UNIVERSITY OF PRIMORSKA
Faculty of Mathematics, Natural Sciences and Information Technologies

Abbreviated Lecture Notes in Measure Theory

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OTHER STUDY TEXTBOOK

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Preface

Measure Theory at UP FAMNIT is a course at the undergraduate study programme Mathematics. In a typical year the students learn the basics about the Lebesgue integral, the construction of the Lebesgue measure and the connection with the Riemann integral. They also learn the basic properties about L^p spaces, complex measures, and the product measure with Fubini's Theorem. The lectures are based on selected topics from the book *Rudin, Walter: Real and complex analysis, Third edition, McGraw-Hill Book Co., New York, 1987* (which contains much more material) and on the lectures given by Prof. Roman Drnovšek, which I attended as a student. These notes provide an overview on the topics/theorems learned at mine course. The lectures themselves contain also most of the proofs and several examples. The notes end with a vocabulary (ENG/SLO) of mathematical words, since the course is often attended by Slovenian and foreign students.

Marko Orel — Koper, May 2022

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CHAPTER 1

Lebesgue Integral

1.1 Measurable Functions

Let X be an nonempty set. A family τ of its subsets is a *topology* on X, if the following three conditions are satisfied:

(i) $\emptyset, X \in \tau$,

(ii) $U_1, \ldots, U_n \in \tau \Longrightarrow \bigcap_{i=1}^{n} U_i \in \tau$,

(iii) $U_\lambda \in \tau$ for all $\lambda \in \Lambda \Longrightarrow \bigcup_{\lambda \in \Lambda} U_\lambda \in \tau$.

Elements of τ are *open sets*. Their complements are *closed sets*. The pair (X, τ) is a *topological space*. A map $f : X \to Y$ between two topological spaces is *continuous* if the following implication holds:

$$U \text{ is open in } Y \Longrightarrow f^{-1}(U) \text{ is open in } X.$$

Let X be an nonempty set. A family $\mathcal{M}$ of its subsets is a *σ-algebra* on X, if the following three conditions are satisfied:

(i) $X \in \mathcal{M}$,

(ii) $A \in \mathcal{M} \Longrightarrow A^c \in \mathcal{M}$,

(iii) $A_1, A_2, \ldots \in \mathcal{M} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$.

Elements of $\mathcal{M}$ are *measurable sets*. The pair $(X, \mathcal{M})$ is a *meausurable space*. Given a topological space Y, a map $f : X \to Y$ is *measurable* if the following implication holds:

$$U \text{ is open in } Y \Longrightarrow f^{-1}(U) \in \mathcal{M}.$$

From the definition of σ-algebra we deduce the following properties:

- $\emptyset \in \mathcal{M}$,
- $A_1, A_2, \dots, A_n \in \mathcal{M} \Longrightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{M}$,
- $A_1, A_2, \dots \in \mathcal{M} \Longrightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{M}$,
- $A_1, A_2, \dots, A_n \in \mathcal{M} \Longrightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{M}$,
- $A, B \in \mathcal{M} \Longrightarrow A \backslash B \in \mathcal{M}$.

Proposition 1.1.1. *Let X be a measurable space and assume that Y and Z are two topological spaces. If a map $f : X \to Y$ is measurable and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is measurable.*

Proposition 1.1.2. *Let X and Y be a measurable space and a topological space, respectively. Assume that real functions $u, v : X \to \mathbb{R}$ are measurable and $\Phi : \mathbb{R}^2 \to Y$ is continuous. Then the function $h : X \to Y$, defined by $h(x) := \Phi\big(u(x), v(x)\big)$, is measurable.*

Remark. In Proposition 1.1.2, sets $\mathbb{R}$ and $\mathbb{R}^2$ are equipped with the usual topology.

Corollary 1.1.3. *Let X be a measurable space.*

(i) If functions $u, v : X \to \mathbb{R}$ are measurable, then $f = u + iv : X \to \mathbb{C}$ is measurable.

(ii) If $f = u + iv : X \to \mathbb{C}$ is measurable, then $u, v, |f| : X \to \mathbb{R}$ are measurable.

(iii) If $f, g : X \to \mathbb{C}$ are measurable, then so are $f + g, f - g, f \cdot g$.

(iv) Given a subset $E \subseteq X$ let $\chi_E : X \to \mathbb{R}$ be its characteristic/indicator function, that is,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Then

$$\chi_E \text{ is measurable} \Longleftrightarrow E \text{ is measurable.}$$

(v) *Constant maps are measurable.*

(vi) *If $f : X \to \mathbb{C}$ is measurable, then there exists a measurable function $\alpha : X \to \mathbb{C}$ such that $|\alpha(x)| = 1$ and $f(x) = \alpha(x)|f(x)|$ for all $x \in X$.*

Remark. In Corollary 1.1.3, sets $\mathbb{R}$ and $\mathbb{C}$ are equipped with the usual topology.

Theorem 1.1.4. *Let $\mathcal{F}$ be a family of subsets of a set $X \neq \emptyset$. Then there exists the smallest σ-algebra on X, denoted by $\sigma(\mathcal{F})$, such that $\mathcal{F} \subseteq \sigma(\mathcal{F})$.*

Let (X, τ) be a topological space. By Theorem 1.1.4 there is the smallest σ-algebra on X that contains τ. It is called *Borel σ-algebra.* Its elements are *Borel sets.*

If X and Y are two topological spaces, where X is equipped with Borel σ-algebra, then a measurable map $f : X \to Y$ is said to be a *Borel map.*

Proposition 1.1.5. *Let $(X, \mathcal{M})$ be a measurable space, Y a topological space, and $f : X \to Y$ a map. Then:*

(i) *The set $\{E \subseteq Y : f^{-1}(E) \subseteq \mathcal{M}\}$ is a σ-algebra on Y.*

(ii) *If f is measurable and $E \subseteq Y$ is a Borel set, then $f^{-1}(E) \in \mathcal{M}$.*

(iii) *If $Y = [-\infty, \infty]$ and $f^{-1}\big((a, \infty]\big) \in \mathcal{M}$ for all $a \in \mathbb{R}$, then f is measurable.*

(iv) *If f is measurable, Z is a topological space, and $g : Y \to Z$ a Borel function, then $g \circ f : X \to Z$ is measurable function.*

For $a_1, a_2, \ldots \in [-\infty, \infty]$ we define the *limes superior* and *limes inferior* as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \big(\sup_{k\geq n} a_k\big) = \inf_n \big(\sup_{k\geq n} a_k\big),$$
$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \big(\inf_{k\geq n} a_k\big) = \sup_n \big(\inf_{k\geq n} a_k\big),$$

respectively. We have $a = \lim_{n\to\infty} a_n$ (i.e. limit in topological space $[-\infty, \infty]$) if and only if $\limsup_{n\to\infty} a_n = a = \liminf_{n\to\infty} a_n$.

