Noncommutative Lattices

Skew Lattices, Skew Boolean Algebras and Beyond

Jonathan E. Leech

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Jonathan Leech graduated from the University of Hawaii and earned a PhD at the University of California, Los Angeles. He has taught mathematics at the University of Tennessee, later at Missouri Western State University and finally at Westmont College in Santa Barbara, California. He was a Visiting Professor at Case Western Reserve University, the Universidad de Granada in Spain and Universidade Mackenzie in Brazil. He has been a scholar in residence at both the University of Sidney and the University of Tasmania in Australia. Throughout his academic career Professor Leech has studied algebraic structures related to semigroups, with much of his emphasis being on the theory of noncommutative lattices and of skew lattices in particular. He laid the foundations of the modern theory of noncommutative lattices in a number of (co)authored seminal publications. His work has inspired many mathematicians around the world to pursue research in this area.

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Famnit Lectures | Famnitova predavanja | 4 | ISSN 2335-3708

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Published by
Založba Univerze na Primorskem/University of Primorska Press
For publisher: Prof. Klavdija Kutnar, PhD, rector
Titov trg 4, SI-6000 Koper
Editor-in-chief
Jonatan Vinkler
Managing editor
Alen Ježovnik

Koper, 2020

ISBN 978-961-293-027-1 (pdf)
http://www.hippocampus.si/ISBN/978-961-293-027-1.pdf
ISBN 978-961-293-028-8 (html)
http://www.hippocampus.si/ISBN/978-961-293-028-8/index.html
DOI: https://doi.org/10.26493/978-961-293-027-1

Kataložni zapis o publikaciji (CIP) pripravili v Narodni in univerzitetni knjižnici v Ljubljani
COBISS.SI-ID=35005187
ISBN 978-961-293-027-1 (pdf)
ISBN 978-961-293-028-8 (html)

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The last 30 years have witnessed sustained research by a number of individuals in skew lattices, a class of noncommutative generalizations of lattices. (A partial list of published work is given in the Bibliography at the end of this monograph.) Papers on noncommutative lattices in general have appeared since the late 1940s. It would not be unfair to say that the more recent research has been both deeper and more fruitful than the earlier work, for several reasons.

To begin, by restricting attention to a particular class of algebras, one is more focused. Indeed, once commutativity is dropped, the possibilities for differing absorption identities that reduce to the familiar identities in the commutative case becomes quite large. Thus, by working within the boundaries of a fixed set of identities, one becomes more concentrated in one's efforts.

Secondly, in more recent times advantage has been made of results in semigroup theory about *bands*, that is, semigroups consisting entirely of idempotents. Indeed the newer research began by studying multiplicative bands of idempotents in rings, and realizing that under certain conditions such bands would also be closed under an "upward multiplication" to yield a skew lattice. Parallel to this was an expanding role of universal algebra, both due to results of a fairly general scope (basic universal algebra) and also results related to structures that were weakened or modified forms of Boolean algebras. This was especially important in the study of skew Boolean algebras. Summing up: there has been a greater awareness of relevant information.

Thirdly, as indicated above, the newer research began with a rich source of motivating examples – bands of idempotents in rings. In particular for bands that were *left regular* ($xyx = xy$), any maximal such band in a ring was also closed under the circle operation $x\circ y = x + y - xy$. And any band closed under both operations satisfied certain absorption identities, e.g., $e(e\circ f) = e = e\circ(ef)$. These observations, along with others related to normal bands of idempotents (that were middle commutative: $xyzw = xzyw$) indicated the presence of structurally enhanced bands with a roughly lattice-like structure. Thus skew lattices arose, along with a number of potential properties first observed in the setting of rings. To this was added a second class of motivating examples, algebras of partial functions P(A, B) between pairs of sets, A and B. These provided examples of skew Boolean algebras and related structures, much as "partial sets" (that is, subsets) led to basic examples of Boolean algebras and distributive lattices.

In addition, there was the effect of computer technology, especially beginning in the late 90s. The internet provided a quick, efficient means of communication, making it easier for like-minded individuals to connect. And computer software made it easier to find examples and initial proofs for some theorems. This continued to impact the development of skew lattice theory in the 21st century.

And finally and most fortunately, skew lattices have attracted the attention of a number very fine mathematicians from around the world, including (to my current awareness) individuals from America, Australia, China, Ethiopia, Europe (including Great Britain ☺), India and Iran.

This present volume is an organized presentation of much that has been published on the subject up through 2017. It is divided into seven chapters, the first four of which form the core of this survey.

The first chapter begins with a review of basic information about lattices and universal algebra. This is followed by a review of results about bands. The reader already familiar with these areas can move to the third section. (Caveat: it is *imperative* that one be grounded in the basic theory of bands, and especially the theory of regular bands, to be comfortable reading this monograph.) The third and final section provides introductory definitions and a few general results about noncommutative lattices. The latter are always given as at algebras (S; ∧, ∨) where ∧ and ∨ are associative, idempotent binary operations that jointly satisfy a set of absorption laws. Two varieties in particular are introduced: the variety of quasilattices and the variety of paralattices. Their intersection, the variety of refined quasilattices, contains the variety of skew lattices, our main topic. Placing skew lattices in a larger context provides a better sense of their place within the pantheon of generalizations of lattices.

In Chapter 2, *Skew Lattices*, we proceed to study skew lattices in earnest, starting with a selection of basic concepts and results in the first two sections. These include two decomposition theorems (Theorems 2.1.2 and 2.1.5.) and initial results about skew lattices of idempotents in rings. In Section 2.3 we study the important class of normal skew lattices for which $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$. (If the idempotents of a ring are closed under multiplication, then they form a normal skew lattice; indeed they form a skew Boolean algebra.) The remaining sections provide a deeper general analysis of the structure of a skew lattice. Sections 2.1 – 2.3 of this chapter are all that is required to read the remaining chapters, except for Sections 5.3 – 5.6.

Chapter 3 is entitled *Quasilattices, Paralattices and Their Congruences*. Sections 3.1 – 3.3 study congruences on quasilattices, thus obtaining results applicable to skew lattices. Section 4 looks at paralattices and especially refined quasilattices. We show the latter to be *roughly* skew lattices in the sense given in Theorem 3.4.14. Section 5 discusses the effects of distributivity, the main results being Theorems 3.5.1 and 3.5.2. Section 6 is recreational.

Skew Boolean algebras are studied in Chapter 4. Skew Boolean algebras are algebras (S; ∧, ∨, \. 0) of type (2, 2, 2, 0) that look and behave in many ways like Boolean algebras, except that they *need not* be commutative. Boolean algebras decompose at will, and so do these algebras. This leads to a crisp description of the finitely generated (and thus finite) algebras, and of finite free algebras in particular. The final two sections are about skew Boolean algebras for which the natural partial order ≥ has a meet called the *intersection* and denoted by ∩. Many skew Boolean algebras have intersections, e.g., all free algebras do. For partial function algebras, ∩ is the standard set-theoretic intersection of the involved partial functions.

As noted above, the core of this monograph is these four chapters. More specialized topics are studied in the last three chapters. Chapter 5 is entitled *Further Topics in Skew Lattices*, Chapter 6 is entitled *Skew Lattices in Rings* and the final chapter is entitled *Further Topics in Skew Boolean Algebras*.

As the reader will see, skew Boolean algebras understandably get a good bit of attention in this monograph. There is other research on these algebras that is not in this monograph. It is typically of more recent vintage, with much being quite good. Hopefully, before too long, some motivated individual or group will produce a monograph devoted to skew Boolean algebras and related topics.

All chapters begin with a fairly detailed introduction and conclude with a list of relevant references, sometimes preceded by historical remarks. Following Chapter 7 is a Bibliography of publications that to my knowledge are either on or are closely related to skew lattices, appearing up through 2017, plus a few beyond. Although some material is not included in this monograph due to limitations on subject matter, this monograph should give the reader a good idea of the extent of activity within the area.

Following the bibliography there is a brief Addendum intended to give the reader a sense of ongoing further research of relevance from the last two years involving newer topics not covered in this monograph.

I: Preliminaries

Noncommutative variants of lattices have been studied for over sixty-five years. The first person, to our knowledge, to engage in their extended study was Pascual Jordan who published numerous articles over a span of seventeen years. Since then papers on this subject have been written by various authors from a variety of perspectives.

Why study noncommutative lattices? One reason comes from an interest in axioms. Clearly many important algebraic structures are characterized by axioms expressed as algebraic identities. In particular, lattices are defined as algebras $(L; \vee, \wedge)$ where $\vee$ and $\wedge$ are binary operations on a set L satisfying the following pairs of associative, absorption and commutative identities.

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c. \qquad a \vee (b \vee c) = (a \vee b) \vee c.$$
$$a \wedge (a \vee b) = a. \qquad a \vee (a \vee b) = a.$$
$$a \wedge b = b \wedge a. \qquad a \vee b = b \vee a.$$

An initial result in lattice theory is that the idempotent identities ($a \wedge a = a = a \vee a$) follow from the two absorption identities above without recourse to either the associative or commutative identities.

If one deletes both commutative identities, then the four remaining identities are satisfied by genuinely noncommutative structures. (Consider, e.g., any set A of size greater than 1. Define $\vee$ and $\wedge$ on A by setting $a \vee b = a = a \wedge b$.) On the other hand, if one deletes the commutative laws, and combines instead the associative identities with a modified and expanded set of absorption identities

$$a \wedge (a \vee b) = a, \qquad (b \vee a) \wedge a = a,$$
$$a \vee (b \wedge a) = a, \qquad (a \wedge b) \vee a = a,$$

then these identities also characterize lattices. (See Theorem 1.3.2 below.) The point is that axiomatic studies of lattices opened the door to considering noncommutative variants of lattices. Indeed, during the period when Jordan studied noncommutative lattices, others were studying axiomatic issues of lattices with an awareness of noncommutative possibilities. Thus an interest in axioms combined with a curiosity about possible noncommutative variations of lattices virtually insured that such variants would appear and then studied to some degree.

A second source of motivation arises from studying the multiplicative semigroups of rings. In the study of rings, idempotents play an important role. In general, given idempotents e and f in a ring, their product ef need not be idempotent (unless, e.g., the ring is commutative). Nonetheless ef is "below and to the right" of e in that $e(ef) = ef$, and at the same time "below and to the left" of f in that $(ef)f = ef$. Dually, e is "below and to the left" of the circle product $e\circ f = e + f - ef$ in that $e(e\circ f) = e$, while f is "below and to the right" of $e\circ f$ in that $(e\circ f)f = f$. On thus has the following picture:

$$\begin{array}{ccc} & e\circ f & \\ \cdot^{\cdot^{\cdot}} & & \ddots \\ e & & f \\ \ddots & & \cdot^{\cdot^{\cdot}} \\ & ef. & \end{array}$$

In general, given an element x in a ring, $x^2 = x$ iff $x\circ x = x$. What is more, for a set of idempotents in a ring that is closed under both multiplication and $\circ$, the following four absorption identities are satisfied.

$$a(a \circ b) = a = (b \circ a)a. \qquad\qquad a \circ (ab) = a = (ba) \circ a.$$

Noncommutative rings that are well endowed with idempotents are rich in such examples. What can one say about their structure? Such ring-based structures will occupy our attention in much of the second and sixth chapters to follow.

Abstracting only slightly, one can think of bands – semigroups of idempotents – that are rich enough in structure to possess an idempotent counter-multiplication. Thus multiplication produces products that are generally "further down" in the band, while the counter-products would be generally "further up" in the band. Such bands exist. What can be said about them?

A third source of motivation comes from universal algebra, especially the study of what may be loosely termed "generalized Boolean phenomena". Do noncommutative generalizations of (generalized) Boolean lattices and algebras exist? If so, what connections exist between them and other structures related to Boolean algebras? Clearly noncommutative lattice theory have something to say about all this? Questions such as these will occupy our attention in the fourth and seventh chapters to follow.

In the meanwhile, we begin this introductory chapter by reviewing a number of concepts about lattices and universal algebra in the first section. In the second section we recall various facts about bands that are pertinent to the rest of the monograph. And then in Section 3 we discuss some "first principals" of noncommutative lattices. All the material in this chapter is foundational to what follows later. The reader well-versed in the material in either of the first two sections can easily skip over one or both of them and then proceed to the third section. We emphasize, however, that a firm grasp of regular bands, and their left and right-sided cases, is crucial to understanding much that will be said about skew lattices.

Left regular bands have received increased attention recently due to their role in combinatorial aspects of algebra and geometry. See, e.g., the introductory remarks in the monograph, *Cell complexes, poset tolopogy and the representation theory of algebras arising in algebraic combinatorics and discrete geometry* by Stuart Margolis, Franco Saliola and Benjamin Steinberg.

1.1 Lattices

Recall that a ***partially ordered set*** or ***poset*** is any pair $(L; \geq)$ where L is a set and $\geq$ is a partial ordering of L, that is, reflexive, anti-symmetric and transitive relation on L. Given $x \geq y$ in L, we think of x as lying above y, or equally of y lying below x.

Given elements x, y in a poset $(L; \geq)$, an element $m \in L$ such that (1) $x \geq m$ and $y \geq m$ and (2) m lies above all other elements lying jointly below x and y is called the ***meet*** of x and y and is denoted by $x \wedge y$. Dually, an element $j \in L$ such that (3) $j \geq x$, $j \geq y$ and (4) j lies below all other elements lying jointly above x and y is called the ***join*** of x and y and is denoted by $x \vee y$.

$$\begin{tikzcd}[column sep=small]
 & x \vee y \arrow[dl] \arrow[dr] & \\
x \arrow[dr] & & y \arrow[dl] \\
 & x \wedge y &
\end{tikzcd}$$

When they exist, $x \wedge y$ and $x \vee y$ are unique with respect to the given x and y. If all pairs $x, y \in$ S have a meet and a join, then $(L; \geq)$ is a ***lattice***. In this case $(L; \vee, \wedge)$ satisfies the following idempotent, commutative, associative and absorption identities:

L0. $x \wedge x = x = x \vee x$.
L1. $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$.
L2. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$.
L3. $x \wedge (x \vee y) = x = x \vee (x \wedge y)$.

Conversely, given $(L; \vee, \wedge)$ with binary operations $\wedge$ and $\vee$ satisfying L0 – L3, a partial order $\geq$ is defined on L by

$$x \geq y \text{ if and only if } x \wedge y = y, \text{ or equivalently, } x \vee y = y.$$

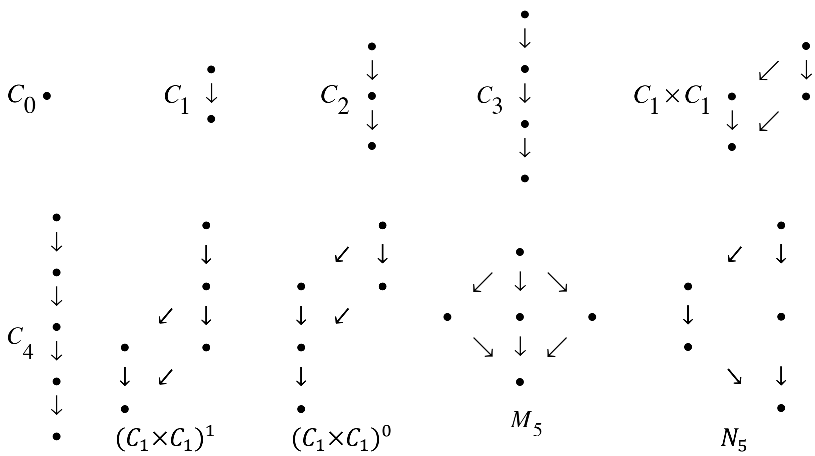
As a poset, $(L; \geq)$ is a lattice whose meets and joins are precisely the given $\wedge$ and $\vee$. Indeed the process of passing from a lattice poset $(L; \geq)$ to an algebra $(L; \vee, \wedge)$ and the reverse process of passing from an algebra $(L; \vee, \wedge)$ satisfying L0-L3 to a lattice poset are reciprocal processes. Thus lattices may be viewed from either a poset perspective or an algebraic perspective.

While not verifying all details, we offer the following remarks. To begin, in any lattice poset, L0 and L1 are clear. L2 refers to the unique elements $x \wedge y \wedge z$ lying maximally below x, y and z and $x \vee y \vee z$ lying minimally above x, y and z. Likewise, the ***absorption identities*** in L3 refer to the fact that $x \vee y \geq x \geq x \wedge y$ in $(L; \geq)$. Conversely, given an algebra $(L; \vee, \wedge)$ satisfying L0 – L3, the derived relation $\geq$ is reflexive by L0 and anti-symmetric thanks to L1. L2 is instrumental in showing that $\geq$ is transitive. L3 yields the basic duality $x \wedge y = y$ if and only if $x \vee y = y$. $(L; \geq)$ is indeed a poset. Its meets and joins are given by $\wedge$ and $\vee$. As stated above, L0 is redundant relative to L1-L3. Indeed:

Lemma 1.1.1. *Given binary operations* $\wedge$ *and* $\vee$ *on a set* L, L3 *implie*s L0.

Proof. Given L3, $x \wedge x = x \wedge [x \vee (x \wedge x)] = x$, and thus $x \vee x = x \vee (x \wedge x) = x$. □

Lattices of small order are easily drawn. A list of all lattices up through order 5 that is complete up to isomorphism follows. The indexing on the totally ordered ***chains*** (C_0, C_1, etc.) corresponds to their length, which is always 1 less than their order.



Recall that a lattice $(L; \vee, \wedge)$ is ***distributive*** when for all $x, y, z \in L$ the following identities hold:

D1. $\quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

D2. $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

All chains are distributive lattices as are $C_1 \times C_1$, $C_1 \times C_1^I$ and $C_1 \times C_1^0$. In general, *a lattice is distributive if and only if it has no sublattice that is a copy of either* $\mathbf{M}_5$ *or* $\mathbf{N}_5$. Distributivity leads us to another fundamental redundancy, whose proof is easily accessible in the literature.

Theorem 1.1.2. *For any lattice* $(L, \wedge, \vee)$, D1 *holds if and only if* D2 *holds*. □

The shaping of distributive identities in noncommutative contexts is an important concern in generalized lattice theory. An important characterizing property of distributivity is:

Theorem 1.1.3. *Distributive lattices are cancellative in that* $x \wedge z = y \wedge z$ *and* $x \vee z = y \vee z$ *together imply* $x = y$. *Conversely, cancellative skew lattices are distributive.*

Proof. Given $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$, then

$$x = x \vee (x \wedge z) = x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = (x \vee y) \wedge (y \vee z) = (x \vee z) \wedge y \leq y$$

and similarly, $y \leq x$, so that $x = y$. Conversely, neither $\mathbf{M}_3$ nor $\mathbf{N}_5$ can be subalgebras of a cancellative slew lattice. □

A lattice $(L; \vee, \wedge)$ is ***complete*** if every subset X of L has a ***supremum*** (an element $u \geq x$ for all x in X, with u being the least such element in L) denoted by $sup(X)$ and an ***infimum*** (an element $v \leq x$ for all x in X, with v being the greatest such element in L) denoted by $inf(X)$. In particular, a complete lattice has a greatest element 1 and a least element 0. Conversely, a lattice with both least and greatest elements 0 and 1 is complete if all subsets have suprema, or equivalently, if all subsets have infima. Finally, in any complete lattice, we let $0 = sup(\varnothing)$ and $1 = inf(\varnothing)$.

Lattices and universal algebra

An ***algebra*** is any system, $\mathcal{A} = (A: f_1, f_2, \ldots, f_r)$, where A is a set and each f_i is an n_i-ary operation on A. If $B \subseteq A$ is such that for all $i \leq r$, $f_i(b_1, b_2, \ldots, b_{n_i}) \in B$ for all $b_1, \ldots, b_{n_i}$ in B, then the system $\mathcal{B} = (B: f_1', f_2', \ldots, f_r')$ where $f_i' = f_i \mid B^{n_i}$ is a ***subalgebra*** of $\mathcal{A}$. (When confusion occurs, subalgebras may be indicated by their underlying sets.) Under inclusion, $\subseteq$, the subalgebras of an algebra $\mathcal{A}$ form a complete lattice $\mathbf{Sub}(\mathcal{A})$ with greatest element A, least element the smallest subalgebra containing $\varnothing$ and meets given by intersection. If none of the operations are nullary, then the least subalgebra is the empty subalgebra, $\varnothing$. If there are no operations, then $\mathbf{Sub}(\mathcal{A})$ is the lattice $\mathbf{2}^A$.

Recall that a ***congruence*** on $\mathcal{A} = (\mathrm{A}: f_1, f_2, \ldots, f_r)$ is an equivalence relation θ on A such that given $i \leq r$ with $a_1 \theta b_1, a_2 \theta b_2, \ldots, a_{n_i} \theta b_{n_i}$ in A, then

$$f_i(a_1, a_2, \ldots, a_{n_i}) \,\theta\, f_i(b_1, b_2, \ldots, b_{n_i}).$$

Under inclusion, $\subseteq$, the congruences on $\mathcal{A}$ form a complete lattice $\mathbf{Con}(\mathcal{A})$. Its greatest element is the universal relation $\nabla = \mathrm{A} \times \mathrm{A}$ relating all elements in A. Its least element is the identity relation Δ. Suprema and infima in $\mathbf{Con}(\mathcal{A})$ are calculated as in the lattice $\mathbf{Equ}(A)$ of all equivalences on A. In particular, infima in $\mathbf{Con}(\mathcal{A})$ are given by intersection. □

Recall that an element c in a lattice $(L; \vee, \wedge)$ is ***compact*** if for any subset X of L, $c \leq supX$ implies that $c \leq supY$ for some finite subset Y of X. (Every cover can be reduced to a finite cover.) An ***algebraic lattice*** is a complete lattice for which every element is a supremum of compact elements. The proof of the following result is easily accessible in the literature

Theorem 1.1.4. *Given an algebra* $\mathcal{A} = (\mathrm{A}: f_1, f_2, \ldots, f_r)$, *both* $\mathbf{Sub}(\mathcal{A})$ *and* $\mathbf{Con}(\mathcal{A})$ *are algebraic lattices.*

Of particular interest is the next result. It's proof may be obtained in any standard text on lattice theory.

Theorem 1.1.5. *Congruence lattices of lattices are distributive.* ☐

A subset U of a poset $(L; \geq)$ is ***directed upward*** if given any two elements x, y in U, a third element z exists in U such that $x, y \leq z$. The proof of the next result is also easily accessible.

Theorem 1.1.6. *Given an algebraic lattice* $(L; \vee, \wedge)$, $a \wedge sup(U) = sup\{a \wedge x \mid x \in U\}$ *holds if* U *is directed upward. This equality holds unconditionally when* $(L; \vee, \wedge)$ *is also distributive.*

Recall that two algebras $\mathcal{A} = (A; f_1, f_2, \ldots, f_r)$ and $\mathcal{B} = (B; g_1, g_2, \ldots, g_s)$ have the ***same type*** if $r = s$ and for all $i \leq r$, both f_i and g_i have the same number of variables, that is, both are say n_i-ary operations. Recall also that a class $\mathcal{V}$ of algebras of the same type is a ***variety*** if it is closed under direct products, subalgebras and homomorphic images. A classic result of Birkhoff is as follows:

Theorem 1.1.7. *Among algebras of the same type, each variety is determined by the set of all identities satisfied by all algebras in that variety. That is, all varieties are equationally determined in the class of all algebras of the same type.* ☐

Let $\mathbf{B} = \{\mathcal{B}_i \mid i \in I\}$ be a set of algebras of the same type. An algebra $\mathcal{A}$ of the same type is a ***subdirect product*** of the $\mathcal{B}_i$ if a monomorphism χ: $\mathcal{A} \to \prod_{i \in I}\mathcal{B}_i$ exists such that for each projection π_i: $\prod_{i \in I}\mathcal{B}_i \to \mathcal{B}_i$, the composite $\pi_i \circ \chi$: A $\to \mathcal{B}_i$ goes *onto* $\mathcal{B}_i$. $\mathcal{A}$ is ***subdirectly irreducible*** if for any subdirect factorization χ: $\mathcal{A} \to \prod_{i \in I}\mathcal{B}_i$ one of the composites $\pi_i \circ \chi$: A $\to \mathcal{B}_i$ is an isomorphism. A second classic result of Birkhoff is as follows:

Theorem 1.1.8. I*n a given variety of algebras* $\mathcal{V}$*, every algebra* $\mathcal{A}$ *in* $\mathcal{V}$ *is a subdirect product of subdirectly irreducible algebras.* ☐

We apply Theorem 1.1.8 to the variety of distributive lattices. But first recall that an ***ideal*** in a lattice L is any subset I of L that is closed joins and given any $x \in I$ and $y \in L$, $x \wedge y \in I$ also. Recall also that a ***filter*** (or ***dual-ideal***) in a lattice L is any subset F of L that is closed under meets and given any $x \in F$ and $y \in$ L, $x \vee y \in F$ also. Given any element $x \in L$, the ***principal ideal*** $x\!\downarrow\, = \{y \in L \mid x \geq y\}$ is the smallest ideal of L containing x. Dually, the smallest filter of L containing x is the ***principal filter*** $x\!\uparrow\, = \{y \in L \mid x \leq y\}$.

Theorem 1.1.9. *Given a distributive lattice* $(L; \vee, \wedge)$ *and an element* $a \in L$, χ: $L \to a\!\downarrow \times\, a\!\uparrow$ *defined by* $\chi(x) = (x \wedge a, x \vee a)$ *is a subdirect decomposition of* $(L; \vee, \wedge)$. *Thus a distributive lattice is subdirectly irreducible if and only if it is a copy of either* C_0 *or* C_1.

Proof. That χ is a homomorphism follows easily from the associative, commutative and distributive laws. By cancellation, χ is one-to-one. Upon composing with either coordinate projections, it clearly it mapped onto each factor. ☐

Corollary 1.1.10. *Every nontrivial distributive lattice is a subdirect product of* C_1. ☐

We return to the variety of all lattices. On any lattice, consider the polynomial $\mathrm{M}(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$ that was implicit in the proof of Theorem 1.5. M satisfies the identities

$$\mathrm{M}(x, x, y) = \mathrm{M}(x, y, x) = \mathrm{M}(y, x, x) = x.$$

Given an algebra $\mathcal{A} = (A; f_1, \ldots, f_r)$ *on which a ternary operation* $\mathrm{M}(x, y, z)$ *satisfying these identities is polynomial-defined using the operations of* $\mathcal{A}$, *then* **Con**$(\mathcal{A})$ *is distributive*. In general, if a ternary function M can be defined from the functions symbols of a variety $\mathcal{V}$ such that M satisfied these identities on all algebras in $\mathcal{V}$, then the congruence lattices of all algebras in that variety are distributive and $\mathcal{V}$ is said to be ***congruence distributive***.

Boolean lattices and Boolean algebras

Given a lattice $(L; \wedge, \vee)$ with maximal and minimal elements 1 and 0, elements x and x' are ***complements*** in L if $x \vee x' = 1$ and $x \wedge x' = 0$. If L is distributive, then the complement x' of any element x is unique. Indeed, let x'' be a second complement of x. Then

$$x'' = x'' \wedge 1 = x'' \wedge (x \vee x') = (x'' \wedge x) \vee (x'' \wedge x') = 0 \vee (x'' \wedge x') = x'' \wedge x'.$$

Similarly, $x' = x' \wedge x''$ and $x' = x''$ follows. Clearly 0 and 1 are mutual complements.

Recall that ***Boolean lattice*** is a distributive lattice with maximal and minimal elements 1 and 0, $(L; \wedge, \vee, 1, 0)$, such that every x in L has a (necessarily unique) complement x' in L. If the operation $'$ is built into the signature, then $(L; \wedge, \vee, ', 1, 0)$ is a ***Boolean algebra***. Boolean algebras are characterized by the identities for a distributive lattice augmented by the identities for maximal and minimal elements and the identities for complementation. They also satisfy the ***DeMorgan identities***: $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$.

Given a Boolean algebra, the ***difference*** (or relative complement) of elements x and y is defined by $x \setminus y = x \wedge y'$. This operation satisfies the ***relative DeMorgan identities***:

$$x \setminus (y \vee z) = (x \setminus y) \wedge (x \setminus z) \text{ and } x \setminus (y \wedge z) = (x \setminus y) \vee (x \setminus z).$$

More generally, given any distributive lattice with a maximum 1 and minimum 0, if x and y have complements, then so do $x \vee y$ and $x \wedge y$ with $(x \vee y)' = x' \wedge y'$ and $(x \wedge y)' = x' \vee y'$.

The classic example of a Boolean algebra is the power set algebra $\mathbf{2}^X$ of all subsets of a given set X with $\vee$ and $\wedge$ being $\bigcup$ and $\bigcap$ respectively, and complementation being ordinary set complementation. More generally recall that a ***ring of sets*** is any family $\mathcal{R}$ of subsets of a given set X that is closed under finite unions and finite intersections. $\mathcal{R}$ is a ***field of sets*** if it is also closed under complementation. Before stating the next lemma, recall that an ideal P in a lattice is a ***prime ideal*** if $x\wedge y \in P$ implies that either $x \in P$ or $y \in P$.

Lemma 1.1.11. *Let I be an ideal and F be a filter that are disjoint in a distributive lattice. Then a prime ideal P exists such that* $I \subseteq P$ *but* $F \bigcap P = \varnothing$.

Proof. Let P be an ideal that is maximal subject to the stated conditions. Suppose that $a\wedge b \in P$ for some $a \notin P$ and $b \notin P$. Let P_1 and P_2 be the ideals generated respectively from $P \bigcup \{a\}$ and $P \bigcup \{b\}$. Since they are properly larger than I, P_1 contains an element $p_1\vee a \in F$ and P_2 contains an element $p_2\vee b \in F$ where $p_1, p_2 \in P$. But then F contains

$$(p_1\vee a)\wedge(p_2\vee b) = (p_1\wedge p_2)\vee(p_1\wedge b)\vee(p_2\wedge a)\vee(a\wedge b)$$

which is also in P, a contradiction. Thus a and b do not exist and P is indeed prime. □

This leads us to the first of several fundamental results about Boolean lattices:

Theorem 1.1.12. (M. H. Stone) *A lattice is (distributive) Boolean if and only if it is isomorphic to a (ring) field of sets.*

Proof. The "if" direction is clear. So suppose that a lattice L is distributive. Let $\mathcal{P}$ denote the set of all nonempty prime ideals of L. To each $x \in L$, set $\pi(x) = \{P \in \mathcal{P} \mid x \notin P\}$. It is easily seen that $\pi(x\vee y) = \pi(x) \bigcup \pi(y)$ and $\pi(x\wedge y) = \pi(x) \bigcap \pi(y)$. Moreover, if $x \neq y$ then either $x{\downarrow} \bigcap y{\uparrow} = \varnothing$ or else $y{\downarrow} \bigcap x{\uparrow} = \varnothing$. In either case, by the lemma, a prime ideal P exists containing exactly one of x and y. Hence $x \neq y$ implies $\pi(x) \neq \pi(y)$. Thus π: L $\to 2^{\mathcal{P}}$ is an embedding of distributive lattices.

If L is Boolean lattice, then first $\pi(0) = \varnothing$. Next, in the Boolean case we consider only proper prime ideals. We still have $\pi(x\vee y) = \pi(x) \bigcup \pi(y)$ and $\pi(x\wedge y) = \pi(x) \bigcap \pi(y)$, but now $\pi(1) = \mathcal{P}$. Moreover, for all x, $\pi(x) \bigcap \pi(x') = \pi(x \wedge x') = \pi(0) = \varnothing$. Likewise, $\pi(x) \bigcup \pi(x') = \pi(x \vee x') = \pi(1) = \mathcal{P}$. Thus $\pi(x)$ and $\pi(x')$ are complements in $2^{\mathcal{P}}$ and so $\pi(x') = \mathcal{P} \setminus \pi(x)$. Thus π is an embedding of Boolean algebras. □

Recall that an ***atom*** in a lattice with 0 is an element $a > 0$ such that no element x exists properly between 0 and a. A lattice is ***atomic*** if every element is a supremum of atomic elements. In particular, each x is the supremum of the set $\alpha(x)$ of all atoms lying beneath x. Again, the proofs of the following three results are easily accessible. We give the proof of the third.

Theorem 1.1.13. *A Boolean lattice* L *is isomorphic the power lattice of some set if and only if it is complete and atomic. If the latter holds, then upon denoting the set of all atoms in* L *by* $\mathcal{A}_L$, *one has* $L \cong \mathbf{2}^{\mathcal{A}L}$ *under the map* $x \to \alpha(x)$. □

Proposition 1.1.14. *For all x in a complete Boolean lattice* L *and all subsets* Y *of* L:

(i) $x \wedge sup(Y) = sup(\{x \wedge y \mid y \in Y\})$ *and* $x \vee inf(Y) = inf(\{x \vee y \mid y \in Y\})$.
(ii) $(sup\, Y)' = inf(\{y' \mid y \in Y\})$ *and* $(inf\, Y)' = sup(\{y' \mid y \in Y\})$.
(iii) $x \setminus sup(\{x \setminus y \mid y \in Y\}) = inf(x \setminus Y)$ *and* $x \setminus inf(Y) = sup(\{x \setminus y \mid y \in Y\})$.
(iv) $sup(Y) \setminus x = sup(\{y \setminus x \mid y \in Y\})$ *and* $inf(Y) \setminus x = inf(\{y \setminus x \mid y \in Y\})$. □

Theorem 1.1.15. *Given a Boolean algebra* $(L, \wedge, \vee, 1, 0, ')$, *let* $a \in L$ *be given. Then both* $a{\downarrow}$ *and* $a{\uparrow}$ *are Boolean lattices and* $\chi: L \to a{\downarrow} \times a{\uparrow}$ *defined by* $\chi(x) = (x \wedge a, x \vee a)$ *is an isomorphism of Boolean lattices.*

Proof. χ is at least a lattice embedding by Theorem 1.1.9. Next, let $(u, v) \in a{\downarrow} \times a{\uparrow}$ be given. If $x = (v \setminus a) \vee u$, then

$$a \wedge x \;=\; a \wedge [(v \setminus a) \vee u] \;=\; (a \wedge (v \setminus a)) \vee (a \wedge u) \;=\; 0 \vee u \;=\; u$$

and

$$a \vee x \;=\; a \vee [(v \setminus a) \vee u] \;=\; a \vee (v \wedge a') \vee a \;=\; (a \vee v) \wedge (a \vee a') \;=\; v \wedge 1 \;=\; v.$$

Thus χ is also surjective and the theorem follows. □

2 denotes the Boolean lattice $\{1 > 0\}$. By mild abuse of notation, **2** also denotes the Boolean algebra $(\{1, 0\}; \vee, \wedge, 1, 0, ')$ again with $1 > 0$. Put otherwise, **2** is the chain C_1 reset in a Boolean context. We have the following sequence of easy corollaries of Theorem 1.1.15 and Corollary 1.10.

Corollary 1.1.16. *Every finite Boolean lattice factors as a finite power of* **2**. □

Corollary1.1.17. *The only nontrivial subdirectly irreducible Boolean algebra is* **2**. □

Corollary1.1.18. *Every distributive lattice can be embedded into a Boolean lattice.* □

By a ***generalized Boolean lattice*** is meant a lattice L with a minimal element 0 such that each principal ideal $x{\downarrow}$ of L is a Boolean lattice. Such a lattice is necessarily distributive; moreover a difference operation on L is given by setting $x \setminus y = x \setminus (y \wedge x)$ in the Boolean lattice $x{\downarrow}$. For Boolean lattices both differences agree. The relative DeMorgan identities also hold for generalized Boolean lattices. Upon including \ in the signature, one has a ***generalized Boolean algebra*** $(L, \wedge, \vee, \setminus, 0)$. It is characterized by the identities for a distributive lattice with a minimal element 0 together with the pair: $(x \wedge y) \wedge (x \setminus y) = 0$ and $(x \wedge y) \vee (x \setminus y) = x$.

Every Boolean algebra $(L, \wedge, \vee, 1, 0, ')$ possesses a generalized Boolean algebra reduct $(L, \wedge, \vee, \setminus, 0)$ with $x \setminus y$ given as $x \wedge y'$. Conversely, any generalized Boolean algebra $(L, \wedge, \vee, \setminus, 0)$

possessing a maximal element 1 becomes a Boolean algebra upon setting $x' = 1 \setminus x$. In particular, every generalized Boolean algebra that is complete as a lattice forms a complete Boolean algebra and thus satisfies all the identities of Theorem 1.1.15. Generalized Boolean algebras play a basic role in the study of skew Boolean algebras in Chapter 5, being both the commutative cases of the latter as well their maximal lattice images.

1.2 Bands

A ***band*** is a semigroup S whose elements are idempotent. Thus $x^2 = x$ (in multiplicative notation) for all x in S. A band that is also commutative is called a ***semilattice***. Clearly:

Lemma 1.2.1. *If* S *is a commutative semigroup, then the set* **E**(S) *of all idempotents in* S *forms a semilattice under the semigroup operation*. □

When a semigroup S is not commutative, **E**(S) need not be closed under multiplication. Closure of **E**(S) is obtained, however, with a weakened version of commutativity. A semigroup is ***mid-commutative*** (or ***weakly commutative***) if it satisfies the identity $uxyv = uyxv$. A mid-commutative band is called a ***normal band***. The next result is trivial.

Lemma 1.2.2. *Given a mid-commutative semigroup S, the set of idempotents* **E**(S) *forms a normal band under the given multiplication*. □

Given a band S, several quasi-orders can be defined on S. To begin, the ***natural partial order*** $\geq$ is defined on S by $e \geq f$ if $ef = f = fe$. The natural partial order refines the ***natural quasi-order*** $\succeq$ on S defined by $e \succeq f$ if $fef = f$. Between $\geq$ and $\succeq$ lie the ***left*** and ***right quasi-orders***, $\succeq_{\mathcal{L}}$ and $\succeq_{\mathcal{R}}$ defined by respectively by: $e \succeq_{\mathcal{L}} f$ if $fe = f$ and $e \succeq_{\mathcal{R}} f$ if $ef = f$.

In the lattice of all quasi-orders on the underlying set of S, $\geq$ is the meet (intersection) of $\succeq_{\mathcal{L}}$ and $\succeq_{\mathcal{R}}$, and $\succeq$ is the join of $\succeq_{\mathcal{L}}$ and $\succeq_{\mathcal{R}}$. That $\geq$ is a partial order, and that $\succeq_{\mathcal{L}}$ and $\succeq_{\mathcal{R}}$ are quasi-orders meeting at $\geq$ are easily verified. To see that of $\succeq$ is a quasi-order that is the join of $\succeq_{\mathcal{L}}$ and $\succeq_{\mathcal{R}}$ we will need the equivalences $\mathcal{L} = \succeq_{\mathcal{L}} \cap \succeq_{\mathcal{L}}^{op}$ and $\mathcal{R} = \succeq_{\mathcal{R}} \cap \succeq_{\mathcal{R}}^{op}$. Alternatively, $\mathcal{L}$ and $\mathcal{R}$ are defined by: $e \,\mathcal{L}\, f$ if both $ef = e$ and $fe = f$, and $e \,\mathcal{R}\, f$ if both $ef = f$ and $fe = e$. $\mathcal{L}$ is a ***right congruence*** ($e \,\mathcal{L}\, f \Rightarrow eg \,\mathcal{L}\, fg$ for all $g \in$ S) and $\mathcal{R}$ is a ***left congruence*** ($e \,\mathcal{R}\, f \Rightarrow ge \,\mathcal{L}\, gf$ for all $g \in$ S). E.g., $e \,\mathcal{L}\, f$ implies $egfg = efgfg = efg = eg$ and likewise $fgeg = fg$. We now state:

Lemma 1.2.3. *For any band* S, $\succeq_{\mathcal{L}} \circ \succeq_{\mathcal{R}} = \succeq_{\mathcal{R}} \circ \succeq_{\mathcal{L}} = \succeq$ *and the result is a quasi-order.*

Proof. If $e \succeq_{\mathcal{L}} \circ \succeq_{\mathcal{R}} f$, then for some g, $ge = g$ and $gf = f$. From this $e \succeq_{\mathcal{R}} ef \succeq_{\mathcal{L}} f$ follows. That $e \succeq_{\mathcal{R}} ef$ is clear. That $ef \succeq_{\mathcal{L}} f$ follows from $(ef)f = ef$ and $g(ef) = f$. Thus $\succeq_{\mathcal{L}} \circ \succeq_{\mathcal{R}} \subseteq \succeq_{\mathcal{R}} \circ \succeq_{\mathcal{L}}$. The reverse inclusion is shown in similar fashion. This commuting composition is thus a quasi-order; moreover, it contains $\succeq$. Indeed, from $e \succeq f$ we obtain $e \succeq_{\mathcal{R}} ef \succeq_{\mathcal{L}} fef = f$, so that $\succeq$ lies in $\succeq_{\mathcal{R}}$

o $\succeq_{\mathcal{L}}$. On the other hand $\succeq_{\mathcal{R}}$ o $\succeq_{\mathcal{L}} \subseteq \succeq$. For suppose $e \succeq_{\mathcal{R}} f \succeq_{\mathcal{L}} g$. But then $f\ \mathcal{R}\ fe$ and thus $g = gf\ \mathcal{R}\ gfe = ge$. It follows that $geg = g$ so that $e \succeq g$. □

$\mathcal{L}$ and $\mathcal{R}$ were originally defined for arbitrary semigroups by J. A. Green, who showed that $\mathcal{L}$o$\mathcal{R}$ = $\mathcal{R}$o$\mathcal{L}$, with the resulting composition also being an equivalence denoted by $\mathcal{D}$. (The proof in the case of bands is similar to that for $\succeq_{\mathcal{L}}$ and $\succeq_{\mathcal{R}}$.) Alternatively, $\mathcal{D} = \succeq \cap \succeq^{\mathrm{op}}$. In the lattice of all equivalences on the underlying set of S, $\mathcal{D}$ is the join of $\mathcal{R}$ and $\mathcal{L}$, while the meet $\mathcal{R}\cap\mathcal{L}$ is the equality relation Δ. To understand the role of these equivalences in the structure of bands we turn to the class of bands that form the "anti-semilattices" amongst bands.

A ***rectangular band*** is a band S satisfying the identity $xyx = x$, or equivalently, the identity $xyz = xz$. (Given the former identity, $xyz = xyzxz = xz$.) Rectangular bands thus form a variety of bands for which $xy = yx$ iff $x = y$. Examples are ***left zero bands*** having the multiplication $xy = x$ and ***right zero bands*** having the dual multiplication $xy = y$. Given a left zero band L and a right zero band R, their direct product L × R is also a rectangular band. The generality of such an example is demonstrated as follows.

Theorem 1.2.4. *Given a rectangular band* S *with* $e \in$ S, L = $\{se \mid s \in$ S$\}$ *is a maximal left zero band in* S *and* R = $\{es \mid s \in$ S$\}$ *is a maximal right zero band in* S; *moreover the map* μ: L × R → S *given by* $\mu(x, y) = xy$ *is an isomorphism*. *Finally*, L *is the* $\mathcal{L}$*-class of* e, R *is the* $\mathcal{R}$*-class of* e *and* S *forms a single* $\mathcal{D}$*-class*.

Proof. Fixing e, since S is rectangular, $sete = se$ so that L is at least a left zero semigroup in S. If L′ were any left zero band in S such that $e \in$ L′, then for all $s \in$ L′, $s = se$ so that $s \in$L. Thus L′ ⊆ L and L is indeed a maximal left zero band in S. Clearly such a subset must be an $\mathcal{L}$-class of S. Similarly, R is a stated. Finally, consider function μ: L × R → S. Viewing L × R as a direct product of bands and applying the defining identity of a rectangular band we get:

$$\mu((re, es)(te, eu)) = \mu((rete, eseu)) = \mu(re, eu) = reu = ru = resteu = \mu(re, es)\mu(te, eu)$$

so that μ is a homomorphism. Since $s = (se)(es)$ for all $s \in$S, μ maps L × R onto S. Finally, let $s = \mu(ue, ev) = uev$. Then $se = ueve = ue$ and $es = euev = ev$ so that $(ue, ev) = (se, es)$. Thus μ one-to-one also and an isomorphism. □

A rectangular band can be represented as a rectangular array with $\mathcal{R}$-equivalent elements comprising the rows and $\mathcal{L}$-equivalent elements comprising the columns.

$$\begin{bmatrix} e & f & g \\ i & j & k \end{bmatrix}$$

Given elements x and y, the product xy is the unique element in the row ($\mathcal{R}$-class) of x and the column ($\mathcal{L}$-class) of y. Thus $ek = g$ while $ke = i$ in this array.

Returning to the broader context of arbitrary bands, we have

Corollary 1.2.5. *Given any band* S, *its* $\mathcal{L}$*-classes are the maximal left zero sub-bands in* S *and its* $\mathcal{R}$*-classes are the maximal right zero sub-bands in* S. □

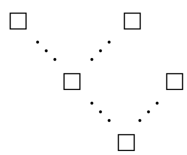
Theorem 1.2.6. (Clifford-McLean) *Given a band* S, *the equivalence* $\mathcal{D}$ *is a congruence on* S. *Its congruence classes form maximal rectangular sub-bands of* S *and the quotient algebra* $S/\mathcal{D}$ *is the maximal semilattice image of* S.

Proof. If $e\,\mathcal{L}\,f$ in S, then in general, $ev\,\mathcal{L}\,fv$ for all v in S. Thus for all u, v in S,

$$(uev)(ufv)(uev) = uevufvuev = uev(fv)ufvuev = uevfvuev = uevuev = uev$$

and similarly $(ufv)(uev)(ufv) = ufv$. Thus $e\mathcal{L}f$ implies $uev\,\mathcal{D}\,efv$ for all u, v in S. In like fashion, $e\mathcal{R}f$ implies $uev\,\mathcal{D}\,efv$ for all u, v in S. Suppose that $e\,\mathcal{D}\,f$ in S. Then $e\,\mathcal{R}\,ef\,\mathcal{L}\,f$ and thus for all u, v in S, $uev\,\mathcal{D}\,uefv\,\mathcal{D}\,ufv$. $\mathcal{D}$ is thus a congruence. Since $ef\,\mathcal{D}\,fe$ for all e, f in S, S/$\mathcal{D}$ is commutative and hence a semilattice. Since each $\mathcal{D}$-class is a maximal subset of S satisfying $xyx = x$, it is a maximal rectangular subalgebra of S. □

In brief, *every band is a semilattice of rectangular bands*. Thus a band has the appearance of a semilattice diagram with each node filled in by a rectangular band.



While multiplication is performed in rectangular fashionwithin $\mathcal{D}$-classes, multiplication between elements from distinct $\mathcal{D}$-classes is another matter. We can, however, be more specific in this regard for the two subvarieties that we consider next.

Unlike $\mathcal{D}$, the relations $\mathcal{L}$ and $\mathcal{R}$ need not be congruences. A band for which $\mathcal{L}$ and $\mathcal{R}$ are full congruences is called ***regular***. Since $\mathcal{D}$ is a congruence, whenever $\mathcal{D} = \mathcal{L}$ (so that $\mathcal{R} = \Delta$) or $\mathcal{D} = \mathcal{R}$ (so that $\mathcal{L} = \Delta$) the band must be regular. When $\mathcal{D} = \mathcal{L}$ the band is called ***left regular*** and when $\mathcal{D} = \mathcal{R}$ it is called ***right regular***. It is both precisely when it is a semilattice. A normal, left (right) regular band is called ***left*** **(*right*)** ***normal*** band.

Theorem 1.2.7. *A band is left regular* [*right regular*] *if and only if* $xyx = xy$ [$xyx = yx$]. *In general, a band is regular, if and only if it satisfies the identity* $xyxzx = xyzx$.

Proof. Given x, y in a band S, $xyx \mathcal{R} xy$. Conversely, given $x \mathcal{R} y$ in S, $x = xyx$ and $y = xy$. Thus S is left regular ($\mathcal{R} = \Delta$ and $\mathcal{D} = \mathcal{L}$) precisely when $xyx = xy$ holds on S. Dually S is right regular ($\mathcal{L} = \Delta$ and $\mathcal{D} = \mathcal{R}$) precisely when $xyx = yx$ holds on S. If S is regular, then consider the canonical epimorphisms $S \to S/\mathcal{L}$ and $S \to S/\mathcal{R}$. From the rectangular structure of $\mathcal{D}$-classes of S, it is follows that $S/\mathcal{L}$ is right regular and $S/\mathcal{R}$ is left regular. Thus both bands satisfy either $xyx = xy$ or $xyx = yx$ and hence the identity $xyxzx = xyzx$. Since the two epimorphisms induce an embedding of S into $S/\mathcal{L} \times S/\mathcal{R}$ which satisfies $xyxzx = xyzx$, so does S. Conversely, let S satisfy $xyxzx = xyzx$. Suppose $u \mathcal{L} v$ in S, so that $uv = v$ and $vu = v$. Then for all w, $(uw)(vw) = uvwuv = uvw = uw$ and likewise $(vw)(uw) = vw$. The assumed identity then gives us $(wu)(wv) = wuwvu = wuw\underline{u}vu = wuwu = wu$ and likewise $(wv)(wu) = wv$ showing that $\mathcal{L}$ is a indeed congruence. In similar fashion $\mathcal{R}$ is seen to be a congruence. ☐

Corollary 1.2.8. *A band* S *being regular is equivalent to either of the following*:

(i) *Given* $e \succeq a, b$ *in* S, $aeb = ab$.

(ii) S *satisfies* $xyx'zx'' = xyzx''$, *given* $x', x'' \mathcal{D} x$.

Proof. If (i) holds, then $xyxzx = (xy)x(zx) = xyzx$ follows and S is regular. Conversely, if S is regular and $e \succeq a, b$ in S, $xyxzx = xyzx$ gives us (i): $aeb = aebaeb = ae\underline{a}ba\underline{b}eb = aababb = ab$. Clearly S is regular if (ii) holds. Conversely, (i) implies $xyx'zx'' = (xy)x'(zx'') = xyzx''$. ☐

The function ζ: $S \to S/\mathcal{L} \times S/\mathcal{R}$ defined by $\zeta(x) = (\mathcal{L}_x, \mathcal{R}_x)$ is always 1-1 for any band. The product $S/\mathcal{L} \times S/\mathcal{R}$ is naturally a band and ζ is a homomorphism (and thus a monomorphism) precisely when S is regular. In this case, the image $\zeta[S]$ is the fibered product $S/\mathcal{L} \times_{S/\mathcal{D}} S/\mathcal{R}$ of $S/\mathcal{L}$ with $S/\mathcal{R}$ over the common maximal semilattice image, $S/\mathcal{D}$. Thus the following commuting diagram of natural epimorphisms is a pullback. The isomorphism $S \cong S/\mathcal{R} \times_{S/\mathcal{D}} S/\mathcal{L}$ is called the *Kimura factorization*, after its discoverer, Naoki Kimura.

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S/\mathcal{L} \arrow[d] \\ S/\mathcal{R} \arrow[r] & S/\mathcal{D} \end{tikzcd}$$

Corollary 1.2.9. *Every normal band is regular. Normal bands are also characterized by the identity* $xyzx = xzyx$. *In particular, normal left [right] regular bands are characterized by the identity* $uxy = uyx$ [$xyu = yxu$].

Proof. Normal bands clearly satisfy this identity, and any band S satisfying this identity is regular: $xyxzx = xxzyx = xzyx = xyzx$. But a left regular band satisfies this identity it and only if it satisfies $xyz = xzy$, thus making it normal. Likewise a right regular band satisfies this identity if and only if it satisfies $yzx = zyx$, also making it normal. Since any regular band S can be embedded in $S/\mathcal{R} \times S/\mathcal{L}$, with the latter being normal if and only if S is, the corollary follows.

The remainder of this section is devoted to assorted further remarks about bands. We begin by defining four canonical sub-bands arising for each e in a band S.

$$e\!\downarrow = \{f \in S \mid f \leq e\}, \quad e\!\Downarrow = \{f \in S \mid f \preceq e\}, \quad {}_{\mathcal{L}}\!\Downarrow\! e = \{f \in S \mid f \,{}_{\mathcal{L}}\!\preceq e\} \quad \text{and} \quad e\!\Downarrow_{\mathcal{R}} = \{f \in S \mid f \,{}_{\mathcal{R}}\!\preceq e\}.$$

Lemma 1.2.10. *Given a normal band* S, *for each* $e \in S$, *under the given operation* $e\!\downarrow$ *is a semilattice; conversely, every band satisfying this property is normal.*

Proof. If S is normal and $f, g \leq e$ in S, then $fg = efge = egfe = gf$. Thus $e\!\downarrow$ is a commutative sub-band of S, that is, a semilattice in S. Conversely, if each $e\!\downarrow$ is commutative, then S at least satisfies the identity $xyxzx = xzxyx$. From this we derive the identity $xyzyxzyzx = xzyzxyzyx$. But since $xyzy$, $zyzx$ and yxz lie in the same $\mathcal{D}$-class, $xyzyxzyzx$ reduces to $xyzyzx = xyzx$. Similarly, $xzyzxyzyx$ reduces to $xzyx$ and $xyzx = xzyx$ follows. ☐

Given $\mathcal{D}$-classes A, B in a band S we write $A \geq B$ if $a \succeq b$ for any (and hence all) pairs $a \in A$ and $b \in B$. When $A \geq B$ but $A \neq B$, we write $A > B$. This reflects, of course, what occurs between the corresponding elements in the underlying lattice $S/\mathcal{D}$. When $A \geq B$ we say that A and B are ***comparable*** $\mathcal{D}$-classes. The following results provide an explicit description of the architecture of a normal band.

Lemma 1.2.11. *Given* $\mathcal{D}$*-classes* $A \geq B$ *in a normal band* S, *for each* $a \in A$ *exactly one* $b \in B$ *exists such that* $a \geq b$. *The function* α: $A \to B$ *determined by* $\alpha(a) = b$ *if* $a \geq b$ *is a homomorphism of rectangular bands; moreover* $\alpha(a) = aba$ *for all* $b \in B$. *Conversely, if* $\geq$ *induces functions in this manner between all pairs of comparable* $\mathcal{D}$*-classes of a band S, then S is normal.*

Proof. Given $a \in A$ and $b \in B$ for comparable $\mathcal{D}$-classes $A \geq B$ in a normal band S, observe that $a \geq aba$ in B. Given $b' \in B$ also, then $bb'b = b$ and $b'bb' = b'$ in B. Normality give us

$$aba = abb'ba = ab'ba = ab'b'ba = ab'bb'a = ab'b'a = ab'a.$$

Thus the procedure $a \to aba$ induces a well-defined map from A to B such that $a \geq aba$ with aba being independent of $b \in B$. Since $b = aba$ whenever $a \geq b$, the first statement of the lemma follows. That α is a homomorphism follows immediately from S being normal: if $a, a' \in A$ and $b \in B$, then $abaa'ba' = aa'bbaa'\ aa'baa'$. Conversely, suppose that $\geq$ always induces a function between comparable pairs of $\mathcal{D}$-classes of a band S in the above manner. Since $xyzx\ \mathcal{D}\ xzyx$ with $x \geq xyzx$ and $x \geq xzyx$ for all $x, y, z \in S$, this assumption gives $xyzx = xzyx$ in S. S is thus normal by Corollary 1.2.9. ☐

Lemma 1.2.12. *For all comparable $\mathcal{D}$-classes* A ≥ B *in a normal band* S, *if* $\alpha^{A}_{B}: A \to B$ *is the homomorphism determined for all* $a \in A$ *by* $a \geq \alpha^{A}_{B}(a) \in B$, *then*:

i) $\alpha^{A}_{A} = id_A$ *for all $\mathcal{D}$-classes* A *in* S, *and*

ii) $\alpha^{B}_{C} \circ \alpha^{A}_{B} = \alpha^{A}_{C}$ *for all comparable $\mathcal{D}$-classes* A ≥ B ≥ C *in* S.

iii) *Given* $a \in A$ *and* $b \in B$, $ab = \alpha^{A}_{M}(a)\,\alpha^{B}_{M}(b)$ *in the meet-class* M *of* A *and* B.

