	1-dim subspaces of V_3	$\mathbf{0} \neq \mathbf{v} = (v_0, v_1, v_2)$
lines:	2-dim subspaces of V_3 $\Leftrightarrow$	
	1-codim subspaces of V_3	$\mathbf{0} \neq \mathbf{u} = (u_0, u_1, u_2)$
incidence:	inclusion	$\sum_{i=0}^{2} u_i v_i = 0$

The relation $\sim$

$$\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists 0 \neq \lambda \in \mathbf{K} \,:\, \mathbf{x} = \lambda \mathbf{y}$$

is an equivalence relation. The equivalence class of the vector $\mathbf{v} \in V_3$ is denoted by $[\mathbf{v}]$.

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Homogeneous coordinates

– of the point represented by the class of vectors $[\mathbf{v}]$: $(v_0 : v_1 : v_2)$,
– of the line represented by the class of vectors $[\mathbf{u}]$: $[u_0 : u_1 : u_2]$.

Collinearity conditions in PG(2, $\mathbf{K}$)

Three distinct points $X = \mathbf{x}$, $Y = \mathbf{y}$ and $Z = \mathbf{z}$ are collinear if and only if their coordinate vectors are linearly dependent.

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$$\begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix} = 0,$$

Cartesian coordinates	homogeneous coordinates
(a, b)	$(1 : a : b)$
(m)	$(0 : 1 : m)$
(∞)	$(0 : 0 : 1)$
$[m, k]$	$[k : m : -1]$
$[c]$	$[c : -1 : 0]$
$[\infty]$	$[1 : 0 : 0]$

PG(n, $\mathbf{K}$)

Let V_{n+1} be an $(n+1)$-dimensional vector space over the field $\mathbf{K}$. The n-dimensional projective space PG($n, \mathbf{K}$) is the geometry whose k-dimensional subspaces are the $(k+1)$-dimensional subspaces of V_{n+1} for $k = 0, 1, \ldots, n$.

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points:	1-dim subspaces of V_{n+1}	$[\mathbf{v}] = (v_0 : v_1 : \ldots : v_n)$
lines:	2-dim subspaces of V_{n+1}	*Plücker*-coordinates
$\vdots$		*Grassmann*-coordinates
hyperplanes:	n-dim subspaces of V_{n+1} $\Leftrightarrow$	
	1-codim subspaces of V_{n+1}	$[\mathbf{u}] = (u_0 : u_1 : \ldots : u_n)$
incidence:	inclusion	
	point-hyperplane:	$\sum_{i=0}^{2} u_i v_i = 0$

Definition

Let $\mathcal{S}$ be a projective space. Its dual space $\mathcal{S}^*$ is the projective space whose k-dimensional subspaces are the $(n-k-1)$-dimensional subspaces of $\mathcal{S}$.
The incidence is defined as

$$\mathcal{S}_k^* \subset \mathcal{S}_\ell^* \in \mathcal{S}^* \iff \mathcal{S}_k \supset \mathcal{S}_\ell \in \mathcal{S}.$$

The principle of duality

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Theorem (Principle of Duality)

If T is a theorem stated in terms of subspaces and incidence, then the dual theorem is also true.

Combinatorial properties of PG(n, q)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^n - q)\dots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\dots(q^k - q^{k-1})}$$

Proposition

The number of k-dimensional subspaces of $PG(n, q)$ is $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$. The number of k-dimensional subspaces of $PG(n, q)$ through a given d-dimensional ($d \leq k$) subspace in $PG(n, q)$ is $\begin{bmatrix} n-d \\ k-d \end{bmatrix}_q$.

Combinatorial properties of PG$(3, q)$

number of	
points	$q^3 + q^2 + q + 1$
lines	$(q^2 + q + 1)(q^2 + 1)$
planes	$q^3 + q^2 + q + 1$
lines through a point	$q^2 + q + 1$
planes through a point	$q^2 + q + 1$
planes through a line	$q + 1$

Definition

Let $\mathcal{S}_i$ $(i = 1, 2)$ be two projective spaces and $\mathcal{P}_i$ be the pointset of $\mathcal{S}_i$. A bijection $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ is called collineation if any three points A, B and C are collinear in $\mathcal{S}_1$ if and only if the points A^ϕ, B^ϕ and C^ϕ are collinear in $\mathcal{S}_2$.

Proposition

A collineation maps any k-dimensional subspace of $\mathcal{S}_1$ into a k-dimensional subspace of $\mathcal{S}_2$.

Collineations

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Simplest example: rotation of the cyclic model

$$P_i \mapsto P_{i+1}.$$

Linear transformations

Let A be an $(n+1) \times (n+1)$ nonsingular matrix over $\mathbf{K}$. Then the mapping

$$\phi : \mathbf{x} \mapsto \mathbf{x}A$$

is a collineation of the projective space $\mathrm{PG}(n, \mathbf{K})$.

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- $\mathbf{x}A = \lambda \mathbf{y}A \iff \mathbf{x} = \lambda \mathbf{y}$ because $\det A \neq 0$, so $[\mathbf{x}] \mapsto [\mathbf{x}A]$.
- $\mathbf{x} = \lambda \mathbf{y} + \mu \mathbf{z} \iff \mathbf{x}A = \lambda \mathbf{y}A + \mu \mathbf{z}$.

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is a collineation of the projective space $\mathrm{PG}(n,\mathbf{K})$.

- $\mathbf{x}A=\lambda\mathbf{y}A \iff \mathbf{x}=\lambda\mathbf{y}$ because $\det A\neq 0$, so $[\mathbf{x}]\ \mapsto\ [\mathbf{x}A]$.
- $\mathbf{x}=\lambda\mathbf{y}+\mu\mathbf{z} \iff \mathbf{x}A=\lambda\mathbf{y}A+\mu\mathbf{z}$.