For a sequence of functions $f_n : X \to [-\infty, \infty]$ we define functions $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$, $\sup_n f_n$, $\inf_n f_n$ of the form $X \to [-\infty, \infty]$

as

$$\begin{aligned}
\big(\limsup_{n\to\infty} f_n\big)(x) &:= \limsup_{n\to\infty} \big(f_n(x)\big),\\
\big(\liminf_{n\to\infty} f_n\big)(x) &:= \liminf_{n\to\infty} \big(f_n(x)\big),\\
\big(\sup_{n\to\infty} f_n\big)(x) &:= \sup_{n} \big(f_n(x)\big),\\
\big(\inf_{n\to\infty} f_n\big)(x) &:= \inf_{n} \big(f_n(x)\big).
\end{aligned}$$

If $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in X$, then f is a *pointwise limit* of the sequence $(f_n)_{n\in\mathbb{N}}$.

Proposition 1.1.6. *If* $f_n : X \to [-\infty, \infty]$ *are measurable, then* $\limsup_{n\to\infty} f_n$, $\liminf_{n\to\infty} f_n$, $\sup_n f_n$, $\inf_n f_n$ *are measurable as well.*

Remark. If f is a pointwise limit of measurable functions f_n, then f is measurable, since $f = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n$.

Corollary 1.1.7. *If* $f, g : X \to [-\infty, \infty]$ *are measurable functions, then* $\inf\{f, g\}$ *and* $\sup\{f, g\}$ *are also measurable functions.*

For $f : X \to [-\infty, \infty]$ we define the *positive part* of f and *negative part* of f as $f^+ = \sup\{f, 0\}$ and $f^- = \sup\{-f, 0\}$, respectively. If f is measurable, the same holds for f^+, f^-. Moreover, $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

A function $s : X \to \mathbb{C}$ is usually defined as simple, if it has a finite image, that is, if $s = \sum_{k=1}^n \alpha_k \chi_{A_k}$, where $s(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $A_k := s^{-1}(\{\alpha_k\})$ (we are assuming that $\alpha_i \neq \alpha_j$ for $i \neq j$). It follows easily that s is measurable if and only if $A_1, \ldots, A_n$ are measurable. Therefore, in measure theory, a *simple function* is a function with a finite image, where the sets $A_1, \ldots, A_n$ are measurable.

Theorem 1.1.8. *Let* $f : X \to [0, \infty]$ *be measurable. Then there exist simple functions* $s_n : X \to [0, \infty)$ *such that*

(i) $0 \leq s_1(x) \leq s_2(x) \leq \cdots$,

(ii) $\lim_{n\to\infty} s_n(x) = f(x)$

for all $x \in X$. *If* f *is bounded, then* $(s_n)_n$ *can be chosen in such way that the convergence in (ii) is uniform.*

1.2 Positive Measure

Let $(X, \mathcal{M})$ be a measurable space. A map $\mu : \mathcal{M} \to [0, \infty]$ is a *positive measure* on $\mathcal{M}$ if it has the following two properties:

(i) For arbitrary disjoint $E_1, E_2, \ldots \in \mathcal{M}$ we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (\textit{countable additivity}).$$

(ii) There exists $A \in \mathcal{M}$ such that $\mu(A) < \infty$.

The triple $(X, \mathcal{M}, \mu)$ is called a *measure space.*

Theorem 1.2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then we have:*

(i) $\mu(\emptyset) = 0$;

(ii) For arbitrary disjoint $E_1, E_2, \ldots, E_n \in \mathcal{M}$ we have

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i) \qquad (\text{finite additivity}).$$

(iii) Measure μ is monotone, that is, $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$ for $A, B \in \mathcal{M}$.

(iv) If $A \subseteq B$ for $A, B \in \mathcal{M}$ and $\mu(A) < \infty$, then $\mu(B \backslash A) = \mu(B) - \mu(A)$.

(v) If $A_1 \subseteq A_2 \subseteq \cdots$, where $A_i \in \mathcal{M}$ for all i, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

(vi) If $A_1 \supseteq A_2 \supseteq \cdots$, where $A_i \in \mathcal{M}$ for all i and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i).$$

(vii) If $A_1, A_2, \ldots \in \mathcal{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i) \qquad (\text{countable subadditivity}).$$

Proposition 1.2.2. *If $f, g : X \to [0, \infty]$ are measurable, then $f + g$ and $f \cdot g$ are measurable.*

1.3 Integral of a Positive Function

Let $(X, \mathcal{M}, \mu)$ be a measure space. For $A, E \in \mathcal{M}$ we define

$$\int_E \chi_A \, d\mu := \mu(A \cap E).$$

If $s : X \to [0, \infty)$ is simple, that is, $s = \sum_{i=1}^n \alpha_i \chi_{A_i}$, where the image $s(X) = \{\alpha_1, \ldots, \alpha_n\}$ consists of n elements and $A_i := s^{-1}(\{\alpha_i\}) \in \mathcal{M}$, then we define

$$\int_E s \, d\mu := \sum_{i=1}^n \alpha_i \mu(A_i \cap E).$$

For an arbitrary measurable function $f : X \to [0, \infty]$ we define

$$\int_E f \, d\mu := \sup \left\{ \int_E s \, d\mu \ : \ 0 \leq s \leq f, \ s : X \to [0, \infty) \text{ is simple} \right\}. \tag{1.1}$$

The integral (1.1) is the*Lebesgue integral* of f over E, with respect to the measure μ. It is a number in $[0, \infty]$.

Proposition 1.3.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Assume all the functions and the sets in the proposition are measurable. Then:*

(i) If $0 \leq f \leq g$, then $\int_E f \, d\mu \leq \int_E g \, d\mu$.

(ii) If $f \geq 0$ and $A \subseteq B$, then $\int_A f \, d\mu \leq \int_B f \, d\mu$.

(iii) If $f \geq 0$ and $c \in [0, \infty)$, then $\int_E cf \, d\mu = c \int_E f \, d\mu$.

(iv) If $f(x) = 0$ for all $x \in E$, then $\int_E f \, d\mu = 0$.

(v) If $\mu(E) = 0$, then $\int_E f \, d\mu = 0$.

(vi) If $f \geq 0$, then $\int_E f \, d\mu = \int_X f \chi_E \, d\mu$.

Proposition 1.3.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. If $s : X \to [0, \infty)$ is a simple function, then*

$$\varphi(E) := \int_E s \, d\mu$$

is a positive measure on $\mathcal{M}$.

Proposition 1.3.3. *Let $(X, \mathcal{M}, \mu)$ be a measure space. If $s, t : X \to [0, \infty)$ are simple functions, then*

$$\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu.$$

Theorem 1.3.4. *(Lebesgue's Monotone Convergence Theorem) Assume a sequence of measurable functions $f_n : X \to [0, \infty]$ satisfy*

$$f_1 \leq f_2 \leq \cdots . \tag{1.2}$$

Then the pointwise limit $f(x) := \lim_{n\to\infty} f_n(x)$ is measurable and

$$\int_X f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$$

Remark. So if a sequence of positive measurable functions satisfy (1.2), then $\int_X (\lim_{n\to\infty} f_n) \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$.

