UNIVERSITY OF PRIMORSKA

Faculty of Mathematics, Natural Sciences and Information Technologies

Introduction to numerical calculation
Collection of colloquiums and exams

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OTHER TEACHING MATERIAL

78 pages

Mathematics and Mathematics in Economics and Finance

Undergraduate study programs

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Foreword

Infront of you there is a collection of colloquiums and written exams, which took place between 2008 and 2019 in the subject of Introduction to numerical calculation, which is a compulsory subject of the second year of the undergraduate study of Mathematics and Mathematics in Economics and Finance at the Faculty of Mathematics, Natural Sciences and Information Technologies at University of Primorska. The topics discussed at this subject and subsequently covered in the material of this script includes solving nonlinear equations and systems of equations, solving linear systems and overdetermined linear systems, computing eigenvalues, interpolation, numerical differentiation and integration and Bézier curves.

Kazalo

POGLAVJE 1

COLLOQUIUMS

COLLOQUIUM

4 December 2008

[10] **1.** For finding the roots of the real function $f : \mathbb{R} \to \mathbb{R}$ we can use different methods.

(a) List at least three of them and describe one in detail.

Find the correct answer on the following statements:

(b) If the simple iterative method converges in the neighborhood of the solution or not is defined by:

(i) first derivative of the iteration function,
(ii) second derivative of the iteration function,
(iii) third derivative of the iteration function.

(c) The method with the quadratic convergence in the neighborhood of the solution in comparison with the method with the linear convergence

(i) converges more slowly,
(ii) converges faster,
(iii) the convergence speed is the same for both methods.

(d) Using the tangent method you have to set at the beginning

(i) the initial approximation,
(ii) two initial approximations,
(iii) the same number of initial approximations as for the secant method.

(e) The tangent method has in the neighborhood of the solution α cubic convergence if

(i) $f(\alpha) = 0,\ f'(\alpha) = 0,$
(ii) $f(\alpha) = 0,\ f'(\alpha) = 0,\ f''(\alpha) = 0,$
(iii) $f(\alpha) = 0,\ f'(\alpha) \neq 0,\ f''(\alpha) = 0.$

[10] **2.** For solving the linear systems $Ax = b$, where A is invertible matrix, we use the LU decomposition ($A = L \cdot U$).

(a) What applies to matrices L and U?

(b) In which case the LU decomposition fails? Which modification of LU decomposition can one use in that case? What one has to do on each step of this procedure?

(c) What is the time complexity of the LU decomposition?

(d) For special matrices, a similar decomposition can be done in half the time. What kind of matrices does this apply to? How we call this decomposition?

[10] **3.** Linear system $Ax = b$ can be solved also iteratively.

(a) List two examples when we use this procedure.

(b) Write down Jacobi's method as an iteration.

(c) Write down Gauss-Seidel's method in an iteration.

(d) Assume that A is upper-triangular matrix. Which one of the methods - (b) or (c) converges in this case faster? Justify the answer.

[10] **4.** Write the number -111.625 in IEEE standard for floating point arithmetic in double precision (64 bits).

[15] **5.** Let $f_\mu(x) = x^{\mu+2} - a\,x^\mu,\ \mu \in \mathbb{R},\ a > 0$. Using the tangent method we would like to compute $\sqrt{a}$ as a zero of f_μ.

(a) Determine the parameter μ such that the order of convergence of the tangent method is at least cubic in the neighborhood of $\sqrt{a}$.

(b) Determine the tangent method for computing μ from the point (a). With the obtained method and the initial approximation $x_0 = 3$ compute $\sqrt{10}$ on 4 decimal places exact.
If you are not able to solve (a) assume that $\mu = -\frac{1}{2}$.

[15] **6.** Let $A \in \mathbb{R}^{n\times n}$ be given as

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Compute $\|A\|_1$, $\|A\|_\infty$ and $\|A\|_F$. Try to derive the best estimation for $\|A\|_2$ in case $n = 4$.

[15] **7.** Let $A \in \mathbb{R}^{n\times n}$. Prove or disprove the following statements:

(a) If $\|A\|_\infty = \|A\|_1$, then A is simetric.

(b) If A is simetric, then $\|A\|_\infty = \|A\|_1$.

(c) If $\|A\|_\infty \le 1$, then $\sum_{i,j=1}^{n} |a_{ij}| \le n$.

[15] **8.** Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & -5 & -7 \\ 1 & 4 & 6 & 4 \\ -2 & -3 & -4 & -8 \end{pmatrix}.$$

Compute the LU decomposition of A without pivoting.

COLLOQUIUM

13 May 2010

[10] **1.** Circle the correct statements:

(a) The bisection is not a suitable method for finding the roots of even degree.

(b) Different iteration functions can have different orders of convergence in the neighborhood of the solutions.

(c) The tangent method has cubic convergence in the neighborhood of the triple zero.

(d) The convergence of the tangent method is assured for an arbitrary initial approximation.

(e) The method that has quadratic convergence in the neighborhood of the solution can far away from the solution converge more slowly than the method that has linear convergence in the neighborhood of the solution.

(f) The Newton method for solving the nonlinear systems is the generalization of the tangent method for solving one nonlinear equation.

(g) For the matrix A we say that it is positive definite if for every vector $x \neq 0$ holds $x^T A^{-1} x > 0$.

(h) The operator norm of the matrices are a special case of matrix norms for which the triangular inequality does not hold.

(i) For the orthogonal matrices it holds that $(AA^T)^{-1} = I$.

(j) For every matrix there exists an eigenvalue that has the absolute value greater than the spectral norm of the matrix.

[15] **2.** Answer the following questions:

(a) For solving the linear systems $Ax = b$, where A is an invertible matrix, we use the LU decomposition ($A = L \cdot U$). What applies to the matrices L and U?

(b) Let A be a diagonal matrix. How does then the matrices L and U in LU decomposition look like? How much operations one needs to compute the LU decomposition in this case?

(c) What is the time complexity for LU decomposition for a general matrix? For special matrices we can do this decomposition in half a shorter time. What kind of matrices does this apply to? How do we call this decomposition?

(d) Can be the matrix A, for which $\det A = -1$, a positive definite matrix? Justify your answer.

(e) If we use the LU decomposition with partial decomposition, then all elements in the matrix L are by the absolute value ≤ 1. Why?

[15] **3.** Write the number -80.34375 in IEEE standard for floating point arithmetic in a single precision (32 bits).

[15] **4.** Compute the fixed points of the iteration function

$$x_{r+1} = g(x_r) = \frac{2x_r - 1}{x_r^2}.$$

Determine which of them are attractive and which are unattractive. Determine the order of convergence in the neighborhood of the attractive fixed points.

[15] **5.** For the following system of nonlinear equations compute one step of Newton's method with the initial approximation $(x_0, y_0)^T = (-1, -2)^T$:

$$x^3 + y^3 = -2, \quad x^2 + xy + y = 1.$$

[15] **6.** Let

$$A_n = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Compute the LU decomposition without pivoting for the matrix A_n. Using the obtained decomposition compute the $\det A_n$.

[15] **7.** Let A_n be the matrix from the task 6. Compute $\|A_n\|_1$, $\|A_n\|_\infty$, $\|A_n\|_F$ and $N_\infty(A_n)$. Try to derive the best approximation for $\|A_n\|_2$ in case $n = 5$.

COLLOQUIUM

4 November 2010

[20] **1.** Circle the correct statements (at most 5 of them):

(a) It is important that the method error D_m is significantly smaller than the irreducible error D_n.

(b) Using the bisection we cannot find the roots of degree 3.

(c) Tangent method has the cubic convergence in the neighborhood of α, if $f(\alpha) = 0$, $f'(\alpha) \neq 0$ and $f''(\alpha) = 0$.

(d) Tangent method converges for an arbitrary initial approximation.

(e) The identity matrix is symmetric positive devinite matrix.

(f) Frobenious norm is not the operator norm but it is the matrix norm.

(g) If A is an invertible matrix, then one can always compute the LU decomposition with partial pivoting.

(h) The system $Ax = b$ is computed such that we compute the inverse of A.

(i) The forward and backward substitution demand less time than LU decomposition.

(j) For the orthogonal matrices Q holds that $Q^{-1} = Q^{-T}$.

[15] **2.** Write the number 2010.65625 in IEEE standard for floating point arithmetic in single precision (32 bits).

[15] **3.** Find the fixed points of the iteration function

$$x_{r+1} = g(x_r) = (\gamma + 1)x_r - x_r^2, \quad \frac{1}{2} \leq \gamma \leq 1.$$

For each of these fixed points determine if it is attractive or unattractive point for the function g. For which values of γ the method has quadratic convergence in the neighborhood of attractive points?

[15] **4.** For the system of linear equation compute one step of the Newton method with the initial approximation $(x_0, y_0)^T = (1, -1)^T$:

$$x^2 + 3y^2 = 1, \quad (x - 2)^2 + (y - 1)^2 = 4.$$

[20] **5.** Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 1 \\ 0 & 1 & 5 \end{pmatrix}.$$

Compute the decomposition of A on a product $A = L \cdot U$, where L is a lower triangular matrix and U is upper triangular matrix in three ways:

(a) the diagonal elements of L are equal to 1,

(b) the diagonal elements of U are equal to 1,

(c) $U = L^T$.

[15] **6.** Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times n}$. Prove or disprove (find the counterexample) of inequalities:

(a) $\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_1 \leq \sqrt{\|A\|_1^2 + \|B\|_1^2}$,

(b) $\left\| \begin{bmatrix} A \\ B \end{bmatrix} \right\|_\infty \leq \sqrt{\|A\|_\infty^2 + \|B\|_\infty^2}$,

COLLOQUIUM

5 December 2011

1. Answer the following question:

   a) You are given the equation $x+x^2+e^x=0$. Which method for solving this equation do you know? Describe in detail one method.

   b) Describe in detail the Cholesky decomposition. What can we say about the matrix A, if the Cholesky decomposition exists for A?

   c) Write the iteration function of Jacobi and Gauss-Seidel method. Describe the difference between them. When this two methods converge?

2. For an arbitrary positive number a we can compute the square root $\sqrt{a}$ using the iteration

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{a}{x_r}\right).$$

   a) Find the fixed points of the iteration and determine if they are attractive or unattractive. Determine the order of convergence in the neighborhood of $\sqrt{a}$.

   b) Write how you would implement this method in Octave.

3. Let

$$A_n = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & n-1 & 1 \\ & & & & 1 & n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

   be tridiagonal symmetric matrix. Compute $\|A_n\|_1$, $\|A_n\|_\infty$, $\|A_n\|_\mathrm{F}$ and $N_\infty(A_n)$. Write the best estimation of the norm $\|A_n\|_2$ in case $n=4$.

4. Compute the LU decomposition with partial pivoting for

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 12 & 1 & 5 \\ 2 & 2 & 3 \end{pmatrix}.$$

   Using the obtained decomposition solve the system $Ax=b$, where $b = (0\ 6\ 12)^\mathrm{T}$.

COLLOQUIUM

13 December 2012

1. [10] Answer the following questions:

   a) Is the matrix
   $$A = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}$$
   symmetric positive definite matrix? Justify your answer. What is the cheapest way to check this?

   b) Which method for computing the QR decomposition you know? Which method would you use to compute the QR decomposition of the matrix
   $$\begin{pmatrix} 1 & 3 & 5 & 0 \\ 5 & -1 & 5 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -4 \end{pmatrix}?$$

   c) Let $I_n \in \mathbb{R}^{n\times n}$ be an identity matrix. Compute $\|I_n\|_1$, $\|I_n\|_\infty$, $\|I_n\|_2$ and $\|I_n\|_F$.

   d) Write the iteration function of the Jacobi and Gauss-Seidel method for the system $Ax = b$, where
   $$A = \begin{pmatrix} -8 & 3 \\ 1 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$
   What is the difference in using these two methods?