If $[\mathbf{u}]$ is a hyperplane then

$$\phi:\ \mathbf{u}\ \mapsto\ \mathbf{u}(A^{-1})^{\mathrm{T}},$$

because

$$\mathbf{x}\mathbf{u}^{\mathrm{T}}=0 \iff (\mathbf{x}A)(\mathbf{u}(A^{-1})^{\mathrm{T}})^{\mathrm{T}}=\mathbf{x}(AA^{-1})\mathbf{u}^{\mathrm{T}}=0.$$

Definition

Let $\mathcal{S}$ be a projective space and $\mathcal{S}^*$ be its dual space. A collineation $\phi : \mathcal{S} \to \mathcal{S}^*$ is called correlation.

If ϕ is a correlation then ϕ maps any $(n-k-1)$-dimensional subspace of $\mathcal{S}$ into a k-dimensional subspace of $\mathcal{S}^*$. Hence ϕ can be considered as an $\mathcal{S}^* \to \mathcal{S}$ collineation, too.

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Definition

A correlation π is called polarity if $(P^\pi)^\pi = P$ holds for each point $P \in \mathcal{S}$.

Definition

Let π be a polarity of the projective space $\mathcal{S}$. If $\mathcal{S}_k$ is any k-dimensional subspace then the $(n-k-1)$-dimensional subspace $\mathcal{S}_k^{\phi}$ is called the polar of $\mathcal{S}_k$.
A k-dimensional subspece $\mathcal{S}_k$ is self-conjugate *if*
– $\mathcal{S}_k \subseteq \mathcal{S}_k^{\pi}$ *if* $k \leq (n-1)/2$,
– $\mathcal{S}_k \supseteq \mathcal{S}_k^{\pi}$ *if* $k \geq (n-1)/2$.

Polarities

Let A be an $(n+1) \times (n+1)$ nonsingular, symmetric matrix over $\mathbf{K}$. Then the mapping

$$\pi : \mathbf{x} \mapsto \mathbf{x}A$$

is a polarity of the projective space $\mathrm{PG}(n, \mathbf{K})$.

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$$\mathbf{x} \mapsto \mathbf{x}A \mapsto (\mathbf{x}A)(A^{-1})^{\mathrm{T}} = \mathbf{x}.$$

because $A^{\mathrm{T}} = A$. This type of polarities is called ordinary polarity. The self-conjugate points form a quadric $\mathbf{x}A\mathbf{x}^{\mathrm{T}} = 0$.

Let A be an $(n+1) \times (n+1)$ nonsingular, antisymmetric matrix over $\mathbf{K}$. $\det A \neq 0$, $A = -A^{\mathrm{T}}$, hence n is odd. Then the mapping

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because $A^{\mathrm{T}} = -A$. This type of polarities is called null polarity. Each point is self-conjugate. If $n = 3$, then there are self-conjugate lines.

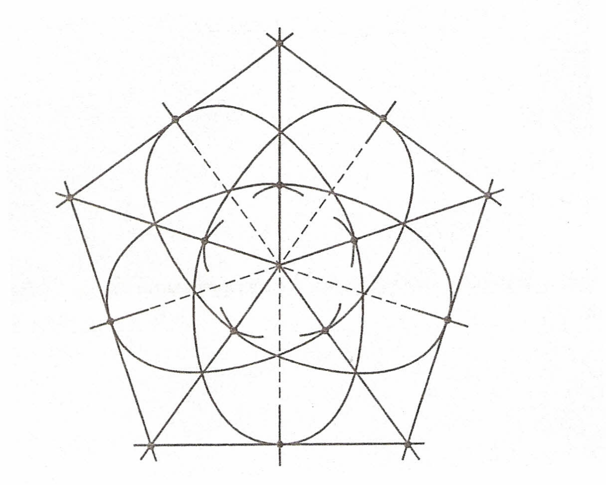
Theorem

Let $\mathcal{Q}$ be a finite, thick generalized quadrangle of order (s, t). Then

1. *For each non-incident point-line pair (P, e) $\exists!$ a point-line pair (R, f) such that $P \mathrm{I} f \mathrm{I} R \mathrm{I} e$,*
2. *$\mathcal{Q}$ contains $(s+1)(st+1)$ points,*
3. *$\mathcal{Q}$ contains $(t+1)(st+1)$ points.*

The smallest example

$s = t = 2$
15 points, 15 lines



A GQ of order (q, q)

Let A be a 4×4 nonsingular, antisymmetric matrix over $\mathrm{GF}(q)$. Then A defines a null polarity π of the projective space $\mathrm{PG}(3, q)$.

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Let A be a 4×4 nonsingular, antisymmetric matrix over $\mathrm{GF}(q)$. Then A defines a null polarity π of the projective space $\mathrm{PG}(3, q)$. A line RS is self-conjugate if and only if

$$\mathbf{r}A\mathbf{s}^{\mathrm{T}} = 0 \iff R \in S^{\pi}.$$

Hence the self-conjugate lines through a point P are the elements of the pencil of lines in P^{π} having carrier P.

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Hence the self-conjugate lines through a point P are the elements of the pencil of lines in P^{π} having carrier P.

Theorem

The points of $\mathrm{PG}(3, q)$ and the self-conjugate lines of a null polarity with the inherited incidence form a generalized quadrangle of order (q, q).

A GQ of order (q, q)

- Each line contains $q + 1$ points.
- There are $q + 1$ lines through each point.
- If (P, ℓ) is a non-incident point-line pair, then $\ell \not\subset P^\pi$. Hence $\exists\,!\;\; Q = \ell \cap P^\pi$. The line PQ is self-conjugate, contains P and meets ℓ.