Theorem 1.3.5. *Assume functions $f_n : X \to [0, \infty]$ are measurable for all $n \in \mathbb{N}$. Then the function $f : X \to [0, \infty]$ defined by $f(x) := \sum_{n=1}^{\infty} f_n(x)$ is measurable and*

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Remark. The content of Theorem 1.3.5 is that for positive functions we can swap the order of the summation and the integration. Clearly, the claim holds also for 'finite' sums.

Corollary 1.3.6. *If $a_{i,j} \in [0, \infty]$ for all $i, j \in \mathbb{N}$, then*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Theorem 1.3.7. *(Fatou's Lemma) Assume that functions $f_n : X \to [0, \infty]$ are measurable for all $n \in \mathbb{N}$. Then*

$$\int_X \liminf_{n\in\infty} f_n \, d\mu \leq \liminf_{n\in\infty} \int_X f_n \, d\mu.$$

Theorem 1.3.8. *Assume that $f : X \to [0, \infty]$ is a measurable function. Then*

$$\varphi(E) := \int_E f \, d\mu$$

is a positive measure on $\mathcal{M}$ and

$$\int_X g \, d\varphi = \int_X f g \, d\mu$$

holds for every measurable function $g : X \to [0, \infty]$.

1.4 Integral of a Complex Function

Let $(X, \mathcal{M}, \mu)$ be a measure space. The members of the set

$$L^1(X, \mathcal{M}, \mu) := \left\{ f : X \to \mathbb{C} \text{ } measurable, \int_X |f| \, d\mu < \infty \right\}$$

are *Lebesgue integrable functions* with respect to μ. Sometimes, where there is no risk of confusion, we write briefly $L^1(\mu)$. For $f \in L^1(\mu)$ write

$$f = u + iv = u^+ - u^- + iv^+ - iv^-,$$

where u and v are the real and the imaginary part of f. The *Lebesgue integral* of f with respect to μ is given by

$$\int_X f \, d\mu = \int_X u^+ \, d\mu - \int_X u^- \, d\mu + i \int_X v^+ \, d\mu - i \int_X v^- \, d\mu.$$

Clearly

$$\begin{aligned} \mathrm{Re}\left(\int_X f \, d\mu \right) &= \int_X \mathrm{Re} f \, d\mu, \\ \mathrm{Im}\left(\int_X f \, d\mu \right) &= \int_X \mathrm{Im} f \, d\mu. \end{aligned}$$

For a measurable function $f : X \to [-\infty, \infty]$ we can define

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu, \tag{1.3}$$

if one of the two integrals on the right hand side of (1.3) is finite.

Proposition 1.4.1. *The set $L^1(\mu)$ form a vector space over $\mathbb{C}$ for the usual operations and*

$$\int_X (\alpha f + \beta g) \, d\mu = \alpha \int_X f \, d\mu + \beta \int_X g \, d\mu$$

holds for all $f, g \in L^1(\mu)$ and $\alpha, \beta \in \mathbb{C}$.

Proposition 1.4.2. *The inequality*

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

holds for all $f \in L^1(\mu)$.

Theorem 1.4.3. *(Lebesgue's Dominated Convergence Theorem) Assume that for a sequence of measurable functions* $f_n : X \to \mathbb{C}$ *the limit* $\lim_{n\to\infty} f_n(x) =: f(x)$ *exists for each* $x \in X$. *Let* $g \in L^1(\mu)$ *be such that*

$$|f_n(x)| \leq g(x)$$

holds for all $x \in X$ *and all* $n \in \mathbb{N}$. *Then*

(i) $f \in L^1(\mu)$,

(ii) $\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$,

(iii) $\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$.

Remark. The main content is that $\int_X (\lim_{n\to\infty} f_n) \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$ if the assumptions in the theorem are satisfied.

A positive measure μ is *complete* if the following implication is satisfied:

$$A \in \mathcal{M}, \mu(A) = 0, B \subseteq A \Longrightarrow B \in \mathcal{M}.$$

Proposition 1.4.4. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. Then there exists a* σ*-algebra* $\mathcal{M}^*$ *on* X *and a complete positive measure* μ^* *on* $\mathcal{M}^*$ *such that* $\mathcal{M} \subseteq \mathcal{M}^*$ *and* $\mu^*(A) = \mu(A)$ *for all* $A \in \mathcal{M}$.

The σ-algebra $\mathcal{M}^*$ from Proposition 1.4.4 is called a μ*-completion* of $\mathcal{M}$.

Let P be some property that an element $x \in X$ has it or not. We say that P holds *almost everywhere* on X if $\{x \in X \ : \ x \text{ has not property } P\}$ is a measurable set with $\mu\big(\{x \in X \ : \ x \text{ has not property } P\}\big) = 0$. Sometimes we abbreviate as a.e. $[\mu]$.

We next redefine a notion of a measurability of a function. Assume that a function f is defined almost everywhere on X, that is, f is defined on $E \in \mathcal{M}$ with $\mu(E^c) = 0$. Then we say that f is *measurable* if $\{x \in E \ : \ f(x) \in U\} \in \mathcal{M}$ whenever U is a open set. If, for example, we define additionally $f(x) = 0$ for $x \in E^c$, we obtain a function that is measurable in 'old sense' with an additional property that $\int_E f \, d\mu = \int_X f \, d\mu$.

Theorem 1.4.5. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. If* f_n *is a complex measurable function defined almost everywhere on* X *for each* $n \in \mathbb{N}$ *and*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty,$$

then the series $f(x) := \sum_{n=1}^{\infty} f_n(x)$ *converges absolutely for almost all* $x \in X$, $f \in L^1(\mu)$, *and*

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Theorem 1.4.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

(i) If $f : X \to [0, \infty]$ is measurable, $E \in \mathcal{M}$, and $\int_E f\, d\mu = 0$, then $f = 0$ a. e. on E.

(ii) If $f \in L^1(\mu)$ and $\int_E f\, d\mu = 0$ for all $E \in \mathcal{M}$, then $f = 0$ a. e. on X.

Theorem 1.4.7. *Let $(X, \mathcal{M}, \mu)$ be a measure space, where μ is a* finite measure, *that is, $\mu(X) < \infty$. Assume that $f \in L^1(\mu)$, $S \subseteq \mathbb{C}$ is a closed set, and*

$$\frac{1}{\mu(E)} \int_E f\, d\mu \in S$$

holds for all $E \in \mathcal{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

Theorem 1.4.8. *Let $(X, \mathcal{M}, \mu)$ be a measure space and assume that measurable sets $E_1, E_2, \ldots$ satisfy $\sum_{n=1}^{\infty} \mu(E_n) < \infty$. Then almost every $x \in X$ is an element of at most finitely many members of the family $\{E_1, E_2, \ldots\}$.*

1.5 Convergence of Sequences of Measurable Functions

Let $(X, \mathcal{M}, \mu)$ be a measure space.

A sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n : X \to \mathbb{C}$ *converges almost everywhere* to a measurable function $f : X \to \mathbb{C}$ if

$$\mu\left(\{x \in X \ : \ f_n(x) \text{ does not converge to } f(x)\}\right) = 0.$$

A sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n : X \to \mathbb{C}$ *almost uniformly converges* to a measurable function $f : X \to \mathbb{C}$ if for every $\varepsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E^c) < \varepsilon$ and $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly on E.

Proposition 1.5.1. *If a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n : X \to \mathbb{C}$ almost uniformly converges to a measurable function $f : X \to \mathbb{C}$, then it converges almost everywhere to f.*

Theorem 1.5.2. *(Egorov's Theorem) Let $\mu(X) < \infty$. Then a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n : X \to \mathbb{C}$ almost uniformly converges to a measurable function $f : X \to \mathbb{C}$ if and only if it converges almost everywhere to f.*

A sequence $(f_n)_{n\in\mathbb{N}}$ of measurable functions $f_n : X \to \mathbb{C}$ *converges in measure* to a measurable function $f : X \to \mathbb{C}$ if

$$\lim_{n\to\infty} \mu\left(\{x \in X \,|\, |f_n(x) - f(x)| \geq \varepsilon\}\right) = 0$$

for every $\varepsilon > 0$.

Proposition 1.5.3. *(i) If* $f_n \to f$ *in measure and* $f_n \to g$ *in measure, then* $f = g$ *a.e.*

(ii) If $f_n \to f$ *almost uniformly, then* $f_n \to f$ *in measure.*

(iii) If $\mu(X) < \infty$ *and* $f_n \to f$ *almost everywhere, then* $f_n \to f$ *in measure.*

CHAPTER 2

Positive Borel Measures

2.1 Topological Preliminaries

Let X be a topological space. A subset $K \subseteq X$ is *compact* if every its open cover contains a finite subcover. Thas is, if G_λ is open for all $\lambda \in \Lambda$ and $K \subseteq \cup_{\lambda\in\Lambda} G_\lambda$, then there exist $\lambda_1, \ldots, \lambda_n \in \Lambda$ such that $K \subseteq \cup_{i=1}^n G_{\lambda_i}$.

A topological space X is *Hausdorff*, if for every pair $a, b \in X$ of distinct points there are a neighborhood U of a and a neighborhood V of b such that $U \cap V = \emptyset$.

A topological space is *locally compact* if every its element has some compact neighborhood.

Given a complex function $f : X \to \mathbb{C}$ on topological space X, let its *support* be the set $\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$. Let $C_c(X)$ be the set of all complex continuous functions on X with compact support. The notation $K \prec f$ means that $f \in C_c(X)$, K is a compact set, $f : X \to [0, 1]$, and $f(x) = 1$ for all $x \in K$. Similarly, the notation $f \prec V$ means that $f \in C_c(X)$, V is an open set, $f : X \to [0, 1]$, and $\operatorname{supp}(f) \subseteq V$.

Theorem 2.1.1. *(Urysohn's lemma) Let X be a locally compact Hausdorff space. If K is a compact set and V is an open set with $K \subseteq V \subseteq X$, then there exists f such that $K \prec f \prec V$.*

Theorem 2.1.2. *(Partition of unity) Let X be a locally compact Hausdorff space. If K is a compact set and $\{V_1, \ldots, V_n\}$ is an open cover of K, then* *there are function* $h_1, \ldots, h_n$ *such that* $h_i \prec V_i$ *for all* i *and* $\sum_{i=1}^{n} h_i(x) = 1$ *for all* $x \in K$.

2.2 Riesz Representation Theorem

Let $\varphi : C_c(X) \to \mathbb{C}$ be a linear functional (i.e. a linear map). We say that φ is *positive* if $\varphi(f) \geq 0$ whenever $f \in C_c(X)$ is a nonnegative function.

Theorem 2.2.1. *(Riesz representation theorem of a positive linear functional on* $C_c(X)$*) Let* X *be a locally compact Hausdorff space and let* $\varphi : C_c(X) \to \mathbb{C}$ *be a positive linear functional. Then there exist a* σ*-algebra* $\mathcal{M}$ *on* X *that contains all Borel sets in* X*, and a unique positive measure* μ *on* $\mathcal{M}$ *such that:*

(i) $\varphi(f) = \int_X f \, d\mu$ *for all* $f \in C_c(X)$*;*

(ii) $\mu(K) < \infty$ *for every compact set* $K \subseteq X$*;*

(iii) $\mu(E) = \inf\{\mu(V) : E \subseteq V, V \text{ is open}\}$ *for all* $E \in \mathcal{M}$*;*

(iv) $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ is compact}\}$ *for all open sets* $E \in \mathcal{M}$ *and for all* $E \in \mathcal{M}$ *with* $\mu(E) < \infty$*;*

(v) μ *is complete.*

2.3 Regular Borel Measures

Given a topological space X, a positive measure is a *Borel measure* if it is defined on a σ-algebra that contains all Borel sets in X. A Borel measure is *regular* if every Borel set satisfies

(i) $\mu(E) = \inf\{\mu(V) : E \subseteq V, V \textit{ is open}\}$ (*outer regularity*),

(ii) $\mu(E) = \sup\{\mu(K) : K \subseteq E, K \textit{ is compact}\}$ (*inner regularity*).

A subset in a topological space is σ-*compact* if it is a countable union of compact sets.

A subset in a measure space is σ-*finite* if it is a countable union of sets with finite measure.

Theorem 2.3.1. *Let* X *be a* σ*-compact locally compact Hausdorff space. If* $\mathcal{M}$ *and* μ *are obtained from the Riesz theorem (for some positive functional* φ*), then:*

(i) *If* $E \in \mathcal{M}$ *and* $\varepsilon > 0$*, then* X *contains a closed set* F *and an open set* V *such that* $F \subseteq E \subseteq V$ *and* $\mu(V \backslash F) < \varepsilon$.

(ii) *Measure* μ *is regular.*

(iii) *If* $E \in \mathcal{M}$*, then* X *contains a* F_σ*-set* A *and* G_δ*-set* B *such that* $A \subseteq E \subseteq B$ *and* $\mu(B \backslash A) = 0$.

Theorem 2.3.2. *Let* X *be a locally compact Hausdorff space with the property that every its open set is* σ*-compact. Let* λ *be a positive Borel measure on* X *with the property that* $\lambda(K) < \infty$ *for every compact set* $K \subseteq X$*. Then* λ *is a regular measure.*

2.4 Lebesgue Measure on $\mathbb{R}^k$

Let $I_1, I_2, \ldots, I_k$ be bounded intervals in $\mathbb{R}$. We call a k-dimensional hyperrectangle $I_1 \times I_2 \times \cdots \times I_k$ a *k-rectangle.* In the case intervals are all open (resp. closed) $I_1 \times I_2 \times \cdots \times I_k$ is an *open k-rectangle* (resp. *closed k-rectangle*). Let $\mathrm{l}(I)$ be the length of the interval I. Then $\mathrm{vol}(P) = \mathrm{l}(I_1)\mathrm{l}(I_2)\cdots\mathrm{l}(I_k)$ is the *volume* of k-rectangle $I_1 \times I_2 \times \cdots \times I_k$.