2. [10] Write the number -20.4 in IEEE standard for floating-point arithmetic in double precision. Determine the sign s, the mantissa m and the exponent e.

3. [10] Approximate the tabularly specified function

x	0.1	0.2	0.5	1
y	100	25	5	2

   with the function $f(x) = \frac{a}{x^2} + b$ using the least squares method.

4. [10] We would like to find the zeros of the function $f(x) = x^n - a$, $a > 0$, $n \in \mathbb{N}$, $n > 1$.

   a) Which method to solve this problem do you know? Describe in detail one among them.

   b) Suppose that we use the tangent method. Determine the iteration function and the fixed point of the iteration. Prove that the method converges for an arbitrary initial approximation $x_0 = (0, \infty)$. What is the order of convergence?

COLLOQUIUM

25 November 2013

1. [2] We are looking for the roots of the function $f(x) = x(x-1)^3(x-2)$ using the bisection.
   To which root the method will converge if we choose the initial interval to be $[-1, 4]$? And what would be the answer if one chooses the initial interval to be $[0.5, 3]$?

2.[4] Which number (in base 10) is in IEEE aritmetics in double precision presented as

$$1 \quad 10000000001 \quad 101\underbrace{0\ldots0}_{49} \quad ?$$

3. [4] Using the tangent method we are looking for the solution of the equation $x^5 = 3$. Write the corresponding iteration function. What can you say about the order of convergence?

4. [7] You are given the iteration function $g(x) = -ax^2 + 2x$, where $a > 0$. Determine the corresponding fixed points. Find the biggest possible interval on which the iteration function will converge to nonzero fixed point. For which initial approximations the convergence theorem assures us the convergence?

5. [8] You are given the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & -1 \\ 1 & 4 & 1 \end{pmatrix}.$$

   a) Determine the values of $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$ and estimate $\|A\|_2$.
   b) Compute the decomposition of a form $A = P^\top LU$, where P is permutation matrix, L is lower triangular matrix with ones on the main diagonal, U is upper triangular matrix and compute the determinant of the matrix A using the obtained decomposition.

6. [5] Let A be symmetric positive definite matrix of dimension $n \times n$. For an arbitrary $x \in \mathbb{R}^n$ define the norm $\|x\| = \sqrt{x^\top Ax}$. Show that $\|\cdot\|$ is norm on $\mathbb{R}^n$.

COLLOQUIUM

9 December 2014

1. [2] Where are the problems with the stability, if you numerically compute the value of the expression $x\sqrt{1+x} - x\sqrt{x}$? Transform the given expression into a stable form.

2. [8] You are given the equation

$$c = \ln x + x^2. \tag{1.1}$$

Let c be determined in such a way that the solution of (2.3) is equal to $\alpha = \frac{1}{3}$. We would like to solve (2.3) with simple iterative method using iterative functions

$$g_1(x) = \sqrt{c - \ln x},$$
$$g_2(x) = \frac{2}{11}\left(x + \frac{9}{2}e^{c-x^2}\right).$$

Show that the fixed point of g_1 and g_2 is the solution of (2.3). For each iterative function check if it assures the convergence in the neighborhood of α and determine the order of convergence.

3. [10] You are given a matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & 3 & 6 \\ 0 & 3 & 10 & 1 \\ 3 & 6 & 1 & 14 \end{pmatrix}.$$

a) Compute $\|A\|_1$, $\|A\|_\infty$ and $\|A\|_F$.

b) With the help of Cholesky decomposition solve the system $Ax = b$, where $b = (1, 0, 1, 0)^\top$.

4. [3] Determine the Givens rotation that rotates the vector $x = (\frac{1}{3}, -1)^\top$ into $y = (\beta, 0)^\top$, $\beta \in \mathbb{R}$. Determine β!

5. [7] Let $\|\cdot\|_p$ be an arbitrary vector norm in $\mathbb{C}^n$. For a given matrix $A \in \mathbb{C}^{n\times n}$ we define

$$\alpha(A) = \min_{\|x\|=1} \|Ax\|_p.$$

Show that if $\alpha(A) > 0$, then the following hold:

a) A is an invertible matrix,

b) $\|A^{-1}\|_p \leq \frac{1}{\alpha(A)}$, where $\|\cdot\|_p$ is the matrix norm induced by the vector $p-$norm.

COLLOQUIUM

9 December 2015

1. [4]
   (a) Find an iteration function for the equation $x + 1 = \tan x$ such that each fixed point is an attractive point.
   (b) Consider equation $f(x) = 0$ with $f(x) = 2 - x + \ln x$. By using tangent method with $x_0 = 3$ as the initial approximation, find the next two approximations.

2. [6] Suppose that the equation $x^2 + ax + b = 0$ has two real roots α and β. Show that the iteration method
   (a) $x_{k+1} = -(ax_k + b)/x_k$ is convergent near $x = \alpha$ if $|\alpha| > |\beta|$.
   (b) $x_{k+1} = -(x_k^2 + b)/a$ is convergent near $x = \alpha$ if $2|\alpha| < |\alpha + \beta|$.

3. [5] Solve the following nonlinear system
$$x^2 + y = 2,$$
$$y^2 - x^2 = 0.$$
   Make two steps of Newton's method with the initial approximation $(x_0, y_0)^T = (-1, -1)^T$.

4. [7]
   (a) We are solving the system of equations $Ax = b$ iteratively as
$$x^{(r+1)} = Rx^{(r)} + b.$$
   Let $A = \begin{pmatrix} 1 & k \\ 2k & 1 \end{pmatrix}$, $k \neq \frac{\sqrt{2}}{2}$, $k \in \mathbb{R}$. Find a necessary and sufficient condition on k for the convergence of the Jacobi method.
   (b) Let now $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$. Does the Gauss-Seidel method converge to the solution of $Ax = b$?

5. [8] Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ -1 & 4 & 1 \end{pmatrix}$.
   (a) Prove that $\|A\|_2 \leq 6$.
   (b) Compute the QR decomposition for A using the Gram-Schmidt orthogonalization.

COLLOQUIUM

25 April 2017

1. [5] Which number (in base-10 numeral system) in IEEE aritmetics in double format is presented as

1	01111111101	$11\underbrace{0\ldots0}_{50}$	?

2. [2] Leu us use the tangent (Newton's) method for finding the roots of $f(x) = x^5 - x^3 + 1$ and let $x_n = -1$. What is the value of the x_{n+1}?

3. [8] Let the following iteration be given:

$$x_{r+1} = (1 + 3x_r)^{\frac{1}{3}}, \quad r = 0, 1, \ldots$$

   a) Show that there exist fixed point of the given iteration on the interval $[1, 2]$.
   b) Show that the given iteration converges for an arbitrary initial approximation $x_0 \in [1, 2]$.
   c) Determine the order of convergence in the neighborhood of fixed point.

4. [8] Compute the LU decomposition with partial pivoting for the matrix

$$A = \begin{pmatrix} 0 & 5 & 5 \\ 2 & 6 & 2 \\ 4 & 10 & 6 \end{pmatrix}.$$

With the help of this decomposition, solve the system $Ax = b$, where $b = (10,\ 0,\ 20)^{\mathrm{T}}$.

5. [7] Let $A \in \mathbb{C}^{m \times n}$. Prove that

$$\|A^H\|_2 = \|A\|_2 \quad \text{and} \quad \|A^H A\|_2 = \|A\|_2^2.$$

COLLOQUIUM

9 April 2018

1. [4] How can we numerically compute in a stable way the value of the function

$$f(x) = \frac{2 - \sqrt{4 - x^2}}{x^2}$$

for small values x? Explain what is the reason for instability? Compute with a calculator the values $f(x)$ for $x = 10^{-5}, 10^{-8}$ using the stable formula and compute the limit $\lim_{x\to 0} f(x)$.

2. [7] (a) Compute $\sqrt[3]{2}$ to three decimal places precise using the tangent method.

   (b) We are solving the equation $x = b\,x(1-x)$ for different values of parameter b. Find the lower and the upper bound on b , such that the iteration

$$x_{n+1} = b\,x_n(1 - x_n),$$

   with a suitable choice of the initial approximation, converges to a positive solution of the equation?

3. [5] Solve the following system of nonlinear equations

$$x^2 + 2y^2 = 2,$$

$$y^2 - 2x = -1.$$

Make one step of Newton's method with initial approximation $(x_0, y_0) = (1, 1)^T$. Sketch both curves. How many real solutions exist?

4. [8] (a) Prove that we can solve the system $Ax = b$, with

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

   using the Jacobi iterative method. Compute the third iteration taking the zero-vector $\mathbf{0}$ as the initial approximation.

   (b) Prove that for any system of size 2×2 Jacobi iterative method converges if and only if Gauss-Seidel iterative method converges.

5. [6] (a) Let $A = \begin{pmatrix} 2 & 0 & 1 \\ 8 & -9 & 4 \\ -3 & 1 & 4 \end{pmatrix}$. Compute $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$ and find an estimation for $\|A\|_2$.

   (b) Prove that for any unitary matrix U it holds $\|UB\|_2 = \|B\|_2$.

COLLOQUIUM

8 April 2019

1. [6] (a) Write the number -2018.75 in IEEE floating-point arithmetic in double precision.

   (b) Which number (in 10-base system) is in IEEE floating-point arithmetic in single precision presented as

1	11111111	$\underbrace{0\ldots0}_{23}$

?

2. [6] Let x and y be two presentable numbers. In floating-point arithmetic with the basic rounding error u we compute the value of z in two ways:

   (a) $z = x^2 - 2xy + y^2$,

   (b) $z = (x - y)^2$

   Analyze both algorithms. Derive a relative error estimate $\left|\frac{fl(z)-z}{z}\right|$. Is any of the algorithms forward/backward stable?

3. [7] We are looking for the roots of the polynomial $f(x) = x^3 - 3x - 1$.

   (a) Show that there exist exactly one positive and two negative roots of f.

   (b) Show that the fixed points of the iteration

$$x_{r+1} = (1 + 3x_r)^{\frac{1}{3}}, \quad r = 0, 1, \ldots$$

   are the roots of f. Show also that the given sequence of iterations converges towards a positive root of f for every $x > 0$ and determine the order of convergence in it's neighborhood.

   (c) Write down the tangent method for searching the roots of f and compute the next approximate value x_1 given by the tangent method, if the initial approximate value is $x_0 = 2$.

4. [6] Let $||\cdot||$ be an arbitrary operator norm. Show that, if $||A|| < 1$, then $I + A$ is nonsingular matrix and the following holds

$$||(I + A)^{-1}|| \leq \frac{||I||}{||I|| - ||A||}.$$

POGLAVJE 2

EXAMS

2.1 Academic year 2008/2009

Academic year 2008/2009

EXAM

February 5th 2009

[10] **1.** Select the right answer for the following statements:

(a) LU decomposition is used for

(i) solving nonlinear equations
(ii) solving linear systems,
(iii) solving linear systems with positive definite matrix,
(iv) computing eigenvalues.

(b) LU decomposition requires the following number of operations

(i) $\mathcal{O}(n^2)$,
(ii) $\frac{1}{3}n^3 + \mathcal{O}(n^2)$,
(iii) $\frac{2}{3}n^3 + \mathcal{O}(n^2)$,
(iv) $n^3 + \mathcal{O}(n^2)$.

(c) How do we solve a linear system $Ax = b$ using LU decomposition?

(i) we decompose A into $A = LU$, and solve systems $Ly = b$ and then $Ux = y$,
(ii) we decompose A into $A = LU$, and solve systems $Uy = b$ and then $Lx = y$,
(iii) we decompose A into $A = LU$, compute $y = Lb$ and then $x = Uy$.
(iv) we decompose A into $A = LU$, compute $y = Ub$ and then $x = Ly$.