A generalized hexagon of order (q, q)

Let A be a 7×7 nonsingular, symmetric matrix over $\mathrm{GF}(q)$. Then A defines an ordinary polarity π of the projective space $\mathrm{PG}(6, q)$.

A generalized hexagon of order (q, q)

Let A be a 7×7 nonsingular, symmetric matrix over $\mathrm{GF}(q)$. Then A defines an ordinary polarity π of the projective space $\mathrm{PG}(6, q)$. The self-conjugate points of π form a parabolic quadric $\mathcal{Q}$. The points of $\mathcal{Q}$ and a subset of the lines contained in $\mathcal{Q}$ with the inherited incidence form a generalized hexagon of order (q, q).

Arcs

Definition

A k-arc is a set of k points no three of them are collinear.
A k-arc is complete if it is not contained in any $(k+1)$-arc.

Definition

Let $\mathcal{K}$ be a k-arc and ℓ be a line. ℓ is called
– a secant to $\mathcal{K}$ if $|\mathcal{K} \cap \ell| = 2$,
– a tangent to $\mathcal{K}$ if $|\mathcal{K} \cap \ell| = 1$,
– an external line to $\mathcal{K}$ if $|\mathcal{K} \cap \ell| = 0$.

Upper bound

Theorem (Bose)

If there exists a k-arc in a finite plane of order n, then

$$k \leq \begin{cases} n+1 & \text{if } n \text{ odd}, \\ n+2 & \text{if } n \text{ even}. \end{cases}$$

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If the points $P_1, P_2, \ldots, P_k$ form a k-arc, then the lines $P_1 P_i$ are distinct lines through P_1. But there are $n+1$ lines through P_1, hence $k \leq n+2$.

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Assume that the points $P_1, P_2, \ldots, P_{n+2}$ form an $(n+2)$-arc $\mathcal{H}$. Then each line of the plane meets $\mathcal{H}$ in either 0 or 2 points, hence $\mathcal{H}$ contains an even number of points, so n must be even.

Definition

An $(n+1)$-arc in a projective plane of order n is called oval.
An $(n+2)$-arc in a projective plane of order n is called hyperoval.

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There are no hyperovals in planes of odd order.

Theorem

There are ovals in $\mathrm{PG}(2,q)$ for all q. If q is even then $\mathrm{PG}(2,q)$ contains hyperovals, too.

The conic $X_1^2 = X_0X_2$ is an oval.

$$\mathcal{C} = \{(1:t:t^2) \,:\, t \in \mathrm{GF}(q)\} \cup \{(0:0:1)\}$$

$$\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{vmatrix} \neq 0,$$

$$\begin{vmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$$

if $t_i \neq t_j$.

The equation of the tangent line to $\mathcal{C}$ at the point (t_0, t_0^2) is

$$Y - t_0^2 = 2t_0(X - t_0).$$

If q even then the equation becomes $Y = t_0^2$. Hence each tangent contains the point $(0 : 1 : 0)$.

Ovals in planes of odd order

Theorem

Let Ω be an oval in the plane Π_n, n odd. Then the points of $\Pi_n \setminus \Omega$ are divided into two classes. There are $(n+1)n/2$ points which lie on two tangets to Ω (exterior points), and there are $(n-1)n/2$ points none of which lie on a tangent to Ω (interior points).

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Let ℓ be the tangent to Ω at P and $P_1, P_2, \ldots, P_n$ be the other points of ℓ. Let t_i be the number of tangents to Ω through P_i. Ω contains an even number of points, hence $t_i > 0$ must be an even number, too. There are n tangents of Ω distinct from ℓ, each of these meets ℓ in a uniqe point, hence $\sum t_i = n$. Thus $t_i = 2$ because of the pigeonhole principle.

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So the number of exterior points is $(n+1)n/2$, while the number of interior points is $n^2+n+1-(n+1)-(n+1)n/2 = (n-1)n/2$.

Caps

Definition

A *k-cap* *is a set of* k *points no three of them are collinear.*
A k*-cap is* *complete* *if it is not contained in any* $(k+1)$*-cap.*

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Theorem

If $\mathcal{K}$ is a k-cap in $\mathrm{PG}(3, q)$ then

$$k \leq q^2 + 1.$$

Proof if q is odd.
If R and S are two distinct points points of $\mathcal{K}$ then each of the $q+1$ planes through the line RS meets $\mathcal{K}$ in an arc. Hence applying the Theorem of Bose we get

$$|\mathcal{K}| \leq 2 + (q+1)(q-1) = q^2 + 1.$$

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The estimate is sharp. The surface

$$\alpha X_0^2 + X_1^2 + X_2^2 + X_3^2 = 0$$

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contains exatly $q^2 + 1$ points if α is a non-square element in $\mathrm{GF}(q)$. Each elliptic quadric contains exatly $q^2 + 1$ points.

One-factorization

Definition

A one-factor of the graph $G = (V, E)$ is a set of pairwise disjoint edges of G such that every vertex of G is contained in exactly one of them. A one-factorization of G is a decomposition of E into edge-disjoint one-factors.

Theorem

The graph $G = (V, E)$ has a one-factor if and only if for each subset $W \subset V$ the number of the components $G - W$ having an odd number of vertices is less than or equal to the number of the vertices W.

One-factorization of K_{2n}

The one-factorizations of K_{2n} have an interesting application. Suppose that several soccer teams play against each other in a league (e.g. 10 teams in Prva Liga).