Theorem 2.4.1. *There exists a* σ*-algebra* $\mathcal{M}$ *on* $\mathbb{R}^k$ *and a complete positive measure* m *on* $\mathcal{M}$ *with the following properties:*

(i) $m(P) = \mathit{vol}(P)$ *for every* k*-rectangle* P.

(ii) *All Borel sets in* $\mathbb{R}^k$ *are members of* $\mathcal{M}$*. Moreover,* $E \in \mathcal{M}$ *if and only if there exist a* F_σ*-set* $A \subseteq \mathbb{R}^k$ *and* G_δ*-set* $B \subseteq \mathbb{R}^k$ *such that* $A \subseteq E \subseteq B$ *and* $m(B \backslash A) = 0$*. Measure* m *is regular.*

(iii) *Measure* m *is translation-invariant, that is,* $m(E+x) = m(E)$ *for every* $x \in \mathbb{R}^k$ *and* $E \in \mathcal{M}$*. Here,* $E + x := \{e + x \,:\, e \in E\}$.

(iv) *If* μ *is a translation-invariant positive Borel measure on* $\mathbb{R}^k$ *such that* $\mu(K) < \infty$ *for all compact sets* $K \subseteq \mathbb{R}^k$*, then there exists a constant* $c \geq 0$ *such that* $\mu(E) = c \cdot m(E)$ *for every Borel set* E.

(v) *If* $T : \mathbb{R}^k \to \mathbb{R}^k$ *is a linear map, then* $m(T(E)) = |\det T| \cdot m(E)$ *for all* $E \in \mathcal{M}$*. Here,* $\det T$ *denotes the determinant of the corresponding matrix with respect to some basis (which is equal in the domain space and in the codomain space).*

Measure m from Theorem 2.4.1 is the *Lebesgue measure* on $\mathbb{R}^k$. Members of $\mathcal{M}$ are *Lebesgue measurable sets* in $\mathbb{R}^k$.

Theorem 2.4.2. *Let $-\infty < a < b < \infty$ and let $f : [a, b] \to \mathbb{R}$ be a function. Then f is Riemann integrable if and only if f is bounded and f is continuous almost everywhere with respect to m. (Here, m is the restriction of the Lebesgue measure in $\mathbb{R}$ to $[a, b]$.)*

2.5 Approximation of Measurable Functions by Continuous Functions

Theorem 2.5.1. *(Lusin's Theorem) Let X be a locally compact Hausdorff topological space. Assume that σ-algebra $\mathcal{M}$ and measure μ are obtained from Riesz Theorem 2.2.1 (induced by some positive functional φ). Let $f : X \to \mathbb{C}$ be a measurable function and $A \in \mathcal{M}$ such that $\mu(A) < \infty$ and $f(x) = 0$ for all $x \in A^c$. Then for every $\varepsilon > 0$ there exists $g \in C_c(X)$ such that*

$$\mu\left(\{x \in X \,:\, f(x) \neq g(x)\}\right) < \varepsilon$$

and $\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|$.

Corollary 2.5.2. *Let X be a locally compact Hausdorff topological space. Assume that σ-algebra $\mathcal{M}$ and measure μ are obtained from Riesz Theorem 2.2.1 (induced by some positive functional φ). Let $f : X \to \mathbb{C}$ be a measurable function and $A \in \mathcal{M}$ such that $\mu(A) < \infty$ and $f(x) = 0$ for all $x \in A^c$. If there exists a constant $M \in (0, \infty)$ such that $|f| \leq M$, then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of functions in $C_c(X)$ such that $\lim_{n\to\infty} g_n(x) = f(x)$ a.e. and $|g_n| \leq M$ for all n.*

CHAPTER 3

L^p spaces

3.1 Convex Functions and Inequalities

Let $-\infty \le a < b \le \infty$. Function $\varphi : (a, b) \to \mathbb{R}$ is *convex* if

$$\varphi\big((1-\lambda)x + \lambda y\big) \le (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

holds for all $x, y \in (a, b)$ and all $\lambda \in [0, 1]$.

Equivalently, $\varphi : (a, b) \to \mathbb{R}$ is convex if

$$\frac{\varphi(t) - \varphi(s)}{t - s} \le \frac{\varphi(u) - \varphi(t)}{u - t}$$

holds for all $a < s < t < u < b$.

Proposition 3.1.1. *Let $-\infty \le a < b \le \infty$. A convex function $\varphi : (a, b) \to \mathbb{R}$ is continuous.*

Theorem 3.1.2. *(Jensen's inequality) Let μ be a probability measure on $(\Omega, \mathcal{M})$, that is, $\mu(\Omega) = 1$. Let $-\infty \le a < b \le \infty$. Assume that $f \in L^1(\mu)$ satisfies $f(x) \in (a, b)$ for all $x \in \Omega$. Then each convex function $\varphi : (a, b) \to \mathbb{R}$ satisfies*

$$\varphi\left(\int_\Omega f\, d\mu\right) \le \int_\Omega \varphi \circ f\, d\mu.$$

From Jensen's inequality we can deduce:

- (Discrete Jensen's inequality) Let $-\infty \leq a < b \leq \infty$ and assume that $\varphi : (a, b) \to \mathbb{R}$ is a convex function. If $\sum_{i=1}^{n} \alpha_i = 1$, where $\alpha_i \geq 0$ and $x_i \in (a, b)$ for all i, then

$$\varphi\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \varphi(x_i).$$

- (Inequality between weighted GM and weighted AM) Let $\sum_{i=1}^{n} \alpha_i = 1$, where $\alpha_i \geq 0$ and $y_i > 0$ for all i. Then

$$y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \leq \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n.$$

- (Innequality between GM and AM) Let $y_1, \ldots, y_n > 0$. Then

$$\sqrt[n]{y_1 y_2 \cdots y_n} \leq \frac{y_1 + y_2 + \ldots + y_n}{n}.$$

- (Young's innequality) Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $x, y > 0$, then

$$xy \leq \frac{1}{p} x^p + \frac{1}{q} x^q.$$

Real numbers $p, q > 1$ that satisfy $\frac{1}{p} + \frac{1}{q} = 1$ are *conjugate exponents.* Values 1 and ∞ are considered as conjugate exponents as well.

Theorem 3.1.3. *Let $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p, q < \infty$. If (X, μ) is a measure space and $f, g : X \to [0, \infty]$ are measurable functions, then*

$$\int_X fg \, d\mu \leq \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} \left(\int_X g^q \, d\mu\right)^{\frac{1}{q}} \qquad (\textit{Hölder's inequality})$$

and

$$\left(\int_X (f+g)^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p \, d\mu\right)^{\frac{1}{p}} \qquad (\textit{Minkowski inequality})$$

Remark. Hölder's inequality is called Cauchy-Bunyakovsky-Schwarz innequality in case $p = 2 = q$.