Proof. (i) is just $aa'a = a$ for all $a' \in A$. (ii) states that $(aba)c(aba) = aca$ for all $a \in A$, $b \in B$ and $c \in C$, a trivial consequence of normality: $abacaba = acbcacacbca = acccccca = aca$. Finally, (iii) follows from the identity $ab = a(ab)ab(ba)b$ holding in all bands. ☐

This leads to the following result of Naoki Kimura and Miyuki Yamada.

Theorem 1.2.13. *Let* T *be a meet semilattice poset* (T; ≥). *Assign a rectangular band* N_a *to each* $a \in T$ *and a homomorphism* $N^{a}_{b}: N_a \to N_b$ *to each pair* $a \geq b$ *in* T, *such that:*

i) *Given* $a \neq b$ *in* T, N_a *and* N_b *are disjoint*.

ii) N^{a}_{a} *is the identity map on* N_a *for all* $a \in T$.

iii) $N^{b}_{c} \circ N^{a}_{b} = N^{a}_{c}$ *for all* $a \geq b \geq c$ *in* T.

Given $x \in N_a$ *and* $y \in N_b$, set $xy = N^{a}_{ab}(x)\, N^{b}_{ab}(y)$ *in* N_{ab}. *Then* $S = \cup_{a \in T} N_a$ *is a normal band with maximal rectangular sub-bands being the* N_a, *with maximal semilattice image being* T *and the canonical homomorphism* τ: S → T *defined by* $\tau(x) = a$ if $x \in N_a$. *Conversely, every normal band arises in this fashion*.

Proof. It is easily verified that the above construction produces a band such that $x \geq y$ for $x \in N_a$ and $y \in N_b$ holds precisely when $a \geq b$ in T and $N^{a}_{ab}(x) = y$. It follows from Lemma 1.2.11 that S is indeed a normal band. The converse assertion follows from Lemma 1.2.12. ☐

What is the picture in the case of regular bands? Or is there one? To begin, given comparable $\mathcal{D}$-classes A ≥ B in a band S and $b \in B$, the set $AbA = \{aba' \mid a, a' \in A\}$ is a ***coset*** of A in B. Similarly the ***left coset*** is $Ab = \{ab \mid a \in A\}$ and the ***right coset*** is $bA = \{ba \mid a \in A\}$.

Proposition 1.2.14. *Let* S *be regular band, with comparable $\mathcal{D}$-classes* A > B. *Then*:

i) *For all* $b, c \in B$, *either* $AbA = AcA$ *or* $AbA \cap AcA = \varnothing$.

ii) *In particular, if* $AbA = AcA$, *then* $aba' = aca'$ *for all* a, a' *in* A.

iii) B *is partitioned into a disjoint union of nonempty subsets* B_i, *each consisting of all* $b \in B$ *inducing the same coset* AbA *of* A in B. *In particular, all* $c \in AbA$ *induce the coset* AbA, *so that* $AbA \subseteq B_i = \{c \in B \mid AcA = AbA\}$.

iv) *For each* $a \in A$ *and each partition cell* B_i, *a unique* $b_i \in B_i$ *exists such that* $a \geq b_i$. *For any* $b \in B_i$, *aba is this* b_i *and* $bab = b$.

Proof. To begin, suppose $AbA \cap AcA \neq \varnothing$ for some $b, c \in C$. Thus $a_1ba_2 = a_3ca_4$ for some a_1, $a_2, a_3, a_4 \in A$. Corollary 1.2.8 then implies that for all a, a' in A,

$$a(a_1ba_2)a' = (aa_1a)b(a'a_2a') = aba'$$

and likewise $a(a_3ca_4)a' = aca'$, so that $aba' = aca'$. From this (i) - (iii) follow. To see (iv), clearly $a \geq aba$ in the cell containing AbA. Assuming $b \in B_i$, then $aca = aba$ for all $c \in B_i$, so that aba is this unique $b_i \leq a$ in B_i. That $bab = b$ follows from $a \succ b$. ☐

Given A > B and cell B_i in B, B_i is called an **A-*cell*** in B and α_i: A → B_i is defined by $\alpha_i(a) = aba$ for all $a \in$A and *any* $b \in B_i$ is the ***cell-map*** of A into B_i. Its image in B is the coset AbA, since by Corollary 1.2.8 again, $aba' = aa'baa'$ for all a, $a' \in$ A and $b \in$ B.

Application. All left regular bands with just two $\mathcal{D}$-classes A > B are constructed as follows.

(1) Given sets A and B, partition B into disjoint nonempty subsets $\{B_i \subseteq B \mid i \in I\}$.

(2) For each cell B_i of the partition, choose a function α_i: A → B_i.

(3) Define multiplication on S = A ∪ B by first imposing left zero multiplications on A and B separately, setting $ba = b$ for all $a \in$ A and $b \in$ B, and finally by setting $ab = \alpha_i(a)$ if $b \in B_i$.

X In the resulting band S = A ⋃ B, the cell decomposition of B is the given partition $B = \cup B_i$ and the cell-maps from A to B are precisely the α_i: A → $B_i \subseteq$ B for each $i \in$ I.

Cell decompositions and cell-maps also determine the multiplication of elements in incomparable $\mathcal{D}$-classes in a left regular band. But first a definition: given $\mathcal{D}$-classes A > B in a band with $a \in$A, the ***image*** of a in B is the set $\{b \in B \mid a \geq b\}$. Put otherwise, this image is aBa = $\{aba \mid b \in B\}$, or when S is left regular, aB = $\{ab \mid b \in B\}$. In the next situation, both $\mathcal{D}$-classes A and B can be incomparable

Proposition 1.2.15. *Given a left regular band* S *and* $\mathcal{D}$*-classes* A *and* B*, with meet* $\mathcal{D}$*-class* M, *and with* $a \in$A *and* $b \in$B, *then*:

(1) *The image* bMb of b *in* M *lies in a unique cell* M_i *of the* A-*decomposition of* M.

(2) *If* α_i: A → $M_i \subseteq$ M *is the corresponding cell-map, then* $ab = \alpha_i(a)$.

(3) *Dual remarks hold for the image class of* a *in* M *and the product* ba.

Conversely, given two left regular bands consisting of $\mathcal{D}$-classes A > M *and* B > M *respectively*, *such that* A ∩ B = ∅, with M *the same for both bands with* A > M *and* B > M *satisfying* (1) – (3), *then the multiplications on each band extend uniquely to a left regular multiplication* on their union A ∪ B ∪ M *such that* M *is the meet class of* A *and* B. (Dual constructions and observations exist for the right regular case. The general case follows using fibered products.)

Proof. Given $b > b' \in$M, $ab' = a(bb') = (ab)b' = ab$. Thus all such $b' < b$ lie in the same cell of the A-decomposition of M and (1) – (3) are now clear. For the second part, given (1) – (3), one needs to show that all possible multiplications of abm (and bam), amb (and bma), mab (and mba),

$aa'b$ (and $bb'a$), aba' (and bab') and baa' (and abb') are unambiguous. We consider only the non-parenthesized cases, of which the amb, mab and baa' cases are trivial.

abm: Since bm is in the A-cell in M containing all images of b in M, $(ab)m = ab = a(bm)$.
$aa'b$: Since ab and $a'b$ lie in a common A-cell in M, $(aa')b = ab$ and $a(a'b)$ are images of a in this cell and so are equal.
aba': Note that ba' is an image of b in M lying in the unique A-cell of M containing all images of B in M. Thus $(ab)a' = ab = a(ba')$. ☐

We consider next a class of necessarily regular bands for which architectural issues are simplified. A band S satisfies the ***class covering condition*** (CCC) if for every comparable pair of $\mathcal{D}$-classes A ≥ B in S, and every $b \in$ B an $a \in$ A exists such that $a \geq b$. Every $b \in$ B is thus "covered" by some a in A. The free left regular band on $\{a, b, c\}$ does not satisfy the CCC.

Theorem 1.2.16. *In a band* S *satisfying the class covering condition, the following hold:*

i) S *is regular*.
ii) *An element x lies in the center of* S *if and only if* $\mathcal{D}_x = \{x\}$
iii) *Given* $x, y \in$S, $\mathcal{D}_{xy}$ *is* $\mathcal{D}_x\mathcal{D}_y = \{uv \mid u \in \mathcal{D}_x \ \&\ v \in \mathcal{D}_y\}$.
iv) *In particular, when* $\mathcal{D}_x$ *and* $\mathcal{D}_y$ *are finite, so is* $\mathcal{D}_{xy}$.

Proof. (i) Given $x, y, z \in$ S, by the CCC, $u, v \in \mathcal{D}_x$ exists such that $u \geq xy$ and $v \geq zx$. Since $uvw = uw$ holds in $\mathcal{D}_x$ we obtain: $xyzx = (xyu)(vzx) = xy(uxv)zx = (xyu)x(vzx) = xyxzx$. Thus S is regular. (ii) If x commutes with all elements in S, then clearly $\mathcal{D}_x = \{x\}$. Conversely, let $\mathcal{D}_x = \{x\}$ and let $y \in$ S be given. By the CCC, $x \geq xy$ and hence $xy = xyx$. Similarly, $x \geq yx$ so that $yx = xyx$. Thus $xy = yx$ and x lies in the center of S. (iii) Clearly $\mathcal{D}_x\mathcal{D}_{\mathrm{y}} \subseteq \mathcal{D}_{\mathrm{xy}}$. Let z in $\mathcal{D}_{\mathrm{xy}}$ be given. Since $\mathcal{D}_x \geq \mathcal{D}_{\mathrm{xy}}$ and $\mathcal{D}_{\mathrm{y}} \geq \mathcal{D}_{\mathrm{xy}}$, by the CCC, $u \in \mathcal{D}_x$ and $v \in \mathcal{D}_y$ exist such that $u \geq z$ and $v \geq z$. Thus $z = uvzuv = uv$ since both z and uv lie in $\mathcal{D}_{xy}$. (iv) is now clear. ☐

The situation regarding cell-maps in the CCC case is much cleaner.

Theorem 1.2.17. *In a band* S *satisfying the class covering condition, the following hold:*

i) *Given* $\mathcal{D}$*-classes* A > B, *the* A-*cells in* B *are precisely the cosets*, AbA.
ii) B *is partitioned by the cosets of* A *in B and the cell-maps from* A *to* B *are collectively surjective in that* B = $\bigcup_{b\in \mathrm{B}}$AbA.
iii) *Given a* $\mathcal{D}$*-class chain* A > B > C, *all compositions of cell-maps from* A *to* B *with cell-maps from* B *to* C *are cell-maps from* A *to* C.
iv) *Conversely, all cell-maps from* A *to* C *are obtained in this fashion*.

Proof. (i) Given $b \in$ B, by the CCC, $a \in$ A exists such that $a \geq b$. Hence $b = aba \in$ AbA and the cell containing b collapses to AbA. (ii) is now clear. (iii) Next, given A > B > C as stated, consider the cell-maps φ: A → Ab_0A ⊆ B and ψ: B → Bc_0B ⊆ C defined by φ(a) = ab_0a and ψ(b) = bc_0b for b_0 ∈ B and c_0 ∈ C. Then for all a ∈ A, ψoφ(a) = $ab_0ac_0ab_0a$ = = $ab_0c_0b_0a$ by Corollary 1.2.8. Thus ψoφ: A → C is a cell-map form A to the cell of $b_0c_0b_0$ in C. Finally, given c ∈ C let ζ: A → C be cell-map defined by ζ(a) = aca. By CCC again, b_0 ∈ B exists such that $b_0 \geq c$. By Lemma 1.2.8 again, for all a ∈ A, aca = ab_0cb_0a = ab_0acab_0a. Thus, if φ: A → Ab_0A ⊆ B and ψ: B → BcB ⊆ C are the cell-maps φ(a) = ab_0a and ψ(b) = bcb, then ψoφ is precisely ζ: A → C. □

The above need not hold in all regular bands, as is seen in $\mathcal{LR}eg_{\{a,\, b,\, c\}}$. The cell-map from {$a$} to the bottom $\mathcal{D}$-class in $\mathcal{LR}eg_{\{a,\, b,\, c\}}$ sending a to abc does not factor through the intermediate class {ac, ca}. Likewise the cell-map sending a to acb does not factor though the class {ab, ba}.

Proposition. 1.2.18. *The class of all bands satisfying the class covering condition are closed under products and homomorphic images.*

Proof. Closure under products is clear. Suppose that band S satisfies the CCC and that f: S → T is a homomorphism. Let $a \succeq b$ in f[S] with $a = f(x)$ and $b = f(y)$ for x, y in S. Then $x \succeq yxy$ in S with $f(yxy) = bab = b$ in T. By CCC, $x' \in \mathcal{D}_x$ exists such that $x' \geq yxy$. Clearly $a' = f(x') \geq b$ and $a' \in \mathcal{D}_a$. □

Thus bands satisfying the class covering condition do not form a variety. This condition holds, however, for both band reducts (S, ∧) and (S, ∨) of any skew lattice (S, ∨, ∧). And skew lattices *do* form a variety.

1.3 Noncommutative lattices – initial observations

In general, a ***noncommutative lattice*** is an algebra (N: ∨, ∧) where both ∨ and ∧ are associative, idempotent binary operations satisfying a specified set of absorption identities. We continue to call ∨ the ***join*** and ∧ the ***meet***. The adjective "noncommutative" is used here in the inclusive sense of "not-necessarily-commutative". Thus lattices will play an important role in the general study of noncommutative lattices. Thus far, nearly all types of noncommutative lattices that have been studied assume absorption identities from among the following:

B_1: $a \wedge (a \vee b) = a$.	C_1: $a \vee (a \wedge b) = a$.
B_2: $(b \vee a) \wedge a = a$.	C_2: $(b \wedge a) \vee a = a$.
B_3: $a \wedge (b \vee a) = a$.	C_3: $a \vee (b \wedge a) = a$.
B_4: $(a \vee b) \wedge a = a$.	C_4: $(a \wedge b) \vee a = a$.

If ∨ and ∧ are commutative, then clearly the B's merge together as do the C's. In general we require that ∨ and ∧ satisfy at least a pair of identities that reduce to B_1 and C_1 when ∨ and ∧ are commutative. We consider several classes of such algebras.

The noncommutative lattices studied most extensively in recent years are ***skew lattices*** that satisfy B_1, B_2, C_1 and C_2. We will see that these identities express the dualities:

$$x \vee y = x \text{ if and only if } x \wedge y = y \text{ and } x \vee y = y \text{ if and only if } x \wedge y = x.$$

(The equivalence of the absorption identities with the stated dualities will be shown shortly.) Likewise, ***skew* lattices*** satisfy the complementary set of identities, B_3, B_4, C_3 and C_4, and are characterized by the pair of dualities: $x{\vee}y = x$ if and only if $y{\wedge}x = y$ and $x{\vee}y = y$ if and only if $y{\wedge}x = x$. Either type of algebra is transformed into the other by replacing $\wedge$ by $\wedge^*$ or by replacing $\vee$ by $\vee^*$ where $a \wedge^* b = b \wedge a$ and $a \vee^* b = b \vee a$. A double replacement yields the original type of algebra. Hence we choose to focus on skew lattices. A detailed study of skew lattices will commence in the following chapter. Two further classes of noncommutative lattices are:

A ***quasilattice*** is a noncommutative lattice satisfying the two-sided absorption identities

$$B_5\text{: } a \wedge (b \vee a \vee b) \wedge a = a. \qquad C_5\text{: } a \vee (b \wedge a \wedge b) \vee a = a.$$

We will see that these identities expresses the duality, $x \wedge y \wedge x = x$ if and only if $y \vee x \vee y = y$, stating that $x \preceq y$ under $\wedge$ if and only if $y \preceq x$ under $\vee$.

A ***paralattice*** is a noncommutative lattice satisfying the absorption identities

$$B_6\text{:} \quad a \wedge (a \vee b \vee a) = a = (a \vee b \vee a) \wedge a.$$
$$C_6\text{:} \quad a \vee (a \wedge b \wedge a) = a = (a \wedge b \wedge a) \vee a.$$

These identities express the duality, $x \wedge y = x = y \wedge x$ if and only if $x \vee y = y = y \vee x$, stating that $x \leq y$ under $\wedge$ if and only if $y \leq x$ under $\vee$.

While B5 – C6 above are not among the previous absorption identities, if combined with flatness (see below), they reduce to identities on the earlier list. (See Theorem 1.3.7 below.)

Quasilattices that are also paralattices are called ***refined quasi-lattices***. Skew lattices and skew* lattices are both refined quasi-lattices. Another significant class of examples is as follows. In Section 3.4 we will see that refined quasi-lattices are closely related to skew* lattices.

An ***antilattice*** is an algebra $(N, \vee, \wedge)$ with associative, idempotent binary operations $\vee$ and $\wedge$ such that both $a \wedge b \wedge a = a$ and $a \vee b \vee a = a$. Lattices and antilattices form antipodal classes of examples having foundational import in the study of noncommutative lattices. Thanks to the simple behavior of rectangular bands, however, antilattices are easily described. A pair of double indexing $\{x_{(\lambda, \rho)} \mid (\lambda, \rho) \in L \times R\}$ and $\{x_{(\lambda^*, \rho^*)} \mid (\lambda^*, \rho^*) \in L^* \times R^*\}$ of the elements of N exist such that:

$$x_{(\lambda, \rho)} \vee x_{(\mu, \sigma)} = x_{(\lambda, \sigma)} \text{ and } x_{(\lambda^*, \rho^*)} \wedge x_{(\mu^*, \sigma^*)} = x_{(\lambda^*, \sigma^*)}.$$

If N is finite, it can be exhibited as a pair of rectangular arrays. Consider the example:

$$\begin{array}{cccc} & a & b & c \\ (\vee) & d & e & f \\ & g & h & i \end{array} \qquad\qquad \begin{array}{cccc} & b & c & f \\ (\wedge) & a & e & i \\ & d & g & h \end{array}$$

The result of either $x \vee y$ or $x \wedge y$ is the element lying in the row of x and the column of y in the relevant array. Thus $a \vee f = c$ while $a \wedge f = i$. In this example, none of the B_i or C_i are satisfied for $i \leq 4$. In particular, we have neither a skew lattice nor a skew* lattice.

If both arrays coincide, then $x \vee y = x \wedge y$ and N is a skew* lattice. If they are *transposes* (so that L = R* and R = L*), then $x \vee y = y \wedge x$ and N is a skew lattice. In both special cases only the $\wedge$-array is needed. (In the transpose case, one has $x_{(\lambda,\rho)} \wedge x_{(\mu,\sigma)} = x_{(\mu,\rho)}$.) In general, both arrays are needed and all we can assert is that N is a refined quasi-lattice.

We return to the first eight absorption identities. Initially, each pair (B_i, C_i) is a dual pair in that either is obtained from the other by switching $\vee$ with $\wedge$. But other forms of duality exist.

Lemma 1.3.1. *Given* (N, $\vee$, $\wedge$) *where both* $\vee$ *and* $\wedge$ *are associative and idempotent*:

B_1 - $a \wedge (a \vee b) = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\vee)}\!\succeq_{\mathcal{R}}\; y$ implies $x \;{}_{\mathcal{L}}\!\preceq_{(\wedge)}\; y$.

B_2 - $(b \vee a) \wedge a = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\vee)}\!\succeq_{\mathcal{L}}\; y$ implies $x \;{}_{\mathcal{R}}\!\preceq_{(\wedge)}\; y$.

B_3 - $a \wedge (b \vee a) = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\vee)}\!\succeq_{\mathcal{L}}\; y$ implies $x \;{}_{\mathcal{L}}\!\preceq_{(\wedge)}\; y$.

B_4 - $(a \vee b) \wedge a = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\vee)}\!\succeq_{\mathcal{R}}\; y$ implies $x \;{}_{\mathcal{R}}\!\preceq_{(\wedge)}\; y$.

C_1 - $a \vee (a \wedge b) = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\wedge)}\!\succeq_{\mathcal{R}}\; y$ implies $x \;{}_{\mathcal{L}}\!\preceq_{(\vee)}\; y$.

C_2 - $(b \wedge a) \vee a = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\wedge)}\!\succeq_{\mathcal{L}}\; y$ implies $x \;{}_{\mathcal{R}}\!\preceq_{(\vee)}\; y$.

C_3 - $a \vee (b \wedge a) = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\wedge)}\!\succeq_{\mathcal{L}}\; y$ implies $x \;{}_{\mathcal{L}}\!\preceq_{(\vee)}\; y$.

C_4 - $(a \wedge b) \vee a = a$ - *asserts that for* $x, y \in$ N, $x \;{}_{(\wedge)}\!\succeq_{\mathcal{R}}\; y$ implies $x \;{}_{\mathcal{R}}\!\preceq_{(\vee)}\; y$.

Thus, we have the following pairs inducing converse implications:

B_1 *and* C_2, $\quad$ B_2 *and* C_1, $\quad$ B_3 *and* C_3 $\quad$ *and* $\quad$ B_4 *and* C_4. ☐

Theorem 1.3.2. (Laslo [1997]) *An algebra* (N, $\vee$, $\wedge$) *where* $\vee$ *and* $\wedge$ *are associative, idempotent binary operations is a lattice if and only if* B_1, B_2, C_3 *and* C_4 *all hold or the complementary set* B_3, B_4, C_1 *and* C_2 *all hold*. (Thus absorption can imply commutativity.)

Proof. B_1, B_2, C_3 and C_4 yield $x \;{}_{(\vee)}\!\succeq_{\mathcal{R}}\; y \Rightarrow x \;{}_{\mathcal{L}}\!\preceq_{(\wedge)}\; y \Rightarrow x \;{}_{(\vee)}\!\succeq_{\mathcal{L}}\; y \Rightarrow x \;{}_{\mathcal{R}}\!\preceq_{(\wedge)}\; y \Rightarrow x \;{}_{(\vee)}\!\succeq_{\mathcal{R}}\; y$. Thus, for each operation, both $\succeq_{\mathcal{R}}$ and $\succeq_{\mathcal{L}}$ reduce to $\geq$ and both natural partial orders dualize each other and we have a lattice. The case for B_3, B_4, C_1 and C_2 is similar. The converse is clear. ☐

Similarly we obtain:

Theorem 1.3.3. *An algebra* (N, ∨, ∧) *where* ∨ *and* ∧ *are both associative and idempotent is a skew lattice (*B_1, B_2, C_1 *and* C_2*) if and only if* ${}_{\wedge}\preceq_{\mathcal{L}}$ *dualizes* ${}_{\vee}\preceq_{\mathcal{R}}$ and ${}_{\wedge}\preceq_{\mathcal{R}}$ *dualizes* ${}_{\vee}\preceq_{\mathcal{L}}$*, that is*:

$$x \vee y = x \text{ iff } x \wedge y = y \text{ and } x \vee y = y \text{ iff } x \wedge y = x.$$

Likewise, N *is a skew* lattice (*B_3, B_4, C_3 *and* C_4*) if and only if* ${}_{\wedge}\preceq_{\mathcal{L}}$ *dualizes* ${}_{\vee}\preceq_{\mathcal{L}}$ and ${}_{\wedge}\preceq_{\mathcal{R}}$ *dualizes* ${}_{\vee}\preceq_{\mathcal{R}}$*, that is*: $x \vee y = x$ *iff* $y \wedge x = y$ *and* $x \vee y = y$ *iff* $y \wedge x = x$. ☐

Laslo has shown that no lesser combination than those given in Theorem 1.3.2 suffice to force (N, ∨, ∧) to be a lattice. Moreover, the effect of Theorems 1.3.2 and 1.3.3 is that the combination of any three of B_1-B_4 with any three of C_1-C_4 guarantees that (N, ∨, ∧) is at least a skew* lattice, if not a lattice.

In the first section on lattices we saw that B_1 and C_1 together were sufficient to force both ∧ and ∨ to be idempotent operations. In general, we have:

Theorem 1.3.4. (Laslo and Cozac [1998]) *Let* B_i *and* C_j *where* $1 \leq i, j \leq 4$ *be a pair of absorption identities and let* (N, ∨, ∧) *be an algebra with binary operations* ∨ *and* ∧ \on N *that satisfy both* B_i *and* C_j. *Unless* B_i *and* C_j *form a converse pair in the above sense, then both* ∨ *and* ∧ *are idempotent. When* B_i *and* C_j *form a converse pair, then even assuming that both operations are associative is not enough to force them to also be idempotent.*

Proof. We have already seen that B_1 and C_1 imply that both operations are idempotent. Given B_1 and C_3 we have $a \wedge a = a \wedge [a \vee (b \wedge a)] = a$ and $a \vee a = a \vee (a \wedge a) = a$. Next, given B_1 and C_4 we have $a \vee a = (a \wedge (a \vee a)) \vee a = a$ and $a \wedge a = a \wedge (a \vee a) = a$. Lastly, both B_1 and C_2 are satisfied by the three-element algebra with binary operations given by tables:

∨	*a*	*b*	*c*
a	*a*	*b*	*c*
b	*c*	*c*	*c*
c	*c*	*c*	*c*

and

∧	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*b*
c	*a*	*a*	*c*

.

Both operations are associative, but clearly not idempotent.

We have settled the case for B_1 and any of the C_j. What about B_2 through B_4? First, suppose we are given algebra (N, ∨, ∧) where ∨ and ∧ are binary operations. For (N, ∨, ∧) to satisfy B_4 it is equivalent that (N, ∨, ∧*) satisfy B_1. (Here $x \wedge^* y = y \wedge x$ as before.) But replacing ∧ by ∧* has the effect of permuting the C_j. In particular, (N, ∨, ∧*) satisfying C_2, corresponds to (N, ∨, ∧) satisfy C_4. Thus, given B_4, all C_j except C_4 induce idempotent operations. Continuing, the B_1-C_2 pairing for (N, ∨*, ∧) corresponds to the B_3-C_3 pairing for (N, ∨, ∧) and the B_1-C_2 pairing for (N, ∨*, ∧*) corresponds to the B_2-C_1 pairing for (N, ∨, ∧). Thus all B_i-C_j pairings insure idempotent operations except for the converse pairings. ☐

Comment. The proof shows that once one of the operations is known to be idempotent, then the nonconverse pairing is enough to force the other operation to be idempotent also.

Thanks to various dualities, coherent quasi-orderings are defined as follows.

1) If (N, ∨, ∧) is a quasilattice, its ***natural quasiordering*** $\preceq$ is just $_\wedge\preceq$, or the dual of $_\vee\preceq$.
2) If (N, ∨, ∧) is a paralattice, its ***natural partial ordering*** $\leq$ is $_\wedge\leq$, or the dual of $_\vee\leq$.
3) If (N, ∨, ∧) is a skew(*) lattice, its left and right quasiorderings, $\preceq_\mathcal{L}$ and $\preceq_\mathcal{R}$, are $_\wedge\preceq_\mathcal{L}$ and $_\wedge\preceq_\mathcal{R}$ respectively, or their duals of the appropriate ∨-quasi-orderings.

Congruences

A ***congruence*** on a noncommutative lattice *N* is an equivalence relation θ on *N* such that for all $a, b, c \in N$,

$$a\,\theta\,b \text{ implies } a\wedge c\,\theta\,b\wedge c,\ c\wedge a\,\theta\,c\wedge b,\ a\vee c\,\theta\,b\vee c \text{ and } c\vee a\,\theta\,c\vee b.$$

In accord with standard notation, Δ denotes the least congruence (equality) and ∇ denotes the greatest congruence. Besides Δ and ∇, two other congruences of interest are:

The ***least lattice congruence*** is the smallest congruence λ on N such that N/λ is a lattice. *N*/λ is thus the maximal lattice image of N. On the interval [λ, ∇], $\theta\cap\sup_i(\theta_i) = \sup_i(\theta\cap\theta_i)$, since [λ, ∇] is lattice isomorphic with **Con**(N/λ), the congruence lattice of the lattice N/λ.

On the other hand, the ***least rectangular congruence***, is the least congruence ρ on *N* for which *N*/ρ is rectangular. Clearly ρ is the congruence generated from the relation $_\vee\leq \cup\, _\wedge\leq$.

Theorem 1.3.5. *If* (N, ∨, ∧) *is a noncommutative lattice with* $\mathcal{D}_\vee$ *and* $\mathcal{D}_\wedge$ *the* $\mathcal{D}$*-congruences for* ∨ *and* ∧*, then* λ *is the congruence on N generated from the relation* $\mathcal{D}_\vee \cup \mathcal{D}_\wedge$.

Proof. If δ is the congruence on *N* generated from $\mathcal{D}_\vee \cup \mathcal{D}_\wedge$, then N/δ is commutative in both ∨ and ∧. Since N satisfies absorption identities inducing B_1 and C_1 on any commutative image, *N*/δ is a lattice. Thus $\lambda \subseteq \delta$. Since $\delta \subseteq \lambda$ is clear, λ = δ follows. ☐

Corollary 1.3.6. *Let* (N, ∨, ∧) *be a noncommutative lattice for which* $\mathcal{D}_\vee = \mathcal{D}_\wedge$. *Then* $\lambda = \mathcal{D}$, *the common* $\mathcal{D}$*-equivalence for both operations*, N *is a quasilattice*, N/$\mathcal{D}$ *is the maximal lattice image of* N *and all the* $\mathcal{D}$*-classes are the maximal antilattices in* N. *Conversely, for all quasilattices*, $\mathcal{D}_\vee = \mathcal{D}_\wedge$.

Proof. If $\mathcal{D}_\vee = \mathcal{D}_\wedge$, then λ is clearly the common $\mathcal{D}$-equivalence. By the Clifford-McLean Theorem for bands, N/$\mathcal{D}$ is the maximal lattice image of N and the $\mathcal{D}$-classes are maximal anti-lattices in N. Hence $a\,\mathcal{D}\,a \wedge (b \vee a \vee b) \wedge a$ in N. But since $a\ _{(\wedge)}\!\geq a \wedge (b \vee a \vee b) \wedge a$ in N, equality follows: $a = a \wedge (b \vee a \vee b) \wedge a$. Similarly, $a = a \vee (b \wedge a \wedge b) \vee a$ so that N is a quasi-lattice. The converse is clear. ☐

Thus, *every quasilattice is a lattice of antilattices*. Hence quasilattices are precisely the noncommutative lattices having a direct analogue of the Clifford-McLean Theorem for bands.

Flatness

A noncommutative lattice is ***flat*** if one of the following holds:

(r, l): $a \vee b \vee a = b \vee a$ and $a \wedge b \wedge a = a \wedge b$.
(l, r): $a \vee b \vee a = a \vee b$ and $a \wedge b \wedge a = b \wedge a$.
(l, l): $a \vee b \vee a = a \vee b$ and $a \wedge b \wedge a = a \wedge b$.
(r, r): $a \vee b \vee a = b \vee a$ and $a \wedge b \wedge a = b \wedge a$.

Thus, being (r, l)-flat means that $\mathcal{D}_{(\vee)} = \mathcal{R}_{(\vee)}$ and $\mathcal{D}_{(\wedge)} = \mathcal{L}_{(\wedge)}$, or equivalently, $\mathcal{L}_{(\vee)} = \mathcal{R}_{(\wedge)} = \Delta$. Modified remarks hold for the other three types of flatness. Clearly:

Proposition 1.3.7. *Given any variety of noncommutative lattices (e.g., paralattices or quasilattices), the flat algebras of a given type form a subvariety*. □

If (N, ∨, ∧) if flat, then (N, ∨*, ∧), (N, ∨, ∧*) and (N, ∨*, ∧*) are also flat, with all four types of flatness represented. Of particular interest is:

Theorem 1.3.8. (l, l)*-flat paralattices are characterized by* B_1, B_4, C_1 *and* C_4:

$$a \wedge (a \vee b) = a = (a \vee b) \wedge a \quad and \quad a \vee (a \wedge b) = a = (a \wedge b) \vee a.$$

Similarly, (l, l)*-flat quasilattices are characterized by* B_1, B_3, C_1 *and* C_3:

$$a \wedge (a \vee b) = a = a \wedge (b \vee a) \quad and \quad a \vee (a \wedge b) = a = a \vee (b \wedge a).$$

Proof. Assuming (l, l)-flatness B_6, B_7, C_6 and C_7 reduce to B_1, B_4, C_1 and C_4. Conversely, from B_1, B_4, C_1 and C_4 we get $a \wedge b \wedge a = a \wedge b \wedge [(a \wedge b) \vee a] = a \wedge b$. Switching ∨ and ∧ (which we can since B_1, B_4, C_1 and C_4 are operational duals) yields $a \vee b \vee a = a \vee b$. Thus B_6, B_7, C_6 and C_7 can be recovered from B_1, B_4, C_1 and C_4. Still assuming (l, l)-flatness, B_5, C_5 and their derived identities, $a \wedge (a \vee b \vee a) \wedge a = a = a \vee (a \wedge b \wedge a) \vee a$, reduce to B_1, B_3, C_1 *and* C_3. From the latter, $a \wedge b \wedge a = a \wedge b \wedge [a \vee (a \wedge b)] = a \wedge b$ follows. Similarly, $a \vee b \vee a = a \vee b$. Thus B_5 and C_5 are recovered from B_1, B_3, C_1 *and* C_3. □

This illustrates a general rule of thumb: *upon assuming flatness,* B_1 - B_4 *and* C_1 - C_4 *usually suffice to describe the dualities encountered in noncommutative lattice theory.*

Flatness was encountered in early work on noncommutative lattices. P. Jordan, who wrote numerous articles on the subject in the 1950s and 1960s, often worked with algebras satisfying B_2, B_3, C_1 and C_4 which characterize (r, l)-flat paralattices. M. D. Gerhardts, who published in the 1960s and early 1970s, studied algebras satisfying B_1, B_3, C_2 and C_4 which characterize (r, l)-flat quasilattices. In a sequence of papers beginning in the 1980s, Gh. Farcas considered systems combining (l, l)-flatness with middle commutativity, $a \vee b \vee c = a \vee c \vee b$ and $a \wedge b \wedge c = a \wedge c \wedge b$, so that both operations were left normal.

Distributive identities

Just as the number of essentially distinct absorption identities proliferates in the absence of commutativity, so do the number of essentially distinct distributive identities. To begin, a noncommutative lattice is ***fully distributive*** if it satisfies the identities:

D_1 : $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. D_1' : $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

D_2 : $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$. D_2' : $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$.

Unlike the case for lattices:

Theorem 1.3.9. *For skew lattices* D_1, D_1', D_2 and D_2' *are mutually independent.*

Proof. A skew lattice satisfying precisely D_1, D_1' and D_2' is given by the tables:

$\vee$	a	b	0
a	a	a	a
b	b	b	b
0	a	b	0

and

$\wedge$	a	b	0
a	a	b	0
b	a	b	0
0	0	0	0

.

That D_2' is not satisfied is seen by $(a \wedge 0) \vee b = b \neq a = (a \vee b) \wedge (0 \vee b)$. Replacing $(\vee, \wedge)$ by $(\vee^*, \wedge^*)$ or $(\wedge, \vee)$ or both $(\wedge^*, \vee^*)$ gives examples of the other ways that a skew lattice can satisfy just three of the four identities above. ☐

Being fully distributive is powerful. To see this, observe first that distributive lattices are fully distributive as well as flat antilattices of any of the four types. Hence any direct product of algebras from these five types is also fully distributive. We shall see in Corollary 3.5.3 that for quasilattices (and hence skew lattices) this is all. Thus, *any fully distributive quasilattice factors as the product of a distributive lattice and up to four flat antilattices*.

We next consider two other sets of distributive identities that while less powerful, have a broader scope. A noncommutative lattice is ***bidistributive*** if it satisfies the slightly weaker pair of identities:

D_3 : $a \wedge (b \vee c) \wedge d = (a \wedge b \wedge d) \vee (a \wedge c \wedge d)$.
D_3': $a \vee (b \wedge c) \vee d = (a \vee b \vee d) \wedge (a \vee c \vee d)$.

Distributive lattices and arbitrary antilattices are bidistributive, as are their direct products. By Theorem 3.5.2 below, this is all among quasilattices. (In this more general case the antilattice factor need not be completely factored into a product of flat antilattices.)

Finally, a noncommutative lattice is ***distributive*** if it satisfies the even weaker pair of identities:

$$\mathrm{D}_4: \quad a \wedge (b \vee c) \wedge a = (a \wedge b \wedge a) \vee (a \wedge c \wedge a).$$
$$\mathrm{D}_4': \quad a \vee (b \wedge c) \vee a = (a \vee b \vee a) \wedge (a \vee c \vee a).$$

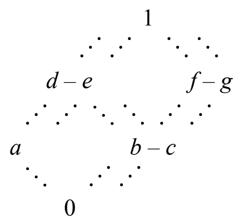
Among skew lattices, D_4 and D_4' play an important role. For instance, skew lattices in rings are distributive, as are skew Boolean algebras. Moreover, as will be seen in Theorem 3.5.1, a distributive, noncommutative lattice is a paralattice if and only if it is a quasilattice. It thus follows that the type of distributivity expressed by D_4 and D_4' is of maximal general use.

Theorem 1.3.10. *Among skew lattices identities* D_3 *and* D_3' *are independent.*

Proof. The skew lattice example of Theorem 1.3.8 satisfies D_3 but not D_3'. Switching the $\wedge$ and $\vee$ tables provides the complementary example satisfying D_3' but not D_3. ☐

Theorem 1.3.11. (Spinks [2000]) *Among skew lattices,* D_4 and D_4' *are independent.*

Proof. Spinks gave the following example of a 9-element skew lattice satisfying D_4 but not D_4'.



($\mathcal{D}$ is indicated by '–'; $\ddots$ and $⋰$ denote $\geq$)

Both 1 and 0 behave as they ought. Otherwise, the operations are described by the partial tables:

$\vee$	a	b	c	d	e	f	g
a	a	d	e	d	e	1	1
b	d	b	b	d	d	f	f
c	e	c	c	e	e	g	g
d	d	d	d	d	d	1	1
e	e	e	e	e	e	1	1
f	1	f	f	1	1	f	f
g	1	g	g	1	1	g	g

$\wedge$	a	b	c	d	e	f	g
a	a	0	0	a	a	0	0
b	0	b	c	b	c	b	c
c	0	b	c	b	c	b	c
d	a	b	c	d	e	b	c
e	a	b	c	d	e	b	c
f	0	b	c	b	c	f	g
g	0	b	c	b	c	f	g

.

In particular, $a \vee (d \wedge g) \vee a = a \vee c \vee a = \mathrm{e} \neq d = d \wedge 1 = (a \vee d \vee a) \wedge (a \vee g \vee a)$. Switching the $\wedge$ and $\vee$ provides a 9-element example satisfying D_4' but not D_4. ☐

Spinks' examples are minimal examples. In all cases of order ≤ 8, D_4 is equivalent to D_4'. Under the added condition of ***symmetry*** ($x \vee y = y \vee x$ iff $x \wedge y = y \wedge x$), Spinks found a machine-generated proof that for skew lattices, D_4 is equivalent to D_4'. In Section 5.2 we give a short "normal" proof of this fact.

Enriched structures

We have already encountered 1 and 0. As is the case for lattices 1, if it exists, is the unique element at the top of the noncommutative lattice and 0, if it exists, is the unique element at the bottom, whatever the (quasi-)ordering may be. This is given expression by the following sets of identities.

$$1 \vee a = 1 = a \vee 1 \text{ and } 1 \wedge a = a = a \wedge 1.$$

$$0 \wedge a = 0 = a \wedge 0 \text{ and } 0 \vee a = a = a \vee 0.$$

For skew lattices (and for paralattices in general) both second pairs of identities is redundant.

Paralattices, and particularly skew lattices, have a coherent natural partial order $\leq$ given as either $\leq_{(\wedge)}$ or the dual of $\leq_{(\vee)}$. It may be that two elements x, y in a paralattice possesses a ***natural meet*** that is maximal among all z such that both $x, y \geq z$. Dually they may also possess a ***natural join*** that is minimal among all z such that both $x, y \leq z$. To distinguish natural meets and natural joins from the given operations $\wedge$ and $\vee$, we shall refer to them as ***intersections*** and ***unions*** of elements respectively, employing the notation $x \cap y$ and $x \cup y$. Unless x and y commute, one has $x \wedge y > x \cap y$ and $x \cup y > x \vee y$.

Theorem 1.3.12. *The natural meet operation in a paralattice is characterized by the identities*:

NM_1: $x \cap x = x$;
NM_2: $x \cap y = y \cap x$;
NM_3: $(x \cap y) \cap z = x \cap (y \cap z)$;
NM_4: $x \wedge (x \cap y) = x \cap y = (x \cap y) \wedge x$;
NM_5: $x \cap (x \wedge y \wedge x) = x \wedge y \wedge x$.

Dual identities characterize a natural joins. ☐

Theorem 1.3.13. *Given a paralattice* (P; $\vee$, $\wedge$) *with a natural meet* $\cap$, *the enriched algebra* (P; $\vee$, $\wedge$, $\cap$) *has a distributive congruence lattice*.

Proof. Setting $m(x, y, z) = (x \cap y) \vee (y \cap z) \vee (x \cap z)$, one has $m(x, x, y) = m(x, y, x) = m(y, x, x) = x$. But this implies its congruence lattice is distributive. (Theorem II.12.3 in Burris and Sankappanavar [1981].) ☐

Rectangular bands and rectangular skew lattices

A ***rectangular skew lattice*** is a skew lattice (S; $\wedge$, $\vee$) for which (S; $\wedge$) and (S; $\vee$) are rectangular bands. Equivalently, its is an antilattice that is also a skew lattice. Given a rectangular skew lattice, from $x \wedge (x \vee y) = x$, it follows that $x \vee y = u \wedge x$ for some u. Similarly, from $(x \vee y) \wedge y = y$, it follows that $x \vee y = y \wedge v$ for some v. Combining we get

$$x \vee y = (x \vee y) \wedge (x \vee y) = y \wedge v \wedge u \wedge x = y \wedge x.$$

Conversely, given an algebra $(S; \wedge, \vee)$ for which both $(S; \wedge)$ and $(S; \vee)$ are rectangular bands and $x \vee y = y \wedge x$ holds on S, it is easily seen that all four relevant absorption identities must hold, making $(S; \wedge, \vee)$ a rectangular skew lattice. We thus have:

Proposition 1.3.14. *The variety of rectangular bands is term equivalent to the variety of rectangular skew lattices. Thus a map between rectangular bands f: A → B is a homomorphism of bands if and only if it is a homomorphism between their induced rectangular skew lattices. Likewise an equivalence on a rectangular band is a band congruence if and only if it is a congruence on the derived skew lattice.* ☐

We continue by considering the case of right zero bands ($xy = y$), and their derived right rectangular skew lattices (where $x \wedge y = y = y \vee x$). The following result is trivial.

Proposition 1.3.15. *Given right zero bands* B *and* B′, *each function f:* B → B′ *is a homomorphism, and thus each equivalence relation on* B *is a congruence*. (Similar remarks hold also for left zero bands where $xy = x$.) ☐

We next consider the case of a factored rectangular band B = L × R, where L and R are left zero and right zero bands, respectively. (All rectangular bands are isomorphic to such a factorization.)

Theorem 1.3.16. *Given a factored rectangular band* B = L × R, *and a left zero band* L′. *Then every homomorphism from* B *to* L′ *has the form $f(l, r) = \lambda(l)$ where λ is any function from* L *to* L′. *Dually, every homomorphism from* B *to a right zero band* R′ *has the form $f(l, r) = \rho(r)$ where ρ is any function from* R *to* R′. *Finally, given a second factored rectangular band* B′ = L′ × R′, *all homomorphisms from* B *to* B′ *are of the form $f(l, r) = (\lambda(l), \rho(r))$ where λ is any function from* L *to* L′ *and ρ is any function from* R *to* R′.

Proof. Given a homomorphism from f: L × R → L′, since homomorphisms send $\mathcal{R}$-classes to $\mathcal{R}$-classes, f is constant on each R-class $\{l\} \times$ R and thus must factor through the left factor L, leading to a chain of homomorphisms L × R → L → L′ whose composite f must be a homomorphism. Here L × R → L is just the projection onto L. The second map is the induced map λ: L → L′ that must be a homomorphism. ☐

Corollary 1.3.17. *Given a factored rectangular band* B = L × R, *each congruence θ on* B *has the form $(l, r)\ \theta\ (l', r')$ iff $l\ \theta_L\ l'$* and *$r\ \theta_R\ r'$ for any pair of equivalences θ_L* and *θ_R on* L *and* R *respectively*. ☐

The internal perspective. Given rectangular bands B and B′, choose elements $b \in$ B and $b' \in$ B′ as base points. Each $x \in$ B is of the unique form $x = lr$ where $l \in \mathcal{L}_b$ and $r \in \mathcal{R}_b$. Indeed, x factors as $(xb)(bx)$ where $xb \in \mathcal{L}_b$ and $bx \in \mathcal{R}_b$. Again, this factorization is unique relative to the base point b. Clearly similar remarks hold for B′ and b'. Next, let λ: $\mathcal{L}_b \to \mathcal{L}_{b'}$ and ρ: $\mathcal{R}_b \to \mathcal{R}_{b'}$ be functions, both of which are trivially homomorphisms between their restricted domains. Finally define f: B → B′ by $f(x) = \lambda(xb)\rho(bx)$, or equivalently, $f(x) = \lambda(xb)b'\rho(bx)$.

Theorem 1.3.18. *As defined, f is a homomorphism from* B *to* B′. *Conversely, every homomorphism from* B *to* B′ *arises in this manner. Finally, f is* 1-1 *or onto if and only if both* λ *and* ρ *are*. (Of course, the choice of b and b' is part of "this manner.")

Proof. We first show that f as defined is a homomorphism from B to B′.

$$f(xy) = \lambda(xyb)b'\rho(bxy) = \lambda(xb)b'\rho(by) = \lambda(xb)\rho(by)$$

whilst

$$f(x)f(y) = \lambda(xb)b'\rho(bx)\lambda(yb)b'\rho(by) = \lambda(xb)\rho(by).$$

Conversely, if f: B → B′ is a homomorphism, then upon picking some b in B and setting $b' = f(b)$, for any x in B we get $f(x) = f(xbbbx) = f(xb)b'f(bx) = \lambda(xb)b'\rho(bx)$ where the functions λ and ρ are the restrictions of f to $\mathcal{L}_b$ and $\mathcal{R}_b$ respectively. ☐

A ***primitive skew lattice*** is a skew lattice consisting of two rectangular algebras A > B, where $b \wedge a \wedge b = b$ or dually $a \vee b \vee a = a$, for all $a \in$ A and $b \in$ B. We are especially interested in the case when B = {0}. We denote such an algebra by A^0. Suppose we are given two such algebras, A^0 and B^0. Then the following assertions follow from the behavior of 0 and the results above

Proposition 1.3.19. *Given primitive skew lattices* A^0 *and* B^0, Hom(A^0, B^0) *consists of* (i) *all constant maps from* A^0 *to* B^0 *and* (ii) *all maps f* from A^0 *to* B^0 *for which* $f(0_A) = (0_B)$ *and the restriction f*|A *is a homomorphism from* A *to* B. (Thus the nontrivial part of f is governed by the conclusions of the preceding results.)

Corollary 1.3.20. *Given a primitive skew lattice* A^0, *the nontrivial congruences on* A^0 (*where not all elements are related*), *are equivalences for which* {0} *forms a single equivalence class and their restriction to* A *is a congruence on the subalgebra* A.

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II: SKEW LATTICES

We take a closer look at skew lattices, i.e., at algebras (S; $\vee$, $\wedge$) with associative binary operations $\vee$ and $\wedge$ satisfying the absorption identities:

$$\begin{array}{llll} B_1: & a \wedge (a \vee b) = a. & C_1: & a \vee (a \wedge b) = a. \\ B_2: & (b \vee a) \wedge a = a. & C_2: & (b \wedge a) \vee a = a. \end{array}$$

By Theorem 1.3.4, both $\vee$ and $\wedge$ are idempotent: $x \vee x = x = x \wedge x$. Equivalently, skew lattices are characterized as the double bands (S; $\vee$, $\wedge$) satisfying the dualities:

$$a \vee b = a \text{ if and only if } a \wedge b = b.$$

$$a \vee b = b \text{ if and only if } a \wedge b = a.$$

In this chapter a basic theory for these algebras is developed. For every statement about skew lattices, clearly a parallel statement holds for skew* lattices. Thus, from this point on, the latter will not be mentioned until more general structures are considered in Chapter 3.

We begin in Section 1 with some fundamental results about skew lattices. Of particular importance are two core structural results for skew lattices: analogues of the Clifford-McLean Theorem and the Kimura Factorization Theorem (Theorems 2.1.2 and 2.1.5), given originally for bands and regular bands respectively (Theorem 1.2.6 and what follows). We also initiate our study of skew lattices of idempotents in rings, a source of both examples and conceptual motivation, in Theorems 2.1.7 and 2.1.9. (In this case, $\wedge$ and $\vee$ are given first as $e\wedge f = ef$ and $e\vee f = e + f - ef$.) Due to the latter theorem, a band can be embedded into some skew lattice as a sub-band of its $\wedge$-reduct (or of its $\vee$-reduct) if and only if the band itself is regular (Theorem 2.1.10).

In Section 2 we consider the role of commutativity in skew lattices. The *center* of a skew lattice S, $\mathcal{Z}(S) = \{e \in S \mid e\vee x = x\vee e \text{ and } e\wedge x = x\wedge e \text{ for all } x \in S\}$ is characterized in Theorem 2.2.2 as the sublattice of all elements that form singleton $\mathcal{D}$-classes. We look at the important property of *symmetry* ($a\vee b = b\vee a$ iff $a\wedge b = b\wedge a$) and some of its consequences. These include Theorem 2.2.10 that asserts that any symmetric skew lattice with a countable maximal lattice image has a lattice section (that is, a sublattice meeting each $\mathcal{D}$-class of S at a single point). Example 2.2.2

exhibits a nonsymmetric 13-element skew lattice with a generating set of 3 mutually commuting elements (under both $\vee$ and $\wedge$), that has noncommuting pairs of elements.

In the third section we consider *normal* skew lattices, i.e., skew lattices whose $\wedge$-reducts are normal in that $x\wedge y\wedge z\wedge w = x\wedge y\wedge z\wedge w$. Of special interest are distributive, symmetric, normal skew lattices characterized in Theorem 2.3.2 by identities $a\wedge(b \vee c) = (a\wedge b) \vee (a\wedge c)$ and $(a \vee b)\wedge c = (a\wedge c) \vee (b\wedge c)$. This strengthened form of distributivity is called *strong distributivity*. Thanks to Theorem 2.3.6, every normal skew lattice of idempotents in a ring is strongly distributive. In this case the operations are given by $e\wedge f = ef$ again, but

$$e\vee f = (e + f - ef)^2 = e + f + fe - efe - fef.$$

Of course when $e + f - ef$ is idempotent, both outcomes agree. Strongly distributive skew lattices are also of interest due to their connections to skew Boolean algebras, the subject of Chapter 4. Suffice it to say here that a skew lattice can be embedded into (the skew lattice reduct of) a skew Boolean algebra precisely when it is strongly distributive.

In Section 4 we engage in a detailed study of the natural partial order $\geq$ on a skew lattice. This study is based on the behavior of *primitive skew lattices* consisting of exactly two $\mathcal{D}$-classes, $A > B$. Primitive skew lattices have a simple description given in terms of *A-cosets* arising in B and *B-cosets* arising in A and the *coset bijections* between these cosets induced by $\geq$. (See Theorem 2.4.1) As a consequence, primitive skew lattices of the most general type are easily manufactured. Moreover, the interaction between the various maximal primitive subalgebras of a skew lattice says much about the behavior of the entire algebra. (See, e.g., Theorem 2.4.9 and its consequences.) We also look at *skew chains* of comparable $\mathcal{D}$-classes A > B > C and consider the case where coset bijections between cosets in the outer classes are compositions of successive intermediate coset bijections (as is the case for skew lattices in rings). When this always occurs in a skew lattice, it is said to be *categorical*.

In Section 5 we continue the analysis of skew lattices by their primitive subalgebras begun in the previous section. But here we pass from coset bijections between comparable pairs of cosets in (usually) distinct $\mathcal{D}$-classes to *coset projections* from an entire $\mathcal{D}$-class onto a coset in a comparable $\mathcal{D}$-class. The individual projections are obtained by combining all coset bijections from one $\mathcal{D}$-class that share a common coset of outputs in the other class. Even if the skew lattice itself is not categorical in the above sense, these downward (or upward) projections taken collectively along with the involved $\mathcal{D}$-classes, form a category. (See Theorem 2.5.4.) All this is developed in the fifth section and then applied to give a general description of a normal skew lattice in Theorem 2.5.7.

In Section 6 we study decompositions of (mostly symmetric) normal skew lattices. After some preliminary cases, theorems of general character are given. For instance, the Reduction Theorem (2.6.9) implies that every symmetric normal skew lattice can be embedded in the product of its maximal lattice image and its maximal distributive image. The Primary Decomposition Theorem (2.6.11) tells how a strongly distributive (hence symmetric and normal) skew lattice with a finite maximal lattice image must factor into a fibered product of primary factors, the latter being algebras of rather simple type. In general, strongly distributive skew lattices can be embedded in powers of a special primitive skew lattice **5**, a noncommutative 5-element variant of the lattice **2** for which the latter is its maximal lattice image. (See Theorem 2.6.12.)

Finally, the material presented in this chapter first appeared in the papers referenced at the end of this chapter.