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Rnk	Team	MP	W	D	L	GF	GA	MP	W	D	L	GF	GA	MP	W	D	L	GF	GA	+/-	Pts
1	NK Maribor	36	26	7	3	88	35	18	14	3	1	50	16	18	12	4	2	38	19	53	85
2	Olimpija Ljubljana	36	19	8	9	60	38	18	11	3	4	30	17	18	8	5	5	30	21	22	65
3	Mura 05	36	18	5	13	52	46	18	10	2	6	26	17	18	8	3	7	26	29	6	59
4	FC Koper	36	16	10	10	48	35	18	8	7	3	28	18	18	8	3	7	20	17	13	58
5	ND Gorica	36	14	11	11	49	37	18	6	8	4	26	20	18	8	3	7	23	17	12	53
6	Rudar Velenje	36	11	10	15	55	54	18	7	3	8	31	26	18	4	7	7	24	28	1	43
7	BST Domzale	36	11	7	18	39	52	18	5	3	10	15	25	18	6	4	8	24	27	-13	40
8	NK Celje	36	9	10	17	44	56	18	3	5	10	20	27	18	6	5	7	24	29	-12	37
9	Triglav Gorenjska	36	9	6	21	22	67	18	4	3	11	14	34	18	5	3	10	8	33	-45	33
10	Nafta Lendava	36	5	10	21	34	71	18	2	5	11	15	36	18	3	5	10	19	35	-37	25

The competition can be represented by a graph with the teams as vertices and edges as games (the edge uv corresponds to the game between the two teams u and v). If every pair of teams plays exactly once, then the graph is complete. Several matches are played simultaneously, every team must compete at once, the set of games held at the same time is called a round. Thus a round of games corresponds to a one-factor of the underlying graph. The schedule of the championship is the same as a one-factorization of K_{2n}.

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The bigger n the more difficult schedule.

Slovenians are lucky, because $10 = 3^2 + 1$ and $10 = 2^3 + 2$.

Italians are also lucky, because $18 = 17 + 1$ and $18 = 2^4 + 2$.

Hungarians are not, because there are 16 teams in NB 1.

Schedule from an oval

Suppose that the projective plane Π_{2n-1} contains an oval $\Omega = \{P_1, P_2, \ldots, P_{2n}\}$. Take the points of Ω as the vertices of K_{2n}. Let E be an external point of Ω. The one-factor $\mathcal{F}$ belonging to E consists of the edges P_jP_k if the points P_j, P_k and E are collinear, and the edge $P_\ell P_m$ if the lines EP_ℓ and EP_m are the two tangent lines to Ω through E.

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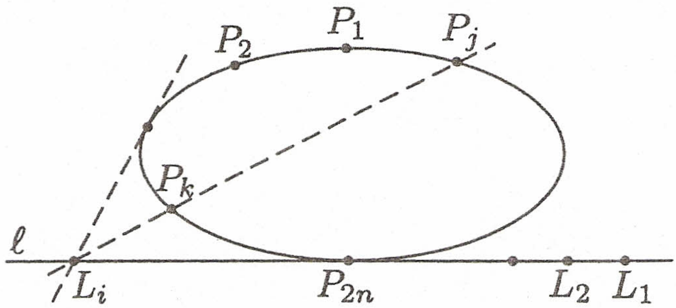
Let e_i be the tangent line to Ω at the point P_{2n}, for $i = 1, 2, \dots, 2n-1$, let L_i be the point $e_i \cap e_0$ and let $\mathcal{F}_i$ be the one-factor belonging to the point L_i.

Lemma

The union of the one-factors $\mathcal{F}_i$ gives a one-factorization of K_{2n}.

The edge $P_{2n}P_\ell$ belongs to $\mathcal{F}_\ell$, and $L_i \neq L_j$ if $i \neq j$. If $i \neq 2n \neq j$, then there is a uniqe intersection point L_k of the lines P_iP_j and e_0, Hence there is a uniqe one-factor $\mathcal{F}_k$ containing the edge P_iP_j.

Schedule from an oval



Schedule from a hyperoval

The following similar construction gives a one-factorization of K_{2n} if there exists a projective plane of order $2n-2$ which contains a hyperoval $\mathcal{H} = \{P_1, P_2, \ldots, P_{2n}\}$.
Take the points $P_1, P_2, \ldots, P_{2n}$ as the vertices of K_{2n}. Let e be an external line to $\mathcal{H}$ and let $L_1, L_2, \ldots L_{2n-1}$ be the points of e

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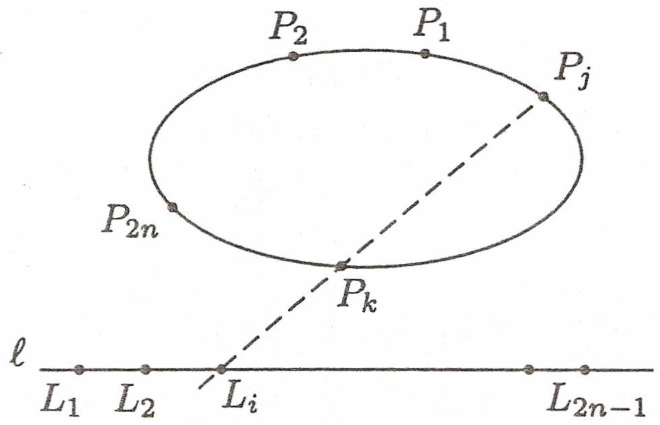
The one-factor $\mathcal{F}_i$ belonging to the point L_i is defined to consist of the edges P_jP_k if the points P_j, P_k and L_i are collinear. The union of the one-factors $\mathcal{F}_i$ is a one-factorization of K_{2n} because there is a unique point of intersection of the lines P_iP_j and e.