3.2 L^p spaces

Let (X, μ) be a measure space, $f : X \to \mathbb{C}$ a measurable function, and $p \in (0, \infty)$. The value

$$\|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

is called the *p-norm* of f (it is not necessarily a norm). Momentarily we define

$$L^p(X, \mu) := \{\text{measurable functions } f : X \to \mathbb{C} \text{ with } \|f\|_p < \infty\}.$$

Given a measurable function $f : X \to [0, \infty]$ let $S := \{\alpha \in [0, \infty) : \mu(\{x \in X : f(x) > \alpha\}) = 0\}$ and let

$$\operatorname{esssup}(f) := \begin{cases} \inf S & \text{if } S \neq \emptyset \\ \infty & \text{if } S = \emptyset \end{cases}$$

be the *essential supremum* of f. If $S \neq \emptyset$, then $\operatorname{esssup}(f) \in S$.

Given a measurable function $f : X \to \mathbb{C}$ define the ∞-*norm* by

$$\|f\|_\infty := \operatorname{esssup}|f|$$

and momentarily let

$$L^\infty(X, \mu) := \{\text{measurable functions } f : X \to \mathbb{C} \text{ with } \|f\|_\infty < \infty\}.$$

Theorem 3.2.1. *If $1 \leq p \leq \infty$, then $L^p(\mu)$ is a vector space. For $f, g \in L^p(\mu)$ and $\lambda \in \mathbb{C}$ we have*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

and

$$\|\lambda f\|_p \leq |\lambda| \cdot \|f\|_p.$$

On $L^p(\mu)$ we define an equivalence relation $\sim$ by

$$f \sim g \iff f = g \text{ a.e.}$$

Then the quotient set of equivalence classes $L^p(\mu)/_\sim = \{[f] : f \in L^p(\mu)\}$, where $[f] := \{g \in L^p(\mu) : g \sim f\}$, becomes a vector space for operations $[f] + [g] := [f + g]$ and $\lambda[f] := [\lambda f]$, where $\lambda \in \mathbb{C}$. Moreover, $\|[f]\|_p := \|f\|_p$ is a norm on $L^p(\mu)/_\sim$.

In the sequel we abuse the notation and denote $L^p(\mu)/_\sim$ and $[f]$ by $L^p(\mu)$ and f, respectively.

Recall that given a normed space X, a sequence $(x_n)_{n\in\mathbb{N}}$ in X is a *Cauchy sequence* if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq n_0$. Each convergent sequence is a Cauchy sequence. The contrary is not necessarily true.

A normed space, possessing the property that each Cauchy sequence is a convergent sequence, is a *Banach space*.

Theorem 3.2.2. *If $1 \leq p \leq \infty$, then $L^p(\mu)$ is a Banach space.*

Theorem 3.2.3. *If $1 \leq p \leq \infty$ and $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mu)$ with limit $f \in L^p(\mu)$, then there is a subsequence of $(f_n)_{n\in\mathbb{N}}$ that converge to f a.e.*

Theorem 3.2.4. *If $1 \leq p, q \leq \infty$ are conjugate exponents and $f \in L^p(\mu)$, $g \in L^q(\mu)$, then $f \cdot g \in L^1(\mu)$ and $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Let $s = \sum_{k=1}^{n} \alpha_k \chi_{A_k}$ be a simple function, that is, $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ are distinct and $A_k := s^{-1}(\{\alpha_k\})$. Then $s \in L^\infty(\mu)$. If $1 \leq p < \infty$, then $s \in L^p(\mu)$ if and only if $\mu\big(\{x \in X \ : \ s(x) \neq 0\}\big) < \infty$. Let

$$\mathcal{S} := \{\text{simple functions } s : X \to \mathbb{C} \text{ with } \mu\big(\{x \in X \ : \ s(x) \neq 0\}\big) < \infty\}.$$

Theorem 3.2.5. *If $1 \leq p < \infty$. Then $\mathcal{S}$ is a dense subset in $L^p(\mu)$.*

Theorem 3.2.6. *Let $1 \leq p < \infty$. If X is a locally compact Hausdorff topological space and μ is obtained from the Riesz's Theorem 2.2.1, then $C_c(X)$ is a dense subspace in $L^p(\mu)$.*

Remark. Recall that a *completion* of a metric space $\mathcal{M}$ is a complete metric space $\mathcal{N}$ such that $\mathcal{M}$ is a dense subset in $\mathcal{N}$. So, Theorems 3.2.2 and 3.2.6 says that the completion of $C_c(X)$, equipped with the p-norm where $1 \leq p < \infty$, is $L^p(\mu)$.

Let X be a locally compact Hausdorff topological space. We say that $f : X \to \mathbb{C}$ *vanishes at infinity* if for each $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \notin K$. The vector space

$$C_0(X) := \{\text{continuous functions } f : X \to \mathbb{C} \text{ that vanishes at infinity}\}$$

contains $C_c(X)$. It is known that $C_0(X)$ is a Banach space for the norm given by

$$\|f\| = \sup_{x\in X} |f(x)|. \tag{3.1}$$

It turns out that if $X = \mathbb{R}^k$ is the Euclidean space, $\mathcal{M}$ is the Lebesgue σ algebra and μ is the Lebesgue measure in $\mathbb{R}^k$, then the ∞-norm and the norm (3.1) equal on $C_0(X)$.

Theorem 3.2.7. *Let X be a locally compact Hausdorff topological space. Then $C_0(X)$ is a completion of $C_c(X)$ equipped with the norm* (3.1).

CHAPTER 4

Complex Measures

4.1 The Total Variation of a Complex Measure

Let $(X, \mathcal{M})$ be a measurable space. At most countable family $\{E_1, E_2, \ldots\}$ of members in $\mathcal{M}$ is a partition of $E \in \mathcal{M}$ if

- $E = \bigcup_n E_n$,
- $E_i \cap E_j = \emptyset$ whenever $i \neq j$.

A map $\mu : \mathcal{M} \to \mathbb{C}$ is a *complex measure* if for each $E \in \mathcal{M}$ and for each partition $\{E_n\}_n$ of E we have

$$\mu(E) = \sum_n \mu(E_n). \tag{4.1}$$

Remark. In (4.1) it is assumed that the series converges. By 'permuting' the members of a partition we obtain the same partition, so the sum (4.1) does not depend on the order of summation. Consequently, the series (4.1) converges absolutely.

Remark. Finite positive measures are examples of complex measures.

Given a complex measure μ, the map $|\mu| : \mathcal{M} \to [0, \infty]$, defined by

$$|\mu|(E) := \sup \left\{ \sum_n |\mu(E_n)| \ : \ \{E_n\}_n \text{ is a partition of } E \right\},$$

is the *total variation* of μ.

Theorem 4.1.1. *The total variation $|\mu|$ of a complex measure μ on a measurable space $(X, \mathcal{M})$ is a positive measure on $(X, \mathcal{M})$.*

Remark. The total variation is the smallest positive measure λ that satisfies $|\mu(E)| \leq \lambda(E)$ for all $E \in \mathcal{M}$.