2.1 Fundamental results

Any noncommutative lattice has two $\mathcal{D}$-equivalences, $\mathcal{D}_{(\vee)}$ and $\mathcal{D}_{(\wedge)}$, with each being a congruence with respect to its associated operation. For skew lattices we have:

Lemma 2.1.1. *Given a skew lattice*, $\mathcal{R}_{(\vee)} = \mathcal{L}_{(\wedge)}$, $\mathcal{L}_{(\vee)} = \mathcal{R}_{(\wedge)}$, *and* $\mathcal{D}_{(\vee)} = \mathcal{D}_{(\wedge)}$. *Thus the common equivalence $\mathcal{D}$ is a congruence of skew lattices.*

Proof. That $\mathcal{R}_{(\vee)} = \mathcal{L}_{(\wedge)}$ and $\mathcal{L}_{(\vee)} = \mathcal{R}_{(\wedge)}$ follow immediately from the above dualities. Hence:

$$\mathcal{D}_{(\vee)} = \mathcal{L}_{(\vee)} \circ \mathcal{R}_{(\vee)} = \mathcal{R}_{(\wedge)} \circ \mathcal{L}_{(\wedge)} = \mathcal{D}_{(\wedge)}. \ \square$$

Thus for any skew lattice (S; $\vee$, $\wedge$) we set $\mathcal{R} = \mathcal{R}_{(\wedge)} = \mathcal{L}_{(\vee)}$ and $\mathcal{L} = \mathcal{L}_{(\wedge)} = \mathcal{R}_{(\vee)}$. A skew lattice is ***rectangular*** if either (S, $\vee$) or (S, $\wedge$) is a rectangular band in which case, thanks to the above dualities, both are rectangular bands with $x \vee y = y \wedge x$. Put otherwise, a rectangular skew lattice is precisely an antilattice for which $\mathcal{R}_{(\vee)} = \mathcal{L}_{(\wedge)}$ and $\mathcal{L}_{(\vee)} = \mathcal{R}_{(\wedge)}$.

Theorem 2.1.2. (The Clifford-McLean Theorem for skew lattices). *Given a skew lattice* (S; $\vee$, $\wedge$), *the equivalence $\mathcal{D}$ is a congruence*, S/$\mathcal{D}$ *is the maximal lattice image of* S *and all congruence classes of $\mathcal{D}$ are maximal rectangular skew lattices in* S.

Proof. This follows from the above lemma and Corollary 1.3.6. $\square$

Lemma 1.2.3, Theorem 1.3.3 and the above result give us:

Lemma 2.1.3. *Given elements a and b in a skew lattice* S,

$$a \geq_{(\wedge)} b \ \textit{ iff } \ b \geq_{(\vee)} a \ \textit{ and } a \succeq_{(\wedge)} b \ \textit{ iff } \ b \succeq_{(\vee)} a. \ \square$$

Thus the ***natural partial order*** on any skew lattice is given by $\geq \, = \, \geq_{(\wedge)} \, = \, \leq_{(\vee)}$ and the ***natural quasiorder*** on any skew lattice is given by $\succeq \, = \, \succeq_{(\wedge)} \, = \, \preceq_{(\vee)}$.

Lemma 2.1.4. *On any skew lattice* S*, both* $\mathcal{R}$ *and* $\mathcal{L}$ *are congruences. In particular, both algebraic reducts* (S, ∨) *and* (S, ∧) *are regular bands and given* $e \succeq a$, $b \succeq f$ *in* S:

$$a \vee f \vee b = a \vee b \text{ and } a \wedge e \wedge b = a \wedge b.$$

Proof. Given a skew lattice S, we first show that (S, ∧) satisfies the class covering condition. Indeed, given comparable $\mathcal{D}$-classes A > B in S with b an arbitrary element in B, then for any a in A, $b \geq_{(\vee)} b \vee a \vee b \in$ A. Thus $b \vee a \vee b \geq_{(\wedge)} b$ and (S, ∧) is seen to satisfy the class covering condition. Thus by Theorem 1.2.16, both $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are ∧-congruences. Similarly, both $\mathcal{L}_{(\vee)}$ and $\mathcal{R}_{(\vee)}$ are ∨-congruences. The first statement follows now from Lemma 2.1.1. Thus both band reducts are regular; moreover, by Theorem 1.2.7, the dual conditional identities must hold. □

A skew lattice for which $\mathcal{D} = \mathcal{R}$ [$\mathcal{D} = \mathcal{L}$] is said to be ***right-handed*** [***left-handed***]. If S is also rectangular, then it is ***right-rectangular*** [***left-rectangular***]. Remarks following Theorem 1.2.27 give us:

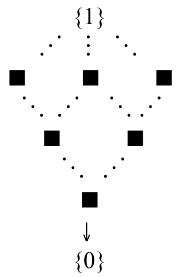
Theorem 2.1.5. (*Kimura factorization for skew lattices*). *Given a skew lattice* S, S/$\mathcal{R}$ *is the maximal left-handed image of* S, S/$\mathcal{L}$ *is the maximal right handed image of* S *and the commuting diagram*

$$\begin{array}{ccc} S & \longrightarrow & S/\mathcal{L} \\ \downarrow & & \downarrow \\ S/\mathcal{R} & \longrightarrow & S/\mathcal{D} \end{array}$$

is a pullback diagram and S *factors as the fibered product* $S \cong S/\mathcal{R} \times_{S/\mathcal{D}} S/\mathcal{L}$. □

An ***embedding*** of a band B into a skew lattice S is a semigroup embedding of B into the band (S; ∧). Clearly for a band to be embedded into a skew lattice, the band must be regular. It turns out that this condition is sufficient. Our easy proof of this fact brings us into the subject of skew lattices in rings. We begin with the following considerations. Given a ring R with set of idempotents **E**(R), a ***band in*** **R** is any subset of **E**(R) that is closed under multiplication. Clearly:

Proposition 2.1.6. *Let* R *be a ring with identity* 1 *such that any descending chain* $e_1 \geq e_2 \geq \ldots$ *in* **E**(S) *eventually stabilizes*: $e_n = e_{n+1} = \ldots$ for some $n \geq 1$. *Then the* $\mathcal{D}$*-classes of any band* B *in* R *such that* 1 ∈ B *are lattice-ordered*. (This is the case for matrix rings over fields.) □



Such cases naturally raise the question as to whether a counter-operation ∨ exists such that the join class $\mathcal{D}_e \vee \mathcal{D}_f$ of two $\mathcal{D}$-classes, $\mathcal{D}_e$ and $\mathcal{D}_f$, is given as $\mathcal{D}_{e\vee f}$. This leads to a simple but important observation:

Theorem 2.1.7. *Any multiplicative band* B *in a ring that is also closed under the circle operation* $x \circ y = x + y - xy$ *has the following properties*:

1. B *is also a band under* $\circ$.
2. $x(x \circ y) = x = (y \circ x)x$ *and* $x \circ (xy) = x = (yx) \circ x$ *hold on* B:
   Thus (B; $\circ$, •) *is a skew lattice*. In this case the elements in B also satisfy the identity:
3. $xyx + yxy = xy + yx$.

Proof. To see (1), note that $\circ$ is always associative and $x \circ x = x + x - x^2 = x$ if and only if $xx = x$. Thus if B is closed under $\circ$, then (B, $\circ$) is indeed a band. (2) is straightforward. For instance,

$$x(x \circ y) = xx + xy - xxy = x + xy - xy = x \text{ and } x \circ (xy) = x + xy - xxy = x + xy - xy = x.$$

Given the assumptions, $x + y - xy = x \circ y = (x \circ y)^2 = (x + y - xy)^2 = x + yx + y - yxy - xyx$, and (3) follows. ☐

The terms in (3) form a rectangular subalgebra for which the diagonal sums are equal.

$$\begin{array}{ccc} xyx & \mathcal{R} & xy \\ \mathcal{L} & & \mathcal{L} \\ yx & \mathcal{R} & yxy \end{array}$$

Two classes of bands closed under $\circ$ are given in the next theorem. But first a lemma:

Lemma 2.1.8. *Let* B *be a right regular band in a ring* R *and let* $e, f \in$ B. *Then*:

1) $e + f - ef \in$ E(R), *and*

2) B $\cup \{e + f - ef\}$ *generates a right regular band.*

Proof. (1) Given $e, f \in$ B, since B is a right-regular band,

$$(e + f - ef)^2 = e + ef - ef + fe + f - fef - efe - ef + ef = e + fe + f - ef - fe = e + f - ef.$$

To see (2), observe first that for all $g, h \in$B,

$$\begin{aligned}(e + f - ef)g(e + f - ef) &= ege + egf - egef + fge + fgf - fgef - efge - efgf + efgef \\ &= ge + egf - gef + fge + gf - gef - fge - egf + gef \\ &= ge - gef + gf \\ &= g(e + f - ef)\end{aligned}$$

Thus any product $g_0(e + f - ef)g_1(e + f - ef)g_2 \ldots g_{n-1}(e + f - ef)g_n$ generated from $\mathrm{B} \cup \{e + f - ef\}$ reduces to $g_0g_1g_2\ldots g_{n-1}(e + f - ef)g_n$. From this observation, (2) follows. □

Theorem 2.1.9. *Every maximal right [left] regular band in a ring* R *is also closed under the circle operation* $e \circ f = e + f - ef$ *and thus forms a skew lattice in* R. □

In general, maximal regular bands in rings need not be closed under $\circ$. We give an example of this in Section 2.3. Nonetheless, what we have seen thus far suffices to prove:

Theorem 2.1.10. *A band can be* $\wedge$*-embedded in a skew lattice if and only if it is a regular band*.

Proof. The condition is clearly necessary. Suppose that B is a regular band. By the Kimura Decomposition B can be embedded in $\mathrm{B}/\mathcal{R} \times \mathrm{B}/\mathcal{L}$. In the semigroup ring $\mathbb{Z}[\mathrm{B}/\mathcal{R}]$, $\mathrm{B}/\mathcal{R}$ generates a regular band C_1 that is closed under the circle operation. Likewise $\mathrm{B}/\mathcal{L}$ generates a regular band C_2 in $\mathbb{Z}[\mathrm{B}/\mathcal{L}]$ that is closed under the circle operation. Thus $\mathrm{B}/\mathcal{R} \times \mathrm{B}/\mathcal{L}$ is a sub-band of $\mathrm{C}_1 \times \mathrm{C}_2$ in $\mathbb{Z}[\mathrm{B}/\mathcal{R}] \times \mathbb{Z}[\mathrm{B}/\mathcal{L}]$. Since $\mathrm{C}_1 \times \mathrm{C}_2$ is closed under the circle operation it is a skew lattice and B is embedded in $\mathrm{C}_1 \times \mathrm{C}_2$. □

Thus, even if a maximal regular band in a ring does not form a skew lattice in that ring, some copy of it will generate a skew lattice under $\circ$ and multiplication in another ring. We complete this section with an important elementary fact about skew lattice in rings. Recall that a skew lattice is distributive if it satisfies identities D3 and D4 in Section 1.3.

Theorem 2.1.11. *Skew lattices in rings (using multiplication and* $\circ$*) are distributive*.

Proof.
$$\begin{aligned}a \wedge (b \vee c) \wedge a = a(b + c - bc)a &= aba + aca - abca \\ &= aba + aca - abaca \quad \text{(regularity)} \\ &= (a \wedge b \wedge a) \vee (a \wedge c \wedge a)\,.\end{aligned}$$

For the dual identity, D4, observe first that $a \vee b \vee a = a + b - ab - ba + aba$. Thus

$$a \vee (b \wedge c) \vee a = a + bc - abc - bca + abca.$$

Expanding $(a \vee b \vee a) \wedge (a \vee c \vee a)$ and then reducing, we get:

$$(a \vee b \vee a) \wedge (a \vee c \vee a) = a + bc - bca - abc - bac + baca + abac.$$

Equating both outcomes and then cancelling corresponding pairs of identical terms we are left with $abca = -bac + baca + abac$. Rearranging gives $abca + bac = baca + abac$, which is true, since it is a case of Theorem 2.1.7(3) with $x = baca$ and $y = abac$, so that $xy = bac$, $yx = abca$, $xyx = baca$ again and $yxy = abac$ again. □

An alternative proof that D4 holds follows from D3 holding and the fact that skew lattices in rings are symmetric (Theorem 2.2.6). This forces D4 to hold also. (See Section 5.2.)

2.2 Instances of commutative behavior

Skew lattices, like many noncommutative structures, can possess abundant instances of commutativity. Indeed, selective instances of commutativity play an important role in their basic theory. We begin a pair of results to this effect.

Theorem 2.2.1. *Let* S *be a skew lattice and let* A *and* B *be* $\mathcal{D}$*-classes in* S *with join class* J = A∨B *and meet class* M = A∧B. *Then given* $v \in$ J *with* $v \geq a \in$ A *and* $v \geq b \in$ B, $a \vee b = v = b \vee a$. *Similarly, given* $m \in$ M *such that* $a \geq m$ *and* $b \geq m$ *for* $a \in$ A *and* $b \in$ B, $a \wedge b = m = b \wedge a$. *Thus*

$$J = \{a \vee b \mid a \in A, b \in B \ \&\ a \vee b = b \vee a\} \quad \text{and} \quad M = \{a \wedge b \mid a \in A, b \in B \ \&\ a \wedge b = b \wedge a\}.$$

Moreover, for every $a \in$ A *there exist* $b, b' \in$ B *such that* $a \vee b = b \vee a$ *in* J *and* $a \wedge b' = b' \wedge a$ *in* M.

Proof. Given $v \in$ J, let $a \in$ A and $b \in$ B be such that $v \geq a, b$. (Both a and b exist since for any x in A and y in B, $v \geq v \wedge x \wedge v$ in A and $v \geq v \wedge y \wedge v$ in B.) For this a and b we have $a \vee b \in$ J so that

$$a \vee b = a \vee b \vee v \vee a \vee b = a \vee v \vee b = v.$$

Similarly, $b \vee a$ equals v also and the assertion about J is seen. The case for M is similar. For the final assertion, pick a in A and let $v \in$ J and $m \in$ M be such that $v \geq a \geq m$. Now apply remarks from the first part of the proof. □

This theorem has the important corollary:

Theorem 2.2.2. *Given an element e in a skew lattice* S, *the following are equivalent*:

1) $\mathcal{D}_e = \{e\}$.
2) *For all* $x \in$ S, $e \vee x = x \vee e$.
3) *For all* $x \in$ S, $e \wedge x = x \wedge e$.

The subset of such elements forms a sublattice $\mathcal{Z}$(S) *of* S (called the ***center*** of S).

Proof. First, if e either $\vee$-commutes with all $x \in$ S or else $\wedge$-commutes with all $x \in$ S, then in particular it does such with all elements in $\mathcal{D}_e$, which forces $\mathcal{D}_e$ to be trivial. Conversely, if $\mathcal{D}_e$ is known to be $\{e\}$, then by the final assertion of Theorem 2.2.1 e must commute under both operations with all elements of S. ☐

While an element that join commutes with *all* elements in a skew lattice also meet commutes with all elements (and conversely), in general two elements commuting under one operation need not commute under the other operation. This is evident in the following example.

Example 2.2.1a. Consider the right-handed skew lattice defined by the Cayley tables

$\vee$	0	$a_{m'}$	$b_{n'}$	$j_{p'}$
0	0	$a_{m'}$	$b_{n'}$	$j_{p'}$
a_m	a_m	a_m	j_m	j_m
b_n	b_n	j_n	b_n	j_n
j_p	j_p	j_p	j_p	j_p

and

$\wedge$	0	$a_{m'}$	$b_{n'}$	$j_{p'}$
0	0	0	0	0
a_m	0	$a_{m'}$	0	$a_{p'}$
b_n	0	0	$b_{n'}$	$b_{p'}$
j_p	0	$a_{m'}$	$b_{n'}$	$j_{p'}$

.

$$\begin{array}{ccccc} & & j_1 - j_2 & & \\ & \therefore & & \ddots & \\ a_1 - a_2 & & & & b_1 - b_2 \\ & \ddots & & \therefore & \\ & & 0 & & \end{array}$$

This skew lattice is jointly determined by being right handed with the displayed Clifford-Mclean picture and having $\geq$ given by $j_n \geq$ both a_n, $b_n \geq 0$ for n = 1 or 2. While a_1 and b_2 $\wedge$-commute, they clearly do not $\vee$-commute. We denote this example by $\mathbf{NS}_7^{\mathcal{R},0}$. (See Section 5.2) ☐

A skew lattice S is ***symmetric*** if for all $e, f \in$ S, $e \vee f = f \vee e$ iff $e \wedge f = f \wedge e$. A symmetric alternative to the above example is as follows:

Example 2.2.1b. The direct product of the right-handed algebras $\begin{array}{c}\{a_1, a_2\} \\ \vdots \\ \{0\}\end{array}$ and $\begin{array}{c}\{b_1, b_2\} \\ \vdots \\ \{0\}\end{array}$ is symmetric. Its $\mathcal{D}$-class diagram (with "redundant" 0-coordinates suppressed) is as follows. The partial ordering between the top class and the intermediate classes given by coordinate projection.

$$\begin{array}{ccc} & \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\} & \\ \therefore & & \ddots \\ \{a_1,\ a_2\} & & \{b_1,\ b_2\} \\ \ddots & & \therefore \\ & 0 & \end{array} \quad . \ ☐$$

Two elements in a skew lattice *commute* if they commute under both operations. If they just commute under a single operation, they are said to $\wedge$-commute or $\vee$-commute, as the case may be. The following statement is easily verified.

Proposition 2.2.3. *Any set of mutually commuting elements in a symmetric skew lattice must generate a sublattice of the skew lattice.* □

This is not the case with nonsymmetric skew lattices.

Example 2.2.2. Consider the 13-element right-handed skew lattice generated from a, b and c'. In the diagram, "---" indicates $\mathcal{R}$-equivalence and the slanted lines indicate the natural partial ordering. Thus in particular $e > a, d$; $e' > a', d'$; $f > d, c$; and $f' > d', c'$.

$$\begin{tikzcd}[column sep=small]
 & & & 1 \arrow[dll, dash, dotted] \arrow[drr, dash, dotted] & & & \\
 & e \arrow[r, dash, dashed] & e' & & f \arrow[r, dash, dashed] & f' & \\
a \arrow[r, dash, dashed] \arrow[urr, dash, dotted] & a' \arrow[u, dash, dotted] & & d \arrow[r, dash, dashed] \arrow[ull, dash, dotted] \arrow[ur, dash, dotted] & d' \arrow[ull, dash, dotted] \arrow[ur, dash, dotted] & c \arrow[r, dash, dashed] \arrow[ul, dash, dotted] & c' \arrow[ul, dash, dotted] \\
 & & & b \arrow[u, dash, dotted] & & & \\
 & & & 0 \arrow[u, dash, dotted] \arrow[uulll, dash, dotted] \arrow[uurrr, dash, dotted] & & &
\end{tikzcd}$$

Here $a \vee b = e = b \vee a$, $b \vee c' = f' = c' \vee b$ and $a \vee c' = 1 = c' \vee a$, while all three pairs of elements meet at 0. Collectively, a, b and c' generate S. But $a \vee b = e$ does not $\wedge$-commute with $b \vee c' = f'$. (We will see in Section 2.4 how the natural partial ordering coupled with the $\mathcal{D}$-class diagram can determine the outcome of either operation.) Since b is central and $\{a, c'\}$ generates the sublattice $\{1, a, c', 0\}$, we see that *a central element and a sublattice of a nonsymmetric skew lattice neednot generate a (possibly larger) sublattice*. □

Theorem 2.2.4. *Symmetric skew lattices form a subvariety of skew lattices characterized by the identities*: $x \wedge y \wedge (x \vee y \vee x) = (x \vee y \vee x) \wedge y \wedge x$ *and* $x \vee y \vee (x \wedge y \wedge x) = (x \wedge y \wedge x) \vee y \vee x$.

Proof. Given elements x and y in a skew lattice, x will $\vee$-commutes with $(x \vee y \vee x) \wedge y \wedge (x \vee y \vee x)$, the join being $x \vee y \vee x$. Moreover, if x $\vee$-commutes with y then $(x \vee y \vee x) \wedge y \wedge (x \vee y \vee x) = y$. Thus all elements in S that $\vee$-commute with x have the form $(x \vee y \vee x) \wedge y \wedge (x \vee y \vee x)$. Likewise, all elements in S that $\wedge$-commute with x are of the form $(x \wedge y \wedge x) \vee y \vee (x \wedge y \wedge x)$. Hence symmetric skew lattices are characterized by the identity

$$x \wedge (x \vee y \vee x) \wedge y \wedge (x \vee y \vee x) = (x \vee y \vee x) \wedge y \wedge (x \vee y \vee x) \wedge x$$

and its dual. Thus the displayed identities follow from absorption. □

Corollary 2.2.5. *Symmetric right-handed skew lattices are characterized by the two identities*: $x \wedge y \wedge (x \vee y) = y \wedge x$ *and* $x \vee y = (y \wedge x) \vee y \vee x$.

Proof. If S is a right-handed symmetric skew lattice, then it satisfies the identities of the above theorem plus the identities $x \vee y \vee x = x \vee y$ and $x \wedge y \wedge x = y \wedge x$ from which the displayed identities follow. Conversely, if the displayed identities hold on a skew lattice S, then one has

$$x \wedge y \wedge x = x \wedge (y \wedge x) = x \wedge [x \wedge y \wedge (x \vee y)] = x \wedge y \wedge (x \vee y) = y \wedge x$$

and similarly $x \vee y \vee x = x \vee y$ so that S is right-handed. But using $x \vee y \vee x = x \vee y$ and $x \wedge y \wedge x = y \wedge x$, the displayed identities can be transformed back into the identities of the previous theorem. ☐

Our interest in symmetry is due further to the following three theorems.

Theorem 2.2.6. *All skew lattices in rings (using multiplication and the circle operation) are symmetric.*

Proof. Obviously $a + b - ab = b + a - ba$ if and only *if* $ab = ba$. ☐

A ***lattice section*** in a skew lattice S is any sublattice T of S having nonempty intersection with each $\mathcal{D}$-class of S, in which case, T ≅ S/$\mathcal{D}$.

Theorem 2.2.7. *If* S *is a symmetric skew lattice for which* S/$\mathcal{D}$ *is countable, then* S *has a lattice section*.

Proof. Let $\mathcal{D}_1$, $\mathcal{D}_2$, … be a listing of all $\mathcal{D}$-classes. Pick $x_1 \in \mathcal{D}_1$. By Theorem 2.2.1 we know that in every $\mathcal{D}$-class of S elements exist that commute with x. By symmetry we know that such commuting is under both operations and, moreover, that the set of all such elements is closed under both operations. Thus $S_1 = \{y \in S \mid y$ commutes with $x\}$ is a sub-skew lattice of S that has nonempty intersection with each $\mathcal{D}$-class of S and for which $\mathcal{D}_{x_1} = \{x_1\}$ as a $\mathcal{D}$-class in S_1. If S_1 is not a lattice (and hence not a lattice section of S) we find an element x_2 in $\mathcal{D}_2 \cap S_1$ and thus not in $\mathcal{D}_{x_1}$ in S_1. The argument repeats. We thus obtain a descending chain of sub-skew lattices

$$S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots$$

each having nonempty intersection with each $\mathcal{D}$-class of S, and such that $\mathcal{D}_j \cap S_i = \{x_j\}$ for all $i \geq j$ where each x_j commutes with all x_i that precede it. Clearly, the full intersection $\bigcap_{n \geq 1} S_n$ is a lattice section of S. ☐

A ***left-handed section*** of S is any left-handed sub-skew lattice S_L of S such that its intersection with each $\mathcal{D}$-class in S is an $\mathcal{L}$-class. A ***right-handed section*** S_R of S is similarly defined. An internal (subobject) variant of the Kimura decomposition is the following result:

Theorem 2.2.8. (Cvetko-Vah) *Given a skew lattice the following are equivalent*:

1) S *has both a left-handed section and a right-handed section*.
2) *Sub-skew lattices* S_L *and* S_R *exist whose intersection with any* $\mathcal{D}$*-class is an* $\mathcal{L}$*-class, or respectively, an* $\mathcal{R}$*-class of* S.
3) S *has a lattice section* S_0.

When these conditions hold, then:

4) *The natural epimorphisms* $S \to S/\mathcal{R}$ *and* $S \to S/\mathcal{L}$ *induce isomorphisms of* S_L *with* $S/\mathcal{R}$ *and* S_R *with* $S/\mathcal{L}$.
5) *Every* $x \in S$ *factors uniquely as* $a = a' \wedge a''$ *with* $a' \in S_L \cap \mathcal{D}_x$, $a'' \in S_R \cap \mathcal{D}_x$. *Under this decomposition*

$$(a'\wedge a'')\wedge(b'\wedge b'') = (a'\wedge b')\wedge(a''\wedge b'') \quad \textit{and} \quad (a'\wedge a'')\vee(b'\wedge b'') = (a'\vee b')\wedge(a''\vee b'').$$

6) *The functions* $\pi_L\colon S \to S_L$ *and* $\pi_R\colon S \to S_R$ *defined by* $\pi_L(a) = a'$ *and* $\pi_R(a) = a''$ *are retractions of* S *upon* S_L *and* S_R *respectively and the commuting composite* $\pi_L\pi_R$ *is a retraction of* S *upon* S_0; *moreover* $\mathcal{R} = \ker(\pi_L)$, $\mathcal{L} = \ker(\pi_R)$ *and* $\mathcal{D} = \ker(\pi_L\pi_R)$.

Proof. (2) clearly implies (1). Given (1), expand S′ to the subset $S_L = \bigcup\{\mathcal{L}_a \mid a \in S'\}$ and expand S″ to the subset $S_R = \bigcup\{\mathcal{R}_a \mid a \in S''\}$. Since $\mathcal{L}$ and $\mathcal{R}$ are congruences, S_L and S_R are as stated in (2). Given (2) again, since every pair of $\mathcal{L}$ and $\mathcal{R}$-classes in a $\mathcal{D}$-class must meet at a single element in that $\mathcal{D}$-class, (3) follows from (2). Conversely, given (3) one has a minimal case of (1). Thus (1) – (3) are equivalent. Since S_L is an $\mathcal{R}$-class cross-section, the map $S \to S/\mathcal{R}$ restricts to an isomorphism of S_L upon $S/\mathcal{R}$. Similar remarks apply to $S_R \to S/\mathcal{L}$ and (4) follows. (5) follows from (4) and the embedding $S \cong S/\mathcal{R} \times_{S/\mathcal{D}} S/\mathcal{L} \subseteq S/\mathcal{R} \times S/\mathcal{L} \cong S_L \times S_R$ given by $a = a'\wedge a'' \to (a', a'')$. To see (6), first set $a_L = a\wedge a_0$ and $a_R = a_0\wedge a$. This gives

$$a\wedge b = (a\wedge a_0\wedge a)\wedge(b\wedge b_0\wedge b)$$

while

$$(a_L\wedge b_L)\wedge(a_R\wedge b_R) = (a\wedge a_0\wedge b\wedge b_0)\wedge(a_0\wedge a\wedge b_0\wedge b) = (a\wedge a_0\wedge b)\wedge(a\wedge b_0\wedge b).$$

But both "middles" are equal since

$$a_0\wedge a\wedge b\wedge b_0 = a_0\wedge a\wedge b\wedge a_0\wedge b_0\wedge a\wedge b\wedge b_0 = a_0\wedge y\wedge a_0\wedge b_0\wedge a\wedge b_0 = a_0\wedge b\wedge a\wedge b_0.$$

Thus $a\wedge b = (a_L\wedge b_L)\wedge(a_R\wedge b_R)$ follows. Recalling that $\mathcal{R}_{(\wedge)} = \mathcal{L}_{(\vee)}$ and $\mathcal{L}_{(\wedge)} = \mathcal{R}_{(\vee)}$, so that $a = a_R\vee a_L$ and $b = b_R\vee b_L$ gives: $a\vee b = (a_R\vee b_R) \vee (a_L\vee b_L) = (a_L\vee b_L) \wedge (a_R\vee b_R)$. □

In the case where S has a lattice section, Theorem 2.2.8 (4)-(6) describes an ***internal (monic) Kimura decomposition*** of S in contrast to the ***external (epic) Kimura decomposition*** of

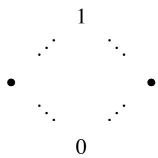
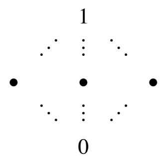
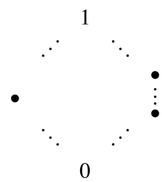
Theorem 2.15. Both decompositions are similar in that they both involve copies of the maximal lattice, left-handed and right-handed images of S. While external decompositions always occur and use quotient objects, internal decompositions that use subobjects need not always occur.

One is led to ask the following question. *Is symmetry really needed for the all this?* Otherwise put, *does a skew lattice* S *exist for which* S/$\mathcal{D}$ *is countable, or even finite, but* S *has no lattice section?*

Example 2.2.2 continued. Returning to the non-symmetric Example 2.2.2, a lattice section is given by the subset $\{0, a, b, c, d, f, 1\}$.

Some elementary, albeit useful, cases where symmetry is not needed are as follows.

Theorem 2.2.9. *A skew lattice* S *will have a lattice section* T *if its maximal lattice image* S/$\mathcal{D}$ *is a copy of one of the following lattices:*

(i) **2 × 2**  (ii) M_3  (iii) N_5 

In these cases, S *also has internal copies of* S/$\mathcal{R}$ *and* S/$\mathcal{L}$, *both occurring as maximal left and right-handed subalgebras of* S.

Proof. First, suppose S/$\mathcal{D}$ is a copy of **2** × **2**. Let a, b, e, f in S be such that e is in the top $\mathcal{D}$-class, f lies in the bottom $\mathcal{D}$-class, and a and b are separately in the two incomparable $\mathcal{D}$-classes. We create a lattice section as follows.

1. Reset a to be $e{\wedge}a{\wedge}e$, b to be $e{\wedge}b{\wedge}e$ and f to be $e{\wedge}f{\wedge}e$ so that $a, b, f < e$.
2. Reset a to be $f{\vee}a{\vee}f$ and b to be $f{\vee}b{\vee}f$ so that $f < a, b < e$ is a lattice section.

Next suppose S/$\mathcal{D}$ is a copy of M_3. Let a, b, c, e, f in S be such that e is in the top $\mathcal{D}$-class, f is in the bottom $\mathcal{D}$-class, and a, b and c lie separately in the three incomparable $\mathcal{D}$-classes. We create a lattice section as follows.

1. Reset all x among a, b, c, f to be $e{\wedge}x{\wedge}e$ so that now $a, b, c, f < e$.
2. Reset all y among a, b, c to be $f{\vee}y{\vee}f$ so that $f < a, b, c < e$ is a lattice section.

Finally suppose S/$\mathcal{D}$ is a copy of N_5. Let a, b, c, e, f in S be such that e and f are as before, a, b and c lie separately in the three middle $\mathcal{D}$-classes, where the $\mathcal{D}$-class of a is incomparable with the $\mathcal{D}$-classes of b and c, but the $\mathcal{D}$-class of b lies above the $\mathcal{D}$-class of c. We create a lattice section by first repeating the first two steps so that $f < a, b, c < e$ and then:

3. Reset c to be $b{\wedge}c{\wedge}b$ so that we also have $c < b$.

The final assertion follows from Theorem 2.2.8. ☐

A skew lattice S is ***quasi-distributive*** if its lattice image $S/\mathcal{D}$ is distributive. *Quasi-distributive skew lattices are a subvariety of skew lattices*. Indeed, since $(x\wedge y) \vee (x\wedge z) \preceq x \wedge (y\vee z)$ holds for all lattices, these skew lattices are characterized by the identity:

$$[x\wedge(y\vee z)] \wedge [(x\wedge y) \vee (x\wedge z)] \wedge [x\wedge(y\vee z)] = x\wedge(y\vee z).$$

We have the following corollary to Theorem 2.2.9.

Theorem 2.2.10. *A skew lattice* S *is quasi-distributive if and only if neither* M_3 *nor* N_5 *is a subalgebra of* S.

2.3 Normal skew lattices

Recall that a band S is ***normal*** if it satisfies any and hence all of:

a) $\forall a, b, c \in S$: $abca = acba$.
a′) $\forall a, b, c, d \in S$: $abcd = acbd$.
b) $\forall a \in S$, $aSa = \{axa \mid x \in S\}$ is a semilattice in S.
b′) $\forall a, b \in S$, $aSb = \{axb \mid x \in S\}$ is a semilattice in S.
c) Given $\mathcal{D}$-classes $A \geq B$ in S, $\forall a \in A, \exists! b \in B, a \geq b$.

Likewise, a skew lattice S is ***normal*** if either (and thus both) of the following hold:

a) $(S: \wedge)$ is a normal band. In particular, $abcd = acbd$ holds on $(S: \wedge)$.
b) $\forall a \in S$, $a\wedge S\wedge a = \{a\wedge x\wedge a \mid x \in S\} = \{b \in S \mid a \geq b\}$ is a sublattice of S.

Clearly: *Normal skew lattices form a subvariety of skew lattices*.

Applying the Kimura decomposition we get the elementary but important:

Theorem 2.3.1. *A skew lattice* S *is normal if and only if its left factor* $S/\mathcal{R}$ *is left normal* ($a\wedge b\wedge c = a\wedge c\wedge b$) *and its right factor* $S/\mathcal{L}$ *is right normal* ($a\wedge b\wedge c = b\wedge a\wedge c$). ☐

Being normal has interesting connections with distributivity.

Theorem 2.3.2. [Leech, 1992] *Given a skew lattice* S*, the following are equivalent:*

1) $a \wedge (b \vee c) \wedge d = (a \wedge b\wedge d) \vee (a \wedge c \wedge d)$ *holds on* S.
2) S *is distributive and normal*.
3) $S/\mathcal{D}$ *is distributive and* S *is normal*.

Proof. First observe that each of these conditions holds on S if and only if it holds on both its left factor $S/\mathcal{R}$ and its right factor $S/\mathcal{L}$. Thus, we may prove the theorem by proving it for all right [left]-handed algebras. So suppose that (1) holds for a right-handed skew lattice, S. We show that if $a \geq b, c$ in S with $b\ \mathcal{R}\ c$, then $b = c$. Indeed, the following instance of (1)

$$b \wedge (a \vee c) \wedge a \quad = \quad (b \wedge a \wedge a) \vee (b \wedge c \wedge a)$$

reduces first to $b \wedge c \wedge a \quad = \quad b \vee c$, and then to $c = b$. Hence S is right normal and thus normal. Next, consider the $\mathcal{D}$-equivalent expressions, $a \vee (b \wedge c) \vee a$ and $(a \vee b \vee a) \wedge (a \vee c \vee a)$. Assuming right normality. They are actually the $\mathcal{R}$-equivalent expressions, $a \vee (b \wedge c)$ and $(a \vee b) \wedge (a \vee c)$. Since $[(a \vee b) \wedge (a \vee c)] \vee (a \vee c) \ = \ a \vee c$ by absorption, and

$$[a \vee (b \wedge c)] \vee (a \vee c) \ = \ a \vee (b \wedge c) \vee c] \ = \ a \vee c.$$

Hence $a \vee c \geq$ both $a \vee (b \wedge c)$ and $(a \vee b) \wedge (a \vee c)$. S being normal with $a \vee (b \wedge c)\ \mathcal{R}\ (a \vee b) \wedge (a \vee c)$ we get $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. Thus (1) implies (2). Next, (2) clearly implies (3). Suppose that (3) holds. Then by normality,

$$a \wedge d \ \geq \ a \wedge (b \vee c) \wedge d,\ a \wedge b \wedge d,\ a \wedge c \wedge d \text{ and hence } a \wedge d \geq (a \wedge b \wedge d) \vee (a \wedge c \wedge d)$$

with $a \wedge (b \vee c) \wedge d\ \mathcal{D}\ (a \wedge b \wedge d) \vee (a \wedge c \wedge d)$ since $S/\mathcal{D}$ is distributive. Whence,

$$a \wedge (b \vee c) \wedge d \quad = \quad (a \wedge b \wedge d) \vee (a \wedge c \wedge d),$$

again by normality. Thus (3) implies (1). ☐

Lemma 2.3.3. *In any normal skew lattice,* $a \vee b = b \vee a$ *implies* $a \wedge b = b \wedge a$.

Proof. Indeed, $a \vee b = b \vee a$ implies $a, b \in (a \vee b) \wedge S \wedge (a \vee b)$, a sublattice, so that $a \wedge b = b \wedge a$. ☐

A skew lattice is ***strongly distributive*** if it satisfies both (4) and (5) below:

4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.
5) $(b \vee c) \wedge d = (b \wedge d) \vee (c \wedge d)$.

Theorem 2.3.4. *A skew lattice* S *is strongly distributive if and only if it is distributive, normal and symmetric*.

Proof. A skew lattice satisfying (4) and (5) also satisfies (1), and thus is distributive and normal. Being normal, $a \vee b = b \vee a$ implies $a \wedge b = b \wedge a$ holds. So let $a \wedge b = b \wedge a$. If $a \vee b \neq b \vee a$, then either $a \wedge (b \vee a) \neq a$ or $(b \vee a) \wedge b \neq b$, for otherwise absorption gives $a \vee b \vee a = b \vee a = b \vee a \vee b$ so that $a \vee b = b \vee a$. Suppose that $a \wedge (b \vee a) \neq a$. But then (4) fails in general since $a \wedge (b \vee a) \neq a$ while

$$(a \wedge b) \vee (a \wedge a) = (a \wedge b) \vee a = (b \wedge a) \vee a = a.$$

Similarly $(b \vee a) \wedge b \neq b$ implies that (5) fails in general. Hence symmetry must also follow from (4) and (5) combined.

Conversely, suppose that S is distributive, symmetric and normal. Assume in addition that S is right-handed. Thus both distributive laws

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \text{and} \quad (b \vee c) \wedge d = (b \wedge d) \vee (c \wedge d)$$

hold by the right-handed version of the standard distributive laws. We consider the status of

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

First, b $\wedge$-commutes with both $a\wedge(b\vee c)$ and $(a\wedge b) \vee (a\wedge c)$ with $a\wedge b$ being their common meet.

$$b\wedge a\wedge(b\vee c) = b\wedge a\wedge b\wedge(b\vee c) = b\wedge a\wedge b = a\wedge b. \qquad a\wedge(b\vee c)\wedge b = a\wedge b\wedge(b\vee c)\wedge b = a\wedge b.$$
$$b\wedge[(a\wedge b) \vee (a\wedge c)] =_{\mathcal{R}} b\wedge[a\wedge(b\vee c)]\wedge[(a\wedge b) \vee (a\wedge c)] = b\wedge(b\vee c)\wedge a\wedge[(a\wedge b) \vee (a\wedge c)]$$
$$= b\wedge a\wedge[(a\wedge b) \vee (a\wedge c)] = b\wedge(a\wedge b)\wedge[(a\wedge b) \vee (a\wedge c)] = b\wedge(a\wedge b) = a\wedge b.$$
$$[(a\wedge b) \vee (a\wedge c)] \wedge b = (a\wedge b) \vee (a\wedge c\wedge b) = (a\wedge b) \vee (a\wedge b\wedge c\wedge b) = (a\wedge b)$$

Next, the necessarily commuting join of b with either $a\wedge(b\vee c)$ or $(a\wedge b)\vee(a\wedge c)$ is $(b\vee a)\wedge(b\vee c)$. Indeed

$$b \vee [a\wedge(b\vee c)] = (b\vee a) \wedge (b\vee b\vee c) = (b\vee a) \wedge (b\vee c)$$

and

$$b \vee [(a\wedge b)\vee(a\wedge c)] = b \vee (b\wedge a\wedge b) \vee (a\wedge c) = b \vee (a\wedge c) = (b\vee a)\wedge(b\vee c).$$

Hence $a\wedge(b\vee c)$ and $(a\wedge b) \vee (a\wedge c)$ are $\mathcal{D}$-equivalent and both are $\leq (b\vee a) \wedge (b\vee c)$. By normality they are equal. We have seen that (4) and (5) above hold when S is distributive, symmetric, normal and right-handed. This must also true in the left-handed case. It follows that they must hold when S is distributive, symmetric and normal. ☐

Corollary 2.3.5. *A normal skew lattice* (S; ○, •) *in a ring is strongly distributive*. ☐

Proof. Indeed, any skew lattice (S; ○, •) is already distributive and symmetric.

The simplest class of normal bands are the rectangular bands that satisfy $abc = ac$.

Example 2.3.1. The set of matrices $\left\{ \begin{bmatrix} 0 & A & AB \\ 0 & I^{j\times j} & B \\ 0 & 0 & 0 \end{bmatrix} \middle|\, A \in \mathcal{F}^{i\times j}, B \in \mathcal{F}^{j\times k} \right\}$ in $\mathcal{F}^{n\times n}$

with $i + j + k = n$ *and* $i + j < n$, for i, j, k fixed is a rectangular band. For such bands

$$\begin{bmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & C & CD \\ 0 & I & D \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & C & AB+CD-AD \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & C & CB \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix}$$

except in special cases. However, $\begin{bmatrix} 0 & C & AB+CD-AD \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & C & CB \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix}$. □

This leads us to define a ***cubic join*** $e\nabla f$ by

$$e\nabla f = e + f + fe - efe - fef.$$

The cubic join extends the ***quadratic join*** given by $\circ$ in the following sense:

1) If $e, f, ef, fe \in$ E(R), then $(e \circ f)^2 = e + f + fe - efe - fef = e\nabla f \in$ E(R) also.

2) Every skew lattice (S; $\circ$, $\bullet$) in a ring is trivially a skew lattice under ∇ and $\bullet$, since $e\nabla f$ reduces to $e \circ f$ whenever the latter is idempotent.

3) Situations occur where $e \circ f$ is not idempotent but $e\nabla f$ is. Indeed, $e\nabla f \in$ E(S) whenever $e, f, ef, fe \in$ E(S). (This combines (1) with the previous example.)

4) **Caveat**: While $\circ$ is always associative, ∇ needn't be, even when giving idempotent closure.

Example 2.3.2. (Karin Cvetko-Vah) Consider the following matrix band with two $\mathcal{D}$-classes:

$$\left\{ \begin{bmatrix} 0 & x_1 & x_2 & x_1y_1 + x_2y_2 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, x_1, x_2, y_1, y_2 \in \mathcal{F} \right\}$$

$$\downarrow$$

$$\left\{ \begin{bmatrix} 0 & u & uv_1 & uv_2 \\ 0 & 1 & v_1 & v_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, u, v_1, v_2 \in \mathcal{F} \right\}$$

This band is also closed under ∇. However, upon setting

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we get

$$A\nabla(B\nabla C) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq (A\nabla B)\nabla C = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \ \square$$

A multiplicative band in a ring R that is closed under ∇ is called a ***∇-band***. Observe that the ∇-band in Example 2.3.2 is not normal. Indeed:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} > \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & v & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ for all } v.$$

For normal ∇-bands in rings the following results hold.

Theorem 2.3.6. *Every normal ∇-band in a ring is a strongly distributive skew lattice.*

Proof. Given e, f, g in a normal ∇-band S, observe that $e\nabla(f\nabla g) = e\nabla(f + g + gf - fgf - gfg)$ calculates to

$$\begin{aligned} & e + f + g + gf - fgf - gfg + fe + ge + \underline{gfe} - fgfe - \underline{gfge} \\ & \quad - e(f + g + \underline{gf} - \underline{fgf} - gfg)e - (f + g + gf - fgf - gfg)e(f + g + gf - fgf - gfg) \\ & = e + f + g + (gf + fe + ge) - (efe + fef + fgf + gfg + ege + geg) \\ & \quad - (fge + gef) + (efge + fegf + gefg) \end{aligned}$$

where in this calculation repeated use is made by the identity $xyzw = xzyw$. Similarly,

$$(e\nabla f)\nabla g = (e + f + fe - efe - fef)\nabla g$$

calculates to the same final expression, showing that $\nabla$is associative. Next, note that

$$e(e\nabla f) = e + ef + efe - eefe - efef = e.$$

Likewise, $e\nabla(ef) = e + ef + efe - eefe - efeef = e$. Similarly $(e\nabla f)f$ and $(ef)\nabla f$ reduce to f. Thus S is a skew lattice. Since $e\nabla f$ and $f\nabla e$ differ only in the respective terms fe and ef, S is symmetric. Finally,

$$\begin{aligned} e(f\nabla g)e &= e(f + g + gf - fgf - gfg)e = efe + ege + egfe - efgfe - egfge \\ &= efe + ege + egefe - efegefe - egefege = (efe)\nabla(ege). \end{aligned}$$

Applying the identity $e\nabla f\nabla e = e + f - fef$ (replace g by e in the above calculation of $e\nabla f\nabla g$ and then simplify) we get

$$\begin{aligned}(e\nabla f\nabla e)\,(e\nabla g\nabla e) &= (e + f - fef)(e + g - geg)\\ &= e + eg - eg + fe + fg - fgeg - fe - feg + fefgeg\\ &= e + fg - feg \;=\; e + fg - fgefg \;=\; e\nabla(fg)\nabla e.\end{aligned}$$

Thus S is also distributive. □

The above theorem has several consequences of significance for skew lattices.

Theorem 2.3.7. *If* R *is a ring for which* **E**(R) *is closed under multiplication, then* **E**(R) *is a normal skew lattice under* ∇ *and* $\bullet$. *In particular,* **E**(R) *is closed under multiplication if* R *satisfies the identity, $abcd = acbd$.* **E**(R) *is a Boolean lattice when* R *has an identity,* 1.

Proof. We prove the last statement first. So let R be a ring with identity 1 for which **E**(R) is closed under multiplication. Then for all $e \in$ **E**(R), $1 - e \in$ **E**(R) also with $e(1 - e) = 0$. Since **E**(R) is a band, for all $f \in$ **E**(R), $ef(1 - e) = 0$ also (The Clifford-McLean Theorem). Thus $ef = efe$ holds for all $e, f \in$ **E**(R). Likewise, from $(1 - e)fe = 0$ we get $fe = efe$. Thus **E**(R) is commutative under multiplication, forcing (**E**(R); $\circ$, $\bullet$, 1, 0) to be a Boolean lattice.

Suppose next that R is any ring for which **E**(R) is a multiplicative band. Given $e \in$ **E**(R), e**E**(R)e = **E**(eRe) is a sub-band that is necessarily commutative since e is the identity of eRe. Thus **E**(R) is a normal band. Moreover, for any $e, f \in$ **E**(R),

$$(e\nabla f)^2 = (e + f + fe - efe - fef)^2 = e + (fe + f - fef) + fe - efe - fe = (e\nabla f)$$

where each term in the third expression is a reduction of $x(e\nabla f)$ where x is one of the terms in $e\nabla f$. Thus **E**(R) is closed under ∇, and being normal forms a skew lattice under ∇ and $\bullet$.

Finally, if R satisfies the identity $abcd = acbd$, then $(ef)^2 = efef = eeff = ef$ for any pair of idempotents e and f. Thus **E**(R) is indeed a multiplicative band and the theorem follows. □

Corollary 2.3.8. *A normal band in a ring generates a normal skew lattice under* ∇ *and* $\bullet$. *A maximal normal band in a ring* R *thus forms a normal skew lattice under* ∇ *and* $\bullet$.

Proof. For such a band B, the subring S generated from B satisfy $xyzw = xzyw$. Thus B $\subseteq$ **E**(S) which is a normal band forming a skew lattice under ∇ and $\bullet$. By maximality, B = **E**(S) . □

Example 2.3.3. If R is the semigroup ring A[B] with A a commutative ring and B a normal band, then **E**(R) is a normal skew lattice. □

Example 2.3.4. [Karin Cvetko-Vah] Consider the band of all $(n+2)\times(n+2)$ matrices of the form:

$$\left\{ \begin{bmatrix} 0 & a_1 & a_2 & a_3 & a_4 & \dots & c \\ 0 & \delta_1 & 0 & 0 & 0 & \dots & b_1 \\ 0 & 0 & \delta_2 & 0 & 0 & \dots & b_2 \\ 0 & 0 & 0 & \delta_3 & 0 & \dots & b_3 \\ 0 & 0 & 0 & 0 & \delta_4 & \dots & b_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix} \;\middle|\; \begin{aligned} \delta_i &= 0, 1 \\ a_i &= a_i \delta_i \\ b_j &= b_j \delta_j \\ c &= \textstyle\sum a_i b_i \end{aligned} \right\}.$$

This is the normal skew lattice **E**(R) for the upper triangular subring R that it ring-generates. ☐

Query: Are maximal regular bands in rings skew lattices under ∇? Do regular bands in rings generate skew lattices under ∇? Do they even generate ∇-bands? That this is not the case in general is guaranteed by:

Example 2.3.5. In the semigroup ring $\mathbb{Z}[\mathbf{Reg}_{a,b,c}]$ of the free regular band on $\{a, b, c\}$, the product $(a \nabla b)c = ac + bc + bac - abac - babc$ fails to be idempotent. ☐

2.4 Primitive skew lattices and skew lattice structure

A ***primitive skew lattice*** is a skew lattice P consisting of exactly two $\mathcal{D}$-classes $A > B$. Primitive skew lattices and their relation to arbitrary skew lattices were studied in Leech [1993]. We present a number of results from that paper.

Given a primitive skew lattice P with $\mathcal{D}$-classes A > B in P/$\mathcal{D}$, a ***coset*** of A in B is any subset of B of the form $A \wedge b \wedge A = \{a \wedge b \wedge a' \mid a, a' \in A\}$ for some fixed $b \in B$. Similarly, a ***coset*** of B in A is any subset of the form $B \vee a \vee B = \{b \vee a \vee b' \mid b, b' \in B\}$ for some $a \in A$. Since both operations are regular, an alternative description of both types of cosets is given by

$$A \wedge b \wedge A = \{a \wedge b \wedge a \mid a \in A\} \text{ and } B \vee a \vee B = \{b \vee a \vee b \mid b \in B\}.$$

Indeed, $\{a \wedge b \wedge a \mid a \in A\} \subseteq A \wedge b \wedge A$ as already defined.. But by regularity,

$$a \wedge b \wedge a' = a \wedge a' \wedge a \wedge b \wedge a' \wedge a \wedge a' = (a \wedge a') \wedge b \wedge (a \wedge a')$$

which is of the form $a \wedge b \wedge a$. The case for $B \vee a \vee B$ is similar. For any $a \in A$, the set

$$a \wedge B \wedge a = \{a \wedge b \wedge a \mid b \in B\} = \{b \in B \mid b \leq a\}$$

is the ***image set*** of a in B. Its elements are the ***images*** of a in B. Dually, given any $b \in B$, the set $b \vee A \vee b = \{a \in A; a \geq b\}$ is the ***image set*** of b in A. We have the following fundamental result:

Theorem 2.4.1. *Let* P *be a primitive skew lattice with $\mathcal{D}$-classes* $A > B$. *Then*

(1) B *is partitioned by the cosets of* A *in* B. *In particular*, $b \in A \wedge b \wedge A$ *for all* $b \in B$ *and if* $x \in A \wedge b \wedge A$ *for some* $x \in B$, *then* $A \wedge x \wedge A = A \wedge b \wedge A$.

(2) *The image set in* B *of any* $a \in A$ *is a transversal of the cosets of* A *in* B.

(3) *Dual remarks hold for cosets and element images of* B *in* A. *Furthermore*:

(4) *Given cosets* $B \vee a \vee B$ *in* A *and* $A \wedge b \wedge A$ *in* B *a natural bijection of both cosets is given by the natural partial ordering: x in* $B \vee a \vee B$ *corresponds to y in* $A \wedge b \wedge A$ *if and only if* $x \geq y$.

(5) *The meet and join on* P *are determined jointly by these coset bijections and the rectangular structure of each $\mathcal{D}$-class.*

Proof. By absorption, $b = (a \vee b) \wedge b \wedge (b \vee a)$ for all a in A so that $b \in A \wedge b \wedge A$. Given $x \in A \wedge b \wedge A$, say $x = m \wedge b \wedge n$ for $m, n \in A$, then for all $a \in A$, $a \wedge x \wedge a = a \wedge m \wedge b \wedge n \wedge a = a \wedge b \wedge a$ where the second identity holds since $\wedge$ is regular and $m, n \succeq a, b$. Thus (1) is seen and this conditional identity also gives us (2). Condition (3) follows by duality. Given cosets $B \vee a \vee B$ and $A \wedge b \wedge A$, for any x in $B \vee a \vee B$, by (2) $x \wedge b \wedge x$ is the unique element y of $A \wedge b \wedge A$ such that $x \geq y$. Dually, for each y in $A \wedge b \wedge A$, $y \vee a \vee y$ is the unique element x in $B \vee a \vee B$ such that $x \geq y$. Between these two cosets, the processes $x \to y \leq x$ and $y \to x \geq y$ are reciprocal and (4) is seen. Finally, given $x, y \in P$, whenever $x \,\mathcal{D}\, y$ then both $x \wedge y$ and $x \vee y$ are given by the rectangular structure of the common $\mathcal{D}$-class of x and y. Otherwise, say $x \in A$ and $y \in B$, we have $x \vee y = x \vee (y \vee x \vee y)$, $y \vee x = (y \vee x \vee y) \vee x$, $x \wedge y = (x \wedge y \wedge x) \wedge y$ and $y \wedge x = y \wedge (x \wedge y \wedge x)$. Since $y \vee x \vee y$ is the image of y in the B-coset in A containing x and $x \wedge y \wedge x$ is the image of x in the A-coset in B containing y, (5) is seen. ☐

Given A > B as above, if A is partitioned by cosets $\{A_i \mid i \in I\}$ and B is partitioned by cosets $\{B_j \mid j \in J\}$, then for each pair of indices i, j let φ_{ji}: $A_i \to B_j$ denote the ***coset bijection*** given by setting $\varphi_{ji}(x) = y$ if for x in A_i, y is the unique element in B_j such that $x \geq y$. Then for all $x \in A_i$ and $y \in B_j$

$$x \vee y = x \vee \varphi_{ji}^{-1}(y),\ \ y \vee x = \varphi_{ji}^{-1}(y) \vee x,\ \ x \wedge y = \varphi_{ji}(x) \wedge y \ \text{ and } \ y \wedge x = y \wedge \varphi_{ji}(x).$$

Thus it seems that any primitive skew lattice should be obtained by a fairly simple construction. To this end we begin by calling a right-handed primitive skew lattice ***right primitive***, with ***left primitive*** skew lattices defined in dual fashion. For right primitive skew lattices the description of a coset can be simplified as

$$A \wedge b \wedge A = b \wedge A = \{b \wedge a \mid a \in A\} \ \text{ and } \ B \vee a \vee B = B \vee a = \{b \vee a \mid b \in B\}$$

for any $a \in A$ and $b \in B$ since $a \wedge b \wedge a = b \wedge a$ and $b \vee a \vee b = b \vee a$. Dual remarks hold in the left-primitive case. All right [left] primitive skew lattices arise as follows.

A **P-graph** is a pair of partitioned disjoint sets $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$, where all A_i and all B_j have a common cardinality, together with a fixed set of bijections φ_{ji}: $A_i \to B_j$. The P-graph is denoted by the triple (A_i, φ_{ji}, B_j). A right primitive structure is induced on $A \cup B$ in the following way. First, $\wedge$ and $\vee$ restricted to either A or B must be $x \wedge y = y$ and $x \vee y = x$. But given $x \in A_i$ and $y \in B_j$ we set

$$x \wedge y = y,\ \ y \wedge x = \varphi_{ji}(x),\ \ x \vee y = x \ \text{ and } \ y \vee x = \varphi_{ji}^{-1}(y)\ .$$

The primitive skew lattice thus obtained, denoted by $P[A_i, \varphi_{ji}, B_j]$, has $\mathcal{D}$-classes $A > B$, with cosets A_i and B_j, and coset bijections φ_{ji}. A left primitive skew lattice $P'[A_i, \varphi_{ji}, B_j]$ is obtained in dual fashion: $x \wedge y = \varphi_{ji}(x)$, $y \wedge x = y$, $x \vee y = \varphi_{ji}^{-1}(y)$ and $y \vee x = x$.