Schedule from a hyperoval



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Turán-type problems

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Graphs are simple (no loops, no multiple edges), finite and connected. (n, e)-graph: graph with n vertices and e edges.

Given a graph F, what is the maximum number of edges of a graph with n vertices not containing F as a subgraph?

- Give estimates on the number of edges.
- Characterize the extremal graphs.

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- $$k \leq \begin{cases} \frac{n^2}{4} & \text{if } n \text{ even}, \\ \frac{(n-1)^2}{4} & \text{if } n \text{ odd}. \end{cases}$$
- The extremal graphs are $K_{n/2,n/2}$ and $K_{(n+1)/2,(n-1)/2}$, respectively.

We investigate the cases $F = C_n$ and $F = K_{s,t}$.
Let us start with $C_4 = K_{2,2}$.

Theorem

Let G be an (n, e)-graph which does not contain C_4. Then

$$e \leq \frac{n}{4}(1 + \sqrt{4n - 3}).$$

C_4-free graphs

Count those pairs of edges of G which has a joint vertex. First consider the "free" ends of the edges. For any pair of free ends there is at most one joint vertex, otherwise a C_4 would appear. Hence the number of pairs is at most $\binom{n}{2}$.

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If we count the pairs at their joint vertex we get the exact number of them:

$$\sum_{v \in V(G)} \binom{\deg(v)}{2}.$$

Applying $\sum_{v \in V(G)} \deg(v) = 2e$, we get

$$\sum_{v \in V(G)} \deg(v)^2 \leq 2e + n(n-1).$$

Because of the well-known inequality between the arithmetic and quadratic means we have

$$\sqrt{\frac{\sum_{v\in V(G)} \deg(v)^2}{n}} \geq \frac{\sum_{v\in V(G)} \deg(v)}{n},$$

and because of $\sum_{v\in V(G)} \deg(v) = 2e$ we get

$$\left(\frac{2e}{n}\right)^2 \cdot n \leq 2e + n(n-1).$$

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$$\left(\frac{2e}{n}\right)^2 \cdot n \leq 2e + n(n-1).$$

$$4e^2 - 2en - n^2(n-1) \leq 0.$$

The solution gives the estimate on e at once.

The extremal graphs are not known in general.
If $n = q^2 + q + 1$ then the polarity graph defined by *Erdős* and *Rényi* is almost optimal.

Definition

Let π be an ordinary polarity of $PG(2, q)$. The vertices of the polarity graph G are the points of the plane, the points P and R are adjacent if and only if $P \,\mathrm{I}\, R^\pi$.

Polarity graph

- $n = q^2 + q + 1$.
- $P \mathrel{\mathrm{I}} P^\pi \iff P$ is a point of the conic defined by π.
- $e = \frac{q^2(q+1)+(q+1)q)}{2} = \frac{q(q+1)^2}{2}$.
- $\frac{n}{4}(1 + \sqrt{4n-3}) = \frac{q^2+q+1}{4}(1 + (2q+1)) = e + \frac{q+1}{2}$.

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Theorem

The polarity graph is C_4-free.

Polarity graph

Suppose that the vertices A, K, B and L form a C_4. Then
– $A \mathrm{I} K^{\pi}$,
– $A \mathrm{I} L^{\pi}$, hence KL is the polar line of A.

Polarity graph

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The polarity graph is not C_3-free.

Theorem (Füredi)

Let $q > 13$ be a prime power, G be a graph with $n = q^2 + q + 1$ vertices which does not contain C_4. Then G has at most $q(q+1)^2/2$ edges.

The problem of Zarankiewicz

What is the maximum number of 1's in an $n \times m$ 0-1 matrix if it does not contain an $s \times t$ submatrix consisting of entirely 1's?

This is a special case of Turán's problem.

What is the maximum number of edges in a $K_{s,t}$-free bipartite graph $K_{n,m}$?

Definition

The Zarankiewicz number $Z_{s,t}(n,m)$ is the maximum number of edges of a $K_{s,t}$-free bipartite graph $K_{n,m}$.

The simplest case: $n = m$ and $t = s = 2$.

Theorem (Reiman)

$$Z_{2,2}(n,n) \le \frac{n}{2}(1 + \sqrt{4n-3}).$$

The first proof

First apply the proof of the previous theorem. Now the graph has $2n$ vertices. The "free" ends of the pairs must come from the same class of the bipartite graph, hence on the right-hand side $2\binom{n}{2}$ stands instead of $\binom{2n}{2}$. Copying the proof finally we get

$$e^2 - ne - n^2(n-1) \leq 0$$

and hence

$$e \leq \frac{n}{2}(1 + \sqrt{4n-3}).$$

The original proof of Reiman

Let $\mathbf{r}_i$ and $\mathbf{c}_j$ be the row and the column vectors of the matrix, respectively. The forbidden 2×2 submatrix means that $\mathbf{r}_i \mathbf{r}_j \leq 1$ and $\mathbf{c}_i \mathbf{c}_j \leq 1$ if $i \neq j$.
If $\mathbf{r}_i^2 = r_i$ and $\mathbf{c}_i^2 = c_i$, then obviously $\sum_{i=1}^n r_i = \sum_{i=1}^n c_i = e$, where e denotes the total number of 1's in the matrix.

$$(\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_n)^2 = c_1^2 + \ldots + c_n^2,$$

and counting in another way

$$(\mathbf{r}_1 + \mathbf{r}_2 + \ldots + \mathbf{r}_n)^2 = (\mathbf{r}_1^2 + \mathbf{r}_2^2 + \ldots + \mathbf{r}_n^2) + 2(\mathbf{r}_1\mathbf{r}_2 + \ldots + \mathbf{r}_{n-1}\mathbf{r}_n) \leq$$

$$(r_1 + r_2 + \ldots + r_n) + n(n-1) = (c_1 + c_2 + \ldots + c_n) + n(n-1).$$

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$$(r_1 + r_2 + \ldots + r_n) + n(n-1) = (c_1 + c_2 + \ldots + c_n) + n(n-1).$$

The inequality between the arithmetic and quadratic means gives

$$c_1^2 + c_2^2 + \ldots + c_n^2 \leq e^2/n,$$

$$\frac{e^2}{n} \leq e + n(n-1),$$

and we have already seen this inequality in the previous proof.