Theorem 4.1.2. *The positive measure $|\mu|$ is finite, that is, $|\mu|(X) < \infty$.*

The set of all complex measures on a measurable space $(X, \mathcal{M})$ form a vector space for the operations $(\mu + \lambda)(E) := \mu(E) + \lambda(E)$ and $(c \cdot \mu)(E) = c \cdot \mu(E)$, where $E \in \mathcal{M}$ and $c \in \mathbb{C}$. Moreover, it is a normed space, where the norm is given by $\|\mu\| := |\mu|(X)$.

If μ is a *real measure*, that is, a complex measure with the image in $\mathbb{R}$, then we define *positive part* and the *negative part* of μ by

$$\mu^+ := \frac{1}{2} (|\mu| + \mu)$$

and

$$\mu^- := \frac{1}{2} (|\mu| - \mu),$$

respectively. Obviously, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. Consequently, since $|\mu(E)| \leq |\mu|(E)$ for each $E \in \mathcal{M}$, we deduce that μ^+ and μ^- are finite positive measures.

4.2 Radon-Nikodym Theorem

Let μ be a positive measure on a σ-algebra $\mathcal{M}$ and let λ be a complex measure on $\mathcal{M}$. We say that λ is *absolutely continuous* with respect to μ, if $E \in \mathcal{M}$ satisfies $\lambda(E) = 0$, whenever $\mu(E) = 0$. In such a case we write

$$\lambda \ll \mu.$$

A complex or a positive measure λ is *concentrated* on $A \in \mathcal{M}$ if $\lambda(E) = \lambda(E \cap A)$ for all $E \in \mathcal{M}$, or equivalently, if $\lambda(E) = 0$ whenever $E \in \mathcal{M}$ satisfies $E \subseteq A^c$.

Two complex or positive measures λ_1 and λ_2 on $\mathcal{M}$ are *mutually singular*, if there are two disjoint sets $A_1, A_2 \in \mathcal{M}$ such that λ_1 is concentrated on A_1 and λ_2 is concentrated on A_2. In such a case we write

$$\lambda_1 \perp \lambda_2.$$

Proposition 4.2.1. *Let $\mathcal{M}$ be a σ-algebra, μ a positive measure on it, and $\lambda, \lambda_1, \lambda_2$ complex measures on it. Then:*

(i) If λ is concentrated on $A \in \mathcal{M}$, then $|\lambda|$ is concentrated on A.

(ii) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(iii) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$.

(iv) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $(\lambda_1 + \lambda_2) \ll \mu$.

(v) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(vi) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(vii) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda \equiv 0$.

Lemma 4.2.2. *Let μ be a positive measure on measurable space $(X, \mathcal{M})$. Then μ is σ-finite if and only if there exists $w \in L^1(\mu)$ such that $w(x) > 0$ for all $x \in X$.*

Given a function $h \in L^1(\mu)$, consider the complex measure

$$\lambda(E) := \int_E h\, d\mu \qquad (E \in \mathcal{M}). \tag{4.2}$$

Then $\lambda \ll \mu$. The function h is the *Radon-Nikodym derivative* of λ with respect to μ. It is denoted by $h = \frac{d\lambda}{d\mu}$ and (4.2) is written shortly as $d\lambda = h d\mu$.

Theorem 4.2.3. *Let $\mathcal{M}$ be a σ-algebra, μ a σ-finite positive measure on it, and λ a complex measure on it. Then:*

(i) (Lebesgue decomposition) There exist unique complex measures λ_a and λ_s on $\mathcal{M}$ such that

$$\lambda = \lambda_a + \lambda_s \quad \textit{and} \quad \lambda_a \ll \mu,\ \lambda_s \perp \mu.$$

(ii) (Radon-Nikodym) There exists a unique $h \in L^1(\mu)$ such that

$$d\lambda_a = h\, d\mu.$$

Remark. If μ is a σ-finite positive measure and λ is a complex measure such that $\lambda \ll \mu$, then Theorem 4.2.3 implies that $d\lambda = hd\mu$. This special case is usually referred as the Radon-Nikodym theorem.

Theorem 4.2.4. *Let $\mathcal{M}$ be a σ-algebra, μ a positive measure on it and λ a complex measure on it. The following two statements are equivalent.*

(i) $\lambda \ll \mu$;

(ii) For every $\varepsilon > 0$ there is some $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ whenever $\mu(E) < \delta$ and $E \in \mathcal{M}$.

Corollary 4.2.5. *Let μ be a positive measure on $\mathcal{M}$ and $f \in L^1(\mu)$. Then for every $\varepsilon > 0$ there is some $\delta > 0$ such that $\int_E |f|\, d\mu < \varepsilon$ whenever $\mu(E) < \delta$ and $E \in \mathcal{M}$.*

Theorem 4.2.6. *(Polar decomposition of a complex measure) Let μ be a complex measure on $\mathcal{M}$. Then there is a measurable function h on X such that $|h(x)| = 1$ for all $x \in X$ and $d\mu = hd|\mu|$.*

Theorem 4.2.7. *Let μ be a positive measure on $\mathcal{M}$ and $g \in L^1(\mu)$. If*

$$\lambda(E) = \int_E g\, d\mu \qquad (E \in \mathcal{M})$$

then

$$|\lambda|(E) = \int_E |g|\, d\mu \qquad (E \in \mathcal{M})$$

Theorem 4.2.8. *(Hahn decomposition of a real measure) Let μ be a real measure on $\mathcal{M}$. Then there exist sets $A, B \in \mathcal{M}$ such that $A \cup B = X$, $A \cap B = \emptyset$, and*

$$\mu^+(E) = \mu(A \cap E) \qquad (E \in \mathcal{M}),$$
$$\mu^-(E) = -\mu(B \cap E) \qquad (E \in \mathcal{M}).$$

CHAPTER 5

Integration on Product Spaces

5.1 Product of Measurable Spaces

Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be two measurable spaces. Given sets $A \in \mathcal{S}$ and $B \in \mathcal{T}$ their cartesian product $A \times B$ is called a *measurable rectangle*. Let $\mathcal{S} \times \mathcal{T}$ denote the smallest σ-algebra on $X \times Y$ that contains all measurable rectangles. If $E = P_1 \cup P_2 \cup \cdots \cup P_n$, where $P_1, P_2, \ldots, P_n$ are pairwise disjoint measurable rectangles, then E is an *elementary set*.

A collection $\mathcal{M}$ of sets is a *monotone class* if the following two implications hold:

$$\begin{aligned} A_1 \subseteq A_2 \subseteq \cdots,\ A_i \in \mathcal{M} \quad &\Longrightarrow \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{M}; \\ B_1 \supseteq B_2 \supseteq \cdots,\ B_i \in \mathcal{M} \quad &\Longrightarrow \quad \bigcap_{i=1}^{\infty} B_i \in \mathcal{M}. \end{aligned}$$

So every σ-algebra is a monotone class.

For each $E \subseteq X \times Y$, $x \in X$, and $y \in Y$ let

$$E_x := \{z \in Y \,:\, (x, z) \in E\}$$

and

$$E^y := \{z \in X \,:\, (z, y) \in E\}$$

be *x-section* of E and *y-section* of E, respectively.