(d) We use LU decomposition with partial pivoting on matrix A

(i) only when all pivots are zero,
(ii) only if all pivots are nonzero,
(iii) if any of the pivots equals zero,
(iv) if any of the pivots is nonzero.

(e) For matrices L and U from the LU decomposition it holds:

(i) that both are upper triangular,
(ii) that both are lower triangular,
(iii) that L is upper triangular and U is lower triangular,
(iv) that L is lower triangular and U is upper triangular.

[10] **2.** There does not exist an exact formula for computing eigenvalues and eigenvectors for matrices of dimension at least 5, so they have to be computed numerically.

(a) Write the definition for an eigenvalue and eigenvector.

(b) One of the methods for computing eigenpairs is the power method. This method gives the approximation for the eigenvector, which corresponds to the dominant eigenvalue. How do we compute the approximation for the eigenvalue using the computed eigenvector?

(c) Which method do we use if we want to compute the eigenvector which corresponds to the non-dominant eigenvalue?

(d) Gerschgorin theorem determines the region in the complex plane where all the eigenvalues are. Write the theorem.

[10] **3.** An important subject in the approximation theory is interpolation.

(a) How many given points can be interpolated in general by a polynomial of degree n?

(b) Write a formula for the interpolating polynomial of degree n.

(c) What is the divided difference? Write the recursive formula for computing it.

(d) Describe the connection between an interpolating polynomial and Newton-Cotes integration rules.

[10] **4.** Let

$$A_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}, \quad n \geq 3.$$

Compute $\|A_n^T A_n\|_\infty$ and $\|A_n^T A_n\|_1$.

[15] **5.** Let us have the following iteration

$$x_{r+1} = g(x_r) = \alpha x_r + \frac{\beta}{x_r}, \quad r = 0, 1 \ldots$$

Determine coefficients α and β such that for any $a > 0$ the sequence $\{x_r\}_{r\in\mathbb{N}}$ converges to the limit $\sqrt{a}$ at least with the quadratic order of convergence.

[15] **6.** Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}.$$

Find the solution of the overdetermined linear system $Ax = b$, $x \in \mathbb{R}^3$, using the least squares method. Use the normal system and solve it using LU decomposition without pivoting.

[15] **7.** Let $f(x) = x^6$. Find the interpolating polynomial p for which

$$p(-1) = f(-1),\ p'(-1) = f'(-1),\ p(0) = f(0),$$
$$p'(0) = f'(0),\ p(1) = f(1),\ p'(1) = f'(1).$$

[15] **8.** Compute the unknown coefficients in the following open type Newton-Cotes integration rule

$$\int_{x_0}^{x_3} f(x)\,dx = Af(x_1) + Bf(x_2) + Cf^{(m)}(\xi).$$

EXAM

June 11th 2009

[10] **1.** Choose the right answers:

(a) Let α be the solution of the equation $x = g(x)$ and let $g^{(k)}(\alpha) = 0$ for $k = 1, 2, \dots, p$, and $g^{(p+1)}(\alpha) \neq 0$. Then we say that the order of convergence for the iteration $x_{r+1} = g(x_r)$ in the neighbourhood of α equals p.

(b) A method with high order of convergence converges everywhere faster then a method with lower order of convergence.

(c) Tangent method has at least quadratic order of convergence for all simple roots.

(d) Matrix A is symmetric iff $A = A^T$.

(e) One of the properties of a matrix norm is $\|A\,B\| \geq \|A\|\|B\|$.

(f) For any matrix norm and for any eigenvalue λ of A it holds $|\lambda| \leq \|A\|$.

(g) LU decomposition with pivoting is used for symmetric positive definite matrices.

(h) Matrix A is symmetric positive definite if $A = A^T$ and $x^T Ax > 0$ for any $x \neq 0$.

(i) Cholesky decomposition requires $\frac{1}{3}n^3 + \mathcal{O}(n^2)$ operations.

(j) The solution of the normal system $A^T Ax = A^T b$ maximizes $\|Ax - b\|_2$.

[10] **2.** A linear system $Ax = b$ can be written as $x = Rx + c$ and solved iteratively as $x^{(r+1)} = Rx^{(r)} + c$.

(a) What is the requirement for the spectral radius $\rho(R) := \max |\lambda(R)|$ of the iteration matrix R, in order to guarantee the convergence of the sequence $x^{(r+1)} = Rx^{(r)} + c$, $r = 0, 1, \dots$, for any initial approximation $x^{(0)}$?

(b) Write the iteration form of the Jacobi method.

(c) Write the iteration form of the Gauss-Seidel method.

(d) Assume A is upper triangular matrix. Which among the methods in (b) and (c) converges faster? Clarify your answer.

[10] **3.** An important subject in the approximation theory is interpolation.

(a) Write a formula for the interpolating polynomial of degree n.

(b) Is the interpolating polynomial an appropriate function to interpolate large number of interpolation points? Clarify your answer.

(c) Describe the connection between an interpolating polynomial and Newton-Cotes integration rules.

(d) What is the difference between open and closed type Newton-Cotes rules? Shortly describe two closed type Newton-Cotes rules.

[15] **4.** Using Gerschgorin theorem define the area $\mathbb{C}$, which contains all the eigenvalues of matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$

To obtain better result use the Gerschgorin theorem also on A^T.

[15] **5.** Show that the order of convergence for the iteration

$$x_{r+1} = g(x_r) := \frac{x_r(1 - \ln x_r)}{1 + 2x_r}, \quad r = 0, 1 \ldots,$$

used to solve the nonlinear equation $e^{2x} - \frac{1}{x} = 0$, is at least two.

[15] **6.** Compute the LU decomposition without pivoting for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -3 & -5 & -10 & -13 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 \end{pmatrix}.$$

[15] **7.** For the function $f(x) = \sin x$ find its interpolating polynomial p, which fulfils

$$p\left(k\frac{\pi}{2}\right) = f\left(k\frac{\pi}{2}\right), \quad p'\left(k\frac{\pi}{2}\right) = f'\left(k\frac{\pi}{2}\right),$$

for $k = -1, 0, 1$.

[10] **8.** We are solving the nonlinear system

$$x^2 + y = 2, \quad y^2 - x^2 = 0.$$

Perform two steps of the Newton method with the initial approximation $(x_0, y_0) = (-1, -1)^T$.

EXAM

June 26th 2009

[10] **1.** There are several methods available to compute zeros of real function $f : \mathbb{R} \to \mathbb{R}$.

(a) List at least three and describe one of them.

Find the right answer for the following statements

(b) Whether the simple iteration converges in the vicinity of the solution depends on

(i) the first derivative of the iteration function,
(ii) the second derivative of the iteration function,
(iii) the third derivative of the iteration function.

(c) Method with the quadratic order of convergence in the vicinity of the solution compared with the cubic convergence method

(i) converges slower,
(ii) converges faster,
(iii) converges with the same speed.

(d) When using the secant method we have to know

(i) one initial approximation,
(ii) two initial approximations,
(iii) the same number of initial approximations as for the tangent method.

(e) Tangent method has a cubic convergence in the vicinity of the solution α if

(i) $f(\alpha) = 0,\ f'(\alpha) = 0,$
(ii) $f(\alpha) = 0,\ f'(\alpha) = 0,\ f''(\alpha) = 0,$
(iii) $f(\alpha) = 0,\ f'(\alpha) \neq 0,\ f''(\alpha) = 0.$

[10] **2.** For solving a linear system $Ax = b$, with the nonsingular matrix A, we use the LU decomposition ($A = L \cdot U$).

(a) What holds for matrices L and U?

(b) When does the LU decomposition fail? What modification of the LU composition do we use in that case? What do we do on every step?

(c) How many operations are needed for the LU decomposition?

(d) For particular matrices a similar decomposition can be done in half time. Describe these matrices and name the simplified decomposition.

[10] **3.** There does not exist an exact formula for computing eigenvalues and eigenvectors for matrices of dimension at least 5, so they have to be computed numerically.

(a) Write the definition for an eigenvalue and eigenvector.

(b) One of the methods for computing eigenpairs is the power method. This method gives the approximation for the eigenvector, which corresponds to the dominant eigenvalue. How do we compute the approximation for the eigenvalue using the computed eigenvector?

(c) Which method do we use if we want to compute the eigenvector which corresponds to the non-dominant eigenvalue?

(d) Gerschgorin theorem determines the region in the complex plane where all the eigenvalues are. Write the theorem.

[10] **4.** Let

$$A_n = \begin{pmatrix} -1 & 1 & & & & \\ & -1 & 1 & & & \\ & & -1 & 1 & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & 1 \\ & & & & & -1 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Compute $\|A_n\|_\infty$, $\|A_n\|_1$ and $\|A_n\|_F$.

[15] **5.** We are solving the following system using the Jacobi iteration

$$\begin{aligned} 4x_1 + 2x_2 - x_3 &= 5 \\ 2x_1 + 6x_2 - 2x_3 &= 6 \\ -x_1 + x_2 + 3x_3 &= 3. \end{aligned}$$

For the initial approximation take $x_1 = 1, x_2 = 1$ and $x_3 = -1$ and compute the next approximation.

[15] **6.** Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 3 & -1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 5 & 6 \\ 2 & 2 & 7 & 5 \\ 2 & 10 & 9 & 16 \end{pmatrix}.$$

Compute the upper triangular matrix U, such that $A = L \cdot U$. Write down all steps of the computation.

[15] **7.** Find the solution of the overdetermined system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},$$

using the least squares method.

[15] **8.** We are solving the nonlinear system

$$xy = 2, \quad xz = 3, \quad yz = 6.$$

Perform one step of the Newton method with the initial approximation $x = y = z = 1$.

EXAM

August 31st 2009

[10] **1.** Choose the right answers:

(a) Tangent method converges with the quadratic order in the vicinity of the double root.

(b) Using the quadratic order method we double the number of precise decimal digits on every step in the vicinity of the solution.

(c) Matrix A is positive definite if for any vector $x \neq 0$ it holds $x^T A\,x > 0$.

(d) Operator norm of matrix A is defined as $\|A\| = \max\limits_{\substack{\|x\|=0 \\ x \in \mathbb{C}^n}} \|Ax\|$.

(e) System $Ax = b$ is preferable to solve as $x = A^{-1}b$.

(f) Linear system $Ax = b$ is solved iteratively if A is large and/or has few nonzero elements.

(g) When solving $A^T A\,x = A^T b$, we say that we solve the normal system that corresponds to $Ax = b$.

(h) Reyleigh quotient is a formula for the approximation of an eigenvector.

(i) Polynomial which fits the given function in some prescribed points is called the interpolation polynomial.

(j) Interpolation with splines is not useful in practice.

[10] **2.** An important subject in the approximation theory is interpolation.

(a) Write a formula for the interpolating polynomial of degree n.

(b) Is the interpolating polynomial an appropriate function to interpolate large number of interpolation points? Clarify your answer.

(c) Describe the connection between an interpolating polynomial and Newton-Cotes integration rules.

(d) What is the difference between open and closed type Newton-Cotes rules? Shortly describe two closed type Newton-Cotes rules.

[10] **3.** There are several methods available to find zeros of nonlinear function $f : \mathbb{R} \to \mathbb{R}$.

(a) Shortly describe the bisection.

(b) Shortly describe the simple iteration method.

(c) Shortly describe the tangent method.

[15] **4.** Using Gerschgorin theorem define the area $\mathbb{C}$, which contains all the eigenvalues of matrix

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

To obtain better result use the Gerschgorin theorem also on A^T.

[15] **5.** Determine coefficients α and β in the iteration

$$x_{r+1} = g(x_r) := \alpha x_r + \beta x_r^2, \quad r = 0, 1 \ldots,$$

such that the sequence $(x_r)_{r \in \mathbb{N}_0}$ converges to the chosen real number $a > 0$ with the quadratic convergence.