Characterization of the extremal graphs

Theorem (Reiman)

If an $n \times n$ 0-1 matrix does not contain a 2×2 submatrix consisting of entirely 1's and it contains exactly

$$\frac{n}{2}(1 + \sqrt{4n - 3})$$

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1's, then the matrix is the incidence matrix of a finite projective plane.

Equality occurs if and only if the scalar product of each pair of rows (and each pair of columns) is equal to 1, and each row and column contains the same number of 1's. This means that the incidence stucture defined by the matrix satisfies the axioms of the finite projective planes.

Some generalizations

Theorem (Kővári, T. Sós, Turán)

If $s \geq t \geq 2$ and G is a $K_{s,t}$-free (n, e)-graph then

$$e \leq \frac{1}{2}((s-1)^{1/t} n^{2-1/t} + (t-1)n).$$

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Theorem (Füredi)

If $s \geq t \geq 2$ and G is a $K_{s,t}$-free (n,e)-graph, then

$$e \leq \frac{1}{2}(s-t+1)^{1/t} n^{2-1/t} + tn + tn^{2-2/t}.$$

In particular if $s = t = 3$, then

$$e \leq \frac{n^{5/3}}{2} + n^{4/3} + \frac{n}{2}.$$

An almost extremal graph

In the case $s = t = 3$ Füredi's bound is asymptotically sharp. The extremal graph was originally constructed by Brown.

Theorem (Brown's construction)

Let k_1, k_2 be such elements of $\mathrm{GF}(q)$, q odd, for which the equation $X^2 + k_1 Y^2 + k_2 Z^2 = 1$ defines an $\mathcal{E}$ elliptic quadric in $\mathrm{AG}(3, q)$.

Let G be the graph whose vertices are the points of $\mathrm{AG}(3, q)$, two points (x, y, z) and (a, b, c) are joined if and only if $(x - a)^2 + k_1(y - b)^2 + k_2(z - c)^2 = 1$.

This graph G has $\sim n^{5/3}/2$ edges and it does not contain $K_{3,3}$ as a subgraph.

An almost extremal graph

$\mathcal{E}$ meets the plane at infinity in a conic. Hence $\mathcal{E}$ contains $q^2 + 1 - (q + 1) = q^2 - q$ affine points. The neighbours of the point $A = (a, b, c)$ are on a translate of $\mathcal{E}$, thus each vertex has degree $q^2 - q$.

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$$e=\frac{1}{2}q^3(q^2-q),\ n=q^3 \Rightarrow e \sim \frac{1}{2}n^{5/3}.$$

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Suppose that G contains a $K_{3,3}$. Let $A=(a,b,c)$, $D=(d,e,f)$ and $G=(g,h,i)$ be the three distinct points of $\mathrm{AG}(3,q)$ and let $\mathcal{E}_A$, $\mathcal{E}_D$ and $\mathcal{E}_G$ be the three translates of $\mathcal{E}$ which contain the neighbours of A, D and G, respectively.

An almost extremal graph

The equations of these quadrics are as follow.

$$(X-a)^2+(Y-b)^2+(Z-c)^2=1,$$

$$(X-d)^2+(Y-e)^2+(Z-f)^2=1,$$

$$(X-g)^2+(Y-h)^2+(Z-i)^2=1.$$

Subtracting the first from the second and also from the third equation we get

$$(d-a)X+(e-b)Y+(f-c)Z+(d^2+e^2+f^2-a^2-b^2-c^2)/2=0,$$

$$(g-a)X+(h-b)Y+(i-c)Z+(g^2+h^2+i^2-a^2-b^2-c^2)/2=0.$$

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Tese are the equations of two non-parallel planes. Hence the common neighbours of A, D and G are incident with both of these planes, hence they are collinear. But the elliptic quadric $\mathcal{E}_A$ contains at most two points of any line, hence the number of the common neighbours of A, D and G is at most two.

Theorem (Damásdi, Héger, Szőnyi)

Assume that a projective plane of order n exists. Then

$$
\begin{aligned}
Z_{2,2}(n^2+n+1-c, n^2+n+1) &= (n^2+n+1-c)(n+1) \\
&\quad \text{if } 0 \le c \le n/2, \\
Z_{2,2}(n^2+c, n^2+n) &= n^2(n+1)+cn \\
&\quad \text{if } 0 \le c \le n+1, \\
Z_{2,2}(n^2-n+c, n^2+n-1) &= (n^2-n)(n+1)+cn \\
&\quad \text{if } 0 \le c \le 2n, \\
Z_{2,2}(n^2-2n+1+c, n^2+n-2) &= (n^2-2n+1)(n+1)+cn \\
&\quad \text{if } 0 \le c \le 3(n-1).
\end{aligned}
$$

C_n-free graphs

Too difficult in general. Some extra conditions are added.

- C_m-free for all $m \leq n$,
- conditions on the vertex degrees,
- regularity.

Definition

A (k,g)-graph is a k-regular graph of girth g. A (k,g)-cage is a (k,g)-graph with as few vertices as possible. We denote the number of vertices of a (k,g)-cage by $c(k,g)$.