Proposition 5.1.1. *If $E \in \mathcal{S} \times \mathcal{T}$, then $E_x \in \mathcal{T}$ for all $x \in X$ and $E^y \in \mathcal{S}$ for all $y \in Y$.*

Theorem 5.1.2. *The σ-algebra $\mathcal{S} \times \mathcal{T}$ is the smallest monotone class that contains all elementary sets in $X \times Y$.*

For each complex function $f : X \times Y \to \mathbb{C}$, $x \in X$, $y \in Y$, define the functions $f_x : Y \to \mathbb{C}$ and $f^y : X \to \mathbb{C}$ by $f_x(y) := f(x, y)$ and $f^y(x) := f(x, y)$, respectively.

Proposition 5.1.3. *Let $f : X \times Y \to \mathbb{C}$ be a $\mathcal{S} \times \mathcal{T}$-measurable function. Then f_x is a $\mathcal{T}$-measurable function for all $x \in X$, and f^y is a $\mathcal{S}$-measurable function for all $y \in Y$.*

5.2 Product Measures

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two measure spaces, where both positive measures μ and λ are σ-finite. Given $Q \in \mathcal{S} \times \mathcal{T}$ denote

$$\varphi(x) := \lambda(Q_x) \quad \textit{and} \quad \psi(y) := \lambda(Q^y).$$

Theorem 5.2.1. *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two σ-finite measure spaces and $Q \in \mathcal{S} \times \mathcal{T}$. Then φ is a $\mathcal{S}$-measurable function, ψ is a $\mathcal{T}$-measurable function, and*

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda. \tag{5.1}$$

Remark. Equation (5.1) can be rewritten as

$$\int_X \left(\int_Y \chi_Q(x, y) \, d\lambda(y) \right) d\mu(x) = \int_Y \left(\int_X \chi_Q(x, y) \, d\mu(x) \right) d\lambda(y).$$

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two σ-finite measure spaces. For $Q \in \mathcal{S} \times \mathcal{T}$ define

$$(\mu \times \lambda)(Q) := \int_X \lambda(Q_x) \, d\mu.$$

Then $\mu \times \lambda$ is a σ-finite positive measure on $\mathcal{S} \times \mathcal{T}$, which is called the *product measure* of μ and λ.

Theorem 5.2.2. *(Fubini's Theorem) Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two σ-finite measure spaces and let f be a $\mathcal{S} \times \mathcal{T}$-measurable function on $X \times Y$.*

(i) *If* $0 \le f \le \infty$, *then* $\varphi(x) := \int_Y f_x \, d\lambda$ *is a* $\mathcal{S}$*-measurable function on* X, $\psi(x) := \int_Y f^y \, d\mu$ *is a* $\mathcal{T}$*-measurable function on* Y, *and*

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda = \int_{X \times Y} f \, d(\mu \times \lambda). \tag{5.2}$$

(ii) *If* f *is a complex function and* $\varphi^*(x) := \int_Y |f|_x \, d\lambda$ *satisfies* $\int_X \varphi^* \, d\mu < \infty$, *then* $f \in L^1(\mu \times \lambda)$.

(iii) *If* $f \in L^1(\mu \times \lambda)$, *then* $f_x \in L^1(\lambda)$ *almost for all* x, $f^y \in L^1(\mu)$ *almost for all* y, $\varphi \in L^1(\mu)$, $\psi \in L^1(\lambda)$, *and (5.2) holds.*

Remark. Equation (5.2) can be rewritten as

$$\begin{aligned} \int_X \left(\int_Y f(x,y) \, d\lambda(y) \right) \, d\mu(x) &= \int_Y \left(\int_X f(x,y) \, d\mu(x) \right) \, d\lambda(y) \\ &= \int_{X \times Y} f(x,y) \, d(\mu \times \lambda)(x,y). \end{aligned}$$

The first two of the above integrals are *iterated integrals*, while the last one is the *double integral*.

5.3 The Completion of a Product Measure

Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be two σ-finite measure spaces, where μ and λ are complete measures. Then $\mu \times \lambda$ is not necessarily complete. Recall that a completion of a measure is defined in Proposition 1.4.4.

Theorem 5.3.1. *Let* m_k *be the Lebesgue measure on* $\mathbb{R}^k$, *where* $k = r + s$ *for some* $r, s \in \mathbb{N}$. *Then* $m_k = (m_r \times m_s)^*$ *is the completion of measure* $m_r \times m_s$.

Theorem 5.3.2. *Let* $(X, \mathcal{S}, \mu)$ *and* $(Y, \mathcal{T}, \lambda)$ *be two* σ*-finite measure spaces, where* μ *and* λ *are complete, and let* f *be a* $(\mathcal{S} \times \mathcal{T})^*$*-measurable function on* $X \times Y$. *Then all conclusions of Fubini's theorem (Theorem 5.2.2) hold, except that* f_x *is* $\mathcal{T}$*-measurable almost for all* $x \in X$, *so* φ *is defined a.e. on* X, *and* f^y *is* $\mathcal{S}$*-measurable almost for all* $y \in Y$, *so* ψ *is defined a.e. on* Y.

Vocabulary

English	**Slovenian**
σ-algebra	σ-algebra
x-section of a set	x-prerez množice
y-section of a set	y-prerez množice
a.e.	s.p.
absolutely continuous	absolutno zvezna
almost everywhere	skoraj povsod
almost uniformly converges	skoraj enakomerno konvergira
Borel σ-algebra	Borelova σ-algebra
Borel map/function	Borelova preslikava/funkcija
Borel set	Borelova množica
bounded function	omejena funkcija
closed set	zaprta množica
complete measure	polna mera
completion	napolnitev
complex measure	kompleksna mera
concentrated	skoncentrirana
conjugate exponents	konjugirana eksponenta
convex function	konveksna funkcija
continuous map/function	zvezna preslikava/funkcija
converges almost everywhere	konvergira skoraj povsod
converges in measure	konvergira po meri
countable additivity	števna aditivnost
countable subadditivity	števna subaditivnost
counting measure	mera, ki šteje
dense subset	gosta podmnožica
elementary set	elementarna množica
essential supremum	bistveni supremum
finite additivity	končna aditivnost
finite measure	končna mera
measurable map/function	merljiva preslikava/funkcija
measurable rectangle	merljiv pravokotnik
measurable set	merljiva množica
measurable space	merljiv prostor
measure space	prostor z mero
monotone class	monoton razred
mutually singular	vzajemno singularna
negative part	negativni del

open set	odprta množica
pointwise limit	limita po točkah
positive part	pozitivni del
positive measure	pozitivna mera
product measure	produktna mera
Radon-Nikodym derivative	Radon-Nikodymov odvod
simple function	enostavna funkcija
support	nosilec
topological space	topološki prostor
topology	topologija
total variation	totalna variacija
uniform convergence	enakomerna konvergenca
vanishes at infinity	gre v neskončnosti proti nič