[20] **6.** Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -2 & -3 & -5 & -7 \\ 1 & 4 & 6 & 4 \\ -2 & -3 & -4 & -8 \end{pmatrix}.$$

Compute matrices L and U defining LU decomposition without pivoting for matrix A. Additionally compute $\|L\|_1 \cdot \|U\|_\infty$.

[20] **7.** For the function $f(x) = \sin x + \cos x - 1$ find the interpolating polynomial p, for which

$$p\left(k\frac{\pi}{2}\right) = f\left(k\frac{\pi}{2}\right), \quad p'\left(k\frac{\pi}{2}\right) = f'\left(k\frac{\pi}{2}\right), \quad p''\left(k\frac{\pi}{2}\right) = f''\left(k\frac{\pi}{2}\right)$$

for $k = 0, 1$.

Academic year 2009/2010

EXAM

June 8th 2010

[10] **1.** Choose the right answers.

(a) Using Newton method a nonlinear system is solved by solving linear system multiple times.

(b) Spectral norm of the symmetric $n \times n$ matrix can be computed as $\|A\|_2 = \min_{i=1,2,\ldots,n} |\lambda_i(A)|$.

(c) Condition number $\kappa_2(A)$ of matrix A can be computed using formula $\kappa_2(A) = \|A\| \cdot \|A\|^{-1}$.

(d) Permutation matrices P have condition number $\kappa_2(P)$ equal to 1.

(e) LU decomposition without pivoting requires $\mathcal{O}(n^3)$ operations.

(f) For lower triangular matrices both Gauss-Seidel and Jacobi methods converge with the same speed.

(g) Solving an overdetermined system using the normal system is numerically very stable.

(h) We can apply the Lagrange formula to express the interpolating polynomial only in the case, when all interpolation points are distinct.

(i) Polynomials of high degree usually perform better in practice as splines.

(j) When using numerical methods for differentiation we have to make sure that the interpolation points are not to close to each other.

[15] **2.** Answer the following questions:

(a) Why do we have to avoid taking interpolation points to close together when considering numerical differentiation formulas?

(b) Is the same true for numerical integration? Clarify your answer.

(c) What is the connection between Newton-Cotes integration rules and polynomial interpolation.

(d) Write down the composite $\frac{3}{8}$-rule for numerical integration.

(e) Describe the Romberg method. What is the main idea of the method?

[15] **3.** Find the lower triangular matrix V from the Cholesky decomposition for the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Additionally compute the determinant of the matrix A using the above decomposition.

[15] **4.** Compute QR decomposition of the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 3 & 7 \\ 1 & -1 & -4 \\ 1 & -1 & 2 \end{pmatrix}.$$

[15] **5.** Find the interpolating polynomial, which fits $f(x) = e^{-2x}$ three times in $x = 0$ and twice in $x = 1$.

[15] **6.** Determine constants A, B, C, D, E and m in the integration rule

$$\int_{x_0}^{x_1} f(x)\,dx = h\left(Af(x_0) + Bf(x_1)\right) + h^2\left(Cf'(x_0) + Df'(x_1)\right) + Ef^{(m)}(\xi).$$

[15] **7.** Find the area $\mathbb{C}$, which contains all the eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

EXAM

June 22th 2010

[10] **1.** Choose the right answers:

(a) Matrix is nonsingular if its determinant is nonzero.

(b) Frobenius norm is not an operator norm.

(c) Decomposition $A = U \cdot L$, where U is an upper triangular matrix and L is a lower triangular matrix with ones on the diagonal is called LU decomposition.

(d) When solving overdetermined systems $Ax = b$ the goal is to minimize $\|Ax - b\|$.

(e) Solving overdetermined system via normal system in more stable then using the QR decomposition.

(f) Singular values are nonnegative real values.

(g) Divided difference is a polynomial of degree 3.

(h) Spline is a C^∞ function.

(i) Simpson's rule is exact for polynomials of degree ≤ 3.

(j) Any Bézier curve interpolates all its control points.

[15] **2.** Answer the following questions:

(a) Write an example of an iteration function for computing roots of $f(x) = x^2 - 3x + 1$.

(b) What is the main difference between Jacobi and Gauss-Seidel methods?

(c) Why the overdetermined systems can not be solved using LU decomposition?

(d) What is the Reyleigh quotient?

(e) Why do we use composite integration rules?

[15] **3.** Solve the linear system $Ax = b$, with

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 4 & 2 \\ 1 & 2 & 4 \\ 1 & 4 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 12 \\ 7 \\ 12 \end{pmatrix}.$$

[15] **4.** Let

$$A_n = \begin{pmatrix} -2 & n-1 & & & & \\ n-1 & -4 & n-2 & & & \\ & n-2 & -6 & n-3 & & \\ & \ddots & \ddots & \ddots & & \\ & & 2 & -2(n-1) & 1 \\ & & & 1 & -2n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Compute $\|A_n\|_1$ and $\|A_n\|_\infty$. Additionally compute also $N_\infty(A_n)$.

[15] **5.** Find fixed points of the iteration

$$x_{r+1} = \frac{1}{2}\left(x_r + \frac{4}{x_r}\right).$$

Compute the order of convergence in the vicinity of these points.

[15] **6.** Determine constants A, B, C, D, E and m in the integration rule

$$\int_0^{3h} f(x)\,dx = Af(h) + Bf(2h) + Cf'(0) + Df'(3h) + Ef^{(m)}(\xi).$$

[15] **7.** We are given the first and the last control point of a cubic Bézier curve $\mathbf{p}$: $\mathbf{b_0}= (1,0)^T$ and $\mathbf{b}_3 = (0,1)^T$. Compute the remaining control points of the form $\mathbf{b}_1 = (1,a)^T$ and $\mathbf{b}_2 = (a,1)^T$, such that the curve will interpolate point $\left(\sqrt{2},\sqrt{2}\right)^T$ at $t = \frac{1}{2}$. Make also the sketch of the curve.

EXAM

August 25th 2010

[10] **1.** Choose the right answers:

(a) Tangent method has a quadratic order of convergence in the vicinity of the double solution.

(b) When using Newton method we solve a nonlinear system by solving a linear system several times.

(c) Permutation matrices P have the condition number $\kappa_2(P) := \|P\|_2 \cdot \|P^{-1}\|_2$ equal to 1.

(d) Matrix A is positive definite if for any vector $x \neq 0$ we have $x^T A\, x > 0$.

(e) System $Ax = b$ is preferable to solve as $x = A^{-1}b$.

(f) When solving $A^T A\, x = A^T b$ we say that we are solving the normal system that corresponds to the system $Ax = b$.

(g) Solving the overdetermined system using the normal system is numerically very stable.

(h) Reyleigh quotient is a formula, which gives a good approximation for an eigenvector.

(i) Simpson's rule is exact for polynomials of degree ≤ 3.

(j) Any Bézier curve interpolates all its control points.

[15] **2.** Answer the following questions:

(a) What is the quantity which determines whether the iteration in the vicinity of the solution converges or not?

(b) Write an example of an iteration function for computing zeros of $f(x) = e^x + \ln x$.

(c) Compare methods with the quadratic and with the linear order of convergence in the vicinity of the solution. Which is faster and where?

(d) How many initial approximations do we have to know for using the tangent method?

(e) What should hold for the tangent method to have the cubic order of convergence in the vicinity of the solution?

[15] **3.** Find the solution of the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 7 \\ 6 \\ 9 \end{pmatrix}.$$

Compute $\|L\|_1 \cdot \|U\|_\infty$.

[15] **4.** Perform one step of the Newton method with the initial approximation $(x_0, y_0)^T = (1,1)^T$ for the following nonlinear system

$$x^2 + y^2 = 2, \quad x^2 - 2xy + y = 2.$$

[15] **5.** Find fixed points of the iteration

$$x_{r+1} = Ax_r - x_r^3.$$

For which values of parameter A are all the fixed points attractive points? For which A is the order of convergence for the sequence $\{x_r\}$ at least quadratic?

[15] **6.** Determine constants A, B, C, D and E in the integration rule

$$\int_{-h}^{h} f(x)\,dx = Af(-h) + Bf(0) + Cf(h) + Df'(-h) + Ef'(h) + Ff^{(m)}(\xi).$$

[15] **7.** We are given the first and the last control point of a cubic Bézier curve $\mathbf{p}$: $\mathbf{b_0} = (1,0)^T$ and $\mathbf{b}_3 = (0,1)^T$. Compute the remaining control points of the form $\mathbf{b}_1 = (1,a)^T$ and $\mathbf{b}_2 = (a,1)^T$, such that the curve will interpolate point $\left(\sqrt{2}, \sqrt{2}\right)^T$ at $t = \frac{1}{2}$. Make also the sketch of the curve.

EXAM

September 8th 2010

[10] **1.** Choose the right answers:

(a) Tangent method has a quadratic order of convergence in the vicinity of the double solution.

(b) When using Newton method we solve a nonlinear system by solving a linear system several times.

(c) Permutation matrices P have the condition number $\kappa_2(P) := \|P\|_2 \cdot \|P^{-1}\|_2$ equal to 1.

(d) Matrix A is positive definite if for any vector $x \neq 0$ we have $x^T A\, x > 0$.

(e) System $Ax = b$ is preferable to solve as $x = A^{-1}b$.

(f) When solving $A^T A\, x = A^T b$ we say that we are solving the normal system that corresponds to the system $Ax = b$.

(g) Solving the overdetermined system using the normal system is numerically very stable.

(h) Reyleigh quotient is a formula, which gives a good approximation for an eigenvector.

(i) Simpson's rule is exact for polynomials of degree ≤ 3.

(j) Any Bézier curve interpolates all its control points.

[15] **2.** Answer the following questions:

(a) What is the quantity which determines whether the iteration in the vicinity of the solution converges or not?

(b) Write an example of an iteration function for computing zeros of $f(x) = e^x + \ln x$.

(c) Compare methods with the quadratic and with the linear order of convergence in the vicinity of the solution. Which is faster and where?

(d) How many initial approximations do we have to know for using the tangent method?

(e) What should hold for the tangent method to have the cubic order of convergence in the vicinity of the solution?

[15] **3.** Find the solution of the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 8 \\ 7 \\ 6 \\ 9 \end{pmatrix}.$$

Compute $\|L\|_1 \cdot \|U\|_\infty$.

[15] **4.** Perform one step of the Newton method with the initial approximation $(x_0, y_0)^T = (1,1)^T$ for the following nonlinear system

$$x^2 + y^2 = 2, \quad x^2 - 2xy + y = 2.$$

[15] **5.** Find fixed points of the iteration

$$x_{r+1} = Ax_r - x_r^3.$$

For which values of parameter A are all the fixed points attractive points? For which A is the order of convergence for the sequence $\{x_r\}$ at least quadratic?

[15] **6.** Determine constants A, B, C, D and E in the integration rule

$$\int_{-h}^{h} f(x)\,dx = Af(-h) + Bf(0) + Cf(h) + Df'(-h) + Ef'(h) + Ff^{(m)}(\xi).$$

[15] **7.** We are given the first and the last control point of a cubic Bézier curve $\mathbf{p}$: $\mathbf{b_0}= (1,0)^T$ and $\mathbf{b}_3 = (0,1)^T$. Compute the remaining control points of the form $\mathbf{b}_1 = (1,a)^T$ and $\mathbf{b}_2 = (a,1)^T$, such that the curve will interpolate point $\left(\sqrt{2}, \sqrt{2}\right)^T$ at $t = \frac{1}{2}$. Make also the sketch of the curve.

Academic year 2010/2011

EXAM

December 3rd 2010

[20] **1.** Choose the right answers:

(a) Gauss-Seidel method always converges faster than the Jacobi method.

(b) Sequence of approximations when iteratively solving linear system does not converge if the supremum norm of the iteration matrix is larger than 1.