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Erdős and Sachs proved that a (k, g)-cage exists with arbitrary prescribed parameters k and g.

Cages

A general lower bound on $c(k, g)$, known as the *Moore bound*, is a simple consequence of the fact that the vertices at distance $0, 1, \ldots, \lfloor (g-1)/2 \rfloor$ from a vertex (if g is odd), or an edge (if g is even) must be distinct.

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Proposition (Moore bound)

$$c(k,g) \geq \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-1}{2}-1} & g \text{ odd;} \\ 2\left(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\frac{g}{2}-1}\right) & g \text{ even.} \end{cases}$$

If $g = 4$ then $c(k, 4) = 2k$, complete bipartite graphs.

Moore graphs

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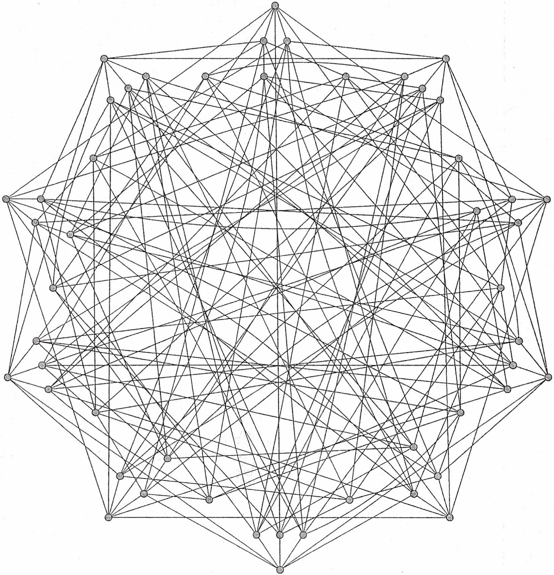
If $g = 5$, then $c(k, 5) = k^2 + 1$. This is known to be attained only if $k = 1$ (trivial) $k = 2$ (almost trivial, pentagon), 3 (Petersen), 7 (Hofman-Singleton) and perhaps 57.

Hofman-Singleton graph



Theorem

If G is a k-regular graph with girth $g = 6$ with $n = 2(1 + (k-1) + (k-1)^2)$ vertices then G is the incidence graph of a finite projective plane.

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Choose an edge of G and colour by black and red its two endpoints. After that colour by red the neighbors of the black vertex and by black the neighbours of the red vertex and continue this process. After the third step each of the $2(1 + (k-1) + (k-1)^2)$ vertices is coloured in such a way that each edge joins one black and one red vertex, thus G is bipartite. G does not contain C_4, hence it has at most $(1 + (k-1) + (k-1)^2)k$ edges. But G is k-regular, thus it contains exactly $(1 + (k-1) + (k-1)^2)k$ edges, so G is the incidence graph of a projective plane.

Incidence graphs of generalized polygons

In the same way it is easy to prove the following theorem.

Theorem

If G is a $(k, 2n)$-graph on $c(k, 2n)$ vertices then G is the incidence graphs of a generalized n-gon.

Definition

A t-good structure in a generalized polygon is a pair $\mathcal{T} = (\mathcal{P}_0, \mathcal{L}_0)$ *consisting of a proper subset of points* $\mathcal{P}_0$ *and a proper subset of lines* $\mathcal{L}_0$*, with the property that there are exactly* t *lines in* $\mathcal{L}_0$ *through any point not in* $\mathcal{P}_0$*, and exactly* t *points in* $\mathcal{P}_0$ *on any line not in* $\mathcal{L}_0$*.*

Cages

Removing the points and lines of a t-good structure from the incidence graph of a generalized n-gon of order q results a $(q+1-t)$-regular graph of girth at least $2n$, and hence provides an upper bound on $c(q+1-t, 2n)$.

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Theorem (Lazebnik, Ustimenko, Woldar)

Let $k \geq 2$ and $g \geq 5$ be integers, and let q denote the smallest odd prime power for which $k \leq q$. Then

$$c(k,g) \leq 2kq^{\frac{3}{4}g-a},$$

where $a = 4,\ 11/4,\ 7/2,\ 13/4$ for $g \equiv 0,\ 1,\ 2,\ 3 \pmod 4$, respectively.

In particular, for $g = 6,\ 8,\ 12$ this gives $c(k,6) \leq 2kq$, $c(k,8) \leq 2kq^2$, $c(k,12) \leq 2kq^5$, where q is the smallest odd prime power not smaller than k. Combined with the Moore bound, this yields $c(k,8) \sim 2k^3$.

(Δ, D)-graphs

A similar problem (with its usual notation, $\Delta = k$, $g \leq 2D + 1$).

Definition

A simple finite graph G is a (Δ, D)-graph if it has maximum degree $\Delta \geq 3$ and diameter at most D.

The degree/diameter problem is to determine the largest possible number of vertices that G can have. Denoted this number by $n(\Delta, D)$, the inequality

$$n(\Delta, D) \leq 1 + \Delta + \Delta(\Delta - 1) + \ldots + \Delta(\Delta - 1)^{D-1} = \\ = \frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2}$$

is also called *Moore bound*.

Moore graphs again

We have already seen the following ($\Delta = k$, $D = (g-1)/2$). This is known to be attained only if either $D = 1$ and the graph is $K_{\Delta+1}$, or $D = 2$ and $\Delta = 1, 2, 3, 7$ and perhaps 57.