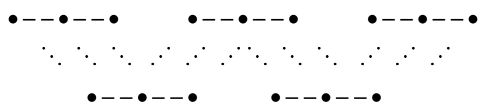
A partial image of a P-graph is given in the following diagram, where the top row of dots represents an upper $\mathcal{D}$-class A of order 9 and the bottom row represents a lower $\mathcal{D}$-class B of order 6. All cosets in this example have size 3, with the members of each coset linked in the diagram. The four arrows represent four of the six coset bijections, with arrows corresponding to φ_{20} and φ_{02} left out.

A: • --- • --- •$_0$ – • --- • --- •$_1$ – • --- • --- •$_2$

↘φ_{00} φ_{01}↙ ↘φ_{11} φ_{12}↙

B: • --- • --- •$_0$ – • --- • --- •$_1$

In terms of the natural partial order, $\geq$, the situation looks more like



where the dotted lines indicate the partial order relationships.

Applying Kimura factorization, every primitive skew lattice is isomorphic to the fibred product of a left primitive skew lattice and a right primitive skew lattice over their maximal lattice image**,** which here is isomorphic to $\mathbf{2} = \{1 > 0\}$. This factorization coupled with the ideas above yields:

Theorem 2.4.2. *Every primitive skew lattice* P *has a fibred product decomposition*,

$$P \cong P[A_i, \varphi_{ji}, B_j] \times_{P/\mathcal{D}} P'[C_k, \psi_{lk}, D_l],$$

where (A_i, φ_{ji}, B_j) *and* (C_k, ψ_{lk}, D_l) *denote* P-*graphs and* $P/\mathcal{D}$ *is isomorphic to the primitive lattice* $1 > 0$. *Both* P-*graphs are unique to within isomorphism of* P-*graphs*. □

The P-graph description of right [left] primitive skew lattices can be refined as follows. A P-graph ***coordinate system*** consists of a sextuple $(I, J, C, G, \mu, \theta)$ where

(i) I, J and C are nonempty sets.
(ii) G is a group and μ: G × C → C is a group action of group G on C.
(iii) θ: J × I → G is a map.

From this data construct a P-graph by first setting A = I × C and B = J × C. The B-cosets in A are the $A_i = \{i\} \times C$ and the A-cosets in B are the $B_j = \{j\} \times C$. Coset bijections are given by $\varphi_{ji}(i, c) = (j, \theta(j, i)c)$. From this data, a right or left primitive skew lattice is constructed to be denoted by P[I, J, C, G, μ, θ] or P′[I, J, C, G, μ, θ].

Every right [*left*] *primitive skew lattice has a coordinatization*. Given a P-graph representation, $P \cong P[A_i, \varphi_{ji}, B_j]$, let $C = A_0$ for some common index 0 in I ⋂ J.

$$\begin{array}{ccccccc} A_0 & & & A_i & & & A_0 \\ \downarrow \varphi_{00} & & \nearrow \varphi_{0i}^{-1} & \downarrow \varphi_{ji} & & \nearrow \varphi_{0j}^{-1} & \\ B_0 & & & B_j & & & \end{array}$$

Next, for each pair (j, i), let $\theta(j, i)$ be the permutation $\varphi_{j0}^{-1}\varphi_{ji}\varphi_{i0}^{-1}\varphi_{00}$ of A_0 and then let G denote the permutation group on A_0 generated collectively by the various $\theta(j, i)$. Here the coordinatization is ***normalized*** in that $\theta(\{0\} \times I \cup J \times \{0\}) = 1$ in G.

In the case where G = C and u is group multiplication, P has a ***coordinatization with group translations***. In this case the data reduces to the indexing sets I, J, the group G and map θ: I ×J → G, and the skew lattice is denoted by P[I, J, G, θ]. By the above discussion we may assume that $0 \in I \cap J$, and $\theta(\{0\} \times I \cup J \times \{0\}) = 1$ in G (or = 0 in the case of additive notation). An instance of this is given by any maximal right [left] primitive skew lattice in a ring.

Example 2.4.1. Given a ring R and an idempotent $e \in \mathbf{E}(R)$, the ***R-set*** of e in R is the set

$$R_e = e + eR(1 - e) = \{x \in R \mid ex = x \text{ and } xe = e\}$$

It is the maximal ***right-0 band*** ($xy = y$) in R containing e. R_e also forms a left-0 band under ○, and thus is a rectangular skew lattice in R. Let f be a second idempotent in R such that $e > f$. Then R_f forms a second rectangular skew lattice and the union $\mathbf{P}_{e>f} = R_e \cup R_f$ is a right primitive skew lattice in R with upper $\mathcal{D}$-class R_e and lower $\mathcal{D}$-class R_f. Its coordinatization is given as follows. To begin, notice that the group $A = eR(1 - e)$ acts simply transitively on R_e and that the group $B = fR(1 - f)$ acts simply transitively on R_f, both under the operation of addition. A and B share the common subgroup

$$G = A \cap B = eR(1 - e) \cap fR(1 - f) = fR(1 - e).$$

A splits as $fR(1 - e) \oplus (e - f)R(1 - e)$ and B splits as $fR(1 - e) \oplus fR(e - f)$. The various summands are naturally arranged in the following array format:

$$\begin{bmatrix} f & f\mathrm{R}(e-f) & f\mathrm{R}(1-e) \\ & e-f & (e-f)\mathrm{R}(1-e) \\ & & 1-e \end{bmatrix}.$$

B-cosets in A are given as

$$\mathrm{A}_c = e + \mathrm{G} + c.$$

for any $c \in (e-f)\mathrm{R}(1-e)$. Similarly, A-cosets in B are

$$\mathrm{B}_d = f + \mathrm{G} + d$$

for any $d \in f\mathrm{R}(e-f)$. Coset maps are given by

$$\varphi_{d,c}(e + g + c) = (f + d)(e + g + c) = f + [g + dc] + d.$$

Identifying A with R_e under $a \to e + a$, and similarly identifying B with R_f under $b \to f + b$, yields the coordinatization. $\mathrm{R}_e \cup \mathrm{R}_f$ corresponds to $\mathrm{G} \oplus (e-f)\mathrm{R}(1-e) \cup \mathrm{G} \oplus f\mathrm{R}(e-f)$. Under this correspondence, for all $c \in (e-f)\mathrm{R}(1-e)$ and all $d \in f\mathrm{R}(e-f)$, $\varphi_{d,c}(g, c) = (g + dc, d)$. Clearly $\theta\colon f\mathrm{R}(e-f) \times (e-f)\mathrm{R}(1-e) \to \mathrm{fR}(1-\mathrm{e}) = \mathrm{G}$ is given by the ring multiplication. □

Another instance involving coordinatization using group translations is as follows. To begin, a connected graph with each vertex having degree 2 is called a ***simple circuit*** in the finite case and an ***infinite simple path*** when infinite. By the ***natural graph*** of a primitive skew lattice **P** we mean the graph with vertices being the elements of P and with edges given by the relation for $> \cup >^{\mathrm{op}}$. That is, $e - f$ is an edge for $e, f \in \mathrm{P}$ if either $e > f$ or $f > e$. We state the following result without proof. (See Leech [1993].)

Theorem 2.4.3. *Let* **P** *be a right primitive skew lattice. Then the natural graph of* **P** *is a simple circuit precisely when* **P** *has a coordinatization by group translations under addition,* $\mathrm{P}[\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_n, \theta]$ *for some* $n \geq 1$, *where* $\theta(j, i) = ji$ *for* i, j *in* $\{0, 1\}$. *The graph is a simple path precisely when* **P** *has a coordinatization by group translations* $\mathbf{P}[\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}\ \theta]$ *where again* $\theta(j, i) = ji$ *in which case* $\mathbf{P}[\mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}, \theta]$ *is an infinite primitive skew lattice on four generators, namely* (0,0), (0,1), (1,0) *and* (1,1). □

Exercise. Show that infinite primitive skew lattices require at least four generators.

Primitive skew lattices, connectedness and maximal rectangular images

In general, the ***natural graph*** of a skew lattice S is the undirected graph on S generated from the natural partial order on S. Thus the vertices are just the elements of S and two vertices e and f are adjacent precisely when either $e > f$ or $f > e$. A ***component*** of S is any maximally connected subset of its natural graph. In particular, a skew lattice is ***connected*** if its natural graph has a single component. Theorem 2.2.1 implies that *each component of S has nonempty intersection with each* $\mathcal{D}$*-class of* S. Clearly the components of S are a partition of the underlying set.

G*iven a primitive skew lattice* P*, each component of* P *has nonempty intersection with each coset of* P. Thus we say that a primitive skew lattice P is ***maximally disconnected*** if distinct elements of each coset of P belong to distinct components of P. P is ***degenerate*** if it is both connected and maximally disconnected. If P has $\mathcal{D}$-class structure A > B, then P is degenerate precisely when $a > b$ for all $a \in$A and all $b \in$B. Finally, a ***coset component*** of P is any (necessarily nonempty) intersection of a coset in P with a component of P. For primitive skew lattices we have the following results:

Lemma 2.4.4. *Given a primitive skew lattice* P, *the cosets of* P *form a congruence partition of* P. *If* ζ *is the corresponding congruence, then the primitive skew lattice* P/ζ *is the maximal degenerate image of* P.

Proof. Given our descriptions of primitive skew lattices in Theorems 2.4.1 and 2.4.2 above, pointwise computations yield $A_i \vee B_j = A_i = B_j \vee A_i$ and $A_i \wedge B_j = B_j = B_j \wedge A_i$. ☐

Theorem 2.4.5. *If* P *is a primitive skew lattice, then its components are the maximal connected subalgebras of* P. *Moreover, they are the congruence classes of the congruence* ρ *whose quotient algebra* P/ρ *is the maximal rectangular image of* P. *Thus the coset components of* P *form the congruence partition for the congruence* $\zeta\cap\rho$ *for which the quotient algebra* P/$\zeta\cap\rho$ *is the maximal disconnected image of* P. *Finally, if* P *is maximally disconnected, then it factors as the product of a degenerate skew lattice with a rectangular skew lattice*.

Proof. The components of P clearly induce an equivalence relation ρ on P. To begin, let σ denote the symmetric closure of $\geq$. Thus $a \;\sigma\; b$ means that either $a \geq b$ or $b \geq a$. Since ρ is the transitive closure of σ, to show that ρ is a congruence we need only show that $a \;\sigma\; b$ and $a' \;\sigma\; b'$ imply that both $a\vee a' \;\rho\; b\vee b'$ and $a\wedge a' \;\rho\; b\wedge b'$. We first show this under the added restriction that P is right primitive. Thus $a\vee a' \geq a$ and $b\vee b' \geq b$ so that $a\vee a' \;\rho\; b\vee b'$ follows. Similarly, $a\wedge a' \leq a'$ and $b\wedge b' \leq b'$ so that $a\wedge a' \;\rho\; b\wedge b'$ also follows and ρ is shown to be a congruence. Similarly, if P is left primitive then again, ρ is a congruence. By the Kimura factorization, ρ is a congruence on any primitive skew lattice. The first part of the theorem follows. Since the coset components arise as congruence classes for the meet congruence $\zeta\cap\rho$, the second part of the theorem is seen. Finally the assumption of being maximally disconnected insures that the induced homomorphism from P to P/ζ × P/ρ with kernel congruence $\zeta\cap\rho$ is an isomorphism. ☐

A skew lattice S is ***bounded*** if it has a maximal class A and a minimal class Z in which case the primitive skew lattice A⋃Z forms the boundary **Bd**(S) of S. A generalization of first part of the above theorem complements the Clifford-McLean Theorem.

Theorem 2.4.6. *The components of a skew lattice* S *are its maximal connected sub-algebras*. *Moreover, their partition of* S *is the congruence class partition into for the congruence* ρ *for which the induced quotient algebra* S/ρ *is the maximal rectangular image of* S. *If* S *is also bounded with boundary algebra* **Bd**(S), *then the inclusion* **Bd**(S) ⊆ S *induces an isomorphism of maximal rectangular images*.

Proof. The theorem holds when S is primitive. Next, assume S is bounded with maximal class A and minimal class Z. Then Bd(S) = A⋃Z decomposes into components $A_i⋃Z_i$. We say that an element x ∈S ***belongs*** to component $A_i⋃Z_i$ if the latter is the unique boundary component such that there exists u ∈A and v ∈Z such that $u \geq x \geq v$. For any $y \in$ S such that either $x \leq y$ or $x \geq y$ it is clear that x and y belong to the same boundary component. Hence the inclusion **Bd**(S) ⊆ S indices a bijection between the classes of components. Next, let x belong to $A_i⋃Z_i$ and y belong to $A_j⋃Z_j$. Pick $u \in A_i$ and $w \in A_j$ such that $u \geq x\vee y\vee x$ and $w \geq y\vee x\vee y$. By the previous theorem, $u\vee w$ ρ $(x\vee y\vee x)\vee(y\vee x\vee y) = x\vee y$. Thus $x\vee y$ lies in the component of S containing $(A_i⋃Z_i) \vee (A_j⋃Z_j)$. Likewise, $x\wedge y$ lies in the component containing $(A_i⋃Z_i) \wedge (A_j⋃Z_j)$. The bounded case of the theorem now follows from the primitive case. The general case follows from the fact that every skew lattice is the directed union of its intervals. ☐

Recall that a noncommutative lattice ***splits*** if it factors as the product of a lattice and an antilattice (here a rectangular skew lattice). Since every component of a skew lattice S meets every $\mathcal{D}$-class of S we have the following corollary.

Corollary 2.4.7. *Given a skew lattice* S, S/($\mathcal{D}$⋂ρ) *is its maximal split image*. ☐

Orthogonal $\mathcal{D}$-classes and the behavior of skew diamonds

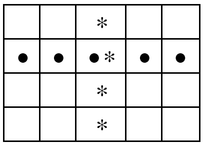
Let A, B and C be $\mathcal{D}$-classes in a skew lattice S such that both A and B are comparable with C. By a ***class component*** of A [or B] in C is meant the intersection of a component of A⋃C [or B⋃C] with C. We say that A and B are ***orthogonal*** in C if each class component of A in C lies in a unique coset of B in C and likewise each class component of B in C lies in a unique coset of A in C where

A and B being orthogonal in C is equivalent to asserting that the image in C of each x in A lies in a unique coset of B in C and likewise the image in C of each y in B lies in a unique coset of A in C. That is, each x in A is ***covered*** in C by a unique coset of B in C and dually each y in B is ***covered*** in C by a unique coset of A in C.

Lemma 2.4.8. *Given $\mathcal{D}$-classes* A, B *and* C *in a skew lattice* S, *if* A *and* B *are orthogonal in* C, *then each coset of* A *in* C *has nonempty intersection with each coset of* B *in* C; *all such coset intersections, moreover, have a common cardinality*.

Proof. Indeed let A_1 and A_2 be cosets of A in C, let B_1 and B_2 be cosets of B in C, and let φ_1 and φ_2 be coset bijections of A_1 and A_2 into a common coset of A in C. (If C lies above A_1 and A_2 then φ_1 and φ_2 are inverses of the downward bijections.) The bijection ${\varphi_2}^{-1}\varphi_1$ and its inverse ${\varphi_1}^{-1}\varphi_2$ keep individual elements in the same class component of A in C. Orthogonality implies that both bijections restrict to an inverse pair of bijections of $A_1⋂B_1$ with $A_2⋂B_1$. Similarly $A_2⋂B_1$ is in 1-1 correspondence with $A_2⋂B_2$, so the assertion is verified. ☐

C:



A-cosets $\cdots$ $a-images*$
B-cosets $\vdots$ $b-images\bullet$

Let the rows in the diagram above represent A-cosets in C and the columns represent B-cosets in C, where A and B are orthogonal in C. Then for any a in A, the images of a in C all lie in different rows, but a single column and for any b in B, the images of b in C all lie in different columns, but a single row. Since the A- and B-cosets all have nonempty intersection by the lemma above, some unique A-C coset intersection contains both an image of a and an image of b.

Theorem 2.4.9. *Let* A *and* B *be* $\mathcal{D}$*-classes in a skew lattice. Then* A *and* B *are orthogonal in both their join class* J *and their meet class* M. *For each* $x \in$ A and $y \in$ B, $x \vee y = x' \vee y'$ *where* x' *is the image of* x *in* J *lying in the unique coset of* A *in* J *covering* y, *and* y' *is the image of* y *in* J *lying in the unique coset of* B *in* J *covering* x. *That is, both* x' *and* y' *lie in the unique* A-B *coset intersection in* J *containing both an image of* x *and an image of* y, *namely* x' *and* y'. *The computation of* $x \wedge y$ *in* M *is determined in dual fashion.*

Proof. Given $x \in$ A and $y \in$ B, for all $u \in$ J such that $u \geq x$ we have

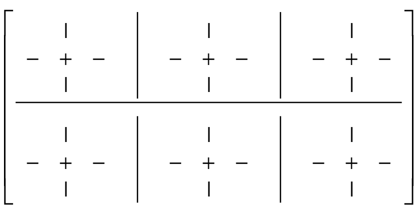
$$y \vee x \vee y = y \vee x \vee x \vee y = y \vee x \vee u \vee x \vee y = y \vee u \vee y.$$

Thus each $x \in$A is covered by a fixed coset of B in J. Likewise each $y \in$ B is covered by a unique coset of A in B. Thus both A and B are orthogonal in J, and similarly, they are orthogonal in M. Since

$$x \vee y \ = \ (x \vee y \vee x) \vee (y \vee x \vee y) \ = \ (x \vee v \vee x) \vee (y \vee u \vee y)$$

for any u, $v \in$J such that $u \geq x$ and $v \geq y$, it follows that indeed $x \vee y = x' \vee y'$ where x' is the image of x in J lin the unique coset A$'$ of A in J covering y, and y' is the image of y in J lying in the unique coset B$'$ of B in J covering x. Clearly both x' and y' lie in A$'\cap$B$'$ but no other pair of x and y images in J can belong to a common A-B coset intersection. ☐

The double partition of either J or M by A-cosets and B-cosets is illustrated below where the partition is further refined by the coset partitions that J and M directly induce on each other. Indeed, if say $m \in$ M, $j \in$ J and $j \geq a \in$ A, then $m \vee j \vee m = (m \vee a) \vee j \vee (a \vee m) \in$ A $\vee j \vee$ A. Thus M $\vee j \vee$ M $\subseteq$ A $\vee j \vee$ A. Likewise M $\vee j \vee$ M $\subseteq$ B $\vee j \vee$ B. Similar remarks hold for cosets in M.



A-cosets $\leftrightarrow$
B-cosets $\updownarrow$

When do these double partitions coincide with the (generally finer) J-M partitions.

Theorem 2.4.10. *A skew lattice* S *is symmetric if and only if for many two equivalence classes* A *and* B: (i) *the double partition of the join class* J *by intersections of* A-*cosets with* B-*cosets equals the partition by cosets of the meet class* M; (ii) *the dual assertion holds for the meet class* M.

Proof. First assume that A-cosets and B-cosets in J and M intersect to J-M cosets. Let $a \in$A and $b \in$B be given with $a \vee b = b \vee a$. By Theorem 2.4.9, $a \wedge b$ and $b \wedge a$ lies in the same A-B coset intersection in M. Thus $a \wedge b$ and $b \wedge a$ belong to the same J-coset in M. Since $a \vee b \geq$ both $a \wedge b$ and $b \wedge a$, we must have $a \wedge b = b \wedge a$. Dually, $a \wedge b = b \wedge a$ implies $a \vee b = b \vee a$.

Suppose instead that say more than J-coset in M lies inside the intersection I of an A-coset with a B-coset. Thus there exists $j \in$ J with at least two distinct images m and m' in I. We form a subalgebra T = J′ ⋃ A′ ⋃ B′ ⋃ M′ inside the subalgebra S′ = $j \wedge$S$\wedge j$ as follows. Set J′ = {j}, M′ = $j \wedge$I$\wedge j$ and set A′ equal to the image of M′ in A∩S′ under a single coset bijection of S′ from M′ to the intermediate class. Define B′ in similar fashion. Thus in T exactly one coset bijection exists from A′ to M′ and likewise exactly one coset bijection exists from B′ to M′. In particular, exactly one $a' \in$A and exactly one $b' \in$B exist such that $a' \geq m$ and $b' \geq m'$. Then $a' \vee b' = b' \vee a' = j$ and by orthogonality $a' \wedge b' = m \wedge m'$ and $b' \wedge a' = m' \wedge m$. Since $m \neq m'$, $m \wedge m' \neq m' \wedge m$ follows and S has an instance of anti-symmetry. ☐

These results have a number of worthwhile corollaries.

Coset bijections in skew chains and as morphisms in categories

A ***skew chain*** is a skew lattice S with finitely many $\mathcal{D}$-classes that is totally quasi-ordered under $\succeq$. Thus it can be viewed as a chain of $\mathcal{D}$-classes A > B > C > … > X in that are totally ordered in S/$\mathcal{D}$. We begin with a near-obvious lemma.

Lemma 2.4.11. *Given a skew chain* A > B > C:

(1) *For each* $c \in$ C, *there is the inclusion of cosets* A$\wedge c \wedge$A ⊆ B$\wedge c \wedge$B *in* C.
(2) *For each* $a \in$ A, *there is the inclusion of cosets* C$\vee a \vee$C ⊆ B$\vee a \vee$B *in* A.
(3) *Given* $a > b > c$ *where* $a \in$ A, $b \in$ B *and* $c \in$ C, *if*

$$\varphi: \mathrm{B}\vee a\vee \mathrm{B} \to \mathrm{A}\wedge b\wedge \mathrm{A},\ \psi: \mathrm{C}\vee b\vee \mathrm{C} \to \mathrm{B}\wedge c\wedge \mathrm{B} \text{ and } \chi: \mathrm{C}\vee a\vee \mathrm{C} \to \mathrm{A}\wedge c\wedge \mathrm{A}$$

are coset bijections between the relevant cosets in the respective $\mathcal{D}$-*classes taking a to b*, *b to c and a to c*, *then* ψoφ ⊆ χ.

Proof. Given $x = a \wedge c \wedge a \in$ A$\wedge c \wedge$A, $b \in$ B exists such that $b \wedge c \wedge b = c$. Hence, $x = a \wedge b \wedge c \wedge b \wedge a$ which us in B$\wedge c \wedge$B and (1) follows. The proof of (2) is dual. To see (3), first observe that the output set of φ and the input set for ψ have the intersection A$\wedge b \wedge$A ⋂ C$\vee b \vee$C within B. Thus (3) follows from the inclusions,

$$[A\wedge b\wedge A \cap C\vee b\vee C]\vee a\vee[A\wedge b\wedge A \cap C\vee b\vee C] \subseteq C\vee b\vee C\vee a\vee C\vee b\vee C = C\vee a\vee C \text{ in A}$$

and

$$[A\wedge b\wedge A \cap C\vee b\vee C]\wedge c\wedge[A\wedge b\wedge A \cap C\vee b\vee C] \subseteq A\wedge b\wedge A\wedge c\wedge A\wedge b\wedge A = A\wedge c\wedge A \text{ in C},$$

where the equalities on the right are due to regularity. Here the expression to the left of $\subseteq$ are the respective input and output sets of $\psi\circ\varphi$. Applying the letter to any x in the input set we get

$$\psi\circ\varphi(x) = (x\wedge b\wedge x)\wedge c\wedge(x\wedge b\wedge x) = x\wedge b\wedge c\wedge b\wedge x = x\wedge c\wedge x = \chi(x). \qquad \square$$

A skew *chain* A > B > C is ***categorical*** if $\psi\circ\varphi = \chi$ always holds in (3) above, in which case every coset bijection χ between a C-coset in A and an A-coset in C must factor as such. Indeed, given $a \in$A and $c \in$C such that χ sends a to c, some $b \in$B exists such that $a > b > c$. (Set $b = a\wedge(c\vee y\vee c)\wedge a$ for some $y \in$B.) If φ and ψ are the A-B and B-C coset bijections sending a to b and b to c, then $\chi = \psi\circ\varphi$. A skew lattice S is ***categorical*** if every chain of $\mathcal{D}$-classes A > B > C in S is categorical. S is ***strictly categorical*** if it is categorical and all such composites $\psi\circ\varphi$ of coset bijections between cosets in comparable $\mathcal{D}$-classes A > B > C are nonempty. The *outer* cosets in a categorical skew chain A > B > C induce virtual versions of themselves in the middle $\mathcal{D}$-class B as follows.

Theorem 2.4.12. (Cvetko-Vah [2005c]) *A skew chain* A > B > C *is categorical if and only if for all* $a \in$ A, $b \in$ B *and* $c \in$ C *that satisfy* $a > b > c$,

$$(C\vee a\vee C) \wedge b \wedge (C\vee a\vee C) = (A\wedge b\wedge A) \cap (C\vee b\vee C) = (A\wedge c\wedge A) \vee b \vee (A\wedge c\wedge A).$$

Proof. In terms of the coset bijections $\varphi\colon A_i \to B_j$, $\psi\colon B'_k \to C_l$ and $\chi\colon A'_m \to C'_n$ such that $\varphi(a) = b$, $\psi(b) = c$ and $\chi(a) = c$ with $\psi\circ\varphi = \chi$, the given equalities are what is needed for $\psi\circ\varphi$ to occur. $\square$

The term "categorical" comes from the following result:

Theorem 2.4.13. *If a skew lattice S is strictly categorical, then a category* **Cat**(S) *is defined by*:

(1) *Letting the objects of* S *to be the* $\mathcal{D}$*-classes of* S.
(2) *For comparable* $\mathcal{D}$*-classes* A ≥ B, Hom(A, B) *consists of all coset bijections from all* B*-cosets in* A *to* A*-cosets in* B. *Otherwise,* Hom(A, B) *is empty.*
(3) *In particular, each* Hom(A, A) *consists of the unique identity bijection on* A.
(4) *Letting morphism composition be the usual composition of partial bijections.*

When S is just categorical, a modified category $\mathbf{Cat}^{\mathbf{0}}$(S) *is given as above, but in addition, for each comparable pair* A ≥ B, Hom(A, B) *contains a labeled copy of the empty bijection* 0_{AB}. $\square$

Theorem 2.4.14.

(1) *A skew lattice* S *is categorical if and only if for all x, p, q, r with* $x\,\mathcal{D}\,p \geq q \geq r$,

$$(x{\wedge}r{\wedge}x) \vee q \vee (x{\wedge}r{\wedge}x) = [(x{\wedge}r{\wedge}x) \vee p \vee (x{\wedge}r{\wedge}x)] \wedge q \wedge [(x{\wedge}r{\wedge}x) \vee p \vee (x{\wedge}r{\wedge}x)].$$

(2) *Categorical skew lattices form a subvariety of skew lattices.*

(3) *A skew lattice* S *is categorical if and only if its left and right factors are categorical.*

Proof. Observe that in any skew lattice S, all p such that $x\,\mathcal{D}p$ arise as $(u{\wedge}x{\wedge}u) \vee x \vee (u{\wedge}x{\wedge}u)$ for u unrestricted. Observe next that for any p, all q, r such that $p \geq q \geq r$ arise $q = p{\wedge}v{\wedge}p$ and $r = q{\wedge}w{\wedge}q$ for v and w unrestricted. Thus the condition stated in (1) can be made unconditional so that (2) follows from (1) and thus (3) follows from (2). To see that (1) holds, consider the case of a nonempty composition of coset bijections ψoφ, where φ is a coset bijection from A to B and ψ is a coset bijection from B to C. Since ψoφ is nonempty, for all elements p in A in the domain of ψoφ, φ send p to some q in B and ψ sends that q to some r in C yielding $p > q > r$. Clearly all triples $p > q > r$ arise in this fashion for some ψ and φ. For the given ψ and φ, letting χ be the coset bijection from A to C sending p to r. Thus at least ψo$\varphi \subseteq \chi$ by Lemma 4.16. The equation of (1) states that

$$\psi^{-1}[A \wedge r \wedge A] \;=\; \varphi[\chi^{-1}[A \wedge r \wedge A]]$$

with A$\wedge r\wedge$A being the image of χ. Since all indicated cosets in this equation lie in the domains of the relevant bijections, this equation is equivalent first to $[A \wedge r \wedge A] \;=\; \psi\text{o}\varphi[\chi^{-1}[A \wedge r \wedge A]]$ and then to $\chi[\chi^{-1}[A{\wedge}r{\wedge}A] = \psi\text{o}\varphi\,[\chi^{-1}[A{\wedge}r{\wedge}A]]$ so that ψoφ must be *all* of χ. ☐

While 2.4.16(1) at first may appear to be an obfuscation of the simple implication, if ψoφ $\subseteq \chi$ then ψo$\varphi = \chi$, it does unpack $\psi\bigcirc\varphi = \chi$ at the element-wise level, thus setting the stage for (2) and (3). In Chapter 5, (strictly) categorical skew lattices will be studied more closely.

We next present several classes of categorical skew lattices.

Theorem 2.4.15. *Skew lattices in rings are categorical.*

Proof. Assume first that S is a right-handed skew lattice in a ring R and that $x\,\mathcal{R}\,p \geq q \geq r$ in S. The conditional identity in of the previous theorem thus reduces to $(r{\wedge}x) \vee q = q \wedge [(r{\wedge}x) \vee p]$. Applying $\bigcirc$ and multiplication, the left side of the equation reduces to $rx + q - rxq = rx + q - r$, and the right side reduces to $q[rx + p - rxp] \;= rx + q - r$ again. Hence S is categorical. Similarly, if S is left handed in R, then S must be categorical.

Next suppose that is neither left nor right-handed with either $\bigcirc$ or ∇ for a join. S will be categorical if all countable subalgebras are thus. So let S′ be a countable subalgebra of S. By symmetry, copies the maximal left and right-handed images of S′ arise as subalgebras of S′ in R and thus these are categorical. By the Theorem 2.4.14(3), S′ is categorical. Since this holds for all countable subalgebras of S, S is also categorical. ☐

Theorem 2.4.16. *Normal skew lattices are strictly categorical.*

Proof. Since the lower $\mathcal{D}$-class in any maximal primitive subalgebra of a normal skew lattice has exactly one coset, the composition of adjacent coset bijections is a nonempty coset bijection. ☐

A third class of categorical skew lattices is as follows.

Theorem 2.4.17. *Every primitive skew lattice is strictly categorical, distributive and symmetric. Thus all skew lattices in the subvariety generated from the class of primitive skew lattices are categorical, distributive and symmetric*.

Proof. Any primitive skew lattice S is trivially strictly categorical. The only nontrivial instances of either $x \vee y = y \vee x$ or $x \wedge y = y \wedge x$ when S is primitive are when $x > y$ or $y > x$. Thus S is also symmetric. Next, consider the equation $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$. It holds trivially when x is in the lower $\mathcal{D}$-class B. So let $x = a$ in the upper class A and let $y = b$ and $z = c$ in B. Then

$$a \wedge (b \vee c) \wedge a = a \wedge (c \wedge b) \wedge a = (a \wedge c \wedge a) \wedge (a \wedge b \wedge a) = (a \wedge b \wedge a) \vee (a \wedge c \wedge a) \text{ in B.}$$

If a in A and say b in B but c in A then $b \vee c$ in A so that $a \wedge (b \vee c) \wedge a = a$, while $(a \wedge b \wedge a) \vee (a \wedge c \wedge a)$ $= (a \wedge b \wedge a) \vee a = a$ in A. The case where the locations of b and c are switched is similar. Finally, one similarly verifies $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$. ☐

We eventually show that all skew lattices in this subvariety are in fact strictly categorical and also cancellative. **Query:** *are skew lattices in this subvariety characterized by being cancellative, distributive, strictly categorical and symmetric?*

Example 2.4.2. [Kinyon and Leech, 2013] A minimal noncategorical skew chain has the following Hasse diagram where $a_1 > b_1, b_3$; $a_2 > b_2, b_4$; $b_1, b_2 > c_1$; $b_3, b_2 > c_2$; and thus both $a_i >$ both c_j.

$$\begin{array}{ll}
\mathrm{A} & \qquad a_1 \; - \; a_2 \\
& \\
\mathrm{B} & b_1 -_{\mathrm{A}} b_2 -_{\mathrm{C}} b_3 -_{\mathrm{A}} b_4 -_{\mathrm{C}} (b_1) \\
& \\
\mathrm{C} & \qquad c_1 \; - \; c_2
\end{array}$$

Instances of left-handed operations are given by $a_1 \vee c_2 = a_2 = a_1 \vee a_2$, $a_1 \wedge b_4 = b_3 \wedge b_4 = b_3$ and $b_1 \vee c_2 = b_1 \vee b_4 = b_4$. ☐

Before moving to the next section we clear up one detail:

Proposition 2.4.18. *Every coset bijection* φ: $\mathrm{A}_i \to \mathrm{B}_j$ *is a skew lattice isomorphism*.

Proof. Pick $b \in B_j$. Then for all $x \in A_i$, $\varphi(x) = x \wedge b \wedge x$. Thus given x, y in A_i we have

$$\varphi(x \wedge y) = (x \wedge y) \wedge b \wedge (x \wedge y) = x \wedge b \wedge y = x \wedge b \wedge b \wedge y = x \wedge b x \wedge y \wedge b \wedge y = \varphi(x) \wedge \varphi(y),$$

due to regularity and Corollary 1.2.8. But since A_i and B_j are rectangular algebras satisfying $x \vee y = y \wedge x$, the proposition follows. □

In the left [right] rectangular case this proposition is trivially true since it is easily seen that any *bijection between left [right] rectangular skew lattices must be an isomorphism*.

2.5 Partial skew lattices and coset projections

Every nonrectangular skew lattice S is the union of its maximal primitive skew lattices and the latter jointly determine its structure. Indeed one could view primitive skew lattices as "lego pieces" that when appropriately "snapped together" produce entire skew lattices. To pursue this perspective, we need a new concept. A ***partial skew lattice*** on a quasi-ordered set $(S, \succeq)$ is a 4-tuple $(S, \succeq, \vee, \wedge)$ where $\vee$ and $\wedge$ are partial binary operations defined for pairs of elements comparable under $\succeq$ in such a way that the union U of any chain of $\succeq$-equivalence classes forms a skew lattice under $\vee$ and $\wedge$ whose natural quasi-order coincides with $\succeq$ over U. In particular, each equivalence class A under $\vee$ and $\wedge$ is a rectangular skew lattice and each comparable pair of such classes, say A > B, forms a primitive skew lattice. That $\vee$ and $\wedge$ are associative on totally quasi-ordered subsets is called ***linear associativity***.

Lemma 2.5.1. *Let* $(S, \succeq, \vee, \wedge)$ *be a partial skew lattice and let* $m \preceq x, y \preceq p$ *in* S. *Then both* $(x \vee p) \vee y = x \vee (p \vee y)$ *and* $(x \wedge m) \wedge y = x \wedge (m \wedge y)$. (This form of associativity, where x and y need not be comparable, is called ***extremal associativity***.)

Proof. $(x \vee p) \vee (p \vee y)$ reduces to both $(x \vee p) \vee y$ and $x \vee (p \vee y)$. Thus the latter must be equal. Similar remarks hold for $(x \wedge m) \wedge y$ and $x \wedge (m \wedge y)$. □

Every skew lattice $(S; \vee, \wedge)$ induces a ***canonical partial skew lattice*** $(S; \succeq, \vee, \wedge)$ upon restricting the given binary operations, $\vee$ and $\wedge$, to $\succeq$-related pairs. *When is a given partial skew lattice* $(S, \succeq, \vee, \wedge)$ *canonical for some skew lattice?* Two conditions are obviously necessary.

I. The $\succeq$-equivalence classes (proto-$\mathcal{D}$-classes of mutually $\succeq$-related elements) must have both join and meet classes.
II. Each pair of equivalence classes must be orthogonal in both their join and meet classes.

Given both conditions, both $\vee$ and $\wedge$ can be extended to full binary operations as follows.

III. Set $x \vee y = (x \vee q \vee x) \vee (y \vee p \vee y)$ for any p, q in the join class of x and y such that $p \geq x$ and $q \geq y$.

IV. Set $x \wedge y = (x \wedge n \wedge x) \wedge (y \wedge m \wedge y)$ for any m, n in the join class of x and y such that $m \leq x$ and $n \leq y$.

Since the equivalence classes of x and y are orthogonal in both their join and meet classes, any p, q, m and n satisfying the stated conditions must produce the same values for $x \vee q \vee x$, $y \vee p \vee y$, $x \wedge n \wedge x$ and $y \wedge m \wedge y$ so that the extended binary operations are uniquely determined. The full algebra (S; $\vee$, $\wedge$) is called the ***algebraic closure*** of (S; $\succeq$, $\vee$, $\wedge$).

Example 2.5.1. Given a ring R, two partial skew lattice structures exist for the entire set of idempotents: ($\mathbf{E}$(R); $\succeq_{\mathcal{L}}$, $\vee$, $\wedge$) and ($\mathbf{E}$(R); $\succeq_{\mathcal{R}}$, $\vee$, $\wedge$). Here $e \succeq_{\mathcal{L}} f$ iff $fe = f$ and $e \succeq_{\mathcal{R}} f$ iff $ef = f$. In both cases $e \vee f = e \circ f$ and $e \wedge f = ef$, given that e and f are $\succeq$-related. While Condition I above is met in many classical rings (e.g., full matrix rings over fields), Condition II is rarely met. ☐

Theorem 2.5.2. *A partial skew lattice* (S; $\succeq$, $\vee$, $\wedge$) *is the canonical partial algebra of a full skew lattice* (S; $\vee$, $\wedge$) *if and only if Conditions I and II above hold, in which case* (S; $\vee$, $\wedge$) *is precisely the algebraic closure of* (S; $\succeq$, $\vee$, $\wedge$).

Proof. Necessity is clear. Moreover, if such a lattice extension exists, it must be the algebraic closure by Theorem 2.4.9. We show that the algebraic closure must be a skew lattice. So let x and y be given. Then $x \vee y$ is calculated as $x' \vee y'$ for some x' and y' in the join class of x and y such that $x \leq x'$ and $y \leq y'$. Thus in the partial skew lattice we have

$$x \wedge (x \vee y) = x \wedge (x' \vee y') = x \wedge (x \vee x' \vee y') = x.$$

The other instances of absorption are similarly seen. Suppose next that both $x, y \preceq s$ in S. By extremal associativity, $x \vee y = x \vee q \vee p \vee y$ for any $p \geq x$ and $q \geq y$ in the join class of x and y. Choose q such that $y \leq q \leq y \vee s \vee y$. (If need be, replace the given q by $(y \vee s \vee y) \wedge q \wedge (y \vee s \vee y)$). Applying linear associativity we get

$$\begin{aligned}(x \vee y) \vee s &= (x \vee q \vee p \vee y) \vee s = (x \vee q) \vee p \vee (y \vee s \vee y) \vee s \\ &= (x \vee q) \vee (y \vee s \vee y) \vee s = x \vee (q \vee y \vee s \vee y) \vee s = x \vee (y \vee s).\end{aligned}$$

We similarly obtain the three other cases of ***outer associativity***: if $r \leq x, y \leq s$, then also

$$s \vee (x \vee y) = (s \vee x) \vee y,\ r \wedge (x \wedge y) = (r \wedge x) \wedge y \text{ and } (x \wedge y) \wedge r = x \wedge (y \wedge r).$$

With the available conditional associative identities we obtain the unconditional identities. For instance, let s lie in the join class of x, y and z. Then $x \wedge (y \wedge (z \wedge s))$ must equal both $((x \wedge y) \wedge z) \wedge s$ and $(x \wedge (y \wedge z)) \wedge s$. Likewise $((s \wedge x) \wedge y) \wedge z$ must equal both $s \wedge ((x \wedge y) \wedge z)$ and $s \wedge (x \wedge (y \wedge z))$. It follows that $x \wedge (y \wedge (z \wedge s)) \wedge ((s \wedge x) \wedge y) \wedge z$ must equal both $((x \wedge y) \wedge z) \wedge s \wedge s \wedge ((x \wedge y) \wedge z)$ which reduces to $(x \wedge y) \wedge z$ and $(x \wedge (y \wedge z)) \wedge s \wedge s \wedge (x \wedge (y \wedge z))$ which reduces to $x \wedge (y \wedge z)$. Thus $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. The dual form of associativity is likewise seen. The algebraic closure is thus indeed a skew lattice and the theorem follows. ☐

Returning to primitive skew lattices, let (S; $\succeq$) be a quasi-ordered set. A ***primitive covering*** $\mathcal{P}$ of (S; $\succeq$) consists of (1) an assignment of a rectangular skew lattice structure to each equivalence class A of (S; $\succeq$) and (2) to any comparable pair of equivalence classes A > B a primitive skew lattice structure is assigned that extends the separate rectangular structures on A and B in such a way that A > B as separate $\mathcal{D}$-classes. Given a partial skew lattice (S; $\succeq$, ∨, ∧), its ***canonical primitive covering*** is the class of all maximal primitive subalgebras of (S; $\succeq$, ∨, ∧).

Given a primitive covering $\mathcal{P}$ of a quasi-ordered set (S; $\succeq$), conditional operations ∨ and ∧ are defined on any $\succeq$-comparable pair of elements e and f in S by letting $e\vee f$ and $e\wedge f$ be the join and meet respectively given in any primitive subalgebra of $\mathcal{P}$ containing both. (This subalgebra is unique if e and f are not equivalent. Otherwise, e and f lie in a common rectangular subalgebra where ∨ and ∧ are defined.) One can ask: *is* (S; $\succeq$, ∨, ∧) *a partial skew lattice*? If "yes", then $\mathcal{P}$ would be its canonical primitive covering. Put otherwise: *when is a primitive covering of given quasi-ordered set* (S; $\succeq$) *the canonical primitive covering of some partial skew lattice* (S; $\succeq$, ∨, ∧) *on* (S; $\succeq$)? Or: *when are the induced operations linearly associative?*

So let a primitive covering $\mathcal{P}$ of (S; $\succeq$) be given and consider equivalence classes A > B. If B_j is an A-coset in B, then the ***lower coset projection*** of A onto B_j is the function p_j: A → B (note that B is the codomain) projecting each element of A onto its unique image in B_j. Clearly $p_j(a) = a\wedge b\wedge a$ for any b in B_j. Similarly for each B-coset A_i in A, an ***upper coset projection*** q_i: B → A is given by $q_i(b) = b\vee a\vee b$ for any a in A_i. When A = B, set $p = q = 1_A$. We let $\mathbf{Proj}_\downarrow$(S; $\succeq$) [respectively, $\mathbf{Proj}_\uparrow$(S; $\succeq$)] denote the family of all lower [upper] coset projections between comparable equivalence classes of (S; $\succeq$). *If composites of lower* [*upper*] *coset projections are also lower* [*upper*] *coset projections, then* $\mathbf{Proj}_\downarrow$(S; $\succeq$) *forms the* ***category of lower coset projections*** *and* $\mathbf{Proj}_\uparrow$(S; $\succeq$) *forms the* ***category of upper coset projections***.

Theorem 2.5.3. *The partial algebra* (S; $\succeq$, ∨, ∧) *induced from a primitive covering* $\mathcal{P}$ *of a quasi-ordered set* (S; $\succeq$) *is a partial skew lattice precisely when both coset projection families,* $\mathbf{Proj}_\downarrow$(S; $\succeq$) *and* $\mathbf{Proj}_\uparrow$(S; $\succeq$), *form categories under ordinary composition of functions*.

Proof. Suppose that $\mathbf{Proj}_\downarrow$(S; $\succeq$) and $\mathbf{Proj}_\uparrow$(S; $\succeq$) are categories under the usual composition of functions. Given comparable classes A > B > C, with $a \in$ A, $b \in$ B and $c \in$ C we first show that $a\wedge(b\wedge c) = (a\wedge b)\wedge c$. First set $d = b\wedge c\wedge b$ in C. Using only primitive operations and projections:

$$a\wedge(b\wedge c) = a\wedge(d\wedge c) = (a\wedge d)\wedge c = p_d(a)\wedge d\wedge c = p_d(a)\wedge c$$

and

$$(a\wedge b)\wedge c = (p_b[a]\wedge b)\wedge c = p_c(p_b[a]\wedge b)\wedge c = p_c p_b[a]\wedge p_c[b]\wedge c == p_c p_b[a]\wedge c.$$

where p_b: A → B, p_c: B → C and p_d: A → C denote coset projections with b, c and d respectively in their images. By our assumption about $\mathbf{Proj}_\downarrow$(S; $\succeq$) the composition $p_c p_b$: A → C is either p_d or their images are disjoint in C. Given $a' \in$ A such that $a' > b$ we have $p_c p_b[a'] = p_c[b] = d$. Thus the images of $p_c p_b$ and p_d overlap and so are equal. Hence $a\wedge(b\wedge c) = (a\wedge b)\wedge c$ for all $a \in$A.

Next observe that $a\wedge(c\wedge b) = a\wedge(c\wedge(c\wedge b)) = (a\wedge c)\wedge(c\wedge b)$ in A∪C while $(a\wedge c)\wedge b = ((a\wedge c)\wedge c)\wedge b = (a\wedge c)\wedge(c\wedge b)$ in B∪C so that $a\wedge(c\wedge b) = (a\wedge c)\wedge b$. Also $c\wedge(a\wedge b) = c\wedge(a\wedge b)\wedge b = c\wedge b$ in B∪C, while $(c\wedge a)\wedge b = c\wedge(c\wedge a)\wedge b = c\wedge b$ so that $c\wedge(a\wedge b) = (c\wedge a)\wedge b$.

The three other cases of potential associativity under $\wedge$ with a, b and c are left-right reflections of cases already considered and must also hold. Finally, the six cases involving $\vee$ are the duals of the cases considered. Hence, using our assumption about $\mathbf{Proj}_{\uparrow}(S;\succeq)$, the dual arguments in all cases for associativity of the $\vee$-product involving a, b and c in some order will be successful. Thus $(S;\succeq,\vee,\wedge)$ is indeed a partial skew lattice.

Conversely, let $(S;\succeq,\vee,\wedge)$ be a partial skew lattice. Conditional regularity yields

$$(a\wedge b\wedge a)\wedge c\wedge(a\wedge b\wedge a) = a\wedge(b\wedge c\wedge b)\wedge a$$

for all $a \succeq b \succeq c$ in S. Thus, $p_c p_b = p_{b\wedge c\wedge b}$ and $\mathbf{Proj}_{\downarrow}(S;\succeq)$ is seen to be a category. In similar fashion, so is $\mathbf{Proj}_{\uparrow}(S;\succeq)$. □

Combining the above two results we have:

Theorem 2.5.4. *A quasi-ordered set* $(S;\succeq)$ *with a covering* $\mathcal{P}$ *of primitive skew lattices is the primitive covering of a (necessarily unique) skew lattice* $(S;\vee,\wedge)$ *if and only if*

i) *Both coset projection families,* $\mathbf{Proj}_{\downarrow}(S;\succeq)$ *and* $\mathbf{Proj}_{\uparrow}(S;\succeq)$, *form categories under the usual composition of functions*.
ii) *The equivalence classes of* $(S;\succeq)$ *form a lattice under their usual partial ordering*.
iii) *Each pair of equivalence classes is orthogonal in both their join and meet classes*. □

The case for normal skew lattices

From the perspective of coset bijections and projections, the significant features of normal skew lattices are (1) that they are strictly categorical and (2) that for each primitive sub-algebra A > B there is exactly one A-coset in the lower $\mathcal{D}$-class B, namely B, and thus exactly *one* projection of A *onto* B. One thus has a situation like following, where the φ_i are coset bijections from individual B-cosets A_i in A onto B.

$$\begin{array}{ccccccccc} A & & A_1 & & A_2 & & A_3 & & \\ \vee & & & \searrow^{\cong} & \downarrow^{\cong} & \swarrow^{\cong} & & & p_B = \varphi_1 \cup \varphi_2 \cup \varphi_3 \\ B & & & & B & & & & \end{array}$$

Clearly upward projections are just the upward coset bijections. We thus modify our discussion of these matters in a way that is more commensurate to the situation for normal skew lattices.

A ***projective pair*** is a pair $(\mathcal{K}, \acute{k})$ where (i) $\mathcal{K}$: A → B is a ***regular epimorphism*** of rectangular skew lattices, that is, a factorization $\varphi: J \times B \cong A$ exists such $\mathcal{K} \circ \varphi$ is the B-coordinate projection of J × B upon B. (*ii*) $\mathcal{k}$ is a set of monomorphisms k_j: B → A called ***injections*** such that the compositions $\varphi^{-1}k_j$ are precisely the canonical injections $b \to (j, b)$ of B into the various $\{j\} \times$ B. Clearly the inverse injections k_j^{-1} jointly decompose the projection $\mathcal{K}$. Any factorization φ for which (*i*) and (*ii*) hold is said to be ***compatible*** with the projection.

Example 2.5.2. $\mathrm{A} = \left[\begin{array}{ccc|ccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right] \to \mathrm{B} = \left[\begin{array}{ccc} \bullet & \bullet & \bullet \end{array}\right]$ with $\mathrm{J} = \left[\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array}\right]$.

$\mathcal{K}$ is the union of the four obvious isomorphisms of each of the four displayed quadrants of A onto B. The four injections are the four embeddings of B upon each of the four quadrants given by the inverse isomorphisms. □

Rectangular skew lattices and projective pairs form a category. Indeed, if $(\mathcal{K}, \mathcal{k})$: A → B and $(\mathcal{L}, \mathcal{l})$: B → C are projections, then the composite projection is $(\mathcal{L}, \mathcal{l})(\mathcal{K}, \mathcal{k}) = (\mathcal{LK}, \mathcal{kl})$: A → C upon setting $\mathcal{kl} = \{k \circ l \mid k \in \mathcal{k}, l \in \mathcal{l}\}$. This composition is well-defined and associative. Moreover if φ: J × B ≅ A is a compatible factorization for $(\mathcal{K}, \mathcal{k})$: A → B and ψ: J′ × C ≅ B is a compatible factorization for $(\mathcal{L}, \mathcal{l})$: B → C, then a compatible factorization for $(\mathcal{LK}, \mathcal{kl})$: A → C is given by

$$\mathrm{J}\times\mathrm{J}'\times\mathrm{C} \xrightarrow[1_{\mathrm{J}}\times\psi]{} \mathrm{J}\times\mathrm{B} \xrightarrow[\varphi]{} \mathrm{A} \quad \text{where } k_j l_{j'}(c) = k_j(\psi(j', c)) = \varphi(j, \psi(j', c)) \text{ for } c \in \mathrm{C}.$$

Lemma 2.5.5. *Let* A > B *be two* $\mathcal{D}$*-classes in a normal skew lattice, let* $\mathcal{K}$: A → B *be the unique projection of* A *onto* B *defined implicitly by* $x \geq \mathcal{K}(x)$ *and let* $\mathcal{k}$ *be the let of all coset bijections from* B *onto cosets of* A. *Together the pair* $(\mathcal{K}, \mathcal{k})$ *forms a projective pair*. (We call this pair the ***natural projective pair*** form A to B.)

Proof. For any b in B, a factorization is given by φ: $b\vee\mathrm{A}\vee b \times \mathrm{B} \to \mathrm{A}$, where $\varphi(x, y) = y\vee x\vee y$. Here $b\vee\mathrm{A}\vee b$ is the image set $\{b\vee a\vee b \mid a \in \mathrm{A}\} = \{a \in \mathrm{A} \mid a \geq b\}$ of b in A that naturally parametrizes the cosets of B in A. Clearly φ is a bijection. Given the rectangular situation one need only show that φ is a ∨-homomorphism; but $(y\vee x\vee y)\vee(y'\vee x'\vee y') = (y\vee y')\vee(x\vee x')\vee(y\vee y')$ is given by regularity, since $x, x' \succeq y, y'$. □

A ***rectangular functor*** on a lattice T is a functor **K** from (T, ≥) to the category **RP** of rectangular skew lattices and projective pairs. **K** is ***separable*** if $\mathbf{K}(s) \cap \mathbf{K}(t) = \varnothing$ for all $s \neq t \in \mathrm{T}$. For $s \geq t \in \mathrm{T}$, the projection from $\mathbf{K}(s)$ to $\mathbf{K}(t)$ is denoted by $\mathbf{K}(s, t)$ and its injections (upward coset bijections) by $k(s, t, i)$ with i parametrizing the various injections.

Lemma 2.5.6. *If* S *is a normal skew lattice with* T = S/$\mathcal{D}$, *then* $\mathcal{K}$: T → **RP** *defined as in the previous lemma is a rectangular functor*.

Proof. This follows from the previous lemma and the fact that any normal skew lattice is strictly categorical. □

Given a lattice T and a separable, rectangular functor **K** from (T, ≥) to the category **RP**, a normal ∧-band may be constructed on $S = \bigcup_{s\in T}\mathbf{K}(s)$ by setting

$$a\wedge b = \mathbf{K}(s, s\wedge t)[a] \wedge \mathbf{K}(t, s\wedge t)[b], \text{ for } a \in \mathbf{K}(s) \text{ and } b \in \mathbf{K}(t).$$

This is, of course, the Yamada-Kimura construction. One would like to be able to use injections to define a join operation ∨ on S and thus turn this normal band into a normal skew lattice. To do so, requires a key concept from Section 2.4.

Given $s, t \in T$ with join $n \in T$ we say that projections $\mathbf{K}(n, s)$ and $\mathbf{K}(n, t)$ are ***orthogonal*** in $\mathbf{K}(n)$ if (i) for each $a \in K(s)$, its pre-image $\mathbf{K}(n, s)^{-1}[a]$ in $K(n)$ lies in the image of a unique injection $k(n, t, j)$ from $\mathbf{K}(t)$ to $\mathbf{K}(n)$ and similarly (ii) for each $b \in K(t)$, its pre-image $\mathbf{K}(n, t)^{-1}[b]$ in $K(n)$ lies in the image of a unique injection $k(n, s, j)$ from $\mathbf{K}(s)$ to $\mathbf{K}(n)$.

Our observations above combined with earlier results in this section yield the following extension of the description of normal bands by Yamada and Kimura [1958]:

Theorem 2.5.7. *Let* **K** *be a separable, rectangular functor defined on a lattice* T *such that for every join situation* $n = s\vee t$ in T *the projections* $\mathbf{K}(n, s)$ *and* $\mathbf{K}(n, t)$ *are orthogonal in* $\mathbf{K}(n)$. *Then* $S = \bigcup_{s\in T}\mathbf{K}(s)$ *becomes a normal skew lattice with* **K** *providing the system of natural projections and coprojections, if given a in* $\mathbf{K}(s)$ *and b in* $\mathbf{K}(t)$, *their meet and join are defined by*

$$a\wedge b = \mathbf{K}(s, m)[a] \wedge \mathbf{K}(t, m)[b] \;\; \textit{and} \;\; a\vee b = k(n, s, i)[a] \vee k(n, t, j)[b]$$

where $m = s\wedge t$, $n = s\vee t$, *the image of* $k(n, s, i)$ *contains* $\mathbf{K}(n, t)^{-1}[b]$ *and the image of* $k(n, t, j)$ *contains* $\mathbf{K}(n, s)^{-1}[a]$. *Conversely, every normal skew lattice arises in this fashion*. □

2.6 Decompositions of normal, symmetric skew lattices

Given rectangular skew lattices I and B, let A be their direct product I × B. A normal primitive skew lattice $P_{I,B}$ with $\mathcal{D}$-class structure A > B is given by letting B be a full A-coset in itself and for each $i \in I$, letting $\{i\} \times B$ be a full B-coset in A, and using the coset bijections: $\varphi_i\colon \{i\}\times B \to B$ and $\varphi_i^{-1}\colon B \to \{i\}\times B$ given by $\varphi_i(i, b) = b$ and $\varphi_i^{-1}(b) = (i, b)$. Clearly $P_{I,B} \cong I^0 \times B$ where I^0 is just I with a zero element 0 adjoined, so that I > {0}. Our first decomposition result states that for normal, primitive skew lattices this is essentially all there is.

Lemma 2.6.1. *Let* P be *a normal, primitive skew lattice with* $\mathcal{D}$*-class structure* A > B, *and let* I *be a set of indices for the* B*-cosets* A_i in A. *Then a rectangular skew lattice structure exists on* I *such that* $I^0 \times B \cong P$. *Given* $b \in B$, I *can be given as the image set* $b\vee A\vee b$ *of b in* A, *in which case* $I^0 \cong I^b = I\cup\{b\} = b\vee S\vee b$. *An isomorphism* $\theta\colon I^b \times B \cong S$ *is given by* $\theta(x, y) = y\vee x\vee y$, *the unique image of y in* $B\vee x\vee B$ *in* S *for all* $(x, y) \in I^b \times B$.

Proof. The basic coset structure insures that θ is at least a bijection. To see that θ as defined is an isomorphism, first observe that for x, x' in I^b and y, y' in B, regularity gives:

$$\theta(x, y) \vee \theta(x', y') = y{\vee}x{\vee}y \vee y'{\vee}x'{\vee}y' = y{\vee}x{\vee}x'{\vee}y' = y{\vee}y'{\vee}x{\vee}x'{\vee}y{\vee}y' = \theta(i{\vee}j, b{\vee}b').$$

Thus θ is at least a $\vee$-isomorphism of skew lattices, in which case it is also a full isomorphism between corresponding $\mathcal{D}$-classes, where $u{\wedge}v = v{\wedge}u$ holds. Suppose say $x = b$. Then we have both $\theta(b, y) \wedge \theta(x', y') = y \wedge y'{\vee}x'{\vee}y' = y{\wedge}y'$ since y' is the unique image of $y'{\vee}x'{\vee}y'$ in B, while $\theta(b{\wedge}x', y{\wedge}y') = (y{\wedge}y'){\vee}(b{\wedge}x'){\vee}(y{\vee}y') = y{\vee}y'$ since $b{\wedge}x' = b$. The case where $x' = b$ is similar. ☐

This simple result can be extended several ways. We begin with normal skew lattices possessing $\mathcal{D}$-classes that are minimal with respect to the partial ordering of $\mathcal{D}$-classes.

Proposition 2.6.2. *Let* S *be a normal skew lattice with a minimal* $\mathcal{D}$*-class* B. *Pick* $b \in$ B *and let* T *be the subalgebra* $b{\vee}\mathrm{S}{\vee}b$ *of* S *given as* $\{b{\vee}x{\vee}b \mid x \in \mathrm{S}\}$ *or equivalently* $\{x \in \mathrm{S} \mid x \geq b\}$. *Then b is the zero element of* T, *and an isomorphism* $\theta\colon \mathrm{T} \times \mathrm{B} \to \mathrm{S}$ *is given by* $\theta(x, y) = y{\vee}x{\vee}y$.

Proof. That $\theta\colon \mathrm{T} \times \mathrm{B} \to \mathrm{S}$ is a bijection, that between corresponding $\mathcal{D}$-classes an isomorphism, follows from the lemma. Given (x, y), (x', y') in $\mathrm{T} \times \mathrm{B}$, that $\theta(x, y) \vee \theta(x', b) = \theta(x{\vee}x', y{\vee}b)$ is seen exactly as above so that θ is at least a $\vee$-isomorphism. Next, observe that

$$\theta(x, y) \wedge \theta(x', y') = (y{\vee}x{\vee}y) \wedge (y'{\vee}x'{\vee} y') \leq (y'{\vee}x'{\vee} y') \vee (y{\vee}x{\vee}y) = y'{\vee}x'{\vee}x{\vee}y$$

since $u{\wedge}v \leq v{\vee}u$ for skew lattices in general. $\theta(x{\wedge}x', y{\wedge} y') = (y{\wedge} y'){\vee}(x{\wedge}x'){\vee}(y{\wedge} y')$, on the other hand, is $\mathcal{D}$-related to $x{\wedge}x'$ and thus to $(y{\vee}x{\vee}y){\wedge}(y'{\vee}x'{\vee} y')$. But $(y{\wedge}y'){\vee}(x{\wedge}x'){\vee}(y{\wedge} y') \leq y'{\vee}x'{\vee}x{\vee}y$ also. Indeed:

$$\begin{aligned}(y{\wedge}y'){\vee}(x{\wedge}x'){\vee}(y{\wedge}y'){\vee}(y'{\vee}x'{\vee}x{\vee}y) &= (y{\wedge}y'){\vee}(x{\wedge}x'){\vee}(y'{\vee}x'{\vee}x{\vee}y) = (y{\wedge}y'){\vee}(x{\wedge}x'){\vee}(x'{\vee}x{\vee}y)\\ &= (y{\wedge}y'){\vee}(x'{\vee}x{\vee}y) = (y{\wedge}y'){\vee}y'{\vee}(x'{\vee}x{\vee}y) = y'{\vee}x'{\vee}x{\vee}y\end{aligned}$$

by a combination of absorption and regularity. Likewise:

$$(y'{\vee}x'{\vee}x{\vee}y) \vee (y{\wedge}y'){\vee}(x{\wedge}x'){\vee}(y{\wedge}y') = y'{\vee}x'{\vee}x{\vee}y.$$

Since $\theta(x, y){\wedge}\theta(x', y')$ is $\mathcal{D}$-equivalent to $\theta(x{\wedge}x', y{\wedge} y')$ in a normal skew lattice with a common upper bound in $(\mathrm{S}, \geq)$, they are equal, i.e., $\theta(x, y) \wedge \theta(x', y') = \theta(x{\wedge}x', y{\wedge} y')$. ☐

As an application, let A > B > C be a normal skew chain. Then first, this skew chain is isomorphic to the product of a skew chain A′ > B′ > {0} with C. But A′ > B′ in turn is isomorphic to the product of a skew chain A″ > {0} with B′. Thus A > B > C essentially is A″×B′×C > B′×C > C with all downward projections and upward coset injections being the coordinate-wise functions: $p(a, b, c)$ projects down to (b, c) or even further down to c, while say $\varphi_a^{-1}(b, c) = (a, b, c)$. In general we have:

Theorem 2.6.3. *For each finite chain* T, *to within isomorphism every normal skew chain* S *with maximal lattice image isomorphic to* T *is obtained as follows. Take a* T-*indexed family of rectangular skew lattices* $\{X(t) \mid t \in T\}$. *For each* $t \in T$, *set* $\mathcal{D}(t) = \prod_{s \le t} X(s)$. *Then for* $t_1 < t_2$, *the primitive subalgebra with* $\mathcal{D}$-*classes* $\mathcal{D}(t_2) = \prod_{s \le t_2} X(s) > \mathcal{D}(t_1) = \prod_{s \le t_1} X(s)$ *is determined by letting the projection from* $\mathcal{D}(t_2)$ *onto* $\mathcal{D}(t_1)$ *and the coset bijections from* $\mathcal{D}(t_1)$ *into* $\mathcal{D}(t_2)$ *be the coordinate-wise projection and injections between these products.* □

Thus given say $t_1 \le t_2$, $(x_1, \dots, x_{t_1}) \wedge (y_1, \dots, y_{t_2}) = (x_1 \wedge y_1, \dots, x_{t_1} \wedge y_{t_1})$ and $(x_1, \dots, x_{t_1}) \vee (y_1, \dots, y_{t_2}) = (x_1 \vee y_1, \dots, x_{t_1} \vee y_{t_1}, y_{1+t_1}, \dots, y_{t_2})$.

Alternatively, such a skew chain S is isomorphic to a fibered product of "near constant" skew chains over a common maximal chain image. In the case of A > B > C above, the skew chain is isomorphic to the fibered product of the following skew chains over 2 > 1 > 0:

$$\begin{array}{ccc} C & B' & A'' \\ \vdots\cong & \vdots\cong & \vdots \\ C & B' & \{0\} \\ \vdots\cong & \vdots & \vdots \\ C & \{0\} & \{0\}. \end{array}$$

We saw in Theorem 2.4.10 that symmetric skew lattices are characterized by the fact that all skew diamonds have some special properties. For symmetric normal skew lattices we have:

Theorem 2.6.4. *A normal skew lattice* S *is symmetric if and only if for every incomparable pair of* $\mathcal{D}$-*classes* A *and* B of S, *the skew diamond* $S_{A\cup B}$ *generated from* $A \cup B$ *is isomorphic to a normal skew diamond of the form*

$$\begin{array}{ccc} & X \times M \times Y & \\ \swarrow & & \searrow \\ A = X \times M & & M \times Y = B \\ \searrow & & \swarrow \\ & M & \end{array}$$

where the downward projections and upward coset-bijections are all given in coordinate-wise fashion. In this case the skew diamond decomposes as a direct product, $M \times X^0 \times Y^0$, *where* X^0 *and* Y^0 *are skew chains* $X > \{0\}$ *and* $Y > \{0\}$. *In particular, if* $A \wedge B = \{0\}$, *then* $S_{A\cup B} \cong A^0 \times B^0$.

Proof. By Proposition 2.5.2, the proof reduces to the special case where $A \wedge B = \{0\}$. So what is the situation in the join class? First it consists of all possible outcomes $a \vee b$. By symmetry, in all cases $a \vee b = b \vee a$. Suppose that $a \vee b = a' \vee b'$. Then since $a \vee b >$ both a and a' in A, $a = a'$. Likewise $b = b'$. Thus the $a \vee b$-outcomes are all distinct, with each pair commuting. Thus the join class factors as A × B with both projections being canonical: $a\vee b \to a$ and $a\vee b \to b$. The isomorphism of $A^0 \times B^0$ with $S_{A\cup B}$ is given by $\theta(x, y) = x\vee y$ for x in A^0 and y in B^0. □

All results in this section thus far involve decompositions. We attempt to develop this theme in what follows. We begin by generalizing the construction of Theorem 2.6.3.

Let T be a lattice and let P be a ***prime filter*** of T. Thus P is a filter and T \ P is an ideal of T. Put otherwise:

(1) For all $p \in P$ and $t \in T$, $p \vee t \in P$.
(2) P is closed under $\wedge$.
(3) If $s \vee t \in P$, then either $s \in P$ or $t \in P$.

Given T, P and a rectangular skew lattice X, let T[X | P] be the normal skew lattice defined on

$$(P \times X) \cup (T \setminus P)$$

by extending the operations on the skew lattice s P × X and the lattice ideal T \ P by setting

$$s \vee (p, x) = (s\vee p, x) = (p, x) \vee s \quad \text{and} \quad s \wedge (p, x) = s\wedge p = (p, x) \wedge s$$

for $(p, x) \in P \times X$ and $s \in S \setminus P$. Any skew lattice isomorphic to T[X | P] is said to be **P-*primary*** over T with fiber X. Prime filters arise as inverse images $f^{-1}(1)$ for lattice epimorphisms f: T → **2** where **2** is the lattice 1^0. Thus T[X | P] may be viewed as the fibered product $T \times_2 X^0$ obtained by pulling the surjection $X^0 \to \mathbf{2}$ back along f: T → **2**.

More generally, let $\mathcal{P}(T)$ be the family of all prime filters of T, including T and let $\{X_P \mid P \in \mathcal{Pr}(T)\}$ be a corresponding family of rectangular algebras. Then the fibered product over T, $\prod_T\{T[X_P \mid P] \mid P \in \mathcal{P}(T)\}$ is both symmetric and normal. To within isomorphism, its rectangular $\mathcal{D}$-classes are given by setting $\mathcal{D}(t) = \prod\{X_P \mid P \in \mathcal{Pr}(T) \;\&\; t \in P\}$ and using canonical coordinate projections and injections. Any skew lattice S isomorphic to such a fibered product is ***decomposable*** with the fibered product being its ***primary decomposition***. The rest of this section is devoted to proving a main result of this section.

Theorem 2.6.5. (*The Decomposition Theorem*) *Every symmetric normal skew lattice with a finitely generated maximal lattice image is decomposable. More generally, a symmetric normal skew lattice with a finite maximal distributive lattice image is decomposable*. □

The first major step in proving this result is our next theorem. But we first need several preliminary lemmas, beginning with:

Lemma 2.6.6. *Let* S *be a symmetric, normal skew lattice and let* A ≥ B *be* $\mathcal{D}$*-classes in* S. *If* ≥ *induces an isomorphism between* A *and* B *as rectangular skew lattices (making both full cosets of each other), then for any* $\mathcal{D}$*-class* C *such that* A ≥ C ≥ B, ≥ *also induces isomorphisms between* A *and* C *and between* C *and* B.

Elements x and y in a normal skew lattice S are ***reflections of each other*** if $\mathcal{D}$-classes A ≥ B exist such that (i) x and y lie in intermediate $\mathcal{D}$-classes, (ii) ≥ induces an isomorphism of A with B and (iii) x and y have the same images in A and in B. (This includes the possibility that either x or y lies in either A or B or both.)

Lemma 2.6.7. *Reflection is an equivalence on symmetric, normal skew lattices.*

Proof. Let x, y and z be given where $a \geq x \geq b$, $a \geq y \geq b$, $c \geq y \geq d$ and $c \geq y \geq d$ with a, b, c and d lying in respective $\mathcal{D}$-classes A, B, C and D such that both A ≅ B and C ≅ D under ≥. It follows that A ≅ Y and C ≅ Y under ≥ where Y = $\mathcal{D}_y$. Thus A ≅ A∧C ≅ C under ≥ also. Setting J = A∨C, J is isomorphic to both A and C under ≥ by Theorem 2.6.4. In similar fashion if M = B⋀D, then B and D are isomorphic to M. Hence x, y and z all lie between the $\mathcal{D}$-classes J and M that are isomorphic under ≥. Clearly x and y and also y and z share a common unique image $a \vee c$ up in J as well as a common unique image $b \wedge d$ down in M. ☐

Lemma 2.6.8. *Given a symmetric, normal skew lattice, reflection is a congruence.*

Proof. If ≥ induces an isomorphism for classes A > B, then for all classes C both A∨C ≅ B∨C and A∧C ≅ B∧C under ≥. In the case of A∨C ≥ B∨C we have the diagram

$$\begin{tikzcd}[column sep=small]
& \mathrm{A} \vee \mathrm{C} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
\mathrm{A} \arrow[dr, dash, dotted] & & \mathrm{B} \vee \mathrm{C} \arrow[dl, dash, dotted] \\
& \mathrm{M} \arrow[d, dash, dotted] & \\
& \mathrm{B} &
\end{tikzcd}$$

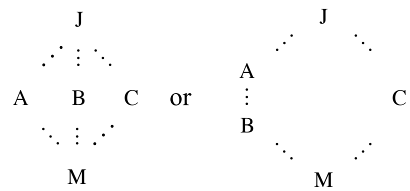
If M is the meet class of A with B⋁C, then M lies somewhere between A and B. (Possibly M = B.) Since A ≅ B under ≥, so is A ≅ M. Since A⋁C = A⋁B⋁C, A⋁C ≅ B⋁C by Theorem 2.5.4 (since if one side of the diagram is an isomorphism, so is the other). Let $b' \in$ M be the unique element such that $b' \geq b$. Since B and M have identical cosets in B∨C, orthogonality yields $b \vee c = b' \vee c$. But Theorem 2.6.4 implies that $a \vee c \geq b' \vee c$, and thus $a \vee c \geq b \vee c$. Likewise, $c \vee a \geq c \vee b$, $a \wedge c \geq b \wedge c$ and $c \wedge a \geq c \wedge b$. (In fact, $a \wedge c \geq b \wedge c$ and $c \wedge a \geq c \wedge b$ follow from $a \geq b$ in any normal skew lattice without the added assumption that A ≅ B under ≥. This implication characterizes normal bands.) It is now clear that reflection must be a congruence. ☐

Reflection is the maximal congruence r on S inducing isomorphisms between $\mathcal{D}$-classes in S and their image $\mathcal{D}$-classes in the quotient skew lattice S/r. The latter is called the ***reduced skew lattice*** of S to be denoted S^{rd}. Clearly $(S^{rd})^{rd} = S^{rd}$. We say that a symmetric, normal skew lattice S is ***reduced*** if $S^{rd} = S$. The following result is of independent interest.

Theorem 2.6.9 (*The Reduction Theorem*) *Let* S *be a symmetric, normal skew lattice with maximal lattice image* T *and reduced skew lattice* S^{rd}. *If* D *is the maximal lattice image of* $\mathbf{S}^{\mathrm{rd}}$. *then the canonical epimorphisms* S → T *and* S → S^{rd} *induce an isomorphism of* S *with the fibered product of* T *and* S^{rd} *over* D, $\mathrm{S} \cong \mathrm{T} \times_{\mathrm{D}} \mathrm{S}^{\mathrm{rd}}$; *both* $\mathrm{S}^{\mathrm{red}}$ *and* D, *moreover, are distributive*.

$$\begin{tikzcd} \mathrm{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "pullback"] & \mathrm{S}^{rd} \arrow[d] \\ \mathrm{T} \arrow[r] & \mathrm{D} \end{tikzcd} \qquad \mathrm{T} = \mathrm{S}/\mathcal{D} \text{ and } \mathrm{D} = \mathrm{S}^{rd}/\mathcal{D}.$$

Proof. We need only show that when S is reduced, it is distributive. We do so by showing that none of the following types of subalgebras can arise in S where in what follows A, B. C, J and M represent distinct $\mathcal{D}$-classes of S.



Suppose first that M is trivial so that in both diagrams each element of C commutes with all elements in A and B. In the left diagram C must also commute with all elements of J so that C must trivial. Similarly in the left diagram A and B must also be trivial and thus the left diagram reduces to a lattice. Dropping the added assumption that M be trivial, we see that the natural rectangular functor K of S when restricted to the left diagram is a functor of isomorphisms between distinct classes. Thus **K** cannot be the rectangular functor of a reduced algebra. In the right subalgebra, each $n \in \mathrm{J}$ has commuting factorizations as $a \vee c$ and $b \vee c$ with $a \in \mathrm{A}$, $b \in \mathrm{B}$ and $c \in \mathrm{C}$ where clearly $a > b$. As a consequence, A ≅ B under ≥. Even if M is not trivial, the skew lattice on the right must factor as M × the M-trivial case. Hence still A ≅ B under ≥, which again cannot happen in a reduced symmetric, normal skew lattice. Thus the right diagram also cannot occur in S. Hence lattice D is distributive and the normal skew lattice S^{rd} must also be distributive by Theorems 2.3.2 and/or 2.3.4. ☐

The structure of symmetric, normal skew lattices has been reduced to that of lattices and symmetric, normal, distributive skew lattices that is, to that of lattices and strongly distributive skew lattices. We now examine the latter, first when $\mathrm{S}/\mathcal{D}$ is finite where we show that S is decomposable, thus proving a significant special case of Theorem 2.5.4.

So let S be strongly distributive with $\mathrm{S}/\mathcal{D}$ finite and denote the latter by T. Being a finite distributive lattice, T has a finite set π of join-irreducible elements, including its minimal element 0. The class of prime filters of T is thus given by

$$\mathcal{Pr}(\mathrm{T}) = \{p \vee \mathrm{T} \mid p \in \pi\}.$$

Recall that the center $\mathcal{Z}(S)$ of S is the union of all trivial $\mathcal{D}$-classes of S. Since S is normal, $\mathcal{Z}(S)$ is a (possible empty) ideal of S that is isomorphic to its image in T which is also an ideal of T. $\mathcal{Z}(S)$ is empty precisely when the minimal $\mathcal{D}$-class of S is nontrivial. In any case, all $\mathcal{D}$-classes of S that are minimal in the complement of $\mathcal{Z}(S)$ correspond to join-irreducible elements in T.

Lemma 2.6.10. *Given* S *and* T *as above, let* X *be a minimal $\mathcal{D}$-class in the complement of* $\mathcal{Z}(S)$, *let x be fixed in* X *and let* P *be the prime filter in* T *induced by the image of* X *in* π. *Set*

$$S' = (S \setminus S\vee x\vee S) \cup x\vee S\vee x$$

and let T[X | P] *be the* P*-primary algebra induced by* X *and* P. *Then*

i) S′ *is a subalgbra of* S *that is also mapped onto* T *by the canonical epimorphism from* S.

ii) *An isomorphism* $\theta\colon S \cong S' \times_T T[X\,|\,P]$ *is given by the rule*

$$\theta(y) = \begin{cases} (x \vee y \vee x,\ y \wedge x \wedge y) & \text{for all } y \in S \vee x \vee S \\ y & \text{otherwise} \end{cases}.$$

iii) *Upon comparison in* T, $\mathcal{Z}(S')$ *is properly larger than* $\mathcal{Z}(S)$.

Proof. S′ is a subalgebra since x commutes with all elements in the complement of S∨x∨S. Thus θ is at least an isomorphism off of S∨x∨S and by Theorem 2.6.3, θ also yields an isomorphism of S∨x∨S with (x∨S∨x) × X. Suppose that $u \in S\vee x\vee S$ and $w \in S \setminus S\vee x\vee S$ are given. Then $\theta(u\vee w) = \theta(u)\vee\theta(w)$ is equivalent to $x\vee u\vee w\vee x = x\vee u\vee x\vee w$ and $(u\vee w)\wedge x\wedge(u\vee w) = (u\wedge x\wedge u)$. Since x commutes with w, the first identity holds. Because $x\wedge w$ lies in $\mathcal{Z}(S)$, $x\wedge u \geq x\wedge w$ and $x\wedge(u\vee w) = x\wedge u$. Similarly, $(u\vee w)\wedge x = u\wedge x$ and the second identity holds. Finally, $\theta(u\wedge w) = \theta(u)\wedge\theta(w)$ is equivalent to $u\wedge w = (x\vee u\vee x)\wedge w$. Distribution in the symmetric, normal case yields

$$(x\vee u\vee x)\wedge w = (x\wedge w) \vee (u\wedge w) \vee (x\wedge w) = u\wedge w$$

since $u\wedge w \succeq x\wedge w$ with $x\wedge w$ in Z(S) implies $u\wedge w \geq x\wedge w$. □

As a consequence we have the following ***Primary Decomposition Theorem***.

Theorem 2.6.11. *Let* S *be a strongly distributive skew lattice with finite maximal lattice image* T *and let* $\mathcal{Pr}$(T) *be the set of prime filters of* T *including* T. *Then to each* P *in* $\mathcal{Pr}$(T) *there corresponds to a rectangular algebra* X_P, *that is unique to within isomorphism, such that* S *is isomorphic to the fibered product* $\prod_T\{T[X_P\,|\,P]\,|\,P \in \mathcal{Pr}(T)\}$. S *is reduced if and only if* X_P *is nontrivial for each proper prime filter* P.

Proof. Repeated applications of Lemma 2.6.10 enable one to pass through the prime filters of T and successfully strip primary factors off of S to obtain the decomposition. To see uniqueness, let X be a $\mathcal{D}$-class corresponding to a join-irreducible element p of T and let P = $p\vee$T be the induced prime filter. If P = T, X was just the minimal $\mathcal{D}$-class of S and $X_P = X$. Otherwise there is a maximal $\mathcal{D}$-class of S lying beneath X, call it Y. Applying the Lemma to the subalgebra X⋃Y, X must factor as $X_P \times Y$ where to within isomorphism X_P is $y\vee X\vee y$ for any y in Y. Thus X_P is indeed unique to within isomorphism. Finally, if any X_P for some P ≠ T is trivial, then for Y < X as just given, $\mathbf{K}_{(X,\, Y)}$ is an isomorphism and S is not reduced. But if no X_P is trivial, except possibly for P = T, then no $\mathbf{K}_{(A,\, B)}$ with A > B can be an isomorphism since then A as an element of T belongs to a prime filter P that excludes B and thus X_P is not trivial so that B is to within isomorphism a proper factor of A. ☐

Theorem 2.6.5 follows immediately from Theorems 2.6.9 and 2.6.11.

What happens when the maximal distributive lattice image D of S is not finite? S still factors as the fibered product of its maximal lattice image T and its maximal reduced image S^{red}: $S \cong T \times_D S^{red}$. Since S^{red} is necessarily distributive, the question reduces to what can be said about arbitrary distributive, symmetric, normal skew lattices. Being a variety of algebras, every such algebra decomposes into a subdirect product of subdirectly irreducible algebras. Which distributive, symmetric, normal skew lattices are the subdirectly irreducible?

To answer this, let **2** denote the lattice 1 > 0, let $\mathcal{R}_2$ [$\mathcal{L}_2$] denote the right [left] rectangular skew lattice on {1, 2} and let $\mathbf{3_R}$ and $\mathbf{3_L}$ be the results of adjoining a zero element 0 to $\mathcal{R}_2$ and $\mathcal{L}_2$, respectively (Thus 1, 2 > 0 in both cases.) Finally, let **5** denote the fibered product algebra $\mathbf{3_L} \times_2 \mathbf{3_R}$ of order 5. Equivalently, $\mathbf{5} = (\mathcal{L}_2 \times \mathcal{R}_2)^{\mathbf{0}}$.

Theorem 2.6.12. *The only nontrivial subdirectly irreducible strongly distributive skew lattices are copies of* **2**, $\mathcal{R}_2$, $\mathcal{L}_2$, $\mathbf{3_R}$ or $\mathbf{3_L}$. *Every strongly distributive skew lattice is thus a subdirect product of some of these algebras and in particular can be embedded in a power of* **5**.

Proof. We show that a nontrivial subdirectly irreducible distributive, symmetric, normal skew lattice S must be a copy of one of the stated algebras. If **S** is such an algebra, a pair of elements $a \neq b$ in S must exist that are congruent under all nonidentity congruences on **S**. If a and b are not $\mathcal{D}$-equivalent, then since $\mathcal{D}$ separates them, $\mathcal{D}$ is the identity Δ and S is a lattice that must be a copy of **2**. So assume that $a \neq b$ in some $\mathcal{D}$-class. We define two congruences on S as follows:

$$x \overset{a}{\sim} y \text{ if } a\wedge x = a\wedge y \quad \text{and} \quad x \overset{a}{\sim}{}^{*} y \text{ if } x\wedge a = y\wedge a$$

Each is clearly an equivalence on S that, thanks to normality and Theorem 2.3.4, is indeed a congruence. Since $a \,\mathcal{D}\, b$ but $a \neq b$, either $\overset{a}{\sim}$ or $\overset{a}{\sim}{}^{*}$ must separate a and b and thus equal Δ. In either case A must be the maximal $\mathcal{D}$-class of S. (Otherwise some $a' > a$ exists in a properly higher $\mathcal{D}$-class, but neither $\overset{a}{\sim}$ nor $\overset{a}{\sim}{*}$ separate a' from a and so neither is Δ.) If S = A, then S is rectangular and so, being subdirectly irreducible, is a copy of either $\mathcal{R}_2$ or $\mathcal{L}_2$. Otherwise lower $\mathcal{D}$-classes exist and in particular there is an element $c < a$. Since both $\overset{c}{\sim}$ and $\overset{c}{\sim}{*}$ identify a with c, both are the trivial congruence, the universal relation ∇. Since $\overset{c}{\sim}$ and $\overset{c}{\sim}{*}$ equal ∇ for all $c < a$, A must have a unique singleton lower class, {0} and thus S = A^0. Since S is subdirectly irreducible, A itself is a copy of either $\mathcal{R}_2$ or $\mathcal{L}_2$ so that S is a copy of either $\mathbf{3_R}$ or $\mathbf{3_L}$. ☐

Corollary 2.6.13. *The variety of strongly distributive skew lattices is generated by* **5**. *The variety of symmetric, normal, skew lattices is generated by* **5** *plus the variety of lattices*. ☐

We saw in Section 2.3 normal skew lattices arise as maximal normal bands in rings. Before proceeding to the next section, we offer an example of somewhat different character, an example that both illustrates much that has occurred in this section and also sets the stage for developments in Chapter 4.

Example 2.6.1. Given sets A and B, let $\mathcal{P}_R$(A, B) be the set of all partial function from A to B. $\mathcal{P}_R$(A, B) becomes a strongly distributive skew lattice upon setting

$$f \vee g = f \cup g\,|\,G\backslash F \quad \text{and} \quad f \wedge g = g\,|\,F \cap G$$

where F and G denote the actual functional domains in A of f and g respectively. $\mathcal{P}_R$(A, B) factors as $\prod_{a\in A} \mathcal{P}_R(\{a\}, B)$ with each factor $\mathcal{P}_R(\{a\}, B)$ being a copy of the primitive algebra B^0, the right rectangular skew lattice on A with 0 adjoined. The various $\mathcal{D}$-classes are indexed by the subset algebra $\mathbf{2}^A$ that forms the maximal lattice image of $\mathcal{P}_R$(A, B). For any subset F of A, its $\mathcal{D}$-class is precisely B^F. If G ⊆ F, $\mathbf{K}_{(B^F, B^G)}$: $B^F \to B^G$ is just the natural coordinate-wise projection. If A is finite, $\mathcal{P}_R$(A, B) has the primary decomposition $\prod_{2^A} \{2^A[B_a \,|P_a] \,\big|\, a \in A\}$ where P_a is the principal filter generated by a. ☐

Example 2.6.2. Given sets A and B, the left-handed variant of the above skew lattice on $\mathcal{P}_L$(A, B) is given by setting $f \vee g = g \cup f\,|\,F\backslash G$ and $f \wedge g = f\,|\,F \cap G$. Remarks similar to those made above can be made here also. ☐

By a ***ring of partial functions*** is meant any subalgebra $\mathcal{P}_R$(A, B) for some pair of sets A and B. The following variation of Theorem 2.6.12 holds. By a ***full ring of partial functions*** is meant a subalgebra S of some $\mathcal{P}_R$(A, B) such that each $a \in$ A correspond to a subset $B_a \subseteq$ B such that the primary decomposition of $\mathcal{P}_R$(A, B) given above restricts to the primary decomposition $\prod_{2^A} \{2^A[B_a \,|P_a] \,\big|\, a \in A\}$.

Theorem 2.6.14. *Every right-handed, strongly distributive skew lattice can be embedded in a power of* $\mathbf{3_R}$. *Equivalently, every such skew lattice can be embedded in some* $\mathcal{P}_R(A, \{1, 2\})$. *Thus, every right-handed strongly distributive skew lattice* S *is isomorphic to a ring of partial functions*. *If* S/$\mathcal{D}$ *is finite, then* S *is isomorphic to a full ring of partial functions*. □

Example 2.6.3 (Example 2.3.4 revisited). This matrix example is isomorphic to the direct product of the primitive algebras $(F \times F)^0$ where F × F is the rectangular algebra, that is, it is isomorphic to the power $[(F \times F)^0]^n$. □

As indicated above, a natural continuation of these ideas is given in Chapter 4, and in particular, in the first section.

Historical remarks

The material in Sections 2.1 and 2.2 appeared in Jonathan Leech's 1989 paper in *Algebra Universalis*. The material in Sections 2.3 and 2.6 appeared in his 1992 paper in the *Semigroup Forum*. The material in Sections 2.4 and 2.5 originated in his 1993 paper in the *Transactions of the American Mathematical Society*. Highlights from these papers appeared later in his 1996 survey article in the *Semigroup Forum*. An important result from his 1993 Transactions paper, not presented here is the fact that the free symmetric skew lattice on two generators is infinite. As a consequence, the free skew lattice on two generators must be infinite. Thus (symmetric) skew lattices are not locally finite. This contrasts with the case for lattices where the free lattice on 2 generators has four elements, but the free lattice on three generators is infinite. In her 2011 paper in *Algebra Universalis*, Karin Cvetko-Vah showed that for symmetric skew lattices that are also categorical, the free algebra on two generators is finite of order 16. (But then thanks to the case for lattices, the free such algebra on three generators must be infinite.) Her result echoed the case for small skew lattices (≤ 2 generators) in rings that had been studied in Leech's 2005 paper below.

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III: QUASILATTICES, PARALATTICES & THEIR CONGRUENCES

We take a closer look at quasilattices, paralattices, and especially refined quasilattices which by definition are simultaneously algebras of both types. Particular attention is given to their congruence lattices and to related topics such as Green's equivalences and simple algebras. Since all skew lattices are refined quasilattices, our study has implications for skew lattices.

In the first section we consider quasilattices where $\mathcal{D}_{(\vee)} = \mathcal{D}_{(\wedge)}$, with $\mathcal{D}$ denoting the common congruence. These are the noncommutative lattices for which the Clifford-McLean Theorem holds: the maximal lattice image of a quasilattice Q is $Q/\mathcal{D}$ and its maximal antilattice subalgebras are its $\mathcal{D}$-classes. The congruence $\mathcal{D}$ also plays an major role in the congruence lattice **Con**(Q) of a quasilattice Q. For instance, given a family of congruences $\{\theta_i\}$, both

$$\mathcal{D} \wedge sup_i(\theta_i) = sup_i(\mathcal{D} \wedge \theta_i)$$

and its dual hold. (Theorem 3.1.1) As a consequence, **Con**(Q) is a subdirect product of the interval sublattices $[\mathcal{D}, \nabla]$ and $[\Delta,\mathcal{D}]$, where ∇ is the universal congruence where Δ is the identity congruence. (Theorem 3.1.2) In particular, **Con**(Q) is naturally a copy of the direct product $[\mathcal{D}, \nabla] \times [\Delta,\mathcal{D}]$ precisely when Q itself factors as the direct product $T \times A$ of a lattice T and an antilattice A. (Theorem 3.1.3) Due to these theorems, the only simple quasilattices (in that the congruence lattice reduces to $\{\Delta, \nabla\}$) are simple lattices and simple antilattices.

Section 2 addresses the topic of simple antilattices. Its main result Theorem 3.2.3 states that finite simple antilattices exist for all composite orders greater than 5. Antilattices of odd prime order are trivially non-simple. In the remaining cases, algebras of orders 1 or 2 are always simple, while simple antilattices of order 4 are shown to be impossible.

In the next section we look at noncommutative lattices that are *regular* in the strongest sense: $\mathcal{L}_{(\wedge)}$, $\mathcal{R}_{(\wedge)}$, $\mathcal{L}_{(\vee)}$ and $\mathcal{R}_{(\vee)}$ are congruences relative to both $\vee$ and $\wedge$. Flat quasilattices (where $\mathcal{D}_{(\vee)}$ is either $\mathcal{L}_{(\vee)}$ or $\mathcal{R}_{(\vee)}$ and likewise $\mathcal{D}_{(\wedge)}$ is either $\mathcal{L}_{(\wedge)}$ or $\mathcal{R}_{(\wedge)}$, with the two remaining Green's relations being equality) are trivially regular since $\mathcal{D}$ is both unambiguous and a congruence on quasilattices (Theorem 3.3.2). Moreover, every regular quasilattice factors as the fibered product of its four possible maximal flat images (Theorem 3.2.4).

In Section 4 we study paralattices, and especially refined quasilattices that are paralattices and quasilattices simultaneously. A paralattice S is also a quasilattice if and only if both $\mathcal{D}_{(\vee)}$ and $\mathcal{D}_{(\wedge)}$ are congruences, in which case they meld into a single relation $\mathcal{D}$. A regular paralattice is thus a refined quasilattice. In fact it is the fibered product of its maximal skew lattice and skew* lattice images. (Corollary 3.4.6.) As a consequence we have the following sublattice of the lattice of quasilattice varieties:

$$\begin{array}{ccc} & \textit{Refined quasilattices} & \\ & \vdots & \\ & \textit{Regular paralattices} & \\ \swarrow & & \searrow \\ \textit{skew lattices} & & \textit{skew* lattices} \\ \searrow & & \swarrow \\ & \textit{lattices} & \end{array}$$

In general a refined quasilattice is partially regular in that $\mathcal{L}_{(\vee)}$ and $\mathcal{R}_{(\vee)}$ are congruences relative to $\vee$, while $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are congruences relative to $\wedge$. Even if not fully regular, refined paralattices behave very much like skew(*) lattices. This aspect is developed in some detail in the discussion from Theorem 3.4.7 through Theorem 3.4.14. The latter asserts that each refined quasilattice, if not isomorphic, is at least *isotopic* to some skew lattice. (The definition of "isotopic" is given in the section). Indeed every refined quasilattice can be viewed as the result of taking a left-handed skew lattice and mildly scrambling its $\vee$- and $\wedge$-computational details. (See the remarks after Theorem 3.4.14.) Thus here is a sense in which refined quasilattices are *roughly* skew lattices.

In Section 3.5 we study the effects of various distributive identities. Theorem 3.5.1 asserts that a double band (S; $\vee$, $\wedge$) satisfying both $a\wedge(b \vee c)\wedge a = (a\wedge b\wedge a) \vee (a\wedge c\wedge a)$ and its dual $a\vee(b \wedge c)\vee a = (a\vee b\vee a) \wedge (a\vee c\vee a)$ is a quasilattice if and only if it a paralattice. Theorem 3.5.2 asserts that such a quasilattice [paralattice] satisfies both strengthened identities $a\wedge(b \vee c)\wedge d = (a\wedge b\wedge d) \vee (a\wedge c\wedge d)$ and its dual $a\vee(b \wedge c)\vee d = (a\vee b\vee d) \wedge (a\vee c\vee d)$ if and only if it factors as the product of a distributive lattice and an antilattice. Finally, any given quasilattice [paralattice] factors as the product of a distributive lattice and a *regular* antilattice if and only if the even stronger identities, $a\wedge(b \vee c) = (a\wedge b) \vee (a\wedge c)$, $(b \vee c)\wedge d = (b\wedge d) \vee (c\wedge d)$ and their duals, hold.

The sixth and final section is rather "recreational" in nature. Extending Section 3.2, we consider ways that magic squares, finite planes and other rectangular designs can be used to create simple antilattices. Many classic magic squares, beginning with the classic *Lo Shu*, provide examples of simple antilattices.

$$\begin{array}{c|c|c} 8 & 1 & 6 \\ \hline 3 & 5 & 7 \\ \hline 4 & 9 & 2 \end{array} \quad \longrightarrow \quad (\wedge)\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad (\vee)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Conditions for simplicity are given for various classes of antilattices derived from finite designs. (See, e.g., Theorems 4.6.10 and 4.6.11.)

3.1 Congruences on quasilattices

Recall that a *quasilattice* is a noncommutative lattice $(Q; \vee, \wedge)$ for which the natural quasi-orders ${}_{(\vee)}\preceq$ and ${}_{(\wedge)}\succeq$ dualize each other: $x\ {}_{(\vee)}\preceq y$ iff $x\ {}_{(\wedge)}\succeq y$, that is, $y \vee x \vee y = y$ iff $x \wedge y \wedge x = x$. Corollary 1.3.5 asserts that a noncommutative lattice is a quasilattice if and only if $\mathcal{D}_{(\vee)} = \mathcal{D}_{(\wedge)}$ with the common relation $\mathcal{D}$ being a congruence, in which case $Q/\mathcal{D}$ is the maximal lattice image of Q and the $\mathcal{D}$-classes are the maximal rectangular subalgebras of Q. Thus quasilattices are precisely the class of those noncommutative lattices that follow the Clifford-McClean Theorem. Indeed we might say that quasilattices are the most natural noncommutative generalizations of lattices in that they have the rough shape of a lattice. They also have some striking properties that have no precursors for regular bands.

Theorem 3.1.1. *For any quasilattice* $(Q; \vee, \wedge)$, *the following hold:*

i) *For all congruences* θ *on* Q, $\mathcal{D} \circ \theta = \theta \circ \mathcal{D} = \theta \vee \mathcal{D}$.

ii) *Given a family of congruences* $\{\theta_i\}$ *on* Q:
$\mathcal{D} \vee inf_i(\theta_i) = inf_i(\mathcal{D} \vee \theta_i)$ *and* $\mathcal{D} \wedge sup_i(\theta_i) = sup_i(\mathcal{D} \wedge \theta_i)$.

Proof. Let $x, y \in Q$ be such that $x\ \mathcal{D}\circ\theta\ y$. Hence $u \in Q$ exists such that $x\ \mathcal{D}\ u\ \theta\ y$. Set $w = (x\vee y\vee x) \wedge y \wedge (x\vee y\vee x)$. Then $x = (x\vee u\vee x)\wedge u\wedge(x\vee u\vee x)\ \theta\ w\ \mathcal{D}\ y$ so that $\mathcal{D} \circ \theta \subseteq \theta \circ \mathcal{D}$. But $\mathcal{D} = \mathcal{D}^{-1}$ and $\theta = \theta^{-1}$, so that $\theta \circ \mathcal{D} = \theta^{-1} \circ \mathcal{D}^{-1} = (\mathcal{D} \circ \theta)^{-1} \subseteq (\theta \circ \mathcal{D})^{-1} = \mathcal{D}^{-1}\circ\theta^{-1} = \mathcal{D} \circ \theta$, and $\mathcal{D} \circ \theta = \theta \circ \mathcal{D}$ follows with $\mathcal{D} \circ \theta$ being precisely $\theta \vee \mathcal{D}$.

Thanks to (i), we next need to show $\mathcal{D} \circ inf_i(\theta_i) = inf_i(\mathcal{D} \circ \theta_i)$. That we have $\mathcal{D} \circ inf_i(\theta_i) \subseteq inf_i(\mathcal{D} \circ \theta_i)$ is clear. So let $x\ inf_i(\mathcal{D} \circ \theta_i)\ y$ in Q. Then for each i, some u_i in Q exists such that $x\ \mathcal{D}\ u_i\ \theta_i\ y$. Set $w = (y\vee x\vee y)\wedge x\wedge(y\vee x\vee y)$. Then

$$y\ \theta_i\ u_i = (u_i\wedge x\wedge u_i)\vee x\vee(u_i\wedge x\wedge u_i)\ \theta\ w\ \mathcal{D}\ x$$

or just, $x\ \mathcal{D}\ w\ \theta_i\ y$. Since w works for all i, $x\ \mathcal{D}\ w\ (inf_i\theta_i)\ y$ and $inf_i(\mathcal{D} \circ \theta_i) \subseteq \mathcal{D} \circ inf_i(\theta_i)$ is seen.

Finally, clearly $sup_i(\mathcal{D} \wedge \theta_i) \subseteq \mathcal{D} \wedge sup_i(\theta_i)$. So let $x\ (\mathcal{D} \wedge sup_i(\theta))\ y$ for some $x, y \in Q$. Thus $x\ \mathcal{D}\ y$ and $u_1,\ldots ,u_n$ exist such that $x = u_0\ \theta_1\ u_1\ \theta_2\ u_2\ \ldots\ u_{n-1}\ \theta_n\ u_n = y$. For $j \leq n$ set $w_j = (u_j\wedge x\wedge u_j)\vee x\vee(u_j\wedge x\wedge u_j)$. Then $x\ \mathcal{D}\ w_j$. In particular $x = w_0$, $y = w_n$, and $w_{j-1}\ \theta_j\ w_j$ for $j \leq n$. Thus

$$x\ \mathcal{D}\cap\theta_1\ w_1\ \mathcal{D}\cap\theta_2\ w_2 \ldots w_{n-1}\ \mathcal{D}\cap\theta_n\ y$$

so that $x\ sup_i(\mathcal{D}\cap\theta_i)\ y$ and $sup_i(\mathcal{D}\wedge\theta_i) = \mathcal{D} \wedge sup_i(\theta_i)$ follows. □

As a consequence we have:

Theorem 3.1.2. *Given a quasilattice* $(Q;\ \vee,\ \wedge)$, *the map* χ: $\mathbf{Con}(Q) \to [\mathcal{D}, \nabla] \times [\Delta, \mathcal{D}]$ *defined by* $\chi(\theta) = (\mathcal{D}\vee\theta,\ \mathcal{D}\cap\theta)$ *is an embedding of complete lattices yielding* $\mathbf{Con}(Q)$ *to be a subdirect product of* $[\mathcal{D}, \nabla]$ *and* $[\Delta, \mathcal{D}]$.

Proof. That χ is a homomorphism of complete lattices follows from the above theorem. To see that χ is 1-1, observe that for any congruence θ on Q,

$$x\theta y \text{ iff } x\theta x\wedge y\wedge x,\ x\wedge y\wedge x\theta y\wedge x\wedge y \text{ and } y\wedge x\wedge y\theta y.$$

In general, $x\wedge y\wedge x\ \theta\ y\wedge x\wedge y$ iff $x\wedge y\wedge x\ \theta\cap\mathcal{D}\ y\wedge x\wedge y$ as $x\wedge y\wedge x\ \mathcal{D}\ y\wedge x\wedge y$. Also $x\ \theta\ x\wedge y\wedge x$ iff $x\ \mathcal{D}\vee\theta\ x\wedge y\wedge x$, and likewise $y\wedge x\wedge y\ \theta\ y$ iff $y\wedge x\wedge y\ \mathcal{D}\vee\theta\ y$. The $\theta \Rightarrow \theta\vee\mathcal{D}$ direction is clear. So suppose that $x\ \mathcal{D}\vee\theta\ x\wedge y\wedge x$. Thus by Theorem 3.1.1(i), $u \in Q$ exists such that $x\ \mathcal{D}\ u\ \theta\ x\wedge y\wedge x$. But then $x = x\wedge u\wedge x\ \theta\ x\wedge(x\wedge y\wedge x)\wedge x = x\wedge y\wedge x$. Similarly $y\wedge x\wedge y\ \mathcal{D}\vee\theta\ y$ implies $y\wedge x\wedge y\ \theta\ y$. Consequently:

$$x\ \theta\ y \text{ if and only if } x\ \mathcal{D}\vee\theta\ x\wedge y\wedge x,\ x\wedge y\wedge x\ \theta\cap\mathcal{D}\ y\wedge x\wedge y \text{ and } y\wedge x\wedge y\ \mathcal{D}\vee\theta\ y.$$

But this means that θ is determined by $\mathcal{D}\cap\theta$ and $\mathcal{D}\vee\theta$, so that χ is an embedding. Since $[\Delta, \mathcal{D}]$ and $[\mathcal{D}, \nabla]$ are sublattices of $\mathbf{Con}(Q)$ with $\mathcal{D}\cap\theta = \theta$ for all $\theta \in [\Delta, \mathcal{D}]$ and $\mathcal{D}\vee\theta = \theta$ for all $\theta \in [\mathcal{D}, \nabla]$, $\mathbf{Con}(Q)$ is embedded as a subdirect product. □

When does $\mathbf{Con}(Q)$ *factor directly as* $[\mathcal{D}, \nabla] \times [\Delta, \mathcal{D}]$? To begin, a quasilattice Q ***splits*** if it is isomorphic with the product $T \times A$ of a lattice T and an antilattice A. In what follows ρ denotes the smallest congruence containing $\geq$.

Theorem 3.1.3. *Given a quasilattice* Q, *the following are equivalent:*

i) *Q splits.*
ii) $\mathcal{D} \cap \rho = \Delta$.
iii) *A congruence* θ *exists such that* $\mathcal{D}\cap\theta = \Delta$ *and* $\mathcal{D}\circ\theta = \nabla$.
iv) *For all* $x, y \in Q$, $x\ \rho\ y$ *iff* $x\vee y = y\vee x$ [*alternatively*, $x\wedge y = y\wedge x$].
v) $\mathbf{Con}(Q) \cong [\mathcal{D}, \nabla] \times [\Delta, \mathcal{D}]$ *under the map* $\theta \to (\theta\circ\mathcal{D},\ \theta\cap\mathcal{D})$.

Proof. Assume (i). Indeed, let $Q = T \times A$ with T and A as above. The $\mathcal{D}$-classes of Q consist of pairs (t, a) having common T-coordinates and the ρ-classes consist of pairs (t, a) having common A-coordinates. From this, (ii) follows. Since $\mathcal{D}\circ\rho = \nabla$ always holds, (ii) implies (iii). Suppose that (iii) holds for a congruence θ. Thus each θ-class meets each $\mathcal{D}$-class at a unique element. Let χ: $Q \to Q/\mathcal{D} \times Q/\theta$ be the homomorphism defined by $\chi(x) = (x\mathcal{D}, x\theta)$, where $x\mathcal{D}$ and $x\theta$ are the respective congruence classes of x. Since $\mathcal{D}\cap\theta = \Delta$, χ is one-to-one. Since each $\mathcal{D}$-class meets each θ-class, every possible pair $(x\mathcal{D}, y\theta)$ is in the image of χ, so that χ is an isomorphism. Since each θ-class meets each maximal rectangular subalgebra of Q, Q/θ is rectangular. Since $Q/\mathcal{D}$ is always a lattice, we have shown that Q splits. Hence (i) through (iii) are equivalent.

From the explicitly split case, $Q = T \times A$, (iv) is easily seen. Conversely, since $x \vee y = y \vee x$ implies $x = y$ in each $\mathcal{D}$-class, (iv) implies that each ρ-class uniquely meets each $\mathcal{D}$-class, that is, (iv) implies (ii). But (ii) in turn implies (v). Indeed, let $(\theta_1, \theta_2) \in [\mathcal{D}, \nabla] \times [\Delta, \mathcal{D}]$. Upon setting $\theta = (\rho \wedge \theta_1) \vee \theta_2$, the distributive properties of $\mathcal{D}$ yield $(\theta \circ \mathcal{D}, \theta \cap \mathcal{D}) = (\theta_1, \theta_2)$. Hence the indicated injective homomorphism is also surjective and (v) follows. Finally, given (v), a congruence θ exists such that $\theta \circ \mathcal{D} = \nabla$ and $\theta \circ \mathcal{D} = \Delta$ and (iii) holds. □

Every split quasilattice is a refined quasilattice. Within the variety of refined quasilattices, split quasilattices are characterized as follows.

Theorem 3.1.4. *Split quasilattices form the subvariety of refined quasilattices S for which* $(S, \vee)$ *and* $(S, \wedge)$ *are normal, in that* $u \vee x \vee y \vee v = u \vee y \vee x \vee v$ *and* $u \wedge x \wedge y 91 \wedge v = u \wedge y \wedge x \wedge v$ *hold,*

Proof. Both identities hold for lattices as well as for rectangular quasilattices, and hence also for split quasilattices. Conversely, suppose that both identities hold on a refined quasilattice. Then $x \vee y = y \vee x$ iff $x \wedge y = y \wedge x$. For $x \vee y = y \vee x$ implies $x \vee y \geq y, x$ so that

$$x \wedge y = (x \vee y) \wedge x \wedge y \wedge (x \vee y) = (x \vee y) \wedge y \wedge x \wedge (x \vee y) = y \wedge x.$$

The other direction is seen in similar fashion. Next, define a binary relation ρ_0 by $x \rho_0 y$ if $x \vee y = y \vee x$ (or equivalently, $x \wedge y = y \wedge x$). Clearly ρ_0 is both reflexive and symmetric. From $x \, \rho_0 \, y \, \rho_0 \, z$, $x \vee y \vee z = y \vee x \vee z \vee y = y \vee z \vee x \vee y = z \vee y \vee x$ follows. Denoting this common value by u we get $u \geq x, z$ and $x \wedge z = u \wedge x \wedge z \wedge u = u \wedge z \wedge x \wedge u = z \wedge x$. Hence ρ_0 is also transitive and thus an equivalence. By the identities in the theorem statement, ρ_0 is seen to be a congruence. Since $\leq \subseteq \rho_0$, we get $\rho \subseteq \rho_0$. Conversely, given $x \, \rho_0 \, y$, from $x \leq x \vee y \geq y$ first $x \, \rho \, y$ and then $\rho_0 = \rho$ follows. Hence Theorem 3.1.3(iv) is satisfied and N splits. □

Corollary 3.1.5. *A noncommutative lattice Q is a split quasilattice if and only if* $\rho \cap \delta = \Delta$. *In general,* $Q/\rho \cap \delta$ *is the maximal split quasilattice image of any quasilattice Q.*

Proof. $Q/\rho \cap \delta$ is isomorphic to a subalgebra of the product $Q/\rho \times Q/\delta$ where Q/ρ is an antilattice and Q/δ is a lattice. Thus $Q/\rho \cap \delta$ splits also and is indeed isomorphic to $Q/\rho \times Q/\delta$ under the map $x(\rho \cap \delta) \to (x\rho, x\delta)$. □

We have seen that $\mathbf{Con}(Q)$ is a subdirect product of $[\Delta, \mathcal{D}]$ and $[\mathcal{D}, \nabla]$. If $\{D_i \mid i \in \mathrm{I}\}$ is the set of all $\mathcal{D}$-classes of Q, then an embedding $\mathcal{D}^*\colon [\Delta, \mathcal{D}] \to \prod_i \mathbf{Con}(D_i)$ of complete lattices is given by

$$\mathcal{D}^*(\theta) = \langle \theta \cap D_i \times D_i) \mid i \in \mathrm{I} \rangle$$

Since $\prod_i \mathbf{Con}(D_i)$ is in turn embedded in $\mathbf{Con}(\prod_i D_i)$, this suggests that $[\Delta, \mathcal{D}]$ shares similarities with congruence lattices of rectangular quasilattices. On the other hand, the interval $[\rho, \nabla]$ is isomorphic to the congruence lattice $\mathbf{Con}(N/\rho)$ of the greatest rectangular image Q/ρ of Q. This leads us to inquire how $[\Delta, \mathcal{D}]$ and $[\rho, \nabla]$ are related.

Consider the embedding χ: **Con**(Q) → [$\mathcal{D}$, ∇] × [Δ, $\mathcal{D}$] given by $\chi[\theta] = (\mathcal{D} \vee \theta, \mathcal{D} \cap \theta)$. Since $\rho \circ \mathcal{D} = \nabla$, restricting χ to [ρ, ∇] yields an embedding χ_0: [ρ, ∇] → [Δ, $\mathcal{D}$] of complete lattices defined by $\chi_0[\theta] = \mathcal{D} \cap \theta$. When is χ_0 an isomorphism? Precisely when $\rho \cap \mathcal{D} = \Delta$. This condition is clearly necessary as it states that Δ lies in the image of χ_0. Given this condition, then for all θ ∈ [Δ, $\mathcal{D}$],

$$\mathcal{D} \cap (\theta \vee \rho) = (\mathcal{D} \cap \theta) \vee (\mathcal{D} \cap \rho) = \theta \vee \Delta = \theta,$$

which implies that χ_0 is also surjective. But $\rho \cap \mathcal{D} = \Delta$ if and only if Q splits. Thus:

Theorem 3.1.6. *The interval* [ρ, ∇] *of rectangular congruences on a quasilattice Q is isomorphic to a complete sublattice of the interval* [Δ, $\mathcal{D}$] *under the map* $\theta \to \mathcal{D} \cap \theta$. *This embedding is an isomorphism if and only if Q splits.* □

To complete our basic picture of quasilattice congruences we give a partial complement of the Clifford-McLean Theorem. In its proof, the terms ∨***–morphism*** and ∧***-morphism*** refer to homomorphisms with respect to the stated operations.

Theorem 3.1.7. *Given a lattice* Λ *with disjoint antilattices* D_λ *assigned to each* λ ∈ Λ, *a quasilattice structure exists on* $Q = \bigcup_\lambda D_\lambda$ *such that the maximal antilattices in Q are precisely the* D_λ *and* $N/\mathcal{D} \cong \Lambda$. *In addition,* ∨ *and* ∧ *can be defined so that* $[\Delta, \mathcal{D}] \cong \prod_\lambda \mathrm{Con}(D_\lambda)$.

Proof. Let $\Phi = \{\varphi_{(\lambda,\mu)}: D_\lambda \to D_\mu \mid \lambda \le \mu\}$ be an ascending family of ∨–morphisms such that for all λ ∈ Λ, $\varphi_{(\lambda,\lambda)}$ is the identity map on D_λ and secondly, $\varphi_{(\mu,\nu)}\varphi_{(\lambda,\mu)} = \varphi_{(\lambda,\nu)}$ for all $\lambda \le \mu \le \nu$ in Λ. Similarly let $\Psi = \{\psi_{(\mu,\lambda)}: D_\mu \to D_\lambda \mid \lambda \le \mu\}$ be a descending family of ∧–morphisms satisfying dual properties. To define ∨, set $x \vee y = \varphi_{(\lambda,\pi)}(x) \vee \varphi_{(\mu,\pi)}(y) \in D_\pi$ if $x \in D_\lambda$, $y \in D_\mu$ and $\pi = \lambda \vee \mu$. Similarly, define ∧ on Q using Ψ in dual fashion. The operations ∨ and ∧ induced from Φ and Ψ yield a quasilattice (Q; ∨, ∧) for which the D_λ are the maximal antilattices and for which $Q/\mathcal{D} \cong \Lambda$ as stated.

To complete the first assertion we need only exhibit at least one pair (Φ, Ψ). This is done as follows. First, to each λ ∈ Λ pick some d_λ in D_λ. Next, for each λ ∈ Λ, let $\varphi_{(\lambda,\lambda)} = \psi_{(\lambda,\lambda)}$ be the identity map on D_λ as required. Finally for each strict comparison λ < μ in Λ, let $\varphi_{(\lambda,\mu)}$: $D_\lambda \to D_\mu$ be the constant map sending all D_λ to d_μ in D_μ. Similarly define $\psi_{(\mu,\lambda)}$: $D_\mu \to D_\lambda$ as the constant map sending all of D_μ onto d_λ. As so defined, (Φ, Ψ) satisfies all the required conditions. Moreover, for such a pair (Φ, Ψ), the embedding $\mathcal{D}^*$: [Δ, $\mathcal{D}$] → $\prod_\lambda \mathrm{Con}(D_\lambda)$ defined by $\mathcal{D}^*(\theta) = \langle \theta \cap D_\lambda \times D_\lambda \mid \lambda \in \Lambda\rangle$ is onto. For given $\langle \theta_\lambda \mid \lambda \in \Lambda\rangle \in \prod_\lambda \mathrm{Con}(D_\lambda)\rangle$, the union $\theta = \bigcup_\lambda \theta_\lambda$ is at least an equivalence on Q. Thanks to the pointed character of (Φ, Ψ) as defined, θ is a congruence inducing $\langle \theta_\lambda \mid \lambda \in \Lambda\rangle$. □

Not all quasilattices $Q = \bigcup_\lambda D_\lambda$ for which $Q/\mathcal{D} \cong \Lambda$ arise from some such pair (Φ, Ψ) of ascending and descending morphisms. Those that do are characterized by both (Q, ∨) and (Q, ∧) being normal. This is just the double assertion of Theorem 1.2.16.

The above construction is relevant to the question: *Given a band B with multiplication* ∧, *is* (B, ∧) *the* ∧*-reduct of some quasilattice* (B, ∨, ∧)?

If (B, ∨, ∧) exists, it is called a ***quasilattice closure*** of B. It is not unique, unless B is a semilattice so that (B, ∨, ∧) is a lattice. Clearly, it is necessary that the maximal semilattice image ($B/\mathcal{D}$, ∧) form a lattice in that given any pair of $\mathcal{D}$-classes M and N in B, a $\mathcal{D}$-class J exists which is the supremum of M and N in ($B/\mathcal{D}$, ∧). This condition is also sufficient. Indeed, assume that B has join classes. On each $\mathcal{D}$-class M define a rectangular join $\vee_M$. (Perhaps, let $x \vee_M y = x$.) Using an ascending family Φ of ∨-morphisms between the M, create an operation ∨ on all of B that yields a quasilattice, (B, ∨, ∧).

Proposition 3.1.8. *A band B is a* ∧*-reduct of a quasilattice if and only if its maximal semilattice image* $B/\mathcal{D}$ *is the* ∧*-reduct of a lattice*.

Returning to quasilattice congruences, note that while the interval [$\mathcal{D}$, ∇] is distributive, in general **Con**(N) need not satisfy any identity beyond those common to all lattices. Indeed [Δ, $\mathcal{D}$] need only satisfy identities common to congruence lattices of antilattices. But consider the antilattice defined on a given set N by $x \wedge y = x = x \vee y$. In this case **Con**($N$) is just the lattice **Equ**(N) of all equivalences on N. But such lattices collectively satisfy only lattice identities common to all lattices. We shall see in the next section that not all antilattices are so behaved.

Consider the following questions. *Given quasilattice Q, when is* **Con**(Q) *distributive*? *Even if* **Con**(Q) *is not distributive, under what conditions will instances of distribution occur*? In response we present three assertions, the first two following immediately from Theorem 3.1.1 and the complete embedding $\mathcal{D}^*$: [Δ, $\mathcal{D}$] in $\prod_i$**Con**(D_i) where the D_i are all maximal antilattices in Q.

1. **Con**(Q) *is distributive precisely when* [Δ, $\mathcal{D}$] *is distributive*. *The latter occurs when each* $\mathcal{D}$*-class of Q has a distributive congruence lattice*. (For example, this is the case if all $\mathcal{D}$-classes are simple as algebras.)
2. *Given particular quasilattice congruences* η *and* θ_i *for* $i \in$ I, $\eta \wedge sup_i(\theta_i) = sup_i(\eta \wedge \theta_i)$ *holds in* **Con**(Q) *if either* η *or at least one of the* θ_i *lies in* [$\mathcal{D}$, ∇].

We present our third assertion as a theorem where we consider the remaining Green's equivalences $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ which are defined for any noncommutative lattice.

Theorem 3.1.9. *If Q is a quasilattice, then $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ permute with any congruence in* $[\Delta, \mathcal{D}]$, *∧-distribute over suprema in* $[\Delta, \mathcal{D}]$ *and ∨-distribute over infima in* $[\Delta, \mathcal{D}]$, *even if these four equivalences are not congruences themselves. If either $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ or $\mathcal{R}_{(\wedge)}$ is also a congruence, then it possesses these distributive properties in* **Con**(Q).

Proof. To begin, $\mathcal{L} \circ \theta = \theta \circ \mathcal{L}$ and $\mathcal{R} \circ \theta = \theta \circ \mathcal{R}$ hold for any congruence θ on a rectangular band. Hence $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ permute with any congruence on an antilattice. Likewise, $\mathcal{L}$ and $\mathcal{R}$ distribute as stated over congruences on a rectangular band. Thus the first statement must hold at least for antilattices. The general statement involving $[\Delta, \mathcal{D}]$ now follows since all calculations take place in $\mathcal{D}$-classes. Since the join of all four equivalences with $\mathcal{D}$ is just $\mathcal{D}$, the second statement follows from the first and Theorem 3.1.2. □

3.2 Antilattices that are simple as algebras

Recall that an algebra **A** is ***simple*** if **Con**(**A**) = $\{\Delta, \nabla\}$. Algebras of order 2 are simple. By Theorem 3.1.2, *a simple quasilattice is either a simple lattice or a simple antilattice*. Our interest is in the latter case. We begin with some remarks about when simplicity *cannot* occur, starting with a definition.

An antilattice N is ***flat*** if $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\wedge)}$ and $\mathcal{R}_{(\wedge)}$ are each either Δ or ∇. In the finite case, N is flat precisely when the arrays defining ∨ and ∧ are either single columns or rows.

Lemma 3.2.1. *All antilattices of prime order are flat; moreover every equivalence on a flat antilattice is a congruence. Thus an antilattice cannot be simple if its order is an odd prime.*

Proof. Given the assumption of prime order, $\mathcal{L}_{(\vee)}$ and $\mathcal{R}_{(\vee)}$ reduce to either Δ or ∇. Hence either $x \vee y = x$ holds uniformly or else $x \vee y = y$ holds uniformly. Similarly, either $x \wedge y = x$ holds uniformly or else $x \wedge y = y$ does. For such operations, every equivalence must be a congruence. □

By contrast, let A be the antilattice on $\{a, b, c, d\}$ determined from the arrays:

$$(\vee) \quad \begin{matrix} a & b \\ c & d \end{matrix} \qquad\qquad (\wedge) \quad \begin{matrix} a & d \\ c & b \end{matrix}.$$

Recall that $x \vee y$ and $x \wedge y$ are the element lying in the row of x and the column of y of the relevant array, where the rows are the $\mathcal{R}$-classes and the columns are the $\mathcal{L}$-classes for the respective operations. Here $\mathcal{L}_{(\vee)} = \mathcal{L}_{(\wedge)}$ is the only proper nontrivial congruence. The lattice of congruences is thus: $\Delta < \mathrm{L} < \nabla$. In general:

Proposition 3.2.2. *No 4-element antiilattice is simple*.

Proof. Any such antilattice A falls into one of three cases. First case: A is flat. In this case **Con**(A) equals **Equ**(A) and has order greater than 2. Second case: either $(A, \vee)$ or $(A, \wedge)$ is flat, but not both. Then both $\mathcal{L}$ and $\mathcal{R}$ for the nontrivial operation are congruences that are distinct from Δ and ∇. Third case: neither $(A, \vee)$ nor $(A, \wedge)$ is flat. Then the elementary combinatorics of 2×2 squares forces one of $\mathcal{L}_{(\vee)}$ or $\mathcal{R}_{(\vee)}$ to equal one of $\mathcal{L}_{(\wedge)}$ or $\mathcal{R}_{(\wedge)}$, with the equated equivalence being a nontrivial congruence. ☐

Thus *simple antilattices of finite order* > 2 *must have composite order greater than* 5. We may now state:

Theorem 3.2.3. *Simple antilattices exist for all composite orders* > 5. *In all such cases*, *none of* $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\wedge)}$ *or* $\mathcal{L}_{(\wedge)}$ *are congruences*.

Proof. Given a set A of order $mn \geq 6$ with $n \geq m \geq 2$, store its elements in each of the following $m\times n$ arrays, so that $a\vee b$ is in the row of a and the column of b of the array on the left and $a\wedge b$ is similarly obtained from the array on the right. (First array indices are used in the second array to show how the elements have been rearranged.)

$$(\vee)\quad \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{array}$$

$$(\wedge)\quad \begin{array}{ccccccccc} a_{11} & a_{12} & \cdots & a_{1,n-m+1} & a_{1,n-m+2} & \cdots & a_{1,\, n-2} & a_{1,n-1} & a_{21} \\ a_{22} & a_{23} & \cdots & a_{2,n-m+2} & a_{2,n-m+3} & \cdots & a_{2,\, n-1} & a_{31} & a_{32} \\ a_{33} & a_{34} & \cdots & a_{3,n-m+3} & a_{3,n-m+4} & \cdots & a_{41} & a_{42} & a_{43} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ a_{m,\, m} & a_{m,\, m+1} & \cdots & a_{m,\, n} & a_{m-1,\, n} & \cdots & a_{3,\, n} & a_{2,\, n} & a_{1,\, n} \end{array}$$

Let θ be a congruence on A and let distinct a, b in A exist such that $a\theta b$. We show that $\theta = \nabla$. To begin, let $a_{11}\theta a_{21}$ be given. Then $(a_{11}\vee a_{1i})\theta(a_{21}\vee a_{1i})$ for $i = 1, 2, \ldots, n$ yields a sequence of congruent pairs $a_{1i}\theta a_{2i}$ in the first two rows of the $\vee$-array. Upon taking $\wedge$-products of these θ-related pairs, all elements in the first two rows of the $\wedge$-array are seen to be θ-related. In particular, $a_{21}\theta a_{31}$. This leads first to a sequence of relations $a_{2i}\theta a_{3i}$ in the $\vee$-array and then to all elements in the first three rows of the $\wedge$-array being θ-related. The process continues until the entire $\wedge$-array, and hence A, is absorbed into a single θ-class, showing that $\theta = \nabla$.

Suppose next that we are just given $a_{ij}\theta a_{kl}$ for some $i < k$. Then

$$a_{i1} = (a_{ij} \vee a_{11})\ \theta\ a_{kl} \vee a_{11} = a_{k1}.$$

Applying $f(x) = (x \wedge a_{11}) \vee a_{11}$ to both sides of $a_{i1}\theta a_{k1}$, as often as needed, eventually yields $a_{11}\theta a_{12}$ and hence $\theta = \nabla$.

Finally, suppose that $a_{hi}\theta a_{hk}$ with $i \neq k$. If a_{hi} and a_{hk} lie in different rows in the $(\wedge)$-array, then $a_{hi} \wedge a_{hk}$ and $a_{hk} \wedge a_{hi}$ are θ–equivalent, but have differing first indices, thus returning us to the previous case to obtain $\theta = \nabla$. Otherwise, look at $a_{m\mathrm{m}} \wedge a_{hi}\ \theta\ a_{mm} \wedge a_{hk}$ in the final row of the $\wedge$-array. If both $\wedge$-products have differing first indices we again return to the previous case. Otherwise, $a_{1i'} = a_{11} \wedge (a_{mm} \wedge a_{hi})\ \theta\ a_{11} \wedge (a_{mm} \wedge a_{hk}) = a_{1k'}$, with $i < i'$ and $k < k'$. We repeat this cycle of calculations until a pair of θ-related elements with distinct first indices is eventually encountered, which must happen due to the design of the $\wedge$-array. We are then returned to the previous case. ☐

By contrast, for any flat antilattice A, **Con**(A) is the lattice **Equ**(A) of all equivalences on the underlying set of A. Flat antilattices of all types together generate the variety of ***regular*** antilattices for which $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ are congruences. Such an algebra A factors as the direct product of flat antilattices of each type, $A \cong A_{(l,\,l)} \times A_{(l,\,r)} \times A_{(r,\,l)} \times A_{(r,\,r)}$ with **Con**(A) correspondingly factoring as

$$\mathbf{Con}(A) \cong \mathbf{Equ}(A_{(l,\,l)}) \times \mathbf{Equ}(A_{(l,\,r)}) \times \mathbf{Equ}(A_{(r,\,l)}) \times \mathbf{Equ}(A_{(r,\,r)}).$$

Flatness and regularity are studied in a broader context in the next section.

3.3 Regular quasilattices

Recall that a noncommutative lattice is ***flat*** if one of the following pairs of identities is satisfied:

(r, l): $a \vee b \vee a = b \vee a$ and $a \wedge b \wedge a = a \wedge b$.
(l, r): $a \vee b \vee a = a \vee b$ and $a \wedge b \wedge a = b \wedge a$.
(l, l): $a \vee b \vee a = a \vee b$ and $a \wedge b \wedge a = a \wedge b$.
(r, r): $a \vee b \vee a = b \vee a$ and $a \wedge b \wedge a = b \wedge a$.

Thus, being (r, l)-flat means that $\mathcal{D}_{(\vee)} = \mathcal{R}_{(\vee)}$ and $\mathcal{D}_{(\wedge)} = \mathcal{L}_{(\wedge)}$, or equivalently, $\mathcal{L}_{(\vee)} = \mathcal{R}_{(\wedge)} = \Delta$. Modified remarks hold for the other three types of flatness.

A noncommutative lattice is ***regular*** if $\mathcal{R}_{(\vee)}$ $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ are all congruences, in which case $\mathcal{D}_{(\vee)}$ and $\mathcal{D}_{(\wedge)}$ are also congruences. While flat quasilattices are regular, this is not the case for flat paralattices. In general:

Theorem 3.3.1. *The condition that $\mathcal{L}_{(\vee)}$ be a congruence determines a subvariety for any variety of noncommutative lattices. It contains as a further subvariety those algebras for which $\mathcal{L}_{(\vee)} = \Delta$. When $\mathcal{L}_{(\vee)}$ is a congruence on a noncommutative lattice N, then $\mathcal{L}_{(\vee)} = \Delta$ on N/θ for any congruence* θ in $[\mathcal{L}_{(\vee)}, \nabla]$. *Similar remarks hold for $\mathcal{D}_{(\vee)}$, $\mathcal{R}_{(\vee)}$, $\mathcal{D}_{(\wedge)}$, $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$.*

Proof. A typical pair of $\mathcal{L}_{(\vee)}$-related elements are $b\vee a$ and $a\vee b\vee a$. $\mathcal{L}_{(\vee)}$ is compatible with $\vee$ and $\wedge$ if and only if the equations,

$$[c\vee(b\vee a)]\vee[c\vee(a\vee b\vee a)] = c\vee(b\vee a) \quad \text{and} \quad [c\vee(a\vee b\vee a)]\vee[c\vee(b\vee a)] = c\vee(a\vee b\vee a)$$

together with their $-\vee c$, $c\wedge-$ and $-\wedge c$ variants, hold. Clearly $\mathcal{L}_{(\vee)}$ is a congruence when $\mathcal{L}_{(\vee)} = \Delta$, a condition that is equivalent to the equation $a\vee b\vee a = b\vee a$ being satisfied. The first assertion follows. If $\mathcal{L}_{(\vee)}$ is a congruence, then $\mathcal{L}_{(\vee)} = \Delta$ on $N/\mathcal{L}_{(\vee)}$. Thus, if θ is as stated, N/θ, being a homomorphic image of $N/\mathcal{L}_{(\vee)}$, must also satisfy $\mathcal{L}_{(\vee)} = \Delta$.

The verifications of the remaining cases are similar, depending on the descriptions of typical $\mathcal{R}_{(\vee)}$-related elements ($a\vee b\vee a$, and $a\vee b$), typical $\mathcal{D}_{(\vee)}$-related elements ($a\vee b\vee a$ and $b\vee a\vee b$) and their $\wedge$-variants. ☐

Theorem 3.3.2. *Flat quasilattices are regular. Regular quasilattices, in turn, form the subvariety of quasilattices generated from all flat quasilattices.*

Proof. In a flat quasilattice, each of $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ is either $\mathcal{D}$ or Δ, and hence is a congruence. Conversely, given a regular quasilattice Q, define κ: $Q \to Q/\mathcal{L}_{(\vee)} \times Q/\mathcal{R}_{(\vee)}$ by $\kappa(x) = (x\mathcal{L}_{(\vee)}, x\mathcal{R}_{(\vee)})$. Since $\mathcal{L}_{(\vee)}\cap\mathcal{R}_{(\vee)} = \Delta$, κ is an embedding of Q into a product of quasilattices for which either $\mathcal{D} = \mathcal{R}_{(\vee)}$ or $\mathcal{D} = \mathcal{L}_{(\vee)}$ holds on each factor. Repeating the process on each factor, but using $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ instead, we obtain a final embedding of Q into a product of four flat quasilattices, each representing a specific type of flatness. The theorem follows. ☐

Given a noncommutative lattice Q, let $\varphi_{(l,\,l)}$ denote the least congruence on θ for which Q/θ is (l, l)-flat. Similarly, let $\varphi_{(l,\,r)}$, $\varphi_{(r,\,l)}$ and $\varphi_{(r,\,r)}$ denote least congruences for the other types of flatness. For regular quasilattices, Theorems 3.1.8 and 4.3.1 yield:

Lemma 3.3.3. *Given a regular quasilattice Q, the equivalences $\mathcal{R}_{(\vee)}$, $\mathcal{L}_{(\vee)}$, $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ all permute with each other. Moreover, $\varphi_{(l,\,l)} = \mathcal{R}_{(\vee)}\circ\mathcal{R}_{(\wedge)}$, $\varphi_{(l,\,r)} = \mathcal{R}_{(\vee)}\circ\mathcal{L}_{(\wedge)}$, $\varphi_{(r,\,l)} = \mathcal{L}_{(\vee)}\circ\mathcal{R}_{(\wedge)}$ and $\varphi_{(r,\,r)} = \mathcal{L}_{(\vee)}\circ\mathcal{L}_{(\wedge)}$.* ☐

Given a regular quasilattice Q, let $Q_{(l,\,l)}$, $Q_{(l,\,r)}$, $Q_{(r,\,l)}$, and $Q_{(r,\,r)}$ denote the maximal flat images of each type. Define Φ: $Q \to Q_{(l,\,l)} \times Q_{(l,\,r)} \times Q_{(r,\,l)} \times Q_{(r,\,r)}$ by:

$$\Phi(x) = (x\varphi_{(l,\,l)}, x\varphi_{(l,\,r)}, x\varphi_{(r,\,l)}, x\varphi_{(r,\,r)}).$$

Φ is a monomorphism, since the involved congruences intersect to Δ. To describe its image, observe that Q shares with its four maximal flat images a common maximal lattice image, $Q/\mathcal{D}$. If $\delta_{(u,v)}$: $Q_{(u,v)} \to Q/\mathcal{D}$, for $(u, v) \in \{(l, l), (l, r), (r, l), (r, r)\}$, are epimorphisms induced by the inclusions of the $\varphi_{(u,\,v)}$ into $\mathcal{D}$, then $\Phi(Q)$ lies in

$$\{(u, v, w, y) \in Q_{(l,\,l)} \times Q_{(l,\,r)} \times Q_{(r,\,l)} \times Q_{(r,\,r)} \mid \delta_{(l,\,l)}(u) = \delta_{(l,\,r)}(v) = \delta_{(r,\,l)}(w) = \delta_{(r,\,r)}(y)\},$$

the fibered product over $Q/\mathcal{D}$ of the various maximal flat images of Q. In fact:

Theorem 3.3.4. *If Q is a regular quasilattice, then Φ is an isomorphism of Q with the fibered product over $Q/\mathcal{D}$ of the four maximal flat images of Q.*

$$Q \cong Q_{(l,\,l)} \times_{Q/\mathcal{D}} Q_{(l,\,r)} \times_{Q/\mathcal{D}} Q_{(r,\,l)} \times_{Q/\mathcal{D}} Q_{(r,\,r)}$$

Proof. Restricting our attention to each $\mathcal{D}$-class A of Q, the respective rectangular band structures yield first $A \cong A/\mathcal{R}_{(\vee)} \times A/\mathcal{L}_{(\vee)}$ and then in turn

$$A/\mathcal{R}_{(\vee)} \cong A/\mathcal{R}_{(\vee)}/\mathcal{R}_{(\wedge)} \times A/\mathcal{R}_{(\vee)}/\mathcal{L}_{(\wedge)} \cong A/\mathcal{R}_{(\vee)}\mathrm{o}\mathcal{R}_{(\wedge)} \times A/\mathcal{R}_{(\vee)}\mathrm{o}\mathcal{L}_{(\wedge)} = A/\varphi_{(l,\,l)} \times A/\varphi_{(l,\,r)}$$

with $A/\mathcal{L}_{(\vee)}$ similarly factoring as $A/\varphi_{(r,\,l)} \times A/\varphi_{(r,\,r)}$. Thus class A factors as a product of its maximal flat images. In passing from $\mathcal{D}$-classes to Q, the result follows. ☐

The above theorem is the quasilattice version of a result of Kimura [1958] about regular bands. For a general discussion of permuting congruences and fibered products, see Grätzer [1979].

3.4 Paralattices and refined quasilattices

Skew lattices are simultaneously quasilattices and paralattices and thus possess the features of both. While quasilattices possess a coherent Clifford-McLean structure, paralattices need not. Nor need (flat) paralattices be regular. Thus some of our basic intuitions about bands and quasilattices no longer always hold. We begin our look at paralattices with a special case.

Unlike skew lattices and more generally quasilattices, it is possible for say $\vee$ to be commutative, but not $\wedge$. (For quasilattices, if say $\vee$ is commutative, then $\mathcal{D}_{\vee} = \Delta$ and hence $\mathcal{D}_{\wedge} = \Delta$, making $\wedge$ commutative.) A ***band with joins***, or a ***$\vee$-band***, is a band B whose natural partial order $\leq$ has natural joins. Thus, for any pair of elements x and y, their supremum $x \vee y$ exists in $(B, \leq)$. Upon denoting the band multiplication by $\wedge$, one obtains a paralattice $\left(B;\, \vee,\, \wedge\right)$ where $(B, \vee)$ is a semilattice, or equivalently, where $\mathcal{D}_{(\vee)} = \Delta$. Thus, in this case $\mathcal{D}$ will denote $\mathcal{D}_{(\wedge)}$.

Viewed as paralattices, $\vee$*-bands form a subvariety characterized simply by* $x \vee y = y \vee x$. Flat $\vee$-bands are $\vee$-bands for which $\wedge$ is either left regular $(x \wedge y \wedge x = x \wedge y)$ or right-regular $(x \wedge y \wedge x = y \wedge x)$. Both types of flat $\vee$-bands form subvarieties. In terms of B_i and C_j identities of Section 1.3, we have the following result and its left dual:

Theorem 3.4.1. *A flat* $\vee$*-band satisfying* $a \wedge b \wedge a = b \wedge a$ *(so that* $\mathcal{D}_{\wedge} = \mathcal{R}_{\wedge}$*) is characterized by the associativity of* $\vee$ *and* $\wedge$ *together with the following four identities,*

B_1 *and* C_2: $\quad a \wedge (a \vee b) = a = (b \wedge a) \vee a.$
B_3 *and* C_3: $\quad a \wedge (b \vee a) = a = a \vee (b \wedge a).$

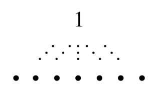
Proof. These identities necessarily hold for such a $\vee$-band, as $b \wedge a \leq a \leq a \vee b = b \vee a$ in the coherent natural partial ordering. Conversely, any associative algebra $(N;\, \vee,\, \wedge)$ satisfying B_1, B_3, C_2 and C_3 is at least a double band by Theorem 1.3.4. In this case B_1 and C_2 together assert that ${}_{\vee}\preceq_{\mathcal{R}}$ dualizes ${}_{\wedge}\preceq_{\mathcal{L}}$ while B_3 and C_3 jointly assert that ${}_{\vee}\preceq_{\mathcal{L}}$ dualizes ${}_{\wedge}\preceq_{\mathcal{L}}$ making ${}_{\vee}\preceq_{\mathcal{R}} = {}_{\vee}\preceq_{\mathcal{L}} = {}_{\vee}\leq$ and thus ${}_{\wedge}\preceq_{\mathcal{L}} = {}_{\wedge}\leq$. Thus N is a flat $\vee$-band of the indicated type. □

Thanks to the dual (union) version of Theorem 1.3.13 we have:

Theorem 3.4.2. *The congruence lattice of a* $\vee$*-band is distributive*. □

How prevalent are $\vee$-bands among bands? Several established classes of bands turn out to be $\vee$-bands.

Example 3.4.1. If B is a rectangular band and set $S = B^1$, the extension obtained by adjoining an identity element 1, then $(B^1, \leq)$ is a join semilattice characterized by $1 \geq b$ for b in B.



Examples 3.4.2. Let B be the free (left, right or 2-sided) normal band $\mathcal{B}_X$ (where $xyzw = xzyw$) on alphabet X. Then B^1 is a $\vee$-band. Indeed elements of B (in the 2-sided case) look like aAc where A is any finite subset of X (thanks to middle commutativity). In $(B, \leq)$, $aAc \leq a'A'c'$ if and only if $a = a'$, $c = c'$ and $A \subseteq A'$. Thus any finite subset of B with a common upper bound in $(B, \leq)$ has a least upper bound:

$$aA_1c \vee aA_2c \vee \ldots \vee aA_nc = a(A_1 \cup A_2 \cup \ldots \cup A_n)c.$$

$(B^1, \leq)$ is hence an upper semilattice. In similar fashion one handles the free left normal case (with expressions aA) and the right normal case (with expressions Ac).

Indeed with a little more work one can show that if B is the free (left, right or 2-sided) *regular* band on X (where $xyxzx = xyzx$), then B^1 is a $\vee$-band. With even further work one can show that if B is a free band on X, then B^1 is a $\vee$-band. □

Dual $\vee$***-bands*** are paralattices for which $\wedge$ is commutative. Dual $\vee$-bands arise as reducts (S; $\vee$, $\cap$) of structurally enriched skew lattices (S; $\vee$, $\wedge$, $\cap$) for which the natural partial ordering $\leq$ has natural meets, with inf(x, y) denoted by $x \cap y$. In particular, they arise in the study of certain types of skew Boolean algebras. (See Section 4.2 below.) For $\vee$-bands and their duals (where $\mathcal{D} = \mathcal{D}_{(\vee)}$) we have:

Theorem 3.4.3. *$\mathcal{D}$ is a congruence on a* [*dual*] *$\vee$-band* B *if and only if $\mathcal{D} = \Delta$ and* B *is a lattice. In general a paralattice is also a quasilattice if and only if both $\mathcal{D}_{(\vee)}$ and $\mathcal{D}_{(\wedge)}$ are congruences. Regular paralattices in particular are quasilattices.*

Proof. Given a $\mathcal{D}$-class A of a $\vee$-band B, pick elements a, b in A. If $\mathcal{D} = \mathcal{D}_{(\wedge)}$ is a congruence, then $a \vee b$ lies in A also with $a \leq a \vee b$ and $b \leq a \vee b$. Being a common $\mathcal{D}$-class, this forces $a = a \vee b = b$. Thus A is a singleton set. Hence $\mathcal{D} = \Delta$ and B is a lattice. More generally, let P be a paralattice for which both $\mathcal{D}_{(\vee)}$ and $\mathcal{D}_{(\wedge)}$ are congruences. Then $P/\mathcal{D}_{(\vee)}$ is a $\vee$-band. Since $\mathcal{D}_{(\wedge)}$ is a congruence on P, $\mathcal{D}_{(\wedge)}$ is also a congruence on $P/\mathcal{D}_{(\vee)}$ by Theorem 3.3.1. Hence $P/\mathcal{D}_{(\vee)}$ is a lattice by our first assertion and $\lambda \subseteq \mathcal{D}_{(\vee)}$. Similarly, $\lambda \subseteq \mathcal{D}_{(\wedge)}$. Since the converse inclusions hold, $\mathcal{D}_{(\vee)} = \mathcal{D}_{(\wedge)}$ and P is a quasilattice by Corollary 1.3.5. Conversely, when P is also a quasilattice, $\mathcal{D}_{(\vee)} = \mathcal{D}_{(\wedge)}$ with the common equivalence being a congruence. □

Corollary 3.4.4. *The class of all regular paralattices is just the class of regular refined quasilattices. In particular, every flat refined quasilattice is necessarily regular and is either a skew lattice or a skew* lattice.*

Proof. The first statement is clear. Next, take, e.g., a (l,l)-flat refined quasilattice, P. Since P is a flat quasilattice, ${}_{\vee}{\preceq}_{\mathcal{L}} = {}_{\vee}{\preceq}$ dualizes ${}_{\wedge}{\preceq}_{\mathcal{L}} = {}_{\wedge}{\preceq}$. Since P is a flat paralattice, ${}_{\vee}{\preceq}_{\mathcal{R}} = {}_{\vee}{\leq}$ dualizes ${}_{\wedge}{\preceq}_{\mathcal{R}} = {}_{\wedge}{\leq}$. Hence N is a flat skew* lattice and thus necessarily regular. □

Theorem 3.4.5. *The variety of regular paralattices is the join of the varieties of skew lattices and skew* lattices. In particular, every regular paralattice factors as the fibred product of its maximal skew lattice image with its maximal skew* lattice images over its maximal lattice image.*

Proof. Clearly skew* lattices are regular paralattices. Thus the join of both varieties lies in the variety of regular paralattices. One the other hand, the larger variety is generated by its four subvarieties of flat algebras by Theorem 3.3.4. But these four subvarieties are either subvarieties of the varieties of skew lattices or the varieties of skew* lattices. The "join" assertion follows. The final remark follows again from the isomorphism of Theorem 3.3.4. ☐

More generally:

Proposition 3.4.6. *Given a refined quasilattice, both* $\mathcal{R}_{(\vee)}$ and $\mathcal{L}_{(\vee)}$ *are congruences with respect to* $\vee$, *while* $\mathcal{R}_{(\wedge)}$ *and* $\mathcal{L}_{(\wedge)}$ *are congruences with respect to* $\wedge$.

Proof. First observe that $\leq$ is surjective between comparable $\mathcal{D}$-classes: given classes, $A \leq B$, then for all $a \in A$ and $b \in B$, there exist $b_A \in A$ and $a_B \in B$ such that $b_A \leq b$ and $a \leq a_B$. Indeed we may choose $b_A = b\wedge a\wedge b$ and $a_B = a\vee b\vee a$. It follows from Theorem 1.2.19 that both $\vee$ and $\wedge$ satisfy the identity $xyxzx = xyzx$. Since $\leq$ is surjective, u and v exist in the $\mathcal{D}$-class of x such that $x\wedge y \leq u$ and $z\wedge x \leq v$. Because u, x and v lie in a common $\mathcal{D}$-class, $u\wedge 101x\wedge 101v = u\wedge 101v$ and thus

$$x\wedge y\wedge x\wedge z\wedge x = (x\wedge y)\wedge u\wedge x\wedge v\wedge(z\wedge x) = (x\wedge y)\wedge u\wedge v\wedge(z\wedge x) = x\wedge y\wedge z\wedge x.$$

From the theory of bands, the identity $x\wedge y\wedge x\wedge z\wedge x = x\wedge y\wedge z\wedge x$ implies $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ are $\wedge$-congruences. Similarly one shows that $\mathcal{R}_{(\vee)}$ and $\mathcal{L}_{(\vee)}$ are $\vee$-congruences. ☐

From refined quasilattices to skew lattices and back again

Many results established for skew lattices extend to refined quasilattices. In particular the following extensions of Theorem 2.2.1 and its immediate consequences hold, with the proofs being essentially the same.

Theorem 3.4.7. *Let* $(S, \vee, \wedge)$ *be a refined quasilattice with* $\mathcal{D}$*-classes* A *and* B. *If* J *and* M *are the respective join and meet classes of* A *and* B *in* $S/\mathcal{D}$, *then*

$$J = \{a\vee b \mid a\in A, b\in B \ \&\ a\vee b = b\vee a\} \text{ and } M = \{a\wedge b \mid a\in A, b\in B \ \&\ a\wedge b = b\wedge a\}. \ \square$$

Corollary 3.4.8. *Given a refined quasilattice* $(N, \vee, \wedge)$ *and an infinite cardinal number* $\aleph_\omega$ *then the union of all* $\mathcal{D}$*-classes of order* [*equal to or*] *less than* $\aleph_\alpha$ *is a subalgebra of* N. *In particular, the union of all finite* $\mathcal{D}$*-classes is a subalgebra of* N. ☐

Corollary 3.4.9. *Let* $(S, \vee, \wedge)$ *be a refined quasilattice with* $a \in S$. *The following are equivalent:*

1. *For all* $b \in S$, $a \vee b = b \vee a$.
2. *For all* $b \in S$, $a \wedge b = b \wedge a$.
3. *The* $\mathcal{D}$*-class of* a *is the singleton*, $\{a\}$.

Thus the union of all singleton $\mathcal{D}$*-classes is the center of* S. □

These results, in contrast with that of Theorem 3.1.7, show the effect of a coherent natural partial order on the structure of a quasilattice. It turns out that (even irregular) refined quasilattices and skew lattices are closely connected in a very precise way. To see this, we borrow a term arising in a number of other contexts both mathematical and scientific.

An ***isotopy of quasi-lattices*** $(Q, \vee, \wedge)$ and $(Q', \vee', \wedge')$ with respective natural pre-orders $\succeq$ and $\succeq'$ is a bijection $f: Q \to Q'$ such that $x \succeq y$ in Q iff $f(x) \succeq' f(y)$ in Q'.

An ***isotopy of paralattices*** $(P, \vee, \wedge)$ and $(P', \vee', \wedge')$ with natural partial orders $\geq$ and $\geq'$ is a bijection $f: P \to P'$ such that $x \geq y$ in P iff $f(x) \geq' f(y)$ in P'.

Finally, an ***isotopy of refined quasi-lattices*** $(S, \vee, \wedge)$ and $(S', \vee', \wedge')$ is any bijection from S to S' that is simultaneously an isotopy of quasi-lattices and an isotopy of paralattices.

In each case, all isomorphisms are isotopies. Also, in each of these cases, given an algebra $(S, \vee, \wedge)$, upon defining $\vee\cdot$ by $x \vee\cdot y = y \vee x$ and $\wedge\cdot$ by $x \wedge\cdot y = y \wedge x$, all three derived algebras $(S, \vee\cdot, \wedge)$, $(S, \vee, \wedge\cdot)$ and $(S, \vee\cdot, \wedge\cdot)$ are all isotopes of $(S, \vee, \wedge)$ under the identity map on S. Indeed $x \vee\cdot y \vee\cdot x = x \vee y \vee x$ and $x \wedge\cdot y \wedge\cdot x = x \wedge y \wedge x$ on S, in the case of quasilattices, thus guaranteeing that $\leq$ is the same for all four algebras. Likewise $x \vee\cdot y = y \vee\cdot x$ iff $x \vee y = y \vee x$ on S with both outcomes agreeing, and $x \wedge\cdot y = y \wedge\cdot x$ iff $x \wedge y = y \wedge x$ on S with both outcomes agreeing in the case of paralattices, thus guaranteeing that $\leq$ is the same for all four algebras. Note that antilattices are isotopic precisely if the have the same size.

In general we have the following assertions, the first of which is near-obvious:

Proposition 3.4.10. *Let quasilattices* $(Q, \vee, \wedge)$ *and* $(Q', \vee', \wedge')$ *be given. Then any isotopy* $f: Q \to Q'$ *induces a unique isomorphism* $f^*: Q/\mathcal{D} \to Q'/\mathcal{D}$ *making the following diagram commute, where* $\delta: Q \to Q/\mathcal{D}$ *and* $\delta' Q' \to Q'/\mathcal{D}$ *denote the canonical epimorphisms.*

$$\begin{array}{ccc} Q & \xrightarrow{f} & Q' \\ \delta\downarrow & & \downarrow\delta' \\ Q/\mathcal{D} & \xrightarrow{f^*} & Q'/\mathcal{D} \end{array}$$

Moreover, bijections f_α: $\mathcal{D}_\alpha \to \mathcal{D}_{\alpha'}$ *between all corresponding* $\mathcal{D}$*-classes of* Q *and* Q' *are induced upon restricting* f *to the various* $\mathcal{D}$*-classes of* S*. Conversely, given quasilattices* $(Q, \vee, \wedge)$ *and* $(Q', \vee', \wedge')$*, suppose that* $f^*: Q/\mathcal{D} \to Q'/\mathcal{D}$ *is an isomorphism and that bijections* f_α: $\mathcal{D}_\alpha \to \mathcal{D}_{\alpha'}$ *exist between all corresponding pairs of* $\mathcal{D}$*-classes (relative to* f^**). Then an isotopy* $f: Q \to Q'$ *is given by* $f = \bigcup f_\alpha$*. Any isotopy of* $(Q, \vee, \wedge)$ *with* $(Q', \vee', \wedge')$ *thus arises in this manner.* □

The existence of an isotopy between quasilattices thus depends on having isomorphic maximal lattice images and corresponding $\mathcal{D}$-classes of the same size. For refined quasilattices more constraints occur.

Proposition 3.4.11. *An isotopy* $f: (S, \vee, \wedge) \to (S', \vee', \wedge')$ *of refined quasilattices preserves both commuting joins and meets. Thus* $a\vee b = b\vee a$ *in* S *implies* $f(a\vee b) = f(a) \vee' f(b) = f(b) \vee' f(a)$ *in* S'*. Dually,* $a\wedge b = b\wedge a$ *in* S *implies* $f(a\wedge b) = f(a) \wedge' f(b) = f(b) \wedge' f(a)$ *in* S'*.*

Proof. Suppose that $a\vee b = b\vee a$ in S. If c denotes this common join, then c is the unique element in the join-class of the $\mathcal{D}$-classes of a and b such that both $c \geq a$ and $c \geq b$. Being an isotropy of paralattices, we have $f(c) \geq$ both $f(a)$ and $f(b)$ in S'. Being an isotropy of quasilattices, we have $f(c)$ lying in the join class of the $\mathcal{D}$-classes of $f(a)$ and $f(b)$ in S'. By uniqueness, $f(a) \vee' f(b) = f(c) = f(b) \vee' f(a)$. Similarly, f preserves commuting meets. □

Michael Kinyon observed (in a personal comminication) that every refined quasilattice is isotopic to a skew lattice in a fairly simple way. To see this first recall a fact about regular bands.

Lemma 3.4.12. *If* $(S, \bullet)$ *is a band and an operation* $\bullet_L$ *is defined on* S *by* $e\bullet_L f = efe$*, then* $(S, \bullet_L)$ *is a band if and only if* $(S, \bullet)$ *is a regular band, in which case* $(S, \bullet_L)$ *is a left regular band. Dually, defining* $\bullet_R$ *on* S *by* $e\bullet_R f = fef$*,* $(S, \bullet_R)$ *is a band if and only if* $(S, \bullet)$ *is a regular band, in which case the band* $(S, \bullet_R)$ *is right regular. Moreover,* $e \leq f$ *in* $(S, \bullet)$ iff $e \leq f$ *in* $(S, \bullet_L)$*, and* $e \preceq f$ *in* $(S, \bullet)$ *iff* $e \preceq f$ *in* $(S, \bullet_L)$*. Similar remarks relate both* $\leq$ *and* $\preceq$ *for* $(S, \bullet)$ and $(S, \bullet_R)$.

Proof. (Here $x\bullet y$ is denoted by xy.) Clearly $\bullet_L$ is idempotent. Is it associative? Note that $a\bullet_L(b\bullet_L c) = a\bullet_L(bcb) = abcba$ while $(a\bullet_L b)\bullet_L c = (a\bullet_L b)c(a\bullet_L b) = abacaba$. Thus $\bullet_L$ is associative iff

$$abcba = abacaba$$

holds on S. Clearly this happens if (S, •) is regular. Conversely, if $(S, \bullet_L)$ is associative, then expanding $a\bullet_L b\bullet_L c$ both ways gives $abcba = abacaba$ for all *a, b, c*. Replace a by ba to get $babcbba = babbacbabba$, which simplifies to

$$(1)\quad babcba = bacba$$

The mirror argument gives

$$(2)\quad abcbab = abcab.$$

Multiply both sides of (1) on the right by bcb to get $b\underline{abcbabcb} = ba\underline{cbabcb}$. Simplifying on the left gives

$$(3)\ \ babcb = bacbabcb.$$

Using (2), the right side of (3) simplifies as follows: $ba\underline{cbabcb} =_{(2)} \underline{bacbac}b = bacb$. Thus (3) becomes $babcb = bacb$, which is regularity.

So assume that $(S, \bullet)$ is regular. Then $a\bullet_L b\bullet_L a = (aba)\bullet_L a = a(aba)a = aba = a\bullet_L b$, making $(S, \bullet_L)$ left-regular. If $e \leq f$ *in* $(S, \bullet)$, then $efe = f = fef$, that is $e\bullet_L f = f = f\bullet_L e$, so that $e \leq f$ in $(S, \bullet_L)$. Conversely $e\bullet_L f = f = f\bullet_L e$ reduces *to* $efe = f = fef$ so that $ef = efefef = f = fefefe = fe$ in $(S, \bullet)$. In similar fashion, $e \preceq f$ *in* $(S, \bullet)$ iff $e \preceq f$ *in* $(S, \bullet_L)$, since $f\bullet_L e\bullet_L f = fefef$ which immediately reduces to fef. Thus $fef = f$ iff $f\bullet_L e\bullet_L f = f$. □

Thus, if $(S, \bullet)$ is regular, then it is isotopic as a band to both $(S, \bullet_L)$ and $(S, \bullet_R)$. Our next theorem rests on the lemma and a special case of Corollary 3.4.4. We give an alternative proof.

Lemma 3.4.13. *If* $(S, \wedge, \vee)$ *is a refined quasilattice that is left-handed in that* $x\wedge y\wedge x = x\wedge y$ *and* $x\vee y\vee x = y\vee x$, *then it is a skew lattice.*

Proof. Being a quasilattice, $x\wedge(x\vee y) = x\wedge(x\vee y)\wedge x = x$. Being a paralattice,

$$(y\vee x)\wedge x = (x\vee y\vee x)\wedge x = x.$$

The dual identities are similarly seen. □

This leads to Kinyon's observation (via email) about refined quasi-lattices.

Theorem 3.4.14. *Given a quasilattice* $(S, \vee, \wedge)$, *it is a refined quasilattice if and only if* $(S, \vee_R, \wedge_L)$ *is a left-handed skew lattice, in which case* $(S, \vee, \wedge)$ *and* $(S, \vee_R, \vee_L)$ *are isotopic under the identity map on* S. *In addition, they share all instances of commutation in which case the outcomes agree for the corresponding pairs of operations*.

Proof. As seen in the proof above, $a\wedge_L b\wedge_L a$ expanded reduces to $a\wedge b\wedge a$. Likewise, $b\vee_R a\vee_R b$ reduces to $b\vee a\vee b$. Thus whenever $(S, \vee_R, \wedge_L)$ is a quasilattice, both algebras share a common natural quasi-order. Also in general, $a\wedge b = b\wedge a$ iff $a\wedge_L b = b\wedge_L a$ with both outcomes being equal, and $a\vee b = b\vee a$ iff $a\vee_R b = b\vee_R a$, with both outcomes being equal. Thus $(S, \vee_R, \wedge_L)$ has a coherent natural partial order iff $(S, \vee, \wedge)$ does, in which case both algebras share the same natural partial order. It follows that if $(S, \vee, \wedge)$ is a refined quasilattice if and only if $(S, \vee_R, \wedge_L)$ is a left-handed skew lattice, thanks to the previous lemmas. □

Every refined quasilattice is thus just a "scrambled skew lattice". Indeed given a left-handed skew lattice (S, ∨, ∧) with $S/\mathcal{D}$ denoted by T, various refined quasi-lattices $(S, \vee^*, \wedge^*)$ may be recovered from it by (1) doing a fibered product factorization $(S_1, \vee_1) \times_T (S_2, \vee_2)$ of the right regular band (S, ∨), replacing $\vee_2$ by its left-handed dual operation and finally shifting the resulting operation on $S_1 \times_T S_2$ back to S to get $\vee^*$; and (2) likewise factoring the left-regular band (S, ∧) as say $(S_3, \wedge_3) \times_{(T, \wedge)} (S_4, \wedge_4)$ and replacing $\wedge_4$ by its right-handed dual and then shifting the resulting operation from $S_3 \times_T S_4$ back to S to get $\wedge^*$. *Every refined quasilattice can in principle be recovered from its derived skew lattice in this manner*. When $S_1 \times_T S_2$ and $S_3 \times_T S_4$ are just reducts of a common fibered factorization of (S, ∨, ∧) as a skew lattice, the algebra (S, $\vee^*$, $\wedge^*$) is also a skew lattice.

Note that results 3.4.7 – 3.4.9 above, are trivial consequences of the above theorem and their skew lattice predecessors in Section 2.2.

3.5 The effects of the distributive identities

Connections between refined quasilattices and distributive properties exist which particularly involve split quasilattices; however, distributive identities, much like absorption identities, proliferate in the absence of commutativity.

To begin, a noncommutative lattice is ***fully distributive*** if it satisfies the identities:

D_1: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$. D_2: $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.
D_3: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. D_4: $(a \wedge b) \vee c = (a \wedge c) \vee (b \wedge c)$.

The four identities are mutually independent, unlike the case for lattices.

Remark. If one replaces (∨, ∧) by (+, •), these identities describe a ***semiring*** that is also ***distributive*** in that addition distributes over multiplication. When both operations are idempotent, one has an ***idempotent, distributive semiring*** or an ***ID-semiring***. That is, ID-semirings are just fully distributive double bands, but in ring notation. ID-semirings were introduced by Pastijn and Romanowska [1982] and studied in several subsequent papers. For instance, varieties of ID-rings where both operations are middle commutative (i.e., normal in band terminology) were classified by Pastijn in [1983]. Two separate lines of research thus meet at fully distributive quasilattices. We will consider connections between ID-semirings and fully distributive quasilattices after Corollary 4.5.3 below.

A noncommutative lattice is ***bidistributive*** if it satisfies a slightly weaker pair of identities:

D_5: $a \wedge (b \vee c) \wedge d = (a \wedge b \wedge d) \vee (a \wedge c \wedge d)$.
D_6: $a \vee (b \wedge c) \vee d = (a \vee b \vee d) \wedge (a \vee c \vee d)$.

Finally, a noncommutative lattice is ***distributive*** if it satisfies an even weaker pair of identities:

D_7: $a \wedge (b \vee c) \wedge a = (a \wedge b \wedge a) \vee (a \wedge c \wedge a)$.
D_8: $a \vee (b \wedge c) \vee a = (a \vee b \vee a) \wedge (a \vee c \vee a)$.

Recall that all skew lattices in rings are distributive as are all skew Boolean algebras.

The power of these varying forms of distribution is seen in the following results.

Theorem 3.5.1. *A distributive, noncommutative lattice is a paralattice if and only if it is a quasilattice.*

Proof. Any paralattice satisfying D_7 and D_8 is a quasilattice. Just expand $a\wedge(b\vee a\vee b)\wedge a$ and $a\vee(b\wedge a\wedge b)\vee a$ using D_7 and D_8 and then simplify using B_6, B_7, C_6 and C_7 to obtain respectively B_5 and C_5. On the other hand, let N be a quasilattice satisfying both D_7 and D_8. To see that N is also a paralattice, suppose to the contrary that there exist $a, b \in N$ such that $a\wedge b = b = b\wedge a$, but either $a\vee b \neq a$ or $b\vee a \neq a$. Then we obtain either

$$a\wedge(a\vee b)\wedge a = a \neq a\vee b = (a\wedge a\wedge a)\vee(a\wedge b\wedge a)$$

or

$$a\wedge(b\vee a)\wedge a = a \neq b\vee a = (a\wedge b\wedge a)\vee(a\wedge a\wedge a)$$

and thus deny D_7. Similarly, if $a,b \in N$ exist such that $a\vee b = b = b\vee a$, but either $a\wedge b \neq a$ or $b\wedge a \neq a$ then we obtain denials of D_8. Hence the two natural partial orderings indeed dualize each other and N is a paralattice. ☐

Passing to D_5 and D_6 we obtain a much stronger result.

Theorem 3.5.2. *A bidistributive quasilattice (paralattice) factors into the product of a distributive lattice and an antilattice. Conversely, every such product is a bidistributive quasilattice (paralattice).*

Proof. Let N be a (necessarily fine) bidistributive quasilattice. We show that if $x \geq y, z$ with $y\ \mathcal{D}\ z$, then $y = z$. Since both $u \geq u\wedge x\wedge y\wedge u$, $u\wedge y\wedge x\wedge u$ and $u\wedge x\wedge y\wedge u\ \mathcal{D}\ u\wedge y\wedge x\wedge u$ in general, the implication yields the identity $u\wedge x\wedge y\wedge u = u\wedge y\wedge x\wedge u$. So let $x \geq y,z$ in N. Since $y\ \mathcal{R}_{(\wedge)}\ y\wedge z\ \mathcal{L}_{(\wedge)}\ z$, with $x \geq y\wedge z$ also, we show that $y = z$ under the added assumption that either $y\ \mathcal{L}_{(\wedge)}\ z$ or $y\ \mathcal{R}_{(\wedge)}\ z$. Let us assume that $y\ \mathcal{L}_{(\wedge)}\ z$. Consider $y\vee z$ and $z\vee y$. D_5 gives

$$y\wedge(y\vee z)\ =\ y\wedge(y\vee z)\wedge x\ =\ (y\wedge y\wedge x)\vee(y\wedge z\wedge x)\ =\ y\vee y\ = y$$

and

$$(y\vee z)\wedge y\ =\ x\wedge(y\vee z)\wedge y\ =\ (x\wedge y\wedge y)\vee(x\wedge z\wedge y)\ =\ y\vee z.$$

Hence $y\ \mathcal{R}_{(\vee)}\ y\vee z\ \mathcal{L}_{(\vee)}\ z$, in a common $\mathcal{L}_{(\wedge)}$-class of N. Thus, we may assume further that in addition to $y\ \mathcal{L}_{(\wedge)}\ z$ either $y\ \mathcal{L}_{(\vee)}\ z$ or $y\ \mathcal{R}_{(\vee)}\ z$. Supposing that $y\ \mathcal{L}_{(\vee)}\ z$, then by D_4,

$$y\ =\ y\vee z\ =\ (x\wedge y\wedge z)\vee(x\wedge x\wedge z)\ =\ x\wedge(y\vee x)\wedge z\ =\ x\wedge x\wedge z\ =\ z.$$

If we take $y\ \mathcal{R}_{(\vee)}\ z$ instead, then D_5 again yields

$$y = z \vee y = (x \wedge x \wedge z) \vee (x \wedge y \wedge z) = x \wedge (x \vee y) \wedge z = x \wedge x \wedge z = z.$$

Thus if $x \geq y,z$ with $y \ \mathcal{L}_{(\wedge)}\ z$, then $y = z$. If we had first supposed that $y \ \mathcal{R}_{(\wedge)}\ z$, then it would follow in similar fashion that $y = z$. Thus $x \geq y,z$ in N with $y \ \mathcal{D}\ z$ implies that $y = z$, from which the identity $u \wedge x \wedge y \wedge u = u \wedge y \wedge x \wedge u$ follows.

Similarly, if $x, y \geq z$ in N with $x \ \mathcal{D}\ y$, then D_6 can be used to show that $x = y$, from which follows the identity $u \vee x \vee y \vee u = u \vee y \vee x \vee u$. By Theorem 3.1.4, N factors into the product of its (necessarily) distributive maximal lattice image and an antilattice.

For the converse, observe that antilattices are always bidistributive as, of course, are distributive lattices. Thus split quasilattices whose lattice factors are distributive are indeed bidistributive quasilattices. ☐

An even stronger result occurs when full distribution is assumed:

Corollary 3.5.3. *A quasilattice is fully distributive if and only if it is the product of a distributive lattice and a regular antilattice. In particular, every fully distributive quasilattice is regular.*

Proof. Let N be a fully distributive quasilattice. N factors as the product of a distributive lattice T and an antilattice A. Being a fine quasilattice, $\mathcal{L}_{(\vee)}$ is a $\vee$-congruence by Theorem 3.4.5. But $\mathcal{L}_{(\vee)}$ is a also a $\wedge$-congruence: given $b \ \mathcal{L}_{(\vee)}\ c$, D_1 implies $a \wedge b \ \mathcal{L}_{(\vee)}\ a \wedge c$ and D_2 implies $b \wedge d \ \mathcal{L}_{(\vee)}\ c \wedge d$. Thus $\mathcal{L}_{(\vee)}$ is a full congruence. Similarly, $\mathcal{R}_{(\vee)}$ is a full congruence. Likewise, D_3 and D_4 imply that $\mathcal{R}_{(\wedge)}$ and $\mathcal{L}_{(\wedge)}$ are full congruences. Thus both N and its rectangular factor A are regular. Conversely, let N factor as the product of a distributive lattice T and a regular antilattice A. Being regular, A factors into a product of flat algebras, $A \cong A_{(l,l)} \times A_{(l,r)} \times A_{(r,l)} \times A_{(r,r)}$, with each factor having operations $\vee$ and $\wedge$ that each satisfy $ab = a$ or $ab = b$. Conversely, identities $D_1 - D_4$ are easily seen to hold on such structures and, of course on T and hence on N. ☐

As mentioned above, two lines of research meet at fully distributive quasilattices. In particular, our Corollary 4.5.3 follows from Theorem 2.6 of Pastijn [1983]. By the latter theorem, the lattice of varieties of all ID-semirings satisfying the identities,

$$x + y + z + w = x + z + y + w \ \text{ and } \ xyzw = xzyw,$$

is described. The sublattice of those varieties that also satisfy semiring versions of B_5 and C_5 is easily seen to be the join of the variety of distributive lattices and the variety of rectangular ID-semirings (satisfying $x + y + z = x + z$ and $xyz = xz$). Thus every fully distributive quasilattice is the subdirect product of a distributive lattice and a rectangular ID-semiring (that is essentially a regular rectangular quasilattice). But such a subdirect product splits by Theorem 3.5.2 and the corollary follows.

Pastijn's theorem also shows that the lattice of all varieties of fully distributive quasilattices is Boolean of order 32. The larger lattice of varieties is also Boolean, but of order 2^{10}. (Pastijn works with identities, $aba + a + aba = a$ and $(a+b+a)a(a+b+a) = a$, the ID-semiring equivalents of B_5 and C_5.)

Returning to refined quasilattices from the last section, one may ask about the effect of distributivity of $(S, \vee, \wedge)$ on its isotopic left-handed skew lattice variant $(S, \vee^*, \wedge^*)$ where $x\vee^*y = y\vee x\vee y$ and $x\wedge^*y = x\wedge y\wedge x$,

Propositon 3.5.4. *If a refined quasilattice* $(S, \vee, \wedge)$ *is distributive, then its isotopic left-handed skew lattice* $(S, \vee^*, \wedge^*)$ *is also distributive*.

Proof. Given that $x\vee^*y = y\vee x\vee y = y\vee^*x\vee^*y$ and $x\wedge^*y = x\wedge y\wedge x = x\wedge^*y\wedge^*x$, suppose that $(S, \vee, \wedge)$ is distributive. Then

$$x \wedge^* (y \vee^* z) \wedge^* x = x\wedge(y \vee^* z)\wedge x = x\wedge(z \vee y \vee z)\wedge x = (x\wedge z\wedge x) \vee (x\wedge y\wedge x) \vee (x\wedge z\wedge x)$$

and

$$(x\wedge^*y\wedge^*x) \vee^* (x\wedge^*z\wedge^*x) = (x\wedge y\wedge x) \vee^* (x\wedge z\wedge x) = (x\wedge z\wedge x) \vee (x\wedge y\wedge x) \vee (x\wedge z\wedge x),$$

so that $(S, \vee^*, \wedge^*)$ satisfies D_7 and likewise its dual D_8. ☐

Conversely, if $(S, \vee^*, \wedge^*)$ is distributive, then $(S, \vee, \wedge)$ is easily seen to satisfy the weaker identity $x\wedge(z\vee y\vee z)\wedge x = (x\wedge z\wedge x)\vee(x\wedge y\wedge x)\vee(x\wedge z\wedge x)$ and its dual. If in addition $(S, \vee, \wedge)$ is already known to be a skew lattice, then one can show it must be distributive.

3.6 Deriving simple antilattices from magic squares

Recall that a ***magic square*** is a square array of distinct numbers where all rows, columns and the two diagonals have a common ***magic sum***. A classic instance is the *Lo-Shu* with a magic sum of 15:

8	1	6
3	5	7
4	9	2

.

Given a magic square, its ***derived antilattice*** arises by letting the given square be the $\wedge$-array and letting the $\vee$-array be the square array storing the same numbers entered in their natural ordering. Thus in the case of the *Lo-Shu* one gets:

$$(\wedge)\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad (\vee)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Square arrays also come from ***finite planes***, that is, vector spaces of dimension 2 over finite fields. For instance, given the field $\mathbb{Z}_5$, the plane $\mathbb{Z}_5 \times \mathbb{Z}_5$ can be represented as the 5×5 array of ordered pairs on the left below, but with parentheses deleted. Alternatively, one could view these pairs as base 5 representations of integers in base 10. Thus 3,2 represents $32_5 = 17_{10}$. The planar array could thus be encoded using the numbers in the right array.

$$\begin{bmatrix} 0,0 & 1,0 & 2,0 & 3,0 & 4,0 \\ 0,1 & 1,1 & 2,1 & 3,1 & 4,1 \\ 0,2 & 1,2 & 2,2 & 3,2 & 4,2 \\ 0,3 & 1,3 & 2,3 & 3,3 & 4,3 \\ 0,4 & 1,4 & 2,4 & 3,4 & 4,4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \\ 20 & 21 & 22 & 23 & 24 \end{bmatrix}$$

In either case, this plane has 25 points and 30 lines, the latter arranged in six classes of five parallel lines each. The rows of the array consist of all lines of slope 0, the columns consist of all lines of undefined slope, the main diagonal plus the four broken descending diagonals yield all five lines of slope 1, and the counter-diagonal plus all four broken ascending counter-diagonals yield all five lines of slope 4. In all, between the rows, columns, diagonals and counter-diagonals, 20 out of 30 lines are accounted for, with only lines of slopes 2 or 3 left out.

Alternatively, $\mathbb{Z}_5 \times \mathbb{Z}_5$ can be represented by storing the five lines of slope 1 in the five rows and the five lines of slope 4 in the five columns in the left array below.

$$\begin{bmatrix} 0,0 & 1,1 & 2,2 & 3,3 & 4,4 \\ 2,3 & 3,4 & 4,0 & 0,1 & 1,2 \\ 4,1 & 0,2 & 1,3 & 2,4 & 3,0 \\ 1,4 & 2,0 & 3,1 & 4,2 & 0,3 \\ 3,2 & 4,3 & 0,4 & 1,0 & 2,1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 6 & 12 & 18 & 24 \\ 13 & 19 & 20 & 1 & 7 \\ 21 & 2 & 8 & 14 & 15 \\ 9 & 10 & 16 & 22 & 3 \\ 17 & 23 & 4 & 5 & 11 \end{bmatrix}$$

In the right array, not only do all rows, columns, the main diagonal and the counter-diagonal sum to 60, but so do all broken diagonals and counter-diagonals. This makes the right array a ***pandiagonal*** magic square. Returning to the left array, the (broken) diagonals and counter-diagonals are precisely the lines of slope 3 and 2 respectively. Indeed, the line arrangement of the left array forces the right array to be pandiagonal. That finite planes can yield pandiagonal squares is well known. Together, the two representations of this plane in integer format describe an antilattice induced from a magic square. As will be shown below, both antilattices derived from the two magic squares thus far encountered are simple. Is this true in general?

To begin, given antilattice **A** and any pair a, $b \in$A, recall that the ***principal congruence*** $\theta_{(a,b)}$ is the smallest congruence on **A** relating a and b. Clearly: $\theta_{(a,b)} = \bigcap\{\theta \in \mathbf{Con(A)} \mid a\ \theta\ b\}$. In particular, $\theta_{(a,b)} = \Delta$ precisely when $a = b$. Clearly:

Lemma 3.6.1. *An algebra* **A** *of any type is simple if and only iff* $\theta_{(a,b)} = \nabla$ *for all* $a \neq b$ *in its underlying set* . □

For antilattices, this obvious criterion can be simplified. Consider an antilattice **A** determined by a pair of rectangular arrays. Let R_0 and C_0 represent a row and a column of, say, the $\vee$-array of **A**. (*Which* array is not important. But R_0 and C_0 must come from the same array.)

Lemma 3.6.2. (*Simplicity Criterion for Antilattices*) *Given an antilattice* A *determined by a pair of rectangular arrays, let* R_0 *and* C_0 *denote respectively a row and a column of the* $\vee$*-array. Then* A *is a simple algebra iff* $\theta_{(a,b)} = \nabla$ *for all* $a \neq b$ *in* R_0 *and all* $a \neq b$ *in* C_0. *In particular, any given* $\theta_{(a,b)}$ *must equal* ∇ *if* A *is generated from* $\{a, b\}$ *using both* $\vee$ *and* $\wedge$.

Proof. The condition is clearly necessary. To see sufficiency, suppose that the condition holds for row R_0 and column C_0 intersecting at element c in the $\vee$-array. Given $a \neq b$ in A, both $c\vee a$ and $c\vee b$ lie in R_0, while $a\vee c$ and $b\vee c$ lie in C_0. Since $a \neq b$, either $c\vee a \neq c\vee b$ in R_0 or $a\vee c \neq b\vee c$ in C_0. Say $c\vee a \neq c\vee b$, so that $\theta_{(c\vee a,c\vee b)} = \nabla$. But since $c\vee a\ \theta_{(a,b)} c\vee b$, $\theta_{(c\vee a,c\vee b)}$ refines $\theta_{(a,b)}$ so that $\theta_{(a,b)} = \nabla$ also. Thus $\theta_{(a,b)} = \nabla$ for all $a \neq b \in A$ and A is simple. Since $\vee$ and $\wedge$ are idempotent, the subalgebra $\langle a, b\rangle$ generated from $\{a, b\}$ lies in the $\theta_{(a,b)}$-class of a. The final statement follows. □

In the case of a square antilattice determined from a pair of $n\times n$ arrays, this theorem says that the number of principal congruences needing to be checked can be reduced from $(n^4 - n^2)/2$ to just $n^2 - n$. Although the check to see that $\theta_{(a,b)} = \nabla$ for $a \neq b$ in either $R_0\times R_0$ or $C_0\times C_0$ can be initially tedious, as the check continues some random recursion enters the process. Thus, if say $\theta_{(a,b)}$ has been shown to equal ∇ and $a\ \theta_{(c,\,d)}\ b$ is encountered in the check of $\theta_{(c,\,d)}$, then one can immediately conclude that $\theta_{(c,\,d)} = \nabla$ also holds.

Example 3.6.1. *The Lo-Shu antilattice* $(\vee)\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix}\ (\wedge)\begin{bmatrix}8 & 1 & 6\\ 3 & 5 & 7\\ 4 & 9 & 2\end{bmatrix}$ *is simple.*

To begin, take $\{1, 2\}$. From the $\wedge$-array, it is clear that $6, 9 \in\langle 1, 2\rangle$, the subalgebra generated from $\{1, 2\}$. But $\{1, 2, 6, 9\}$ clearly generates the $\vee$-array and thus the algebra. Hence $\theta_{(1,2)} = \nabla$. Similar remarks hold for any other pair $a \neq b$ in any row or column of *either* array. □

These remarks deserve a more precise analysis. Given distinct elements a and b in a common row (column) of a 3×3 array, the elements c and d lying in neither the row (column) or the two columns (rows) of a and b is called the ***dual pair***. The relationship is symmetrical. Thus $\{1, 2\}$ and $\{6, 9\}$ form dual pairs in the $\vee$-array above, but not in the $\wedge$-array. *Any pair of dual pairs in a* 3×3 *array generates the entire array under the ambient idempotent operation*. Given two distinct elements in a common row or column of one of the above *Lo-Shu* arrays, this pair immediately generates its dual pair in the opposite array. In this sense, *these two arrays are* ***complementary*** 3×3 *arrays*, *so that any pair of elements lying in a common row or column in either array generates the entire antilattice which thus is simple.*

What can be said in general about congruences on an antilattice?

Given a rectangular array A, a ***cartesian partition*** of A is a partition $\mathcal{P}$ that is induced in cartesian fashion from a partition of the rows and a partition of the columns of A. For example, one cartesian partition of

$$\begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ k & l & m & n & o \end{bmatrix}$$

is given by:

$$\begin{bmatrix} \begin{bmatrix} a & b & c \\ f & g & h \end{bmatrix} & \begin{bmatrix} d & e \\ i & j \end{bmatrix} \\ \begin{bmatrix} k & l & m \end{bmatrix} & \begin{bmatrix} n & o \end{bmatrix} \end{bmatrix}.$$

In all, 260 = 52 × 5 cartesian partitions of this array are given by the fifty-two partitions of $\{a, b, c, d, e\}$ and the seven partitions of $\{a, f, k\}$.

Given such a partition $\mathcal{P}$, an equivalence $\mathcal{P}^{\#}$ on A is given by $a\ \mathcal{P}^{\#}\ b$ if a and b lie in the same $\mathcal{P}$-class. Such an equivalence is called a ***cartesian equivalence*** on A. In the case of rectangular bands, the congruences on an array that are consistent with a single rectangular band operation (using just one of ∨ or ∧) are precisely its cartesian equivalences. Thus:

Proposition 3.6.3. *Given an antilattice* A, *its congruences arise from pairs of cartesian partitions of its two arrays sharing the same equivalence classes*. □

The general 3×3 *case.*

The *Lo-Shu* is one of infinitely many possible 3×3 magic squares that can arise if we agree to store integers *besides* 1 - 9. Others include the following two squares:

$$\begin{bmatrix} 71 & 89 & 17 \\ 5 & 59 & 113 \\ 101 & 29 & 47 \end{bmatrix} \qquad \begin{bmatrix} 252 & 171 & 363 \\ 373 & 262 & 151 \\ 161 & 353 & 272 \end{bmatrix}$$

The magic square on the left consists entirely of primes, with a magic sum of 177, the least possible such sum for any magic square of primes. The magic square on the right consists of 3-digit palindromes. Like the Lo Shu case, both examples induce simple antilattices. Is this true for all 3 × 3 magic squares? To answer this, we begin with the following result.

Lemma 3.6.4. *Given* 3×3 *arrays*, A *and* A′, *each storing the same* 9 *distinct elements*, *the following assertions are equivalent*:

1. A *and* A′ *form a complimentary pair of* 3×3 *arrays*.
2. *If two distinct elements are either row-related or column-related in either array*, *they are unrelated in either sense in the other array*.
3. *The rows [columns] in* A *either all become (extended) diagonals in* A′ *or all become extended counterdiagonals in* A′; *similar remarks hold in passing from* A′ *to* A.

Proof. Clearly (1) implies (2). For the converse, observe that the status of (2) is unchanged if either array undergoes row or columns interchanged! Thus, we assume (2) in the case where elements a and b lie in a common row of A, as in $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Assertion (2) implies that A′ is of form $\begin{bmatrix} a & f & x \\ i & b & y \\ u & v & w \end{bmatrix}$ or $\begin{bmatrix} a & i & x \\ f & b & y \\ u & v & w \end{bmatrix}$. Applying (2) further, A′ must be either $\begin{bmatrix} a & f & h \\ i & b & d \\ e & g & c \end{bmatrix}$ or $\begin{bmatrix} a & i & e \\ f & b & g \\ h & d & c \end{bmatrix}$ its trnaspose. In either case we have a complementary pair of arrays. Similarly, assuming a and b lie in the same column of A, (2) forces A′ to be a complementary array. Likewise, if a and b are row-[column-] related in A′, then (2) forces A to be a complementary to A′. Thus, (1) and (2) are equivalent. Clearly (3) implies (1) and (2). Given the latter, every row/column in either array must be an (extended) [counter]diagonal in the other array. This can only happen if (3) holds. ☐

We are ready to state our main result about antilattices induced from 3×3 magic squares.

Theorem 3.6.5. *Given a* 3×3 *array* A = $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ *consisting of distinct positive integers in their natural* (*increasing*) *order and a second* 3 × 3 *magic square* A′ *storing the same integers, then either* A *and* A′ *are complementary or* A′ *is a dihedral variation of* $\begin{bmatrix} b & i & c \\ f & e & d \\ g & a & h \end{bmatrix}$.

(An instance of the latter is the pair A = $\begin{bmatrix} 1 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 11 \end{bmatrix}$ and A′ = $\begin{bmatrix} 3 & 11 & 4 \\ 7 & 6 & 5 \\ 8 & 1 & 9 \end{bmatrix}$.)

Proof. Using a dihedral replacement of A′ if need be, distinct β > γ > 0 and α > β + γ exist such that:

$$\mathrm{A}' = \alpha\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \beta\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \gamma\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \alpha-\beta & \alpha+\beta+\gamma & \alpha-\gamma \\ \alpha+\beta-\gamma & \alpha & \alpha-\beta+\gamma \\ \alpha+\gamma & \alpha-\beta-\gamma & \alpha+\beta \end{bmatrix}.$$

Clearly $a = \alpha - \beta - \gamma$ and $b = \alpha - \beta$. What is the next smallest element? If it is $\alpha - \gamma$, then the ascending sequence

$$\alpha-\beta-\gamma < \alpha-\beta < \alpha-\gamma < \alpha-\beta+\gamma < \alpha < \alpha+\beta-\gamma < \alpha+\gamma < \alpha+\beta < \alpha+\beta+\gamma$$

must occur. In this case we have the displayed array. Otherwise, we must have:

$$\alpha-\beta-\gamma < \alpha-\beta < \alpha-\beta+\gamma < \alpha-\gamma < \alpha < \alpha+\gamma < \alpha+\beta-\gamma < \alpha+\beta < \alpha+\beta+\gamma$$

yielding an array complementary to A. □

This theorem has the following consequence:

Corollary 3.6.6. *Given the arrays* A *and* A′ *of the prior theorem, the induced antilattice* A *is simple if and only if* A *and* A′ *are complementary. Otherwise,* **Con**(A) *is a* 3-*element chain. In general, all antilattices induced from* 3 × 3 *magic squares are congruence distributive.*

Proof. In the complementary case, any pair of distinct elements generates A, which thus is simple. Otherwise, a single nontrivial, proper congruence is given by [*a, b, c, g, h, i* | *d, e, f*]. □

The 4 × 4 *case*

We next consider antilattices induced from 4 × 4 magic squares storing 1 - 16. While just one 3×3 magic square stores 1 - 9 (with eight dihedral variations), 880 essentially distinct magic squares store 1 to 16. A list of all 880 squares was given by Bernard Frénicle de Bessy in a posthumous 1693 publication. A mathematical analysis was given in the 1983 paper of Dame Kathleen Ollerenshaw and Sir Hermann Bondi [**5**]. Thanks to the following observation, these 880 cases decompose into 220 classes of 4.

Lemma 3.6.7. *Given a* 4 × 4 *magic square* A*, let squares* B, C *and* D *be induced from* A *by simultaneous row and column permutations determined by* (2 3), (1 2)(3 4) *and* (1 3 4 2) *respectively. Then* A – D *are all magic squares, but none are dihedrally equivalent. Moreover all four squares induce the same antilattice.* □

Thus one can get by checking the leading array in each row of four squares in the Ollerenshaw-Bondi list. Among these, the nonsimple cases are easily spotted, thanks to a theorem about semimagic squares (all rows and columns add up to 34). In its statement, the ***index*** of a congruence μ counts its number of congruence classes.

Theorem 3.6.8. *If a semi-magic square* A *storing* 1 – 16 *induces a nonsimple antilattice* A, *then* A *has a maximal congruence* μ *of index* 2 *whose corresponding congruence class partition is either*

$$\pi_R = \{1-4,\ 13-16 \mid 5-12\}$$

or

$$\pi_C = \{1, 4, 5, 8, 9, 12, 13, 16 \mid 2, 3, 6, 7, 10, 11, 14, 15\}$$

where π_R *and* π_C *are outer/inner partitions splitting rows[columns]* 1 & 4 *against rows[columns]* 2 & 3 *in the standard array*.

Examples 3.6.2. Consider the following magic squares in the Ollerenshaw-Bondi listing:

$$(1)\begin{bmatrix}1 & 7 & 12 & 14\\ 10 & 16 & 3 & 5\\ 15 & 9 & 6 & 4\\ 8 & 2 & 13 & 11\end{bmatrix}\qquad (9)\begin{bmatrix}1 & 4 & 15 & 14\\ 13 & 16 & 3 & 2\\ 12 & 9 & 6 & 7\\ 8 & 5 & 10 & 11\end{bmatrix}\qquad (25)\begin{bmatrix}1 & 16 & 6 & 11\\ 13 & 4 & 10 & 7\\ 12 & 5 & 15 & 2\\ 8 & 9 & 3 & 14\end{bmatrix}.$$

Square (1) induces a simple antilattice because 1 - 4 lie in distinct rows and columns (denying π_R) and 1, 5, 9, 13 lie in distinct rows and columns (denying π_C). By contrast both π_R and π_C work for (9), while π_C works, but not π_R, for square (25). Thus both (9) and (25) are nonsimple. (*Caveat.* In the Ollerenshaw-Bondi list, the arrays actually store 0 – 15, instead of 1 – 16, and do so in base 4 notation.) □

A survey of the 220 leading squares in the Ollerenshaw-Bondi list yields, upon applying the test of Theorem 3.6.8, the following statistic:

Corollary 3.6.9. *Of the* 880 *magic squares storing* 1 – 16, 416 *cases yield simple antilattices and* 464 *yield nonsimple antilattices, giving a breakdown of* 47.27% to 52.73%. □

Proof of Theorem 3.6.8. (All arrays in this proof are identified to within row and column permutations.) To begin, all possible cartesian partitions of a 4×4 square with distinct elements can only have indices among the following: 1, 2, 3, 4. 6, 8, 9, 12, 16. Thus if A is nonsimple, the index $|\mu|$ of its maximal proper congruence μ must lie among 2, 3, 4, 6, 8, 9, 12.

If $|\mu| = 2$, then any cartesian partition of the standard array is one of four cases: one row and three rows, or one column and three columns, or two rows and two rows, or two columns and two columns. The first two cases are impossible when A is included, as no row or column in the standard array has the magic sum of 34. In the final cases, the sum of each pair of rows or columns must be 2 × 34 = 68. This occurs only for {row 1 ∪ row 4 | row 2 ∪ row 3} or {column 1 ∪ column 4 | column 2 ∪ column 3}, just as stated.

$|\mu| = 3$ is impossible in the antilattice context since that would mean a row or column in the standard array would sum to 34 (because it would appear as a row or column in A), which is impossible.

$|\mu| = 4$ is possible. But in this case, the quotient algebra A/μ would have order 4 and thus be nonsimple by Proposition 3.2.2. Hence μ was not really maximal after all.

$|\mu| = 6$ is also possible with the cartesian partition of the standard array having either template $\begin{bmatrix}[1\times 2] & [1\times 2]\\ [1\times 2] & [1\times 2]\\ [2\times 2] & [2\times 2]\end{bmatrix}$ or template $\begin{bmatrix}[1\times 1] & [1\times 3]\\ [1\times 1] & [1\times 3]\\ [2\times 1] & [2\times 3]\end{bmatrix}$ or a transpose. In all these cases, the two bottom cells would still be adjacent in the cartesian partition of the magic square, thus inducing a congruence of index two.

Similarly for the remaining indices of 8, 9 and 12, congruences with these indices must be refined by a properly larger congruence of index 2, thus returning us to the $|\mu| = 2$ case. □

Examples 3.6.3. Pandiagonal magic squares first appear in the 4×4 case. In considering pandiagonal magic squares in general, two such squares of the same dimension are ***equivalent*** if either is obtained from the other using a combination of dihedral operations, along with cyclic permutations of the rows and/or the columns. Any given pandiagonal square storing 1 through 16 is thus one of $8 \times 4 \times 4 = 128$ equivalent pandiagonal magic squares. In the 4×4 case, 48 dihedrally distinct pandiagonal squares exist, all being equivalent to exactly one of the following three pandiagonal magic squares:

$$\begin{bmatrix} 1 & 8 & 10 & 15 \\ 12 & 13 & 3 & 6 \\ 7 & 2 & 16 & 9 \\ 14 & 11 & 5 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 11 & 14 \\ 12 & 13 & 2 & 7 \\ 6 & 3 & 16 & 9 \\ 15 & 10 & 5 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 14 & 4 & 15 \\ 8 & 11 & 5 & 10 \\ 13 & 2 & 16 & 3 \\ 12 & 7 & 9 & 6 \end{bmatrix}.$$

The first two pandiagonal magic arrays induce simple antilattices; but not the third array. □

Two classic constructions

In 1693, Simon de la Loubère gave the following rule for constructing magic squares for any odd order n. Again we will work with base n ordered pairs, often in abbreviated notation.

De la Loubère's Rule. *Place* 00 *in the middle of the first row. In ascending (broken) diagonal fashion place in order the remaining* 01 *through* 0, n–1. *Beneath* 0, n–1 *place* 10 *and again in ascending diagonal fashion place* 11 *through* 1, n–1. *Beneath* 1, n–1 *place* 20, *and repeat the process until an entire n×n array is filled. The resulting array is a magic square of odd order n storing* 0 *through* n^2–1 *in base n.*

The array to the right is the $n = 5$ case in base 5 notation storing 0 – 24.

$$(\vee) \begin{bmatrix} 00 & 01 & 02 & 03 & 04 \\ 10 & 11 & 12 & 13 & 14 \\ 20 & 21 & 22 & 23 & 24 \\ 30 & 31 & 32 & 33 & 34 \\ 40 & 41 & 42 & 43 & 44 \end{bmatrix} \qquad (\wedge) \begin{bmatrix} 31 & 43 & 00 & 12 & 24 \\ 42 & 04 & 11 & 23 & 30 \\ 03 & 10 & 22 & 34 & 41 \\ 14 & 21 & 33 & 40 & 02 \\ 20 & 32 & 44 & 01 & 13 \end{bmatrix}$$

Theorem 3.6.10. *The magic squares of odd order n given by De la Loubère's rule along with their corresponding standard array yield simple antilattices precisely when n is prime.*

Proof. Suppose that $x\ \theta\ y$ where $x \neq y$. By Lemma 3.6.2, we may assume that x and y either lie in a common row or in a common column of the $\vee$-array. If x and y are in the same column, then their meets yield θ-related elements u and v in distinct columns of the $\vee$-array. From $\{u, v, u\vee v, v\vee u\}$ we gain a pair of distinct θ-related elements lying in a common row of the $\vee$-array.

Thus at the outset we may assume that $x\ \theta\ y$ with x and y distinct elements in a common row of the $\vee$-array. If the order of the magic square is $p = 2n + 1$ (and the order of the algebra is p^2), then $n0 \vee x$ and $n0 \vee y$ must be distinct θ-related elements in the middle row of the $\vee$-array, say ni and nj. But ni and nj are also lie the main ascending diagonal of the $\wedge$-array and from them we can generate *via* $f(\mathrm{X}, \mathrm{Y}) = ni \vee (\mathrm{X} \wedge \mathrm{Y})$, all $n, i\pm mk$ where $k = j - i$. If p is prime, the main ascending diagonal in the $\wedge$-array must lie in a common θ-class. Since this diagonal generates the entire algebra, $\theta = \nabla$.

If p is composite, say $p = ab$ with $1 < a, b < p$, then define an equivalence α by $ij\ \alpha\ kl$ if both $i \equiv k \pmod a$ and $j \equiv l \pmod a$. That α is a $\vee$-congruence is clear. In the case of $\wedge$, observe that in the $\wedge$-array any horizontal or vertical displacement of a positions from any starting position yields an α-related element. Conversely, any pair of α-related elements are connected by a sequence of such displacements. Thus given $x\ \alpha\ y$ and $u\ \alpha\ v$, the $\wedge$-columns of $x\wedge u$ and $y\wedge v$, being the $\wedge$-columns of u and v, differ in their position by a multiple of a. Likewise the $\wedge$-rows of $x \wedge u$ and $y \wedge v$, being the $\wedge$-rows of x and y, differ in their position by a multiple of a. It follows that $x \wedge u\ \alpha\ y \wedge v$ so that α is a $\wedge$–congruence also. Clearly α is neither Δ or ∇. ☐

A variation of the above rule had been given previously by Claude Gaspar Bachet de Méziriac (who in 1621 published the famous edition of Diophantus' *Arithmetica*).

Bachet de Méziriac's Rule. *Place* 00 *directly above the middle position of an* $n{\times}n$ array. *In ascending* (*broken*) *diagonal fashion place in order*, *the remaining* 01 *through* 0, n–1. *Next, place* 10 two rows directly above 0, n–1. *In ascending* (*broken*) *diagonal fashion place, in order*, 11 *through* 1, n–1. *Next, place* 20 *two rows directly above* 1, n–1. *Repeat the process until an entire* $n{\times}n$ *array is filled to produce a magic square storing* 0 *through* n^2–1 *in base* n.

Theorem 3.6.11. *The magic squares of odd order given by Bachet de Méziriac's rule induce simple antilattices precisely when the order is prime*.

$$\begin{bmatrix} 00 & 01 & 02 & 03 & 04 & 05 & 06 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 20 & 21 & 22 & 23 & 24 & 25 & 26 \\ 30 & 31 & 32 & 33 & 34 & 35 & 36 \\ 40 & 41 & 42 & 43 & 44 & 45 & 46 \\ 50 & 51 & 52 & 53 & 54 & 55 & 56 \\ 60 & 61 & 62 & 63 & 64 & 65 & 66 \end{bmatrix} \qquad \begin{bmatrix} 63 & 20 & 54 & 11 & 45 & 02 & 36 \\ 26 & 53 & 10 & 44 & 01 & 35 & 62 \\ 52 & 16 & 43 & 00 & 34 & 61 & 25 \\ 15 & 42 & 06 & 33 & 60 & 24 & 51 \\ 41 & 05 & 32 & 66 & 23 & 50 & 14 \\ 04 & 31 & 65 & 22 & 56 & 13 & 40 \\ 30 & 64 & 21 & 55 & 12 & 46 & 03 \end{bmatrix}$$

(Standard array for 0 – 48, base 7) (de Méziriac array for 0 – 48. base 7)

Proof. Suppose first that p is prime with $p = 2n + 1$ and let θ be a congruence with $x\ \theta\ y$ where $x \neq y$. As with the previous theorem, things may be reduced to the case where $x = ni$ and $y = nj$ in the middle row of the $\vee$-array and the ascending diagonal of the $\wedge$-array. If $F(X, Y) = n0 \vee (X \wedge Y)$, then $F(ni, nj) = n\ (i + j)/2$ in the same θ-class as $ni = n0 \vee ni$. (Here $(i + j)/2$ is calculated in $\mathbb{Z}_p$.) Since $n0, n1, \ldots, n\ p{-}1$ generates the algebra, simplicity follows if we can show that from ni and nj one can F-generate the entire n^{th} $\vee$-row. This is equivalent to showing that from any two $i \neq j$ in $\mathbb{Z}_p$, all of $\mathbb{Z}_p$ is generated via the function $f(x, y) = (x + y)/2$. Let S be the set of all numbers in $\mathbb{Z}_p$ thus generated. If $0 \in S$, then S is closed under addition and thus is a nontrivial subgroup of $\mathbb{Z}_p$ which forces $S = \mathbb{Z}_p$. Indeed $f(0, (x + y)/2) = (x + y)/4$, $f(0, (x + y)/4) = (x + y)/8$, etc. Hence all $(x + y)/2^n$ lie in S. Since some power of 2 equals 1 in $\mathbb{Z}_p$, we get $x + y \in S$ so that S is as claimed. Otherwise, suppose $0 \notin S$. From $f(x + k, y + k) = f(x, y) + k$, the general case can be shifted to the 0-case, so that no matter what pair i, j is given, the f-generated set is all of $\mathbb{Z}_p$.

If p is composite, say $p = ab$ with $1 < a, b < p$, then define an equivalence α by $ij\ \alpha\ kl$ if both $i \equiv k \pmod{a}$ and $j \equiv l \pmod{a}$. The argument that α is a congruence is identical to that in the case of de la Loubère's rule. ☐

A number of further examples of magic squares that induce simple antilattices are given in Leech [2005b].

Historical remarks

Nearly all results in the first five sections appeared in a 2002 paper by Gratiela Laslo and Jonathan Leech that studied congruences on noncommutative lattices. The paper was written while Laslo was working her dissertation at the University of Cluj-Napoca; several results are from that dissertation. The material on isotopy is of more recent vintage, and was developed from remarks in an email from Michael Kinyon. It appears here for the first time. The final section on recreational mathematics and antilattices appeared Leech's 2005 paper.

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IV: SKEW BOOLEAN ALGEBRAS

In this chapter such fundamental concepts as normal bands, skew lattices and generalized Boolean algebras are integrated. We have already seen in Theorem 2.3.7 that maximal normal bands in rings form strongly distributive skew lattices that are characterized by

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad (a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c).$$

More is true: any such maximal normal band S forms a noncommutative variant of a generalized Boolean algebra called a *skew Boolean algebra*. In particular S possesses a zero element 0 for which $0 \vee a = a = a \vee 0$ and $0 \wedge a = 0 = a \wedge 0$ hold; S is also closed under a difference operation $a \setminus b$ given by $a - aba$. Such a system (S; $\vee$, $\wedge$, $\setminus$, 0) satisfies many identities, a subset of which provides a defining set of identities for a skew Boolean algebra.

The strongly distributive skew lattices of partial functions encountered Section 2.6 give archetypal examples: to $\vee$ and $\wedge$ we adjoin the empty partial function $\varnothing$ as the zero and the difference $f \setminus g$ given by the restriction of f to $\mathrm{F} \setminus \mathrm{F} \cap \mathrm{G}$, where F and G are the respective supports of f and g (their sets of actual inputs). In this chapter we will work with the right-handed case $\mathcal{P}_{\mathrm{R}}(\mathrm{A}, \mathrm{B})$. The skew Boolean version (with the expanded signature) is denoted by $\mathcal{P}(\mathrm{A}, \mathrm{B})$.

In addition to the above classes of examples, every primitive skew lattice with a singleton lower class $\mathrm{A} > \{0\}$ is strongly distributive and has a zero element 0; moreover a difference operation is given by the simple rule: $x \setminus y = x$, if $y = 0$, and 0 otherwise. These primitive algebras play a basic role in the theory, doing for skew Boolean algebras what the Boolean algebra **2** and its isomorphic copes do for (generalized) Boolean algebras.

In the Section 4.1, skew Boolean algebras (S; $\vee$, $\wedge$, $\setminus$, 0) are formally defined as structural enhancements of strongly distributive skew lattices. Variants of some of the familiar results about generalized Boolean algebras are then proved. In particular, skew Boolean algebras are shown to be subdirect products of primitive skew Boolean algebras; moreover every skew Boolean algebra can be embedded into a power of **5**, a 5-element primitive algebra. (See Corollaries 4.1.6 and 4.1.7.) Not surprisingly, every right-handed skew Boolean algebra can be embedded in some partial function algebra $\mathcal{P}(\mathrm{A}, \mathrm{B})$.

In Section 4.2, special attention is given to classifying finite algebras, and in particular, to classifying finitely generated (and thus finite) free skew Boolean algebras. In the process, not only do we look at an important class of examples, we also engage in some of the basic algebraic procedures of skew Boolean algebras. A fundamental concept in this section is that of an *orthosum*, both an orthosum of elements and an orthosum of subalgebras. The main results are Theorems 4.2.2 and 4.2.6, with the latter describing the structure of finitely generated free algebras.

A skew Boolean algebra is, of course, just a strongly distributive skew lattice with added operations and axiomatic constraints. While section 4.3 focuses mostly on the relation between skew Boolean algebras and strongly distributive skew lattices, this is also a convenient context to consider alternative characterizations of skew Boolean algebras. Theorem 4.3.5 gives an independent set of five identities that characterizes all right-handed skew Boolean algebras.

In the Section 4.4 we look at skew Boolean algebras with finite intersections $\cap$, that is, algebras for which the natural partial order $\geq$ has meets that are called *intersections* and denoted by $x \cap y$. For a partial function algebra $\mathcal{P}(\mathrm{A}, \mathrm{B})$, $f \cap g$ is the usual intersection of partial functions f and g viewed as subsets of $\mathrm{A} \times \mathrm{B}$. All skew Boolean algebras S for which $\mathrm{S}/\mathcal{D}$ is finite have intersections as do, more generally, all complete skew Boolean algebras. Indeed, having at least finite intersections is often the rule. Here, similarities with Boolean algebras are much tighter: if $\cap$ is included in the signature, then all congruences are determined by their kernel ideals – the congruence classes of 0 – and thus their congruence lattices are distributive (Theorem 4.4.8). We also show that free skew Boolean algebras have finite intersections (Theorem 4.4.18). The lattice of all subvarieties of these algebras is described in the section's concluding Theorem 4.4.24.

Sections 4.1 – 4.4 give a structural hierarchy lying at the core of this chapter's subject.

skew Boolean algebras
with finite intersections

⋮

skew Boolean algebras

⋮

strongly distributive
skew lattices

In Section 4.5 we study a functor $\boldsymbol{\omega}$ that object-wise takes a generalized Boolean algebra B and constructs a skew Boolean cover of B, that is, a skew Boolean algebra S_{B} such that (1) $\mathrm{S}_{\mathrm{B}}/\mathcal{D} \cong \mathrm{B}$ and (2) S_{B} has a trivial center $\{0\}$. Its underlying set $\omega(\mathrm{B})$ consists of all naturally ordered pairs in B, $\{(b, b') \mid b \geq b'\}$. Initially, $\omega(\mathrm{B})$ has obvious join and meet operations:

$$(b, b') \vee (c, c') = (b \vee c, b' \vee c') \text{ and } (b, b') \wedge (c, c') = (b \wedge c, b' \wedge c').$$

We "twist" these outcomes to produce noncommutative variants of $\vee$ and $\wedge$. This provides another class of skew Boolean algebras that have finite intersections as well as other properties that we investigate. The process also gives us a "workout" of much that was discussed in previous sections.

In Chapter 6, where we return to skew lattices in rings, special attention is given to skew Boolean algebras that occur when the set of idempotents in a ring is closed under multiplication, in which case the idempotents form such an algebra. In particular we will examine how this occurrence affects the structure of the ring. All this occurs in Sections 6.4 to 6.7.

In the seventh and final chapter, we examine further the role of skew Boolean algebras in universal algebra, in particular in the study of what might be termed "generalized Boolean phenomena," a topic of continuing interest in universal algebra. In particular we will present a number of results by Robert Bignall and his student, Matthew Spinks. Both gentlemen have made significant contributions in this area.

This chapter concludes with a discussion of historical aspects of this topic and references.

4.1 Skew Boolean algebras

We have seen that any maximal normal band S in a ring R is a skew lattice in R under multiplication and the cubic join ∇. Indeed, every such band is the full set of idempotents in the subring it generates in R. There is more.

To begin, if **E**(R) is commutative, then **E**(R) is a generalized Boolean lattice since for each e in **E**(R), the principal lattice ideal $\lceil e \rceil = \{f \mid f \leq e\}$ is a Boolean lattice. Indeed, if $e \geq f$ in **E**(R), then $e - f \in$ **E**(R) also; moreover $e \geq e - f$ with $f \wedge (e - f) = 0$ and $f \vee (e - f) = e$. All this is true even when **E**(R) is not commutative, provided it forms a band under multiplication. Since **E**(R) is then a normal skew lattice by Theorem 2.3.7, each $\lceil e \rceil$ is a sublattice; what is more, for all $f \in \lceil e \rceil$ one has $e - f \in \lceil e \rceil$ with $f \vee (e - f) = e$ and $f \wedge (e - f) = 0$. This leads us to a definition:

A ***skew Boolean algebra*** is an algebra (S; $\vee$, $\wedge$, \, 0) such that

(i) (S; $\vee$, $\wedge$, 0) is a strongly distributive skew lattice with 0 and

(ii) \ is a binary operation satisfying $(e \wedge f \wedge e) \vee (e \backslash f) = e$ and $(e \wedge f \wedge e) \wedge (e \backslash f) = 0$.

By Theorem 2.3.4, (i) is equivalent to (S; $\vee$, $\wedge$) being distributive, symmetric and normal and having the 0-identities hold. (i) and (ii) together imply $e \wedge f \wedge e$ and $e \backslash f$ commute. They also imply that each $\lceil e \rceil$ is a Boolean sublattice of S with $e \backslash f$ being the unique complement of $e \wedge f \wedge e$ in $\lceil e \rceil$. Conversely given (i), if each [e] forms a Boolean lattice, then (ii) follows. These observations give:

Theorem 4.1.1. *Skew Boolean algebras form a variety; moreover, every congruence on the skew lattice reduct of a skew Boolean algebra is also a skew Boolean algebra congruence. In particular, the Green's equivalences are all skew Boolean algebra congruences.* ☐

This variety will be denoted by $\mathcal{SBA}$. The broader perspective of skew lattices yields:

Theorem 4.1.2. *A normal, symmetric skew lattice with* 0 *forms a skew Boolean algebra if and only if its maximal lattice image* S/$\mathcal{D}$ *is a generalized Boolean lattice, in which case* \ *is implicitly determined by* (ii) *above*.

Proof. Let S be a normal, symmetric skew lattice with 0 such that S/$\mathcal{D}$ is a generalized Boolean lattice. Then S is distributive by Theorem 2.3.2. Since each $\lceil e \rceil$ is a Boolean lattice, \ is implicitly determined by (ii). The converse is clear. □

Examples 4.1.1. The following skew lattices are examples of skew Boolean algebras:

(a) Maximal normal bands in rings form skew Boolean algebras upon setting $e \wedge f = ef$, $e \vee f = e \nabla f$ and $e \backslash f = e - efe$. (See Cvetko-Vah and Leech, [2011] and [2012].)

(b) Any partial function set $\mathcal{P}$(A, B) with ∨ and ∧ defined as before and with 0 = ∅ and $f \backslash g = f|\,F \backslash G$ where F and G are the supports in A for f and g respectively

(c) More generally, any ring of partial functions provided the underlying lattice of subsets of A is a generalized Boolean lattice and contains the empty set.

(d) Given a rectangular skew lattice D, a primitive skew Boolean algebra is formed by D^0 upon setting $x \backslash y = x$ if $y = 0$, but 0 otherwise. □

Theorem 4.1.3. *Skew Boolean algebras satisfy*:

(iii) $e \backslash f = e \backslash (e \wedge f \wedge e)$.

(iv) $e \backslash (f \vee g) = (e \backslash f) \wedge (e \backslash g)$ *and* $e \backslash (f \wedge g) = (e \backslash f) \vee (e \backslash g)$.

(v) $e \backslash (e \backslash f) = e \wedge f \wedge e$.

Proof. These identities are immediate consequences of each inner ideal $\lceil e \rceil$ being a Boolean lattice. □

Given a skew lattice S, an ***ideal*** of S is a subset I of S such that given x, y in I, and z in S $x \vee y$, $z \wedge x$ and $x \wedge z$ are in I. Given any element a in a skew lattice S, the ***principal ideal*** of a is the set $(a) = \{x \in S \mid x \preceq a\}$. Clearly, $x \in (a)$ if and only if $x \preceq b$ for all $b \in \mathcal{D}_a$. If S has a zero element 0, the ***annihilator*** of a is the set $ann(a) = \{x \in S \mid x \wedge a = 0\}$. Due to Theorem 2.1.2, $x \in ann(a)$ if and only if $x \wedge b = 0 = b \wedge x$ for all $b \in \mathcal{D}_a$. If the skew lattice is distributive, then $ann(a)$ is easily seen to be an ideal. Clearly (a) and $ann(a)$ can also be parameterized by the relevant $\mathcal{D}$-class $A = \mathcal{D}_a$ as (A) and $ann(A)$ respectively since due to Theorem 2.1.2, any b in $\mathcal{D}_a$ induces the same pair of sets,. When S is a skew Boolean algebra, the situation can be sharpened to give a decomposition of primary importance in understanding these algebras.

Theorem 4.1.4. *Given a* $\mathcal{D}$*-class A of a skew Boolean algebra* S*, then:*

(i) *Both* (A) *and* $ann(A)$ *are ideals of* S.

(ii) *All elements of* (A) *commute with all elements of* $ann(A)$.

(iii) *In particular, for all* $u \in (A)$ *and all* $v \in ann(A)$, $u \wedge v = 0 = v \wedge u$.

(iv) *The map* $\mu\colon (A) \times ann(A) \to S$ *defined by* $\mu(e_1, e_2) = e_1 \vee e_2$ *is an isomorphism.*

Proof. As stated above, (i) holds for all distributive skew lattices with zero elements. From $v \wedge b = 0 = b \wedge v$ for all $v \in ann(A)$ and all $b \in \mathcal{D}_a$, (iii) follows by Theorem 2.12. Next, (ii) follows from (iii) and symmetry. Given $(e_1, e_2), (f_1, f_2) \in (A) \times ann(A)$, $(e_1 \vee e_2) \vee (f_1 \vee f_2) = (e_1 \vee f_1) \vee (e_2 \vee f_2)$ follows from (ii). By Theorem 2.3.4, $(e_1 \vee e_2) \wedge (f_1 \vee f_2)$ expands to

$$(e_1 \wedge f_1) \vee (e_1 \wedge f_2) \vee (e_2 \wedge f_1) \vee (e_2 \wedge f_2) = (e_1 \wedge f_1) \vee (e_2 \wedge f_2).$$

Hence μ is at least a homomorphism. For any $s \in S$, the decomposition $s = (s \wedge a \wedge s) \vee (s \setminus a)$ represents s as the join of an element in S_1 with an element in S_2. Hence μ is also "onto". Finally, suppose that $(e_1 \vee e_2) = (f_1 \vee f_2)$ for e_1, f_1 in S_1 and e_2, f_2 in S_2. Letting u be this common join we have $u \geq e_1, f_1, e_2$ and f_2. Thus

$$e_1 = e_1 \wedge u = e_1 \wedge (f_1 \vee f_2) = (e_1 \wedge f_1) \vee (e_1 \wedge f_2) = (e_1 \wedge f_1) \vee 0 = e_1 \wedge f_1$$

and in similar fashion $f_1 = e_1 \wedge f_1$. Likewise $e_2 = f_2$ so that μ is indeed an isomorphism. □

Corollary 4.1.5. *Every skew Boolean algebra* S *with a finite maximal lattice image is isomorphic to a product of primitive skew Boolean algebras that are determined to within isomorphism by its minimal non-*0 *$\mathcal{D}$-classes*. □

Recall that **2** is the Boolean lattice $\{1 > 0\}$, $\mathbf{3_L}$ is the left-handed primitive skew Boolean algebra $\{1 \mathcal{L} 2 > 0\}$, $\mathbf{3_R}$ is its right-handed variant and **5** is the fibered product, $\mathbf{3_L} \times_{\mathbf{2}} \mathbf{3_R}$.

Corollary 4.1.6. *The nontrivial directly irreducible skew Boolean algebras are the primitive algebras. The nontrivial subdirectly irreducible skew Boolean algebras are* **2**, $\mathbf{3_L}$ *and* $\mathbf{3_R}$. *Thus every skew Boolean algebra is a subdirect product of copies of* **2**, $\mathbf{3_L}$ *and* $\mathbf{3_R}$. □

Proof. The first statement is clear by Theorem 4.1.4. A nontrivial, subdirectly irreducible algebra is thus primitive, and must be either left-or right handed, thanks to the factorization of Theorem 2.1.5. If both, it is a copy of **2**. Otherwise, it is a copy of $\mathbf{3_L}$ or $\mathbf{3_R}$ by Theorem 2.6.12. □

Corollary 4.1.7. *Every skew Boolean algebra can be embedded in a power of* **5**. *Every left-handed (right-handed) skew Boolean algebra can be embedded in a power of* $\mathbf{3_L}$ *(of* $\mathbf{3_R}$*). Alternatively, every right-handed skew Boolean algebra can be embedded in some algebra of partial functions with codomain* $\{1, 2\}$. □

Corollary 4.1.8. *A (quasi-)identity in* ∨, ∧ *and* \ *holds for all skew Boolean algebras if and only if it holds on* **5**. *It holds for all left-handed (right-handed) skew Boolean algebras if and only if it holds on* $\mathbf{3_L}$ *(or* $\mathbf{3_R}$*). The question of when a (quasi-)identity holds in any of these varieties is thus decidable*. □

The above results reveal the following simple lattice. Here $\langle \mathbf{A} \rangle$ denotes the subvariety generated by the algebra **A**.

Theorem 4.1.9. *The lattice of all subvarieties of skew Boolean algebras is given by*:

$$\begin{tikzcd}[column sep=small] & \langle \mathbf{5} \rangle & \\ \langle \mathbf{3_L} \rangle \arrow[ur, dotted, dash] & & \langle \mathbf{3_R} \rangle \arrow[ul, dotted, dash] \\ & \langle \mathbf{2} \rangle \arrow[ul, dotted, dash] \arrow[ur, dotted, dash] & \\ & \langle \mathbf{1} \rangle \arrow[u] & \end{tikzcd}$$

where **1** *is the subvariety of trivial algebras,* $\langle \mathbf{5} \rangle$ *is the variety of all skew Boolean algebras, and* $\langle \mathbf{3_L} \rangle$, $\langle \mathbf{3_R} \rangle$ *and* $\langle \mathbf{2} \rangle$ *are the respective subvarieties of all left- and right-handed skew Boolean algebras and generalized Boolean algebras*. ☐

Recall that an algebra is ***locally finite*** if every finite subset generates a finite subalgebra. A variety of algebras is called locally finite if each of its algebras are locally finite.

Theorem 4.1.10. *Skew Boolean algebras are locally finite*.

Proof. Let a skew Boolean algebra S be generated from a finite set X. Clearly only finitely many distinct functions from X to **5** exist. Let $\varphi = \prod_{i\in I}\varphi_i : S \to \mathbf{5}^I$ be an embedding of S into a power of **5**. Since each φ_i is determined by its behavior on X, only finitely many of the φ_i's can be distinct, in which case only finitely many are needed for an embedding, since in a family of functions that collectively separate points in their common domain, repetition is superfluous. Put otherwise, we may assume that |I| is finite, in which case the embedding gives $|S| \leq 5^{|I|}$. ☐

Complete algebras

We now turn to issues of completeness. A symmetric skew lattice is ***join*** [***meet***] ***complete*** if all commuting subsets have suprema [minima] in the natural partial ordering. It is ***complete*** if it is both join and meet complete. By a ***maximal element*** in a skew lattice is meant any element in the maximal $\mathcal{D}$-class of S, should the latter exist. The *unique* maximal element, if it exists, is essentially the constant 1.

Lemma 4.1.11. *Join complete, symmetric skew lattices have maximal* $\mathcal{D}$*-classes*. *In particular, every join complete symmetric, normal skew lattice* S *has lattice sections, all given as* $m \wedge S \wedge m$ *for some maximal element* m *of* S.

Proof. For then every maximal totally ≤-ordered subset must contain a maximal element m that in turn is ≤-maximal in the skew lattice. The rest is clear. ☐

A nontrivial skew Boolean algebra is ***completely reducible*** if it is isomorphic to a product of primitive skew Boolean algebras. Since primitive algebras are trivially complete, so is any completely reducible algebra. A ***primitive class*** in a skew Boolean algebra is any $\mathcal{D}$-class lying directly above 0. An ***atom*** or ***primitive element*** is any element in a primitive class. A skew Boolean algebra is ***atomic*** if for every nonzero $x \in$S an atom $a \in$S exists such that $a \leq x$.

Theorem 4.1.12. *A skew Boolean algebra is completely reducible if and only if it is complete and atomic*.

Proof. Completely reducible algebras are clearly complete and atomic. For the converse, let S be a complete atomic skew Boolean algebra. For any $x \in$S, the set $x \wedge S \wedge x$ is a complete atomic Boolean algebra with (unique) maximal element x. Thus each $x > 0$ in S is the supremum of its underlying atoms, all of which commute, with distinct sets of commuting atoms giving distinct suprema. Next, let $\{X_j\}$ be an indexed collection of all primitive classes of S and let $P_j = X_j^0$ be the corresponding primitive skew Boolean algebras. Define σ: $\prod_j P_j \rightarrow S$ by setting $\sigma[\langle e_j \rangle] = sup\langle e_j \rangle$. By our remarks, σ is at least a bijection.

Let $\langle e_j \rangle, \langle f_j \rangle \in \prod_j P_j$ be given with $\sup_j\langle e_j \rangle = e$ and $\sup_j\langle f_j \rangle = f$. Then $e \vee f \geq e_j \vee f_j$ for each j. Indeed,

$$(e\vee f) \wedge (e_j \vee f_j) = (e\wedge e_j) \vee (f\wedge e_j) \vee (e\wedge f_j) \vee (f\wedge f_j) = e_j \vee (f\wedge e_j) \vee (e\wedge f_j) \vee f_j = e_j \vee (f\wedge e_j) \vee f_j.$$

Case 1. Both $e_j, f_j \neq 0$. Here $e_j \vee (f\wedge e_j) \vee f_j = e_j \vee f_j$ since all $\vee$-factors are $\mathcal{D}$-equivalent.

Case 2. $f_j = 0$. Then $f\wedge e_j = 0$ since its image in S/$\mathcal{D}$ is 0. Hence $f_j \vee (f\wedge e_j) \vee f_j = e_j = e_j \vee f_j$.

Case 3. $e_j = 0$. Here $e_j \vee (f\wedge e_j) \vee f_j = f_j = e_j \vee f_j$ again.

Similarly, $(e_j \vee f_j) \wedge (e\vee f) = (e_j \vee f_j)$ so that $e\vee f \geq e_j \vee f_j$ is verified. Each primitive join $e_j \vee f_j$ is either an atom of S or 0. Claim $e\vee f = \sup(e_j \vee f_j)$. Clearly $e\vee f \geq \sup(e_j \vee f_j)$. If the inequality is strict, this means that $e\vee f \geq$ an atom g_k that does not lie below $\sup(e_j \vee f_j)$. Thus for this index k, $e_k = 0 = f_k$. But this yields a contradiction in the Boolean lattice S/$\mathcal{D}$ where $\mathcal{D}_{g_k}$ is an atom and neither $\mathcal{D}_e \geq \mathcal{D}_{g_k}$ nor $\mathcal{D}_f \geq \mathcal{D}_{g_k}$, yet $\mathcal{D}_e \vee \mathcal{D}_f \geq \mathcal{D}_{g_k}$. Thus $e\vee y = \sup(e_j \vee f_j)$ as claimed. Next, by mid-commutativity, $(e\wedge f) \wedge (e_j \wedge f_j) = (e\wedge e_j) \wedge (f\wedge f_j) = e_j \wedge f_j$, and similarly $(e_j \wedge f_j) \wedge (e\wedge f) = e_j \wedge f_j$. Thus $(e\wedge f) \geq (e_j \wedge f_j)$ for each j and hence $(e\wedge f) \geq \sup(e_j \wedge f_j)$. An argument similar to that for the join guarantees that $(e\wedge f) = \sup(e_j \wedge f_j)$. Thus σ: $\prod_j P_j \rightarrow S$ is an isomorphism of skew lattices. Since \ is implicitly determined by the skew lattice structure, σ is an isomorphism and S is seen to be completely reducible. □

Theorem 4.1.13. *Complete skew Boolean algebras satisfy the identities*:

1) $e \wedge \sup(f_i) = \sup(e \wedge f_i)$.
2) $e \vee \inf(f_i) = \inf(e \vee f_i)$.
3) $e \setminus \sup(f_i) = \inf(e \setminus f_i)$.
4) $e \setminus \inf(f_i) = \sup(e \setminus f_i)$.

Proof. In each case the obvious inequality ($\leq$ or $\geq$) becomes equality in its complete maximal Boolean algebra image. Thus one already has equality in the skew Boolean algebra. □

Given complete skew Boolean algebras S and T, any homomorphism f: S → T sends commutative subsets to commutative subsets. If f also preserves suprema and infima, then f is a ***complete homomorphism*** of complete skew Boolean algebras. We state the following fuller description of the $\mathcal{P}$(A, B) process.

Theorem 4.1.14. *Given maps* $\alpha: A' \to A$ *and* $\beta: B \to B'$, $\mathcal{P}(\alpha, \beta): \mathcal{P}(A, B) \to \mathcal{P}(A', B')$ *defined by* $\mathcal{P}(\alpha, \beta)f = \beta f \alpha$, *for all* $f \in \mathcal{P}(A, B)$, *is a homomorphism of complete Boolean skew algebras. Moreover, if* **Set** *and* **CSBA** *denote the respective categories of sets and complete skew Boolean algebras, then* $\mathcal{P}$: **Set** × **Set** → **CSBA** *is a bifunctor from the category of sets to the category of complete skew Boolean algebras.*

4.2 Finiteness, orthosums and free algebras

We recast the assertions of Theorem 4.1.4 and Corollary 4.1.5 in more detail for the case where $S/\mathcal{D}$ is finite and especially when the algebra S itself is finite. It will be helpful to explore things in a slightly more general context, proceeding as follows. Two element a and b in a skew Boolean algebra are ***orthogonal*** if $a \wedge b = 0$ (and thus $b \wedge a = 0$ and also $a \vee b = b \vee a$). A finite set of elements $\{a_1, \dots, a_n\}$ is an ***orthogonal set*** if the a_i are pairwise orthogonal, so that

$$a_1 \vee \dots \vee a_n = a_{\sigma 1} \vee \dots \vee a_{\sigma n}$$

for all permutations σ on $\{1, 2, \dots, n\}$. In this situation, $a_1 \vee \dots \vee a_n$ is denoted by $a_1 + \dots + a_n$ or $\sum_1^n a_i$. (Indeed, such notation will assume orthogonality.) Such a sum is referred to as an ***orthogonal sum***, or an ***orthosum*** for short.

A family of $\mathcal{D}$-classes $\{D_1, \dots, D_r\}$ is orthogonal when elements from distinct classes are orthogonal. For this it is sufficient that some transversal set $\{d_1, \dots, d_r\}$ is orthogonal. When this occurs, *the D_i are the primitive $\mathcal{D}$-classes in the subalgebra they generate*. In general:

Proposition 4.2.1. *Given an orthogonal family of nonzero $\mathcal{D}$-classes $\{D_1, \dots, D_r\}$ and two orthosums $a_1 + \dots + a_r$ and $b_1 + \dots + b_r$ where $a_i, b_i \in D_i$:*

i) $(a_1 + \dots + a_r) \vee (b_1 + \dots + b_r) = (a_1 \vee b_1) + (a_2 \vee b_2) + \dots + (a_r \vee b_r)$;

ii) $(a_1 + \dots + a_r) \wedge (b_1 + \dots + b_r) = (a_1 \wedge b_1) + (a_2 \wedge b_2) + \dots + (a_r \wedge b_r)$;

iii) $(a_1 + \dots + a_r) \setminus (b_1 + \dots + b_r) = (a_1 \setminus b_1) + (a_2 \setminus b_2) + \dots + (a_r \setminus b_r)$;

iv) $a_1 + \dots + a_r = b_1 + \dots + b_r$ *iff* $a_1 = b_1, a_2 = b_2, \dots,$ *and* $a_r = b_r$.

Indeed (i) – (iv) extend to the subalgebra $\sum_1^r D_i^0 = \{x_1 + \dots + x_r \mid x_i \in D_i^0\}$ *generated from the union* $D_1 \cup \dots \cup D_r$ *where elements from distinct* D_i^0 *are also orthogonal.* □

$\sum_1^r D_i^0$ is an internal direct product of the primitive subalgebras $D_1^0, \ldots, D_r^0$ that we call the ***orthosum*** of the D_i^0. Of course, $a_i \setminus b_i = 0$ whenever $a_i \mathcal{D} b_i$. A special case occurs when the D_i are *atomic* $\mathcal{D}$-classes lying directly over the class $\{0\}$. Here $a_i \wedge b_j = 0$ for elements a_i and b_i from distinct atomic $\mathcal{D}$-classes, making them orthogonal. In general, nonzero orthogonal $\mathcal{D}$-classes $D_1, \ldots, D_r$ are the atomic $\mathcal{D}$-classes of the generated subalgebra $\sum_1^r D_i^0$. In any case we have a basic result for skew Boolean algebras with only finitely many $\mathcal{D}$-classes:

Theorem 4.2.2. *If S is a nontrivial skew Boolean algebra with finitely many $\mathcal{D}$-classes and having atomic $\mathcal{D}$-classes $\{D_1, \ldots, D_r\}$, then **S** is the orthosum $\sum_1^r D_i^0$ of primitive subalgebras $\{D_1^0, \ldots, D_r^0\}$.* □

The above decomposition is an ***internal*** form of the ***atomic decomposition*** of a skew Boolean algebra ***S***, which must occur when ***S***/$\mathcal{D}$ is finite. This internal form is unique. The ***external*** form, given as a direct product, is unique to within isomorphism. In the *left-handed case* for finite ***S***, the ***standard*** atomic decomposition is

$$\boldsymbol{S} \cong \boldsymbol{n}_{1L} \times \boldsymbol{n}_{2L} \times \ldots \times \boldsymbol{n}_{rL} \text{ with } 2 \le n_1 \le n_2 \le \ldots \le n_r,$$

with $\boldsymbol{n_L}$ being the unique left-handed primitive algebra on $\{0, 1, 2, \ldots, n-1\}$ with 0 being the 0-element. Standard decompositions are also unique. Consider $\mathbf{2} \times \mathbf{2} \times \mathbf{4}_L \times \mathbf{5}_L \times \mathbf{5}_L$ or more briefly $\mathbf{2}^2 \times \mathbf{4}_L \times \mathbf{5}_L^2$. In this instance $\mathbf{2}^2$ provides the center of the algebra where "L" is superfluous. Similar remarks hold in the right-handed case. In the two-sided general case one uses notation such as $\mathbf{3}_L\bullet\mathbf{5}_R$ to represent the primitive algebra $\mathbf{3}_L\times_2\mathbf{5}_R$ given by the fibered product, as in: $\boldsymbol{S} \cong \mathbf{2}^3 \times \mathbf{3}_L\bullet\mathbf{5}_R \times \mathbf{5}_L\bullet\mathbf{4}_R \times \mathbf{7}_L\bullet\mathbf{7}_R$. In this case a standard decomposition could be given by lexicographically ordering the factors. In any case, a finite skew Boolean algebra is classified when its standard atomic decomposition is given.

Example 4.2.1. Partial function algebras serve as primary examples of SBAs. Note that

$$\begin{aligned}\mathcal{P}_{\mathrm{L}}(\{1, \ldots, n\},\{1, \ldots, m\}) &\cong \textstyle\prod_{i=1}^n \mathcal{P}_{\mathrm{L}}(\{i\}, \{1, 2, \ldots, m\}) \\ &\cong \left(\mathcal{P}_{\mathrm{L}}(\{1\}, \{1, 2, \ldots, m\})\right)^n \cong (\boldsymbol{m}+\mathbf{1})_L^n.\end{aligned}$$

In particular, $\mathcal{P}_{\mathrm{L}}(\{1, \ldots, n\},\{1\}) \cong \mathbf{2}^n$. In this case each partial function f is determined by choosing a subset of $\{1, 2, \ldots, n\}$ to be $f^{-1}(1)$, resulting in a bijection between $\mathcal{P}_{\mathrm{L}}(\{1, \ldots, n\},\{1\})$ and the power set of $\{1, 2, \ldots, n\}$ that preserves the generalized Boolean operations.

We characterize congruences on and homomorphisms between skew Boolean algebras with finitely many $\mathcal{D}$-classes.

Proposition 4.2.3. *Let θ be a congruence on a skew Boolean algebra A viewed as an orthosum $\sum_1^r D_i^0$ of primitive subalgebras D_i^0, the D_i being the atomic $\mathcal{D}$-classes. Then:*

1) *If $d\,\theta\,0$ for $d \in D_i$, then $D_i \subseteq [0]_\theta$, the congruence class of* 0.

2) *If $d_1\ \theta\ d_2$ with $d_1 \in D_i$, $d_2 \in D_j$, but $D_i \neq D_j$, then $D_i \cup D_j \subseteq [0]_\theta$.*

3) *Thus if some $D_i \subseteq [0]_\theta$, then upon re-indexing, $D_1 \cup \ldots \cup D_k \subseteq [0]_\theta$, with the remaining D_i refined by θ-classes and*

$$\left(\sum\nolimits_1^r D_i^0\right)\Big/\theta \cong \sum\nolimits_{k+1}^r D_i^0 \big/ \theta_i$$

where $\theta_i = \theta \mid D_i^0 \times D_i^0$ and D_i^0 / θ_i is primitive for each $i \geq k+1$.

Thus, given $\boldsymbol{S} = \sum_1^r D_i^0$ and a homomorphism of skew Boolean algebras $f: \boldsymbol{S} \to \boldsymbol{T}$:

4) *$f[\boldsymbol{S}]$ is an orthosum with summands $f\left[D_i^0\right]$, each of which is either primitive or else just $\{0_{\boldsymbol{T}}\}$. In the former case, $f[D_i]$ is at least atomic in $f[\boldsymbol{S}]$.*

5) $f\left[D_i^0\right] \cap f\left[D_j^0\right] \neq \{0_T\}$ *implies $i = j$ so that $D_i^0 = D_j^0$.*

Finally, in the purely primitive case:

6) *Given left-[right]handed primitive algebras D_1^0* and *D_2^0, a non-*0 *homomorphism from D_1^0 to D_2^0 is any map sending* 0 *to* 0, *and elements in D_1 to elements in D_2.*

7) *In general, all non-0 homomorphisms f: $D_1^0 \to D_2^0$ are obtained as follows*:
   (a) Send 0 to 0.
   (b) *Pick $a \in D_1$ and $b \in D_2$ and any map λ: $\mathcal{L}_a \to \mathcal{L}_b$ and any map ρ: $\mathcal{R}_a \to \mathcal{R}_b$.*
   (c) *Finally set $f(0) = 0$ and for all $x \in \mathcal{L}_a$ and $y \in \mathcal{R}_a$, set $f(x \wedge y) = \lambda(x) \wedge \rho(y)$.*

Proof. (1) should be clear. (2) For $d_1 = d_1 \wedge d_1$ is θ-related to $d_1 \wedge d_2 = 0$ and likewise $d_2\ \theta\ 0$. The conclusion now follows from (1). (3) – (5) should also be clear. (6) and (7) recall some of our remarks in the final part of Section 1.3. ☐

As a consequence, we have the following particularly crisp result:

Theorem 4.2.4. *Given a skew Boolean algebra* $\boldsymbol{S} = \sum_1^r D_i^0$ *with finitely many* $\mathcal{D}$*-classes and with the* D_i *being its atomic* $\mathcal{D}$*-classes, then its congruence lattice decomposes as follows:* $\boldsymbol{Con}(S) \cong \boldsymbol{Con}(D_1^0) \times \ldots \times \boldsymbol{Con}(D_r^0)$. ☐

Recall that for a primitive algebra D^0, $\boldsymbol{Con}(D_r^0)$ is essentially the lattice of rectangular partitions of D, augmented by a bottom element corresponding to the universal congruence on D^0.

Free algebras: the finite case

Given a non-empty set X:

$\mathbf{SBA}_X$ is the free skew Boolean algebra on X.
${}_{\mathcal{R}}\mathbf{SBA}_X$ is the free right-handed skew Boolean algebra on X.
${}_{\mathcal{L}}\mathbf{SBA}_X$ is the free left-handed skew Boolean algebra on X.
$\mathbf{GBA}_X$ is the free generalized Boolean algebra on X.

Free algebras are, of course, unique to within isomorphism. Thus if we say "*the* free" we have in mind a particular concrete instance, from which we are free (in an alternative sense) to find other isomorphic variants. In this paper, the default free algebra $\mathbf{F}_X$ on an alphabet X is the algebra of all terms (or polynomials) in X. In the current context, the terms are defined inductively as follows.

1) Each x in X is a term, as is the constant 0.
2) If u and v are terms, so are $(u \vee v)$, $(u \wedge v)$ and $(u \setminus v)$.

Two terms, u and v, are equivalent in $\mathbf{F}_X$ iff $u = v$ is an identity in the given variety of algebras. Clearly these criteria for equivalence differ among the four varieties of interest. Given an $\mathcal{SBA}$ equation of terms in X, $u = v$, one can check if it is a left-handed identity (or right-handed identity) by seeing if it holds for all evaluations on 3_L (or on 3_R). It is an $\mathcal{SBA}$ identity precisely when it holds for all evaluations on both 3_L and 3_R. Finally, it is a $\mathcal{GBA}$ identity if and only if it holds for all evaluations on 2. In our considerations, we are free to relax aspects of the syntax for parentheses if all ways of reinserting them lead to equivalent expressions. E.g., that would happen with $x \vee y \vee z$, but not with $x \wedge y \vee z$.

Given the universal character of the homomorphisms involved in the Clifford-McLean and the Kimura Factorization theorems for skew Boolean algebras:

$$\mathbf{GBA}_X \cong \mathbf{SBA}_X/\mathcal{D} \cong {}_{\mathcal{R}}\mathbf{SBA}_X/\mathcal{D} \cong {}_{\mathcal{L}}\mathbf{SBA}_X/\mathcal{D}.$$
$${}_{\mathcal{R}}\mathbf{SBA}_X \cong \mathbf{SBA}_X/\mathcal{L} \text{ and } {}_{\mathcal{L}}\mathbf{SBA}_X \cong \mathbf{SBA}_X/\mathcal{R}.$$
$$\mathbf{SBA}_X \cong {}_{\mathcal{L}}\mathbf{SBA}_X \times_{\mathbf{GBA}_X} {}_{\mathcal{R}}\mathbf{SBA}_X.$$

(Indeed, let $\mathcal{V}$ be *any* variety of algebras with $\mathcal{W}$ a subvariety of $\mathcal{V}$. For each algebra $\mathbf{A}$ in $\mathcal{V}$, let $\theta_{\mathbf{A}}$ be the congruence on $\mathbf{A}$ such that $\mathbf{A}/\theta_{\mathbf{A}}$ is in $\mathcal{W}$ and the induced map $\varphi_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A}/\theta_{\mathbf{A}}$ is a universal homomorphism from **A** to $\mathcal{W}$. Then if **A** is a free $\mathcal{V}$-algebra on generating set X, then $\mathbf{A}/\theta$ is a free $\mathcal{W}$-algebra on generating set $\varphi_{\mathbf{A}}[X]$. In the above context, $\theta_{\mathbf{A}} = \mathcal{D}$, $\mathcal{L}$ or $\mathcal{R}$ as appropriate, with X and $\varphi_{\mathbf{A}}$[X] being equipotent under $\varphi_{\mathbf{A}}$.)

In what follows we first consider $\mathbf{SBA}_n$, ${}_{\mathcal{L}}\mathbf{SBA}_n$, etc. which denote $\mathbf{SBA}_X$, ${}_{\mathcal{L}}\mathbf{SBA}_X$ *etc*. on alphabet $X = \{x_1, x_2, \dots, x_n\}$. Their standard atomic decomposition is given in Theorem 4.2.6 below. But to obtain the latter we need to understand their atomic structure. The case for ${}_{\mathcal{L}}\mathrm{SBA}_n$ and for ${}_{\mathcal{R}}\mathrm{SBA}_n$ is described in Theorem 4.2.5, the content of which is our immediate goal. We focus on ${}_{\mathcal{L}}\mathbf{SBA}_n$. Since ${}_{\mathcal{L}}\mathbf{SBA}_n$ has finitely many generators it will be finite and thus is determined by its atomic $\mathcal{D}$-classes, all lying just above the class $\{0\}$. We first describe these classes. Each class consists of atoms all sharing a common form. The justification that they are indeed the atomic $\mathcal{D}$-classes will follow. They are the $2^n–1$ classes of one of the forms below where $\{y_1, y_2, \dots, y_n\}$ in the table represents an arbitrary permutation of $\{x_1, x_2, \dots, x_n\}$. A typical class arises from a partition {L|M} of $\{x_1, x_2, \dots, x_n\}$ with $k \geq 1$ elements in L and $n–k$ elements in M used to form the term $(y_1\wedge \dots \wedge y_k) \setminus (y_{k+1}\vee\dots\vee y_n)$. This partition is ordered in that {L|M} is distinct from {M|L}. Thus, e.g., $\{1, 2|3, 4\} \neq \{3, 4|1, 2\}$.

Form Type	*Number of Classes of this Form*	*Class Sizes*
$y_1 \setminus (y_2 \vee y_3 \vee \cdots \vee y_n)$	$n = \binom{n}{1}$	1
$(y_1 \wedge y_2) \setminus (y_3 \vee y_4 \vee \cdots \vee y_n)$	$\binom{n}{2}$	2
$(y_1 \wedge y_2 \wedge y_3) \setminus (y_4 \vee y_5 \vee \cdots \vee y_n)$	$\binom{n}{3}$	3
…	…	…
$y_1 \wedge y_2 \wedge \cdots \wedge y_n$	$1 = \binom{n}{n}$	n

Given the **left-handed identity** $x\wedge y\wedge z = x\wedge z\wedge y$ and the **2-sided identities**

$$x \setminus (y \vee z) = x \setminus (z \vee y) = (x \setminus y) \setminus z = (x \setminus z) \setminus y,$$

(easily checked on $\mathbf{3}_{\mathrm{L}}$ or on both $\mathbf{3}_{\mathrm{L}}$ and $\mathbf{3}_{\mathrm{R}}$ respectively), $(y_1\wedge \dots\wedge y_k)\setminus(y_{k+1}\vee\dots\vee y_n)$ is invariant in outcome under any permutation of $y_2, \dots, y_k$ or of $y_{k+1}, \dots, y_n$. What *does* distinguish the elements in each class is the left-most element or variable, y_1. In all, a total of $n2^{n-1}$ essentially distinct atoms exist to produce $n2^{n-1}$ n-variable functions on $\mathbf{3}_{\mathrm{L}}$ (or on $\mathbf{3}_{\mathrm{R}}$). This is verified in the remarks below. But first, an example:

Example 4.2.2. For X = $\{x, y, z, w\}$, the 15 atomic classes and the $4(2^3) = 32$ atoms are:

$\{x \setminus (y \vee z \vee w)\}$ $\{y \setminus (x \vee z \vee w)\}$ $\{z \setminus (x \vee y \vee w)\}$ $\{w \setminus (x \vee y \vee z)\}$

$\{(x\wedge y)\setminus(z\vee w), (y\wedge x)\setminus(z\vee w)\}$ $\{(x\wedge z)\setminus(y\vee w), (z\wedge x)\setminus(y\vee w)\}$
$\{(x\wedge w)\setminus(y\vee z), (w\wedge x)\setminus(y\vee z)\}$ $\{(y\wedge z)\setminus(x\vee w), (z\wedge y)\setminus(x\vee w)\}$
$\{(y\wedge w)\setminus(x\vee z), (w\wedge y)\setminus(x\vee z)\}$ $\{(z\wedge w)\setminus(x\vee y), (w\wedge z)\setminus(x\vee y)\}$
$\{(x\wedge y\wedge z)\setminus w, (y\wedge z\wedge x)\setminus w, (z\wedge x\wedge y)\setminus w\}$ $\{(x\wedge y\wedge w)\setminus z, (y\wedge w\wedge x)\setminus z, (w\wedge x\wedge y)\setminus z\}$
$\{(x\wedge z\wedge w)\setminus y, (z\wedge w\wedge x)\setminus y, (w\wedge x\wedge z)\setminus y\}$ $\{(y\wedge z\wedge w)\setminus x, (z\wedge w\wedge y)\setminus x, (w\wedge y\wedge z)\setminus x\}$

$\{x\wedge y\wedge z\wedge w, y\wedge z\wedge w\wedge x, z\wedge w\wedge x\wedge y, w\wedge x\wedge y\wedge z\}$. ☐

Theorem 4.2.5. *Given the free left-handed skew Boolean algebra* $_{\mathcal{L}}\mathbf{SBA}_n$ *on* $\{x_1, \dots, x_n\}$:

i) $_{\mathcal{L}}\mathbf{SBA}_n$ *is a finite algebra whose atoms are the terms* $(y_1\wedge\ldots\wedge y_k)\setminus(y_{k+1}\vee\ldots\vee y_n)$ *for* $k \geq 1$ *and where* $(y_1, \dots, y_n)$ *is a permutation of* $\{x_1, \dots, x_n\}$.

ii) *Atoms* $(y_1\wedge\ldots\wedge y_k)\setminus(y_{k+1}\vee\ldots\vee y_n)$ *and* $(z_1\wedge\ldots\wedge z_l)\setminus(z_{l+1}\vee\ldots\vee z_n)$ *lie in the same atomic class if and only if* $k = l$, $(z_1, \dots, z_k)$ *is a permutation of* $\{y_1, \dots, y_k\}$ *and thus* $(z_{l+1}, \dots, z_n)$ *is a permutation of* $\{y_{k+1}, \dots, y_n\}$.

iii$_{\mathcal{L}}$) $(y_1\wedge\ldots\wedge y_k)\setminus(y_{k+1}\vee\ldots\vee y_n) = (z_1\wedge\ldots\wedge z_l)\setminus(z_{l+1}\vee\ldots\vee z_n)$ *if besides* (ii), $y_1 = z_1$.

For the free right-handed dual algebra $_{\mathcal{R}}\mathbf{SBA}_n$, (i) *and* (ii) *again hold along with*:

iii$_{\mathcal{R}}$) $(y_1\wedge\ldots\wedge y_k)\setminus(y_{k+1}\vee\ldots\vee y_n) = (z_1\wedge\ldots\wedge z_l)\setminus(z_{l+1}\vee\ldots\vee z_n)$ *if in addition to* (ii), $y_k = z_k$.

Proof. We consider the left-handed case. The right-handed assertion is similar. To begin, given a permutation $(z_1, \dots, z_k)$ of $\{y_1, \dots, y_k\}$, $(z_1\wedge\ldots\wedge z_k)\setminus(y_{k+1}\vee\ldots\vee y_n)$ and $(y_1\wedge \ldots\wedge y_k)\setminus(y_{k+1}\vee\ldots\vee y_n)$ are $\mathcal{L}$-related; they are not equal if $z_1 \neq y_1$. Indeed $y_1\wedge\ldots\wedge y_k \;\mathcal{L}\; z_1\wedge\ldots\wedge z_k$ plus $(a\setminus c)\wedge(b\setminus c) = (a\wedge b)\setminus c$ implies they are $\mathcal{L}$-related; they are not equal if $z_1 \neq y_1$ since they are not equal when operating as functions on $\mathbf{3}_{\mathrm{L}}$. Give y_1 and z_1 values 1 and 2 respectively, the remaining front variables 1, and all $n–k$ back variables 0. The outcome for $(y_1\wedge \ldots\wedge y_k)\setminus(y_{k+1}\vee\ldots\vee y_n)$ is 1 and for $(z_1\wedge\ldots\wedge z_k)\setminus(y_{k+1}\vee\ldots\vee y_n)$ is 2.

In general, given distinct partitions {L|M} and {L′|M′} of $\{x_1, x_2, \dots, x_n\}$ with L and L′ nonempty, some element m lies in L∩M′ or in L′∩M, say the former. Viewing m as a generator, given any {L|M}-term u and any {L′|M′}-term v, we have $u\wedge m = u$ but $v\wedge m = 0 = m\wedge v$. Thus $u\wedge v = u\wedge m\wedge v = u\wedge 0 = 0 = v\wedge u$. Thus all {L|M}-terms are orthogonal to all {L′|M′}-terms. Since $(x_1\wedge \ldots\wedge x_k)\setminus(x_{k+1}\vee\ldots\vee x_n) = 0$ is not an identity in $\mathbf{3}_{\mathrm{L}}$ for $k \geq 1$, all {L|M}-classes are non-0 classes and distinct {L|M}-classes are orthogonal. Returning to the example above, $\{(x\wedge y)\setminus(z\vee w), (y\wedge x)\setminus(z\vee w)\}$ is disjoint from $\{(x\wedge w)\setminus(y\vee z), (w\wedge x)\setminus(y\vee z)\}$ with pairs from distinct classes being orthogonal.

To see that they are full $\mathcal{D}$-classes of $_{\mathcal{L}}\mathbf{SBA}_n$, and that they (all the) atomic $\mathcal{D}$-classes, observe first that *they are the atomic* $\mathcal{D}$*-classes in the subalgebra of* $_{\mathcal{L}}\mathbf{SBA}_n$ *that they generate*.

We need to show that this subalgebra is in fact all of ${}_{\mathcal{L}}\mathbf{SBA}_n$. We do so by showing that each generator x_k of ${}_{\mathcal{L}}\mathbf{SBA}_n$ is in the generated subalgebra. The identities above give us:

$$\begin{aligned} x_1 = (x_1\wedge x_2) + (x_1\backslash x_2) &= (x_1\wedge x_2\wedge x_3) + ((x_1\wedge x_2)\backslash x_3) + ((x_1\backslash x_2)\wedge x_3) + (x_1\backslash x_2)\backslash x_3) \\ &= (x_1\wedge x_2\wedge x_3) + ((x_1\wedge x_2)\backslash x_3) + ((x_1\wedge x_3)\backslash x_2) + (x_1\backslash(x_2\vee x_3)) \\ &= (x_1\wedge x_2\wedge x_3\wedge x_4) + ((x_1\wedge x_2\wedge x_3)\backslash x_4) + \ldots \end{aligned}$$

The process keeps repeating on each new term until generator x_1 is resolved into an orthosum of 2^{n-1} {L|M}-type terms – indeed into all the {L|M}-type terms with left-most entry x_1. Similar calculations work for the remaining generators. Thus the 2^n-1 distinct {L|M}-classes are all the atomic $\mathcal{D}$-classes of ${}_{\mathcal{L}}\mathbf{SBA}_n$. □

In the generalized Boolean case, all atomic terms resulting from the same {L|M} decomposition are equated. Thus the particular left-most generator/variable no longer differentiates among outcomes. In the 2-sided case, in the Kimura fibered product construction each left-handed atomic class is matched off with the right-handed atomic class with the same {L|M} partition. In this case the data of $((y_1\wedge\ldots\wedge y_k), (y'_1\wedge\ldots\wedge y'_k))$ can be combined as $y_1\wedge\ldots\wedge y_k\wedge y'_1\wedge\ldots\wedge y'_k$ and then reduced via 2-sided normality. Returning to the previous example:

Example 4.2.2 continued. These previous terms describe the atomic classes of both the left- and right-handed free algebras on $\{x, y, z, w\}$. E.g. $\{(\boldsymbol{x}\wedge y)\backslash(z\vee w), (\boldsymbol{y}\wedge x)\backslash(z\vee w)\}$ works in the left-hand case, while $\{(x\wedge \boldsymbol{y})\backslash(z\vee w), (y\wedge \boldsymbol{x})\backslash(z\vee w)\}$ works in the right-hand case. In both finite cases it is possible to describe the "atomic" terms using cyclic permutations in a way that the terms do double duty. But that won't "stretch" to the 2-sided case. Here we adjoin both $(x\wedge y\wedge x)\backslash(z\vee w)$ and $(y\wedge x\wedge y)\backslash(z\vee w)$ to the class to get:

$$\{(x\wedge y)\backslash(z\vee w), (y\wedge x)\backslash(z\vee w), (x\wedge y\wedge x)\backslash(z\vee w), (y\wedge x\wedge y)\backslash(z\vee w)\}.$$

For two terms to be equal in value, both end variables in the left part would have to agree. In general, the corresponding atomic classes would be squared in size. □

We thus obtain precise structural descriptions of all four relevant free algebras. In what follows $D_{\{L|M\}}$ is the {L|M}-induced $\mathcal{D} = \mathcal{L}$-class, $\mathbf{P}^{\mathcal{L}}_{\{L|M\}}$ is the left-handed primitive algebra $D_{\{L|M\}}{}^0$ and $\mathbf{P}^{\mathcal{R}}_{\{L|M\}}$ is its right-handed counterpart. Also, given primitive algebras $\boldsymbol{P}$ and $\boldsymbol{Q}$, $\boldsymbol{P}\bullet\boldsymbol{Q}$ *denotes their fibered product over* $\mathbf{2}$, $\boldsymbol{P}\times_{\mathbf{2}}\boldsymbol{Q}$. In the next theorem, the trivial algebra $\mathbf{1}$ on $\{0\}$ is included to allow the full distribution of binomial coefficients. This factor corresponds to the front-empty partition $\{\varnothing|X\}$. Hence:

[2.2]

Theorem 4.2.6. *The free left-handed skew Boolean algebra* ${}_{\mathcal{L}}\mathbf{SBA}_n$ *on* $\{x_1, \ldots, x_n\}$ *is a direct sum of the primitive algebras* $\mathbf{P}^{\mathcal{L}}_{\{L|M\}}$ *where* $\{L|M\}$ *ranges over partitions* $\{L|M\}$ *of* $\{x_1, \ldots, x_n\}$ *where* $L \neq \varnothing$. *Thus:*

$$ {}_{\mathcal{L}}\mathbf{SBA}_n \cong \mathbf{1}^{\binom{n}{0}} \times \mathbf{2}^{\binom{n}{1}} \times \mathbf{3}_L^{\binom{n}{2}} \times \mathbf{4}_L^{\binom{n}{3}} \times \ldots \times (\boldsymbol{n}+\mathbf{1})_L^{\binom{n}{n}} .$$

Dually, the free right-handed skew Boolean algebra ${}_{\mathcal{R}}\mathbf{SBA}_n$ *on* $\{x_1, \ldots, x_n\}$ *is a direct sum of the primitive algebras* $\mathbf{P}^{\mathcal{R}}_{\{L|M\}}$ *where* $\{L|M\}$ *shares the same range. Thus:*

$$ {}_{\mathcal{R}}\mathbf{SBA}_n \cong \mathbf{1}^{\binom{n}{0}} \times \mathbf{2}^{\binom{n}{1}} \times \mathbf{3}_R^{\binom{n}{2}} \times \mathbf{4}_R^{\binom{n}{3}} \times \ldots \times (\boldsymbol{n}+\mathbf{1})_R^{\binom{n}{n}} .$$

Finally, the free skew Boolean algebra $\mathbf{SBA}_n$ *on* $\{x_1, \ldots, x_n\}$ *is a direct sum of the primitive algebras* $\mathbf{P}^{\mathcal{L}}_{\{L|M\}} \bullet \mathbf{P}^{\mathcal{R}}_{\{L|M\}}$ *where* $\{L|M\}$ *again shares the same range. Thus:*

$$ \mathbf{SBA}_n \cong \mathbf{1}^{\binom{n}{0}} \times \mathbf{2}^{\binom{n}{1}} \times \left(\mathbf{3}_L \bullet \mathbf{3}_R\right)^{\binom{n}{2}} \times \left(\mathbf{4}_L \bullet \mathbf{4}_R\right)^{\binom{n}{3}} \times \ldots \times \left((\boldsymbol{n}+\mathbf{1})_L \bullet (\boldsymbol{n}+\mathbf{1})_R\right)^{\binom{n}{n}} . \ \square $$

Corollary 4.2.7. *For all* $n \geq 1$:

i) $|{}_{\mathcal{L}}\mathbf{SBA}_n| = 2^{\binom{n}{1}} 3^{\binom{n}{2}} 4^{\binom{n}{3}} \cdots (n+1)^{\binom{n}{n}} = |{}_{\mathcal{R}}\mathbf{SBA}_n|.$

ii) $|\mathbf{SBA}_n| = 2^{\binom{n}{1}} 5^{\binom{n}{2}} 10^{\binom{n}{3}} \cdots (n^2+1)^{\binom{n}{n}}.$

Moreover, if $\alpha_L(n)$, $\alpha_R(n)$ *and* $\alpha(n)$ *denote the number of atoms in* ${}_{\mathcal{L}}\mathbf{SBA}_n$, ${}_{\mathcal{R}}\mathbf{SBA}_n$ *and* $\mathbf{SBA}_n$, *respectively, then:*

iii) $\alpha_L(n) = \binom{n}{1}1 + \binom{n}{2}2 + \ldots + \binom{n}{n-1}(n-1) + \binom{n}{n}n = \alpha_R(n)$.

iv) $\alpha(n) = \binom{n}{1}1 + \binom{n}{2}4 + \cdots + \binom{n}{n-1}(n-1)^2 + \binom{n}{n}n^2$. $\square$

Standard combinatorial arguments give the following simplifications:

Corollary 4.2.8. *Given* $\alpha_L(n)$, $\alpha_R(n)$ *and* $\alpha(n)$ *as above*:

$\alpha_L(n) = \alpha_R(n) = n2^{n-1}$ *and* $\alpha(n) = n(n+1)2^{n-2}$, *so that* $\alpha(n) = \frac{n+1}{2}\alpha_L(n)$.

Proof. To see $\binom{n}{1}1+\binom{n}{2}2+\ldots+\binom{n}{n}n = n2^{n-1}$, differentiate the binomial expansion of $(1 + x)^n$ and set $x = 1$. Setting $x = 1$ again in the second derivative of the binomial expansion of $(1 + x)^n$ gives: $\binom{n}{2}2\bullet 1+\binom{n}{3}3\bullet 2 +\cdots+ \binom{n}{n}n(n-1) = n(n-1)2^{n-2}$.

Adding the equality of the previous expansion to this and simplifying gives

$$\binom{n}{1}1+\binom{n}{2}4+\cdots+\binom{n}{n}n^2 = n2^{n-1} + n(n-1)2^{n-2} = n(n+1)2^{n-2}. \ \square$$

A short table of values follows with the sizes for $n = 5$ given to 4-digit accuracy.

n	$\lvert {}_{\mathcal{L}}\mathbf{SBA}_n \rvert$	$\alpha_L(n)$	$\lvert \mathbf{SBA}_n \rvert$	$\alpha(n)$
2	12	4	20	6
3	864	12	10,000	24
4	14,929,920	32	425×10^8	80
5	3.715×10^{16}	80	3.017×10^{25}	240

Since $\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n} = 2^n - 1$, we have:

Corollary 4.2.9. *A free (left-handed, right-handed or two-sided) skew Boolean algebra on n generators has* $2^n - 1$ *primitive factors in its atomic decomposition. Thus any skew Boolean algebra on n generators has at most that many. Any generalized Boolean algebra on n generators thus has* $\leq 2^n - 1$ *atoms, and is free if and only if it has exactly* that *many.* □

Theorems 4.2.2 and 4.2.6 also lead to:

Corollary 4.1.20. *Every finite skew Boolean algebra can be embedded in a finite free skew Boolean algebra. Every finite left-handed [right-handed] skew Boolean algebra is isomorphic to a direct factor of a finite free left-handed [right-handed] skew Boolean algebra.*

4.3 Connections with strongly distributive skew lattices

A skew lattice is ***embedded*** in a skew Boolean algebra (S; ∨, ∧, \, 0) if it is embedded in its skew lattice reduct (S; ∨, ∧). Since such reducts are strongly distributive, theorems from the previous section and Section 2.6 lead to the following results.

Theorem 4.3.1. *A skew lattice can be embedded in a skew Boolean algebra if and only if it is strongly distributive*.

Proof. (Only if) Skew lattice reducts of skew Boolean algebras are strongly distributive and strongly distributive skew lattices form a skew lattice subvariety. (If) By Theorem 2.6.12 every strongly distributive skew lattice can be embedded in a power of the skew lattice **5**, which however, is the reduct of *that* power of the skew Boolean algebra **5**. ☐

Theorem 4.3.2. *For skew lattice S the following are equivalent:*

i) *S can be embedded in a right-handed skew Boolean algebra.*
ii) *S is right-handed and strongly distributive*.
iii) *S can be embedded in a slew lattice of partial functions* $\mathcal{P}_R(A, B)$.

Proof. The equivalence of (i) and (ii) is seen as in the previous proof, but with **5** replaced by **3**. If S can be embedded in some partial function skew lattice $\mathcal{P}_R(A, B)$, then since the latter is right-handed and strongly distributive so it S. The converse follows from Theorem 2.6.14. ☐

Theorem 4.3.3. *Given an identity in* $\vee$ *and* $\wedge$ *the following are equivalent*:

i) *The identity holds in all skew Boolean algebras.*
ii) *The identity holds in all strongly distributive skew lattices*.
iii) *The identity holds in the skew lattice* **5**.

Proof. Both (i) and (ii) are equivalent by Theorem 4.2.1, and each clearly implies (iii). Conversely, any identity holding on **5** must also hold on any sub-skew lattice of any power of **5**, and extending via isomorphism, it must hold on any strongly distributive skew lattice. ☐

Of course, one has the right-handed specialization of the above.

Theorem 4.3.3R. *Given an identity in* $\vee$ *and* $\wedge$ *the following are equivalent*:

i) *The identity holds in all right-handed skew Boolean algebras.*
ii) *The identity holds all right-handed strongly distributive skew lattices*.
iii) *The identity holds in the skew lattice* $\mathbf{3_R}$.
iv) *The identity holds in all partial function algebras* $\mathcal{P}(A, B)$. ☐

Proof. The equivalence of (i) – (iii) is seen similarly as in the general case. Their equivalence with (iv) comes from Theorem 2.6.14. ☐

The override and the update operations

J. Berendsen, D. Jansen, J. Schmaltz and F. Vaandrager in their 2010 paper, *The axiomatization of overriding and update* (Journal of Applied Logic, **8** (2010), 141-150.), gave an approach to studying partial function algebras that is similar in many ways to ours, but with some differences. To begin, given a pair of partial functions f and g in the partial function set P(A, B) with subsets F and G of A being their respective support, the authors considered the following operations:

Override: $f \triangleright g = f \cup g|(G \setminus F)$.

Update: $f[g] = g|(F \cap G) \cup f|(F \setminus G)$.

Minus: $f - g = f|(F \setminus G)$.

The override $f \triangleright g$ is clearly the join $f \vee g$, while the minus $f - g$ is just $f \setminus g$. Stated in our notation, the authors defined the update by $f[g] = (f \wedge g) \vee (f \setminus g)$ where $f \wedge g$ is $g \setminus (g \setminus f)$, the latter holding for all right-handed skew Boolean algebras. For right-handed, strongly distributive skew lattices in general, $f[g]$ is given also as either $(f \wedge g) \vee f$ or $f \wedge (g \vee f)$. Indeed, using any of these three ways, on $\mathbf{3}_R$ one has:

$x[y]$	0	1	2
0	0	0	0
1	1	1	2
2	2	1	2

.

Thus all three evaluations of $x[y]$ *in any skew Boolean algebra agree*. (The equation

$$(x \wedge y) \vee x = x \wedge (y \vee x)$$

and its $\vee$-$\wedge$ dual are examined in Section 5.1.)

One can consider algebras with signature ([], $\vee$) as was suggested by Berendsen *et al* in their paper. The problem of interest for those authors was that of determining identities in [ ] and $\vee$ that hold for all partial function algebras (and thus more generally, for all right-handed, strongly distributive skew lattices). Clearly all identities in [ ] and $\vee$ that hold in all partial function algebras follow from the defining identities for right-handed skew Boolean algebras; more generally, they follow from the defining identities for right-handed, strongly distributive skew lattices. In their paper, the authors were interested particularly in finding a set of identities in just [ ] and $\vee$ that (1) held for all partial function algebras and (2) was powerful enough so that all identities in $\vee$ and [ ] that hold in all partial function algebras were consequences of the given identities. The search for such a set of identities is apparently still open. However, if **3*** denotes the algebra ({0, 1, 2}; [ ], $\vee$) defined by the Cayley tables

$x \vee y$	0	1	2
0	0	1	2
1	1	1	1
2	2	2	2

and

$x[y]$	0	1	2
0	0	0	0
1	1	1	2
2	2	1	2

that reflect what occurs on $\mathbf{3}_R$, either as a skew lattice or as a skew Boolean algebra, then along with Theorem 4.3.3R, we are led to the following result of Cvetko-Vah, Leech and Spinks:

Theorem 4.3.4. *An identity in* [ ] *and* ∨ *holds in all right-handed skew Boolean algebras, or more broadly in all right-handed strongly distributive skew lattices, if and only if it holds in* **3***. *Thus, the question of whether an identity in* [ ] *and* ∨ *holds in these classes of algebras, or in particular in all partial function algebras of the form* $\mathcal{P}$(A, B) *is decidable*.

Proof. The condition is clearly necessary. Since every right-handed SBA [strongly distributive skew lattice] is a subalgebra of a power of $\mathbf{3}_R$ [under the appropriate signature] the condition is also sufficient. The final statement is now clear. ☐

Our interest in such identities and especially in [ ] is due to the role of the latter in right-handed strongly distributive skew lattices. Returning to Section 2.7, the free categorical and symmetric right-handed skew lattice on generators *x* and *y* can be presented as follows:

$$\begin{array}{ccccccccc} & & x \vee y & -_R & y \vee x & & & & \\ & \iddots & \iddots & \ddots & & \ddots & & & x[y] = x \wedge (y \vee x) = (x \wedge y) \vee x. \\ x \; -_R & x[y] & & & & y[x] & -_R \; y & \text{where} & \\ & \ddots & \ddots & \iddots & & \iddots & & & y[x] = y \wedge (x \vee y) = (y \wedge x) \vee y. \\ & & y \wedge x & -_R & x \wedge y & & & & \end{array}$$

In particular, this skew diamond and its subalgebras describe the 2-generator subalgebras that can occur in right-handed skew Boolean algebras or in right-handed strongly distributive skew lattices in general or, for that matter, in right-handed skew lattices in rings. Thus, just as skew joins/overrides can be seen as biased unions, and meets as biased intersections, the two other terms in this diagram besides *x* and *y* can be viewed as the outcomes of updating *x* and *y* relative to each other. More will be said about the update operation in Section 5.1.

Axioms for (right-handed) skew Boolean algebras

In their paper, Berendsen *et al* introduced five identities that describe the behavior of ▹, – and ∅ in combination. Using automated reasoning software they showed their equations to be independent and obtained a number of derived equations. They also raised questions about the algebras that satisfy these equations.

In a responding paper by Cvetko-Vah, Leech and Spinks (*Skew lattices and binary operations on functions*, Journal of Applied Logic, **11** (2013), pp. 253-265), the variety of algebras satisfying those five identities was shown to be term equivalent to the variety of right-handed skew Boolean algebras. Thus the five identities given in the first paper lead to the following equational basis (or set or characterizing identities) for right-handed skew Boolean algebras.

Theorem 4.3.5. *An independent set of identities that characterize the variety of all right-handed skew Boolean algebras* (S; $\wedge$, $\vee$, $\backslash$, 0) *is given by:*

i) $x \vee x \approx x$.
ii) $x \setminus x \approx 0$.
iii) $x \vee y \approx (y \setminus x) \vee x$.
iv) $x \setminus (y \setminus z) \approx (x \setminus y) \vee (x \setminus (x \setminus z))$.
v) $(x \vee y) \setminus z \approx (x \setminus z) \vee (y \setminus z)$.

Here the meet $\wedge$ *is defined by* $x \wedge y := y \setminus (y \setminus x)$. ☐

An independent set of six identities characterizing the variety of skew Boolean algebras was given in Spinks' dissertation: (See also Spinks [**1998**] Prop. LBSL-RC-7.)

i) $x \wedge (x \vee y) \approx x$;
ii) $(y \wedge x) \vee x \approx x$;
iii) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;
iv) $(x \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z)$;
v) $((x \wedge y) \wedge x) \vee (x \setminus y) \approx x$;
vi) $(x \setminus y) \wedge y \approx 0$.

These contrast with the 12 identities implicit in the definition of a skew Boolean algebra in Section 4.1. As of this writing, neither of these two sets of identities have been bettered with respect to size. Of interest in the first set is the fact that along with 0 which is just used once, the binary operations \ and $\vee$ suffice in their statements.

Could $\wedge$ be eliminated from the second set of 2-sided identities? Actually yes, thanks to the identity:

$$x \wedge y = (y \vee x) \setminus \{[(y \vee x) \setminus x] \vee [(y \vee x) \setminus y]\}.$$

This identity is easily seen to hold on primitive algebras, and hence on all skew Boolean algebras. Returning to all six identities above, it can be used to first define $\wedge$ in terms of $\vee$ and \, and then to eliminate all occurrences of $\wedge$ in the six identities. A reduced signature ($\vee$, \, 0) is thus possible for skew Boolean algebras, but not convenient. As with generalized Boolean algebras, $\vee$ cannot be eliminated in favor of $\wedge$, \ and 0. A closer look at the role of \ occurs in the final chapter.

4.4 Skew Boolean algebras with intersections

Recall that a skew lattice (S; $\vee$, $\wedge$) *has finite intersections* if every pair $e, f \in$S possesses a *natural meet* with respect to the natural partial order $\geq$ on S. When it occurs, the natural meet of any e and f in S is denoted by $e \cap f$. We say that (S; $\vee$, $\wedge$) *has intersections* if the natural partial order $\geq$ on S has infima for arbitrary subsets. In general the following must hold:

Lemma 4.4.1. *For any pair e, f in a skew lattice* S *the following are equivalent*:

i) $e \wedge f = f \wedge e$.
ii) $e \wedge f = e \cap f$, *where* $e \cap f$ *exists*.
iii) $f \wedge e = e \cap f$, *where* $e \cap f$ *exists*.

Proof. That (i) implies (ii) and (iii) is clear. Conversely, given say (ii) one has

$$f \wedge e = f \wedge e \wedge f \wedge e = f \wedge (e \cap f) \wedge e = e \cap f = e \wedge f$$

and (i) follows. The equivalence of (i) and (iii) is similar. ☐

Lemma 4.4.2. *A skew lattice having finite intersections is an algebra* S = (S; ∨, ∧, ∩) *such that* (S; ∩) *is a meet semilattice*, (S; ∨, ∧) *is a skew lattice and the following identities hold:*

$$e \cap (e \wedge f \wedge e) = e \wedge f \wedge e \ \text{ and } \ e \wedge (e \cap f) = e \cap f = (e \cap f) \wedge e.$$

Skew lattices with finite intersections thus form a variety of algebras.

Proof. The identities state in essence that the two partial orders on S induced by ∧ and ∩ must contain each other and thus coincide. ☐

Theorem 4.4.3. (Theorem 1.3.11 restated) *The variety of all skew lattices having finite intersections is congruence distributive*. ☐

A skew lattice (S; ∨, ∧) is ***initially finite*** if for all $e \in$ S, $\lceil e \rceil = \{f \mid f \leq e\}$ is finite. Clearly:

Theorem 4.4.4. *If* (S; ∨, ∧, 0) *is a normal, symmetric skew lattice with a zero*, *then*:

i) *If* S *is join complete*, *then* S *is complete*.
ii) *If* S *is complete*, *then* S *has intersections*.
iii) *If* S *is initially finite*, *then* S *has intersections*. ☐

A skew Boolean algebra with finite intersections is called a ***skew Boolean ∩-algebra***. (Read "skew Boolean intersection algebra".) These algebras form a variety of algebras denoted by $\mathcal{SBA}^{\cap}$. We next scan examples of skew Boolean algebras to see which are ∩-algebras.

Example 4.4.1. Finite intersections trivially exist for generalized Boolean algebras.

Example 4.4.2. A 0-primitive skew Boolean algebra S = D^0 has arbitrary intersections. Given A ⊆ S, *inf*A is the sole element of A when A is a singleton set, and 0 otherwise.

Example 4.4.3. Every completely reducible skew Boolean algebra, being the product of primitive algebras, has arbitrary intersections.

In particular, a partial function algebra $\mathcal{P}$(A, B) is completely reducible and thus must have arbitrary intersections. Indeed, upon viewing partial functions from A to B as subsets of A × B, intersections in our sense are just intersections of subsets: $f \cap g = f \cap g$.

In particular, a skew Boolean algebra S for which S/$\mathcal{D}$ is finite has arbitrary intersections.

Example 4.4.4. Let R be a C-ring: an associative ring such that every for each $x \in$R a central idempotent C(x) exists such that xC(x) = x and C(x) is the least central idempotent with respect to this property. Cornish [1980] showed that a left-handed skew Boolean algebra can be defined from any C-ring upon setting: $x \wedge y = x\mathrm{C}(y)$, $x \vee y = x + y - x\mathrm{C}(y)$ and $x \setminus y = x - x\mathrm{C}(y)$. The intersection of a pair of elements is given by $x \cap y = [1 - \mathrm{C}(x - y)]x$.

Example 4.4.5. Kudryavtseva and Leech [2016] have shown that free skew Boolean algebras have finite intersections. This is discussed later in this section.

Example 4.4.6. Given any maximal normal band S in a ring R, we have seen that S forms a skew Boolean algebra upon setting $e \wedge f = ef$, $e \vee f = e \nabla f$ and $e \setminus f = e - efe$. They have intersections if the ring is semisimple and Artinian, or in particular, is a matrix ring over a field.

Example 4.4.7. (Counterexample) Let ℕ be the set of natural numbers and let S denote the subset of $\mathcal{P}$(ℕ, {0, 1}) consisting of all partial functions having domains that are either finite or cofinite (in that the complement in ℕ is finite). It is easily verified that S is closed under the skew Boolean operations, ∨ and ∧ and \. Thus S is a skew Boolean algebra. Clearly $f \cap g$ does not exist for the partial functions $f, g \in$ S, both with full domain ℕ, but with

$$f(n) = 0 \quad \text{and} \quad g(n) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}.$$

Recall that an ideal is a nonempty subset I of a skew lattice S such that for all $x, y \in I$ and all $z \in$S, $x \vee y$, $x \wedge z$ and $z \wedge x$ all lie in I. While arbitrary sub-skew Boolean algebras of skew Boolean ∩-algebras need not have finite intersections, ideals of skew Boolean ∩-algebras do.

Extending Theorem 4.1.4, we have:

Theorem 4.4.5. *Given a principal ideal* $\langle a \rangle$ *in a skew Boolean algebra* S *and its associated annihilator ideal ann*(a), *S has finite intersections if and only if both* $\langle a \rangle$ *and ann(a) have finite intersections, in which case the map* $\mu: \langle a \rangle \times ann(a) \to$ S *defined by* $\mu(x, y) = x \vee y$ *is an isomorphism of skew Boolean ∩-algebras. In particular, given x, y in* $\langle a \rangle$ *and u, v in ann*(a),

$$(x \vee u) \cap (y \vee v) = (x \cap y) \vee (u \cap v). \ \square$$

As a consequence we have:

Theorem 4.4.6. *In the variety of skew Boolean* $\cap$*-algebras the following assertions hold.*

i) *The primitive algebras are precisely the nontrivial simple algebras.*
ii) *The primitive algebras are the nontrivial directly irreducible algebras.*
iii) *The primitive algebras are the nontrivial subdirectly irreducible algebras.*

Proof. By Theorem 4.4.5, a nontrivial simple algebra must be primitive. Conversely, let S be a primitive algebra and let θ be a congruence on S. Suppose that $e\ \theta\ f$ with $e \neq f$ in S. Then e and f are also θ-congruent to $e \cap f = 0$. Since either $e \neq 0$ or $f \neq 0$, this forces every element of the primitive algebra to be congruent to 0. Hence θ is the universal congruence. The only other congruence possible is thus the identity congruence Δ. Thus all primitive algebras are simple and (i) holds. (ii) is clear. Given (i) and (ii), (iii) easily follows. ☐

Given a primitive skew Boolean algebra P and an $\cap$-preserving homomorphism φ from P to a skew Boolean $\cap$-algebra S, it follows that φ is either the 0-homomorpism, or an embedding.

The ideals of any skew lattice S form a complete lattice. The canonical map π: S $\to$ S/$\mathcal{D}$ induces an isomorphism of this lattice with the lattice of ideals of S/$\mathcal{D}$, via the map $I \to I/\mathcal{D}$ = S/$\mathcal{D}$. If S/$\mathcal{D}$ is finite, then all ideals of both S and S/$\mathcal{D}$ are principal with both lattices of ideals being isomorphic to S/$\mathcal{D}$. In the current context, ideals serve as the kernels of homomorphism.

Lemma 4.4.7. *Let* θ *be a congruence on a skew Boolean* $\cap$*-algebra* S *and let* $\theta[0]$ *be the congruence class of* 0. *Then the class* $\theta[0]$ *is an ideal of* S *and for all* $e, f \in$ S,

$$e\ \theta\ f \text{ if and only if } (e \setminus e\cap f) \vee (f \setminus e\cap f) \in \theta[0].$$

Conversely, if I is an ideal of S, *then the relation* $\theta = \theta_I$ *defined as above is a congruence on* S. *Moreover the maps* $\theta \to \theta[0]$ *and* $\mathrm{I} \to \theta_{\mathrm{I}}$ *form an inverse pair of bijections.*

Proof. Observe that on any skew Boolean $\cap$-algebra, $e = f$ if and only if $(e \setminus e\cap f) \vee (f \setminus e\cap f) = 0$. This is certainly true in the primitive case. Thus it is true for all such algebras by Theorem 4.3.6 and the first assertion of the lemma follows. For the converse, first let I be a principal ideal $\langle a \rangle$ with complement $ann(a)$. By Theorem 4.4.5, an $\cap$-algebra homomorphism φ of S onto $ann(a)$ exists for which $\langle a \rangle = \varphi^{-1}(0)$. If θ is the congruence associated with φ, then

$$\begin{array}{lll} e\ \theta\ f \text{ in S} & \text{iff} & \varphi(e) = \varphi(f) \text{ in } ann(a) \\ & \text{iff} & (\varphi(e) \setminus \varphi(e)\cap\varphi(f)) \vee (\varphi(f) \setminus \varphi(e)\cap\varphi(f)) = 0 \text{ in } ann(a) \\ & \text{iff} & (e \setminus e\cap f) \vee (f \setminus e\cap f) \in \langle a \rangle. \end{array}$$

In general, any ideal I is the directed union of principal ideals: $\mathrm{I} = \bigcup\!\uparrow \{\langle a \rangle \mid a \in \mathrm{I}\}$. The θ_{I} as defined above is the corresponding directed union $\bigcup\!\uparrow \{\theta_{\langle a \rangle} \mid a \in \mathrm{I}\}$ of "principal congruences" and thus is also a congruence. That both processes are reciprocal is clear. ☐

Theorem 4.4.8. *Given a skew Boolean ∩-algebra* **S***, its congruence lattice* $\mathbf{Con}_{\cap}(\mathbf{S})$ *on is isomorphic to the lattice of ideals of S (or of its maximal generalized Boolean image* $S/\mathcal{D}$.*). Finitely generated congruences correspond to the principal ideals of* $S/\mathcal{D}$ *and thus form a generalized Boolean sublattice