(c) When using the least squares method for solving the linear system $Ax = b$ we are computing vector x, such that $(Ax - b)^T(Ax - b)$ is minimal.

(d) Classical and modified Gram-Schmidt approach are the same when using the exact computation.

(e) For each Givens rotation R it holds $R^T R = I$.

(f) Overdetermined system can be solved also via singular value decomposition.

(g) Reyleigh quotient is the best approximation for the eigenvector, which corresponds to the dominant eigenvalue.

(h) When increasing the number of interpolation points, interpolating polynomial converges to the given interpolated function.

(i) Lagrange basis polynomials form a partition of unity.

(j) Using Romberg method we can highly improve the approximation for the exact value of the integral. But since the approach is very expensive we do not use it in practice.

[15] **2.** Find matrices L, U and P from the LU decomposition with partial pivoting for the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{pmatrix}.$$

Assuming $\kappa_\infty(A) = 435$, compute $\|A^{-1}\|_\infty$.

[15] **3.** Using Givens rotations compute the QR decomposition of the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 0 & 12 \\ 6 & 3 \\ 0 & -9 \end{pmatrix}.$$

[15] **4.** Let $M \in \mathbb{R}^{n\times n}$ be a matrix with nonnegative elements for which

$$\sum_{j=1}^{n} m_{ij} = 1, \quad i \in \{1, 2, \dots, n\}.$$

Using the Gerschgorin theorem, prove that all eigenvalues of matrix M are contained in the unit circle $\{z \in \mathbb{C},\ |z| \le 1\}$.

If you fail to prove the statement in general, prove it at least for the matrix

$$M = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.2 & 0.5 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}.$$

Show that the same result holds also for matrices with nonnegative elements for which $\sum_{i=1}^{n} m_{ij} = 1$ for any $j \in \{1, 2, \ldots, n\}$.

[20] **5.** We interpolate function

$$f(x) = \frac{5 - x}{5 + x}$$

in distinct points $x_0, x_1, \ldots, x_n$.

(a) Derive the closed form formula for the divided difference $f[x_0, x_1, \ldots, x_n]$.

(b) What is the interpolation error in point x?

Advice: Use the formula $f(x) - I_n(x) = f[x_0, x_1, \ldots, x_n, x]\,(x - x_0)(x - x_1)\cdots(x - x_n)$.

[15] **6.** Determine constants A, B, C, D and m in the rule for numerical integration:

$$f''(x_1) = Af(x_0) + Bf(x_1) + Cf(x_2) + Df^{(m)}(\xi).$$

EXAM

January 28th 2011

[20] **1.** Choose the correct statements:

(a) The length of the mantisa for the double precision is 64 bits.

(b) Irremovable error is an error, which is the consequence of the error of the initial approximations.

(c) When using the bisection method we can always find the zero of the function, if we only start on the interval, where the function has different signs on both interval ends.

(d) All fixed points are unattractive points for the tangent method.

(e) For every symmetric matrix there exists an eigenvalue, which absolute value is larger than the supremum norm of the matrix.

(f) Solving a linear system with a lower triangular matrix only requires $\mathcal{O}(n)$ operations.

(g) Matrix of the normal system is symmetric matrix.

(h) Using the inverse iteration we can find the smallest eigenvalue considering absolute values.

(i) Gerschgorin theorem precisely determines the smallest and the largest eigenvalue.

(j) Convex hull of control points of a Bézier curve contains all control points of the curve.

[15] **2.** Compute $\|A_n\|_1, \|A_n\|_\infty, N_\infty(A_n)$ and $\|A_n\|_F$ for the matrix

$$A_n = \begin{pmatrix} -n & 0 & 0 & \cdots & 0 & 1 \\ 0 & -n & 0 & \cdots & 0 & 1 \\ 0 & 0 & -n & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -n & 1 \\ 1 & 2 & 3 & \cdots & n-1 & -n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

[15] **3.** Using the Gram-Schmidt orthogonalization find the QR decomposition of matrix A and then solve the overdetermined system $Ax = b$, where

$$A = \begin{pmatrix} 8 & 4 \\ 0 & 12 \\ 6 & 3 \\ 0 & -9 \end{pmatrix}, \quad b = \begin{pmatrix} 15 \\ 0 \\ 5 \\ -25 \end{pmatrix}.$$

[20] **4.** Find all fixed points of the iteration function

$$x_{r+1} = g(x_r) = \frac{x(x^2+12)}{3x^2+4}.$$

For each of them find out whether they are attractive or unattractive for the function g. What is the order of convergence in the vicinity of attractive points?

[15] **5.** Write the interpolating polynomial of degree 5, which interpolates function $f(x) = -x^7 + x^6$ three times in point 0 and twice in point 1.

[15] **6.** We are given a parametric polynomial curve

$$p(t) = \begin{pmatrix} t^3 + 2t^2 + 1 \\ 2t^2 - t + 2 \end{pmatrix}, \quad 0 \le t \le 1.$$

Write it in the Bézier form.

EXAM

June 28th 2011

[20] **1.** Choose the correct statements:

(a) Error of the method is the error which is the consequence of rounding the numbers.

DA NE

(b) Using the bisection we find all zeros on the chosen interval.

DA NE

(c) Second derivative of the iteration function is the one that decides whether we have a convergence or not of the iteration method in the vicinity of a solution.

DA NE

(d) The convergence of the tangent method highly depends on the initial approximation.

DA NE

(e) All eigenvalues of the Cholesky factor are positive.

DA NE

(f) Gauss-Seidel method does not necessarily converge faster than the Jacobi method.

DA NE

(g) Matrix Q of the QR decomposition is symmetric matrix.

DA NE

(h) Gerschgorin theorem determines the area in the complex plane which contains all eigenvectors of a particular matrix.

DA NE

(i) Divided difference is a symmetric function of its arguments.

DA NE

(j) If all the control points of a Bézier curve lie on a line, then also the whole Bézier curve is contained in this line.

DA NE

[15] **2.** Compute the LU decomposition without pivoting for a matrix

$$A = \begin{pmatrix} 4 & 3 & 1 & 4 & 3 \\ 4 & 7 & 3 & 6 & 5 \\ 0 & 16 & 10 & 12 & 9 \\ 0 & 0 & 6 & 13 & 4 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}.$$

Compute $\|U\|_\infty$, $\|U\|_1$ in $\|L\|_F$.

[15] **3.** Perform two steps of the Newton method for the system of nonlinear equations

$$x^2 + 2y^2 = 2, \quad x^2 - xy + y = 0,$$

with the initial approximation $(x_0, y_0)^T = (1, 1)^T$.

[20] **4.** Find fixed points of iteration function

$$x_{r+1} = g(x_r) = x_r(3 - 3\,a\,x_r + a^2 x_r^2), \quad r = 0, 1, \ldots, \qquad a \neq 0.$$

For each of fixed points compute whether they are attractive or unattractive points for the function g. What is the order of convergence in the vicinity of attractive fixed points. Selecting $x_0 = 0.25$ and $a = 3$ compute the first two approximations.

[15] **5.** Determine constants A, B, C, D and m of the integration rule

$$\int_0^4 f(x)\,dx = A\,f(0) + B\,f(2) + C\,f(4) + D\,f^{(m)}(\xi).$$

[15] **6.** Find conditions that have to be fulfilled for a Bézier curve of degree 4 to be a curve of degree 1.

EXAM

September 5th 2011

[20] **1.** Choose the correct statements:

(a) Error of the method is the error which is the consequence of rounding the numbers.

DA NE

(b) Using the bisection we find all zeros on the chosen interval.

DA NE

(c) Second derivative of the iteration function is the one that decides whether we have a convergence or not of the iteration method in the vicinity of a solution.

DA NE

(d) The convergence of the tangent method highly depends on the initial approximation.

DA NE

(e) All eigenvalues of the Cholesky factor are positive.

DA NE

(f) Gauss-Seidel method does not necessarily converge faster than the Jacobi method.

DA NE

(g) Matrix Q of the QR decomposition is symmetric matrix.

DA NE

(h) Gerschgorin theorem determines the area in the complex plane which contains all eigenvectors of a particular matrix.

DA NE

(i) Divided difference is a symmetric function of its arguments.

DA NE

(j) If all the control points of a Bézier curve lie on a line, then also the whole Bézier curve is contained in this line.

DA NE

[15] **2.** Compute the LU decomposition without pivoting for a matrix

$$A = \begin{pmatrix} 4 & 3 & 1 & 4 & 3 \\ 4 & 7 & 3 & 6 & 5 \\ 0 & 16 & 10 & 12 & 9 \\ 0 & 0 & 6 & 13 & 4 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}.$$

Compute $\|U\|_\infty$, $\|U\|_1$ in $\|L\|_F$.

[15] **3.** Perform two steps of the Newton method for the system of nonlinear equations

$$x^2 + 2y^2 = 2, \quad x^2 - xy + y = 0,$$

with the initial approximation $(x_0, y_0)^T = (1, 1)^T$.

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$$x_{r+1} = g(x_r) = x_r(3 - 3\,a\,x_r + a^2 x_r^2), \quad r = 0, 1, \ldots, \qquad a \neq 0.$$

For each of fixed points compute whether they are attractive or unattractive points for the function g. What is the order of convergence in the vicinity of attractive fixed points. Selecting $x_0 = 0.25$ and $a = 3$ compute the first two approximations.

[15] **5.** Determine constants A, B, C, D and m of the integration rule

$$\int_0^4 f(x)\,dx = A\,f(0) + B\,f(2) + C\,f(4) + D\,f^{(m)}(\xi).$$

[15] **6.** Find conditions that have to be fulfilled for a Bézier curve of degree 4 to be a curve of degree 1.

2.4 Academic year 2011/2012

Academic year 2011/2012

EXAM

30 January 2012

1. You are given a points in $\mathbb{R}^2$:

$$(-2,-6),\ (0,2),\ (1,0),\ (2,-8).$$

   Approximate the given points with parabola $y = ax^2 + b$ by the least squares method.

2. a) What is the divided difference? How we compute it?

   b) Write the interpolating polynomial p for which

$$p(0) = -2,\ p'(0) = 2,\ p''(0) = 6,\ p(2) = 6,\ p(3) = -2$$

   and compute it's value in points 1 and 2.

3. You are given equidistant points $x_i = x_0 + ih$, $i = 1, 2, 3$, and let $f_i := f(x_i)$.
   Using the method of undetermined coefficients derive the Milne rule

$$\int_{x_0}^{x_4} f(x)dx = \frac{4}{3}h\big(2f_1 - f_2 + 2f_3\big) + \frac{14}{45}h^5 f^{(4)}(\xi).$$

4. Let A and B be square matrices, $A, B \in \mathbb{R}^{n\times n}$. Prove the following:

   a) The matrices AB and BA have the same eigenvalues.

   b) If AB and BA are symmetric matrices, then

$$\|AB\|_F \leq \|BA\|_F.$$

Academic year 2012/2013

EXAM

24 January 2013

1. [9] Answer the following questions:

   a) We are given 6 points on the plane. We would like to construct the polynomial that interpolates these points. What would be the degree of this polynomial? How would you determine it?

   b) We are given 100 points on the plane. What would you use to interpolate these points - would be interpolating polynomial good, or would you use something else? Justify your answer.

   c) What is the difference between interpolation and approximation, which includes also solving the overdetermined systems?

2. [13] You are given a matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 5 & 4 & 5 \\ 0 & 4 & 8 & 6 \\ 1 & 5 & 6 & 7 \end{pmatrix}.$$

   Find the matrix V from the Cholesky decomposition of A. Compute also $\|V\|_1$ and $\|V\|_\infty$ and estimate $\|V\|_2$.

3. [13] The following points are given, $(0,0)^\top$, $(1,1)^\top$, $(3,-1)^\top$. Write the quadratic Bézier curve that interpolates these points at parameters 0, $\frac{1}{3}$, 1. Sketch the corresponding control polygon and the obtained curve.

4. [15] Let the points be equidistant. Derive the quadrature rule

$$\int_{x_0}^{x_2} f(x)dx = h\left(Af(x_0)+Bf(x_1)+Cf(x_2)\right)+h^2\left(Df'(x_0)+Ef'(x_2)\right)+Ff^{(m)}(\xi),$$

   where $x_0 < \xi < x_2$, using the method of undetermined coefficients, namely, the constants A, B, C, D, E, F and m should be determined in such a way that the rule would be of the highest possible degree.

EXAM

7 February 2013

1. [10] a) Write the Newton interpolating polynomial p for which:

$$p(0) = 1, \quad p'(0) = 0, \quad p(3) = 0, \quad p'(3) = 1.$$

   b) Describe two properties of the Bézier curves. Does the Bézier curve interpolate it's control points?

2. [30] We would like to compute the $\sqrt[3]{17}$. Which method would you choose:

$$\text{a) } x_{n+1} = \left(\frac{17}{x_n}\right)^{\frac{1}{2}}, \quad \text{b) } x_{n+1} = x_n - \frac{x_n^3 - 17}{3x_n^2}, \quad \text{c) } x_{n+1} = x_n - \frac{x_n^4 - 17x_n}{x_n^2 - 17}?$$

   Justify your answer.

3. [30] Using the LU decomposition without pivoting solve the system of linear equations:

$$\begin{aligned} 2x_1 + x_2 &= 2, \\ x_1 - x_2 + \frac{1}{2}x_3 &= 3, \\ 2x_1 - \frac{1}{2}x_2 + 3x_3 &= -1. \end{aligned}$$

4. [30] In the following table, you are given a function values:

x	1.9	2.0	2.1
$f(x)$	12.703	14.778	17.149

   We would like to compute the value of the second derivative of the function f in point $x = 2.0$. Derive the appropriate formula for numerical differentiation and compute the approximation for $f''(2.0)$.

   Hint: Using the method of undetermined coefficients derive formula for equidistant points:

$$f''(x_1) = Af(x_0) + Bf(x_1) + Cf(x_2) + Df^{(m)}(\xi),$$

   where $x_0 < \xi < x_2$.

EXAM

3 June 2013

1. [10] Three points P_0, P_1, P_2 and the corresponding tangent vectors d_0, d_1, d_2 are interpolated by Bézier curve.

   a) What is the degree of the obtained curve? Is it important what is the order in which you interpolate given data?

   b) What is the condition on first two control points of the adjacent Bézier curve in order that the spline of these two Bézier curves would be G^1?

2. [30] You are given a matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -2 \\ 1 & -3 & 1 \end{pmatrix}.$$

   a) Compute the decomposition of a form $A = P^\top LU$, where P is permutation matrix, L is lower triangular matrix with ones on the main diagonal and U is upper triangular matrix. Justify, why the elements of the matrix L are ≤ 1 (by the absolute value).

   b) Using a) compute the determinant of A.

3. [30] Consider the function

$$f(t) = \cos\left(\frac{\pi t}{2}\right)$$

and the end of the robot arm that is moving on the plane. Determine polynomial p along which it must move such that it's path would match f in points $t_0 = 0$, $t_1 = 2/3$ and $t_2 = 1$. Estimate also the error $\max|f(t) - p(t)|$ for $t \in [0, 1]$!

4. [30] We would like to determmine $x \in \mathbb{R}$ for which

$$\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt = 0.31. \qquad (2.1)$$

   a) Use the tangent (Newton) method write the iteration formula for solving the equation (2.1).

   b) Let the first approximation of the obtained iteration formula be equal to $x_0 = 0.5$. Using the composite Simpson's rule ($n = 4$) compute the next approximation x_1.

Academic year 2013/2014

EXAM

27 January 2014

1. [10] a) We would like to solve the system

$$\begin{aligned} 8x_1 + 2x_2 + x_3 &= 5 \\ 5x_2 + x_3 &= 1 \\ 6x_3 &= 2 \end{aligned}$$

iteratively. Which suitable methods do you know? Which one would converge the fastest?

b) Using the divided differences write the polynomial p for which

$$p(0) = 1, \quad p'(0) = 2, \quad p(2) = -1, \quad p'(2) = 1.$$

What would be the degree of the interpolating polynomial, if you would additionally interpolate also the values of the second derivatives?

c) The quadrature rule $\int_{-1}^{1} f(x)dx \doteq Af(-1) + Bf(0) + Cf(1)$ is exact for the polynomials of degree ≤ 2. Determine the constants A, B, C.

2. [10] For the matrix

$$A = \begin{pmatrix} 4 & -2 & 0 & 2 \\ -2 & 5 & 2 & -1 \\ 0 & 2 & 10 & 0 \\ 2 & -1 & 0 & 2 \end{pmatrix}$$

compute the matrix V from the Cholesky decomposition. What the result tells you about the matrix A? Estimate also the eigenvalues of A.

3. [15] Using the method of undeterined coefficients derive the formula for the numerical differentiation:

$$f'(x_2) = \frac{1}{h}\left(\frac{1}{2}f(x_0) - 2f(x_1) + \frac{3}{2}f(x_2)\right) + \frac{h^3}{3}f^{(3)}(\xi), \quad x_0 < \xi < x_2.$$

Then, with the help of the obtained rule estimate $f'(1.2)$, if you know the values of f in three points, namely, $f(0.8) = 4.01$, $f(1) = 5.44$, $f(1.2) = 7.30$.

4. [15] Let for the cubic Bézier curve $\mathbf{p}$ the following hold:

$$\mathbf{p}(0) = (0,0)^\top, \quad \mathbf{p}\left(\frac{1}{3}\right) = \left(\frac{26}{27}, \frac{4}{3}\right)^\top, \quad \mathbf{p}(1) = (2,0)^\top.$$

Determine the control points of $\mathbf{p}$, if the second and the third control points are of a form $\mathbf{b}_1 = (x_1, 2)^\top$ and $\mathbf{b}_2 = (2, y_2)^\top$. Define the corresponding Bézier curve and sketch it.

EXAM

10 February 2014

1. [5] Write the general form of a Bézier curve. Can any polynomial curve be written in a Bézier form? List three properties of Bézier curves.

2. [15] We would like to compute $\sqrt[5]{15}$. You are given two methods:

$$a)\ x_{r+1} = x_r - \frac{x_r^5 - 15}{40}, \quad b)\ x_{r+1} = x_r - \frac{x_r^5 - 15}{5x_r^4}.$$

Check if any of these two methods would converge and determine the order of convergence.
For a tangent method define the area in which the method would converge.

3. [10] Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

   a) Using Gram-Schmidt algorithm compute QR decomposition of A.
   b) Using the decomposition obtained in a) compute products Q^TQ and QQ^T. What can you observe?

4. [10] In statistics we often use matrix H,

$$H = X(X^TX)^{-1}X^T, \quad X \in \mathbb{R}^{m\times n},\ m \geq n,\ \operatorname{rang} X = n,$$

where in most cases we are interested only in the diagonal elements of this matrix.
Write the economical algorithm for computing the diagonal elements. Hint: Use the QR decomposition of X.

5. [10] Determine te interpolation polynomial p, that matches the function $f(x) = e^x$ in a value, first and second derivative at $x = 0$ and $x = 1$. Estimate the error.

EXAM

11 June 2014

1. [7] What is the divided difference? For $f(x) = x\cos(x)$ compute $f[0,0,\pi,\pi]$. Explain what the obtained divided difference represents.

2. [14] Using the Givens rotations and the QR decomposition solve the system

$$\begin{aligned} x - y &= 0, \\ -x + 2y &= 1, \\ -y + 2z &= 1. \end{aligned}$$

3. [14] We would like to solve the equation $e^x = \frac{1}{x}$.

   a) Which numerical methods for solving the given equation do you know? Describe one of them in detail.

   b) Supose that we use the iteration formula

   $$x_{r+1} = \frac{x_r^2 + x_r}{1 + 2x_r + \ln x_r}.$$

   Determine the order of convergence.

4. [15] We are given equidistant points. Derive the quadrature rule

$$\int_{x_0}^{x_2} f(x)dx = h\left(Af(x_0)+Bf(x_1)+Cf(x_2)\right)+h^2\left(Df'(x_0)+Ef'(x_2)\right)+Ff^{(m)}(\xi),$$

where $x_0 < \xi < x_2$, using the method of undetermined coefficients: the constants A, B, C, D, E, F and m should be determined such that the rule would be of the highest possible degree.

EXAM

1 September 2014

1. [5] Let α be a solution of equation $x = g(x)$. When do we say that the order of convergence of the iteration $x_{r+1} = g(x_r)$ in the neighborhood of α is equal to 4?

2. [12] For
$$A = \begin{pmatrix} -2 & 3 & 4 \\ -4 & 7 & 8 \\ 0 & 4 & 4 \end{pmatrix}$$
compute the LU decomposition with partial pivoting. Why the elements of L are ≤ 1 by the absolute value? Compute also $\|L\|_1 \|U\|_\infty$.

3. [10] Write the interpolating polynomial that matches function $f(x) = e^{-x}$ three times (function value and values of the first and second derivative) in $x = 0$ and two times (value and value of first derivative) in $x = 1$.

4. [10] Using the Newton method find the first approximation for the solution of the system
$$xy = 4, \quad xz = 12, \quad yz = 6.$$
Take $(x, y, z) = (1, 1, 1)$ as the initial approximation.

5. [13] Derive the quadrature rule
$$\int_{-h}^{h} f(x)dx = Af(-h) + Bf(0) + Cf(h) + Df'(-h) + Ef'(h) + Ff^{(m)}(\xi),$$
where $-h < \xi < h$, using the method of undetermined coefficients: the constants A, B, C, D, E, F and m should be determined such that the rule would be of the highest possible degree.

2.7 Academic year 2014/2015

Academic year 2014/2015

EXAM

19 January 2015

1. [4] Let $A \in \mathbb{R}^{n\times n}$. Show that for $n \times n$ unitary matrices Q and R holds:

$$\|QAR\|_2 = \|A\|_2.$$

2. [4] Let $A \in \mathbb{R}^{m\times n}$, $m > n$, $\operatorname{rang}(A) = n$. Show that the system

$$\begin{pmatrix} A & I \\ 0 & A^\top \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

has a solution that matches the solution of overdetermined system $Ax = b$ by the least squares method.

3. [8] Write one step of the Jacobi and Gauss-Seidel iteration with the initial approximation $(0, 0, 0)^\top$ for the system

$$\begin{pmatrix} 5 & 2 & -1 \\ 3 & 7 & 3 \\ 1 & -4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Do both of the methods converge for an arbitrary initial approximation?

4. [10] You are given a function $f(x) = \sqrt{1 - 3x^2}$. Find the interpolation polynomial p, that matches f two times (in point and derivative) at 0 and $\frac{1}{2}$. Estimate the error if the interpolation.

5. [9] Determine the control points of the cubic Bézier curve b, for which:

$$b(0) = (0, 0)^\top, \quad b'(0) = (6, 6)^\top, \quad b''(0) = (0, -12)^\top, \quad b'''(0) = (-6, 6)^\top.$$

6. [15]
   a) Show that one step of the Romberg extrapolation of the (composite) trapezoidal rule returns the (composite) Simpson's rule.
   b) Determine the weights in a scheme obtained after one step of the Romberg extrapolation of the Simpson rule. Explain, if we get another Newton-Cotes rule by this approach. If the answer is yes, explain if there exist also other examples when one step of Romberg extrapolation of the Newton-Coses rule returns another Newton-Cotes rule?

EXAM

4 February 2015

1. [4] Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & -2 & 1 \\ 2 & 1 & -4 & 2 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & -5 & 2 \end{pmatrix}.$$

   a) Compute LU decomposition of A. Can we use the LU without pivoting or we have to use decomposition with partial pivoting?

   b) Using the obtained decomposition compute the determinant of A.

   c) What is the number of operations that are needed to compute the determinant of $n \times n$ matrix using LU decomposition?

2. [16] We are given cubic polynomial p with real coefficients.

   a) Let p has three simple roots α, β and γ. Show that using the tangent method on the initial approximation $x_0 = \frac{1}{2}(\alpha + \beta)$ we get the root γ after one step.

   b) Suppose that we have one double root, e.g. $\beta = \gamma$. Explain why there exists exactly one initial approximation $\xi, \xi \neq \beta$, for which the tangent method fails. To what values the tangent method converge for the initial approximations on the interval $(-\infty, \xi)$ and to what values if the initial approximations belong to (ξ, ∞)?

   c) Suppose that all three roots are simple. Using the consideration in b) explain that in this case there exist infinetly many initial approximations for which the tangent method fails.

3. [9] For computing the value of the integral

$$\int_0^1 \frac{1}{1+2x} dx$$

   use the composite trapezoidal rule, where you should take $h = \frac{1}{2}$. Estimate also the error.

4. [9] Find the control points of the Bézier curve b, such that

$$b(0) = (0,0)^\top, \quad b'(0) = (0,3)^\top, \quad b\left(\frac{1}{2}\right) = (2,4)^\top, \quad b(1) = (4,0)^\top.$$

   Sketch the obtained curve and the corresponding control polygon.

EXAM

18 June 2015

1. [4] Where are the problems with the stability, if we want to compute

$$\frac{1}{x}\sqrt{1-x+x^2}-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}},$$

numerically at $x = 10^{-16}$. Write the expression in a more stable form.

2. [16] Consider the matrix $A \in \mathbb{R}^{n\times n}$,

$$\begin{aligned} a_{i,i} &= 3n-2i, \quad i=1,\dots,n, \\ a_{i+1,i} &= a_{i,i+1} = -i, \quad i=1,\dots,n-1, \end{aligned}$$

where the rest of the elements are equal to 0 :

$$A = \begin{pmatrix} 3n-2 & -1 & & & & \\ -1 & 3n-4 & -2 & & & \\ & -2 & 3n-6 & -3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -n+2 & n+2 & -n+1 \\ & & & & -n+1 & n \end{pmatrix}.$$

Compute $\|A\|_1, \|A\|_\infty, \|A\|_F$ and derive the best estimation for $\|A\|_2$.
Hint:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

3. [18] We would like to compute the zeros of

$$f(x) = e^{-x} - 3x$$

numerically. Justify, why f has exactly one zero α on the interval $[0, \frac{1}{3}]$. Then check if the simple iteration

$$x_{r+1} = \frac{x_r(1-\ln(3x_r))}{1+x_r}$$

converge to α and determine the order of convergence.

4. [12] Consider the Bézier curve b with control points

$$\mathbf{b}_0 = (0,0)^\top, \mathbf{b}_1 = (h,0)^\top, \mathbf{b}_2 = (0,h)^\top, \mathbf{b}_3 = (0,0)^\top,$$

where $h > 0$.

   a) Determine $b\left(\frac{1}{2}\right)$. Sketch the curve.
   b) Find the value of the parameter t, at which the x coordinate of the curve is the greatest and determine the corresponding point on the curve.
   c) For what value of the parameter h the curve would the curve best fit into a unit square?

2.8 Academic year 2015/2016

Academic year 2015/2016

EXAM

26 January 2016

1. [10] a) Show that the Newton method for finding reciprocals by solving $\frac{1}{x} - c = 0$, $c \neq 0$, results in the iteration $x_{n+1} = x_n(2 - cx_n)$, for $n \geq 0$. Also show that this iteration converges quadratically close to a fixed point.

   b) Show that a solution of the equation $x = c^{1/5}$ is a fixed point of iteration $x_{n+1} = cx_n^{-4}$, for $n \geq 0, c > 0$. Moreover show that the iteration fails in finding the fifth root of c.

2. [10] Solve the following system of equations

$$\begin{aligned} x - y &= 0, \\ -x + 2y &= 1, \\ -y + 2z &= 1, \end{aligned}$$

   by using Givens rotations and QR-decomposition.

3. [8] Derive a method for approximating the second derivative of f:

$$f''(a) \approx Af(a-h) + Bf(a) + Cf(a+h) + Df(a+2h), \quad A, B, C, D \in \mathbb{R},$$

   by requiring that the method is exact for polynomials of degree ≤ 3.

4. [12] a) Let $f(x) = \sqrt{1-x}$ and $x_0 = 0, x_1 = \frac{1}{2}$. Compute the divided difference on four points $f[x_0, x_0, x_1, x_1]$.

   b) Using composite Simpson's rule determine the largest appropriate step size h and the corresponding number of subintervals n to approximate the integral

$$\int_0^2 \frac{1}{x+4}\, dx$$

   with error smaller than 10^{-5}. Moreover compute this approximation.

5. [10] a) Write the following parametric polynomial curve

$$p(t) = (t^3 + 2t^2 + 1, 2t^2 - t + 2)^T, \quad 0 \leq t \leq 1,$$

   in the Bézier form.

   b) Prove that $B_n(x^2; x) = x^2 + \frac{x(1-x)}{n}$, where

$$B_n(f; x) := \sum_{i=0}^{n} f(i/n) \binom{n}{i} x^i (1-x)^{n-i}.$$

EXAM

19 February 2016

1. [8] a) Prove that $0.1 = \sum_{i=1}^{\infty}(2^{-4i} + 2^{-4i-1})$.

   b) Using a) prove that the binary representation for $x = 0.1$ is equal to $0.000\overline{1100}_2$ (the last four digits are repeated).

2. [10] Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 5 & 3 & 6 \\ 0 & 3 & 10 & 1 \\ 3 & 6 & 1 & 14 \end{pmatrix}.$$

   a) Compute $\|A\|_1, \|A\|_\infty$ and $\|A\|_F$.

   b) Using Cholesky decomposition solve the system $Ax = b$, where $b = (1, 0, 1, 0)^T$.

3. [10] Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} -5 & 2 & 1 \\ 1 & -10 & 1 \\ 1 & 1 & -4 \end{pmatrix}, \quad b = \begin{pmatrix} -3 \\ 27 \\ 4 \end{pmatrix}.$$

   a) Find Jacobi iteration matrix R_J and show that Jacobi method converges.

   b) Use Jacobi method to find approximation $x^{(1)}$ of the linear system by using initial approximation $x^{(0)} = (-0.5, -2.5, -1.5)^T$.

4. [10] For the function $f(x) = \sin x + \cos x - 1$ find interpolating polynomial p of degree 5, such that

$$p\left(k\frac{\pi}{2}\right) = f\left(k\frac{\pi}{2}\right), \quad p'\left(k\frac{\pi}{2}\right) = f'\left(k\frac{\pi}{2}\right), \quad p''\left(k\frac{\pi}{2}\right) = f''\left(k\frac{\pi}{2}\right),$$

   for $k = 0, 1$.

5. [12] Area inside the closed curve $y^2 + x^2 = \cos x$ is given by

$$A = 4\int_0^{\alpha} (\sqrt{\cos x - x^2})dx$$

   where α is the positive root of the equation $\cos x = x^2$.

   a) Compute α to three correct decimals using tangent method.

   b) Compute area A by using simple Simpson's rule.

EXAM

20 June 2016

1. [10] Let $a \in \mathbb{R}, a \neq 0$, be a given number and let

$$x_{r+1} = \frac{1}{8}\left(15x_r - 10\,a\,x_r^3 + 3\,a^2 x_r^5\right), \quad r = 0, 1, \ldots$$

be an iteration sequence.

   a) Find all fixed points for this iteration. Which are attractive and which not? Compute the order of convergence in a neighborhood of all attractive fixed points.
   b) Let $a = 5$. Choosing $x_0 = \frac{1}{2}$ calculate the next two approximations to a corresponding fixed point and then calculate the error of the approximation x_2.

2. [10] Using Newton's method solve the system of nonlinear equations

$$\sin(\pi x)\cos(\pi y) = \cos(\pi x),$$

$$\sin(\pi x)\sin(\pi y) = \frac{1}{2}.$$

Taking initial approximation $\left(\frac{1}{2}, \frac{1}{4}\right)^T$ calculate new approximation of the solution.

3. [10] Calculate singular value decomposition for the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

4. [10] Let $(x_1, y_1)^T, (x_2, y_2)^T, \ldots, (x_m, y_m)^T$ be points in the plane.

   a) Using the least squares method find a line $y = kx + n$, which approximates the points in a best way.
   b) Use the result from a) for the points $(1, 2)^T, (2, 3)^T, (3, 5)^T, (4, 8)^T$.

5. [10] Let $(0, 0)^T, (1, 1)^T, (3, -1)^T$ be given points. Write quadratic Bézier curve, which interpolates these points at parameters $0, \frac{1}{3}, 1$. Sketch also the obtained control polygon and the curve.

2.9 Academic year 2016/2017

Academic year 2016/2017

EXAM

21 June 2017

1. [20] For computing $\sqrt[4]{\alpha}$, $\alpha > 0$, we use the iteration

$$x_{r+1} = \frac{4x_r^5}{5x_r^4 - \alpha}.$$

   a) Show that $\sqrt[4]{\alpha}$ is the fixed point of the above iteration function and determine the order of convergence in the neighborhood of this point.
   b) Let $\alpha = 9$ and $x_0 = 2$. Compute the next approximation x_1 for $\sqrt[4]{9}$.

2. [20] Using the QR decomposition find the solution of the overdetermined system

$$\begin{aligned} 9x_1 + 26x_3 &= 15, \\ 12x_1 - 7x_3 &= 0, \\ 4x_2 + 4x_3 &= -5, \\ -3x_2 - 3x_3 &= 5, \end{aligned}$$

   by the least squares method.

3. [20] Derive the formula for the approximation of the second derivative $f''(x_0)$ such that you use only the function values $f(x_0-h)$, $f(x_0)$ and $f(x_0+3h)$. Determine also the error of the obtained rule.

4. [20] Find the control points of the cubic Bézier curve $\mathbf{b}$ for which

$$\mathbf{b}(0) = (0,0)^\top, \quad \mathbf{b}'(0) = (6,6)^\top, \quad \mathbf{b}''(0) = (0,-12)^\top, \quad \mathbf{b}'''(0) = (-6,6)^\top.$$

   Sketch carefully the obtained Bézier curve.

EXAM

6 July 2017

1. [20] We are interested in the roots of the function $f(x) = x^3 - 2x$.

   a) Check that $g(x) = \frac{x(x^2+6)}{3x^2+2}$ is the iteration function for f. Compute the solutions of the equation $f(x) = 0$.

   b) What can you say about the order of convergence of the iteration function g in the neighborhood of each solution of $f(x) = 0$ using only the first derivative of function g?

2. [20] Using LU decomposition without pivoting find the solution of the following system,

$$\begin{aligned} x_1 + 3x_2 + x_4 &= 1, \\ x_1 + 2x_2 - 2x_3 + x_4 &= 2, \\ -2x_1 - 4x_2 + 3x_3 &= 3, \\ 2x_1 + 5x_2 + 3x_4 - 1 &= 2x_3. \end{aligned}$$

3. [20] Let $f : [a, a+h] \to \mathbb{R}$, $h > 0$, be at least three times continuously differentiable function. Determine the coefficients α_0 and α_1 in the integration rule

$$\int_a^{a+h} f(x)\,dx = \alpha_0 f(a) + \alpha_1 f\left(a + \frac{2}{3}h\right) + Rf$$

   such that it would be exact for the polynomials of as highest degree as possible. Determine also the error Rf with respect to appropriate derivative of function f and the constant h.

4. [20] You are given a cubic Bézier curve $\mathbf{b}$ with the control points:

$$\mathbf{b}_0 = (0,0)^\top, \quad \mathbf{b}_1 = (a,b)^\top, \quad \mathbf{b}_2 = (27,9)^\top, \quad \mathbf{b}_3 = (9,0)^\top$$

   that interpolates the point $(\frac{19}{3}, 6)^\top$ at parameter $t = \frac{1}{3}$.

   a) Determine the unknown control point $\mathbf{b}_1$, such that you use De Casteljau's algorithm.

   b) Graphically illustrate De Casteljau's algorithm for the obtained Bézier curve $\mathbf{b}$ at parameter $t = \frac{1}{3}$ and sketch carefully the curve $\mathbf{b}$.

EXAM

22 August 2017

1. [20] We would like to solve the equation

$$e^{2x} + x = 0. \tag{2.2}$$

   a) Check that
   $$g(x) = \frac{e^{2x}(2x-1)}{2e^{2x}+1}$$
   is the iteration function for solving the (2.3). For the initial approximation $x_0 = 0$ compute the next approximate value x_1, that is returned by iteration function g.

   b) Does the iteration using the iteration function g converge in the neighborhood of the solution of equation (2.3)?

2. [20] Let
$$A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 14 \end{pmatrix}.$$
Using QR decomposition find the solution of the overdetermined system $Ax = b$ by the least squares method.

3. [20] Derive the formula for the approximation of the first derivative $f'(x_0)$ such that you use only the function values $f(x_0 - h), f(x_0)$ and $f(x_0 + 3h)$. Determine also the error of the rule.

   Then determine the whole error (ignore the rounding error) and the optimal step h, if for the computed function values the following holds: $|f(x_i) - \tilde{f}(x_i)| < 5 \cdot 10^{-6}$ where $\|f^{(3)}\|_\infty \leq 10$.

4. [20] You are given the cubic Bézier curve $\mathbf{b}$ with the control points:
$$\mathbf{b}_0 = (1, a)^\top, \quad \mathbf{b}_1 = (a+b, 1)^\top, \quad \mathbf{b}_2 = (1, a+b)^\top, \quad \mathbf{b}_3 = (b, 0)^\top,$$
where $a, b \in \mathbb{R}$.

   a) Determine the point on the Bézier curve $\mathbf{b}$ at the parameter $t = \frac{1}{2}$, such that you use the De Casteljau's algorithm. Using the triangular scheme determine also $\mathbf{b}'(\frac{1}{2})$.

   b) Determine the values a and b such that the point $\mathbf{b}(\frac{1}{2})$ will lie on the midpoint of the line segment between the second and third control point. Graphically illustrate the De Casteljau's algorithm for the obtained Bézier curve $\mathbf{b}$ at the parameter $t = \frac{1}{2}$ and sketch carefully the curve $\mathbf{b}$.

EXAM

5 September 2017

1. [20] For the system of equations

$$\begin{aligned} 2y^2 - x^2 + xy &= 3y - 2, \\ 3x^2 + 2y^2 - 2xy &= 8 - x, \end{aligned}$$

do one step of the tangent (Newton method) with initial approximation $x_0 = 0$ and $y_0 = 1$. Then write the system of equations that one must solve on the second step of the tangent method (you do not have to solve this system).

2. [20] Compute the matrices L and U from the LU decomposition of a matrix

$$A = \begin{pmatrix} 6 & 6 & 8 & 8 \\ 0 & 1 & 9 & 18 \\ 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \end{pmatrix}$$

with partial pivoting. Then compute the $\|L\|_\infty$ and $\|U\|_1$.

3. [20] For computing the zeros of the function f we use the iteration

$$x_{r+1} = x_r - \frac{f(x_r)}{p_r'(x_r)}, \quad r \geq 2,$$

where p_r denotes the polynomial of degree ≤ 2, that interpolates the value of the function f in points x_r, x_{r-1} and x_{r-2}.

   a) Using the divided differences and the Newton form of the interpolation polynomial p_r determine $p_r'(x_r)$.
   b) Let $f(x) = x^3 - 7$ be given and also the initial approximations $x_0 = 5, x_1 = 4, x_2 = 3$. Using the given method, determine the next approximation x_3.

4. [20] Let $f : [a, b] \to \mathbb{R}$ be at least three times continuously differentiable function. Determine the coefficients A, B and C in the quadrature rule

$$\int_a^b f(x)\, dx = A\, f(a) + B\, f\left(a + \frac{2}{3}(b-a)\right) + C\, f(b) + Rf$$

such that it would be exact for the polynomials of as high degree as possible. Determine also the error Rf of this rule.

2.10 Academic year 2017/2018

Academic year 2017/2018

EXAM

7 June 2018

1. [10] The iteration
$$x_{r+1} = 2 - (1+c)x_r + cx_r^3$$
for some values of the parameter c and initial approximation x_0 close enough to $a = 1$ converges to a.

   a) For which values of the parameter c the convergence in the vicinity of the fixed point a is guaranteed?
   b) For which value of the parameter c, the order of the convergence in the vicinity of the fixed point a is quadratic?
   c) For the value of the parameter c from (b) perform the first iteration step with initial approximation $x_0 = \frac{2}{3}$.

2. [10] Let
$$A = \begin{pmatrix} 3 & -1 & -2 \\ 3 & 3 & 5 \\ 2 & 0 & 4 \end{pmatrix}.$$

   a) Compute `LU` decomposition of A using partial pivoting.
   b) Compute the determinant of A using the above decomposition.

3. [10] We interpolate function $f(x) = \sin(x)$ at the points $x_0 = 0$ and $x_1 = h$ with polynomial of third degree. At the point x_0 we interpolate the value of the function, the first derivative of the function and its second derivative, while at the point x_1 we only interpolate the value of the function. Estimate h such that the interpolation error on the interval $[0, h]$ is less than 10^{-8}.

4. [10] Assume that function f has a local minimum on the interval $x_{n-1} \leq x \leq x_{n+1}$ where $x_k = x_0 + kh$, $k = 0, 1, \ldots, n+1$. We can approximate this local minimum by computing the local minimum of the quadratic polynomial which interpolates f at points x_{n-1}, x_n and x_{n+1}. Show that the approximation is of the form
$$f(x_n) - \frac{1}{8}\left(\frac{(f(x_{n+1}) - f(x_{n-1}))^2}{f(x_{n+1}) - 2f(x_n) + f(x_{n-1})}\right).$$

5. [10]
   a) Write the parametric representation of the Bézier curve in Bernstein form determined by the control points $(1,2)^T, (3,4)^T, (6,-6)^T$ and $(10,8)^T$. Draw the sketch of the curve by computing its values at parameters $t = 0$, 0.2, 0.4, 0.6, 0.8, 1.
   b) Using the definition for linear independence, prove that the Bernstein polynomials $B_0^n, B_1^n, \ldots, B_n^n$ are linearly independent.

EXAM

22 June 2018

1. [10] In order to solve the equation $x^2 - a = 0$, $a \in \mathbb{R}$, $a \neq 0$, we use the iteration

$$x_{r+1} = \frac{x_r^3}{Ax_r^2 + B}, \quad r = 0, 1, \dots$$

Find the unknown coefficients A and B such that the order of convergence in the neighborhood of the solution $\sqrt{a}$ is at least quadratic. What is the exact order of convergence?
With the above iterative formula calculate $\sqrt{5}$ on five decimal places precise using the initial value $x_0 = 2$.

2. [10] Using `QR` decomposition with Givens rotations solve the overdetermined system $Ax = b$ given by

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Compute

$$\min_x \| Ax - b \|_2 .$$

3. [10] We interpolate the function $f(x) = \sin(x)$ at the points $x_0 = 0$ and $x_1 = h$ with a cubic polynomial. Moreover, at x_0 we interpolate the value of the function, its first and its second derivative. At the point x_1 we interpolate only the value of the function. Estimate the parameter h, such that the error of the interpolation on the interval $[0, h]$ is smaller than 10^{-8}.

4. [10] Using the method of undetermined coefficients find coefficients a, b, c, d and unknown order n in the quadrature formula

$$\int_{x_0}^{x_1} f(x)dx = h(af(x_0) + bf(x_1) + cf(x_2)) + df^{(n)}(\xi),$$

where $x_i = x_0 + ih$ and $x_0 \leq \xi \leq x_2$.

5. [10]
   a) Construct the Bézier curve of order three given with four control points $(0,0)^T, (1,2)^T, (3,2)^T$ and $(2,0)^T$. Generate at least 5 points on the curve.
   b) Prove that any Bernstein polynomial of degree $< n$ can be expressed as a linear combination of Bernstein polynomials of degree n.

Academic year 2018/2019

EXAM

12 June 2019

1. [13] You are given a matrix A and the vector b,

$$A = \begin{pmatrix} 6 & 6 & 8 & 8 \\ 0 & 1 & 9 & 18 \\ 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \end{pmatrix}, \quad b = \begin{pmatrix} 44 \\ 4 \\ 56 \\ 28 \end{pmatrix}.$$

Compute the LU decomposition with the partial pivoting for the matrix A. Then, using the obtained decomposition, compute (economically) the value of the expression $b^T A^{-1} b$.

2. [11] Define a vector in a subspace that is spanned by the columns of the matrix

$$A = \begin{pmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{pmatrix}.$$

which is on at least distance (in the second norm) from the vector $(4, 0, 2, 14)^T$.

3. [13] Find the polynomial p, that matches function $f(x) = \frac{1}{x^2}$ three times (namely, in the value, first and second derivative) in points 1 and 2. Show that f and p do not absolutely differ by more than $\frac{1}{8}$ on the interval $[1, 2]$.

4. [13] Let $f : [a, b] \to \mathbb{R}$ be at least three times continuously differentiable function. Determine the coefficients A, B and C in the integration rule

$$\int_a^b f(x)\,dx = A\,f(a) + B\,f\left(a + \frac{2}{3}(b-a)\right) + C\,f(b) + Rf$$

such that it will be exact for polynomials of as high as possible degree. Then, specify the error Rf of this rule.

EXAM

26 August 2019

1. [5] Which number (in base 10) is in IEEE standard for floating point arithmetic in double precision presented as

1	10000000101	$10100101\underbrace{0\ldots0}_{44}$	?

2. [6] Find the interpolating polynomial p, for which

$$p(0) = 1, \quad p(2) = -1, \quad p'(2) = 3, \quad p''(2) = 8, \quad p(3) = 4.$$

3. [13] You are given the iteration

$$x_{r+1} = x_r(3 - 3ax_r + a^2x_r^2), \quad r = 0, 1, 2, ...$$

where $a \neq 0$.

a) Compute the fixed points of the iteration and determine if they are attractive or not.

b) In the neighborhood of the attractive fixed points determine the order of convergence and the initial approximate values for which the sequence of iterations certainly converges.

c) Test the given iteration in case $a = 5$ in $x_0 = 0.1$. Comment on the result.

4. [13] You are given a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ with LU decomposition $A = LU$ and the vectors $a, b \in R^n$.

(a) Build an economical algorithm to calculate the solution of the system

$$\begin{pmatrix} U & I \\ A & -L \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{2.3}$$

and count the number of operations.

(b) Compute the LU decomposition without pivoting of the matrix A and solve the system (2.3) for

$$A = \begin{pmatrix} 12 & 0 & 8 \\ 96 & -4 & 72 \\ 0 & 32 & -48 \end{pmatrix}, \quad a = \begin{pmatrix} -43 \\ 10 \\ 98 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 14 \\ 46 \end{pmatrix}.$$

5. [13] Determine the function $f(x) = a + bx^2 + c\sin(\frac{\pi x}{3})$, that best matches the points

$$(-2, 4), \quad (-1, 0), \quad (1, 2), \quad (2, 8)$$

by the least-squares method. To solve the given task use the QR decomposition that you should compute by the Gram-Schmidt process.