The only known general lower bound is given as

$$(\Delta, 2) \geq \left\lfloor \frac{\Delta + 2}{2} \right\rfloor \cdot \left\lceil \frac{\Delta + 2}{2} \right\rceil. \tag{1}$$

This is obtained by choosing G to be the Cayley graph $\mathrm{Cay}(\mathbb{Z}_a \times \mathbb{Z}_b, S)$, where $a = \lfloor \frac{\Delta+2}{2} \rfloor$, $b = \lceil \frac{\Delta+2}{2} \rceil$, and $S = \{ (x, 0), (0, y) \mid x \in \mathbb{Z}_a \setminus \{0\}, y \in \mathbb{Z}_b \setminus \{0\} \}$.

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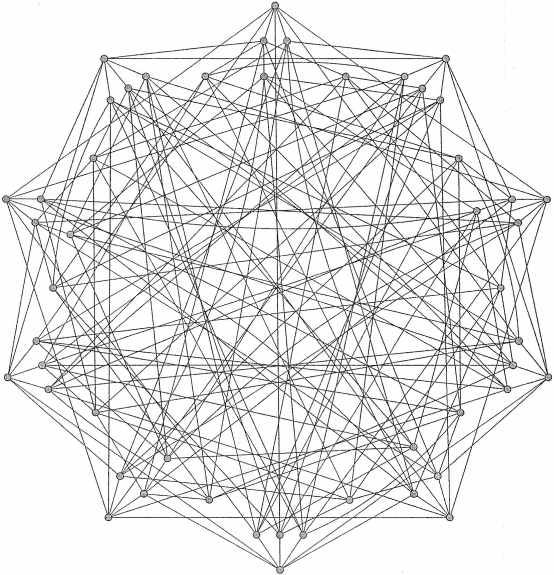
If $\Delta = kD + m$, where k, m are integers and $0 \leq m < D$, then a straightforward generalization of this construction results in a Cayley (Δ, D)-graph of order

$$\left\lfloor \frac{\Delta + D}{D} \right\rfloor^{D-m} \cdot \left\lceil \frac{\Delta + D}{D} \right\rceil^{m}. \tag{2}$$

Hofman-Singleton graph



Linear Cayley graphs

Let V_n denote the n-dimensional vector space over $\mathrm{GF}(q)$. For $S \subseteq V$ such that $0 \notin S$, and $S = -S := \{-x \mid x \in S\}$, the *Cayley graph* $\mathrm{Cay}(V, S)$ is the graph having vertex-set V and edges $\{x, x+s\}$, $x \in V$, $s \in S$. A Cayley graph $\mathrm{Cay}(V, S)$ is said to be *linear*, if $S = \alpha S := \{\alpha x \mid x \in S\}$ for all nonzero scalars $\alpha \in \mathrm{GF}(q)$. In this case $S \cup \{0\}$ is a union of 1-dimensional subspaces, and therefore, it can also be regarded as a point set in the projective space $\mathrm{PG}(n-1, q)$. Conversely, any point set $\mathcal{P}$ in $\mathrm{PG}(n-1, q)$ gives rise to a linear Cayley graph, namely the one having connection set $\{x \in V \setminus \{0\} \mid \langle x \rangle \in \mathcal{P}\}$. We denote this graph by $\Gamma(\mathcal{P})$.

Given an arbitrary point set $\mathcal{P}$ in $\mathrm{PG}(n,q)$, $\langle\mathcal{P}\rangle$ denotes the projective subspace generated by the points in $\mathcal{P}$, and $\binom{\mathcal{P}}{k}$ ($k \in \mathbb{N}$) is the set of all subsets of $\mathcal{P}$ having cardinality k.

Proposition

Let $\mathcal{P}$ be a set of k points in $\mathrm{PG}(n,q)$ with $\langle\mathcal{P}\rangle = \mathrm{PG}(n,q)$. Then $\Gamma(\mathcal{P})$ has q^{n+1} vertices, with degree $k(q-1)$ and with diameter

$$D = \min\left\{\ d \mid \cup_{\mathcal{X}\in\binom{\mathcal{P}}{d}}\langle\mathcal{X}\rangle = \mathrm{PG}(n,q)\ \right\}. \tag{3}$$

Saturating sets

Once the number of vertices and the diameter for $\Gamma(\mathcal{P})$ are fixed to be q^{n+1} and D, respectively, our task becomes to search for the smallest possible point set $\mathcal{P}$ for which

$$\cup_{\mathcal{X}\in\binom{\mathcal{P}}{D}}\langle\mathcal{X}\rangle = \mathrm{PG}(n,q).$$

A point set having this property is called a *(D-1)-saturating set*.

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If $D=2$, then a 1-saturating set $\mathcal{P}$ is a set of points of $\mathrm{PG}(n,q)$ such that the union of lines joining pairs of points of $\mathcal{P}$ covers the whole space.
In the plane: complete arcs, double blocking sets of Baer subplanes.
In $\mathrm{PG}(3,q)$: two skew lines.
In $\mathrm{PG}(n,q)$: complete caps.

Theorem (Gy. K. I. Kovács, K. Kutnar, J. Ruff, and P. Šparl)

Let $\Delta = 27 \cdot 2^{m-4} - 1$ *and* $m > 7$. *Then*

$$n(\Delta, 2) \geq \frac{256}{729}(\Delta + 1)^2.$$

Lower estimates

Theorem (Gy. K. I. Kovács, K. Kutnar, J. Ruff, and P. Šparl)

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Theorem (Gy. K. I. Kovács, K. Kutnar, J. Ruff, and P. Šparl)

Let $q > 3$ *be a prime power and let* $\Delta = 2q^2 - q - 1$*. Then*

$$n(\Delta, 2) > \frac{1}{4}\left(\Delta + \sqrt{\frac{\Delta}{2} + \frac{5}{4}}\right)^2.$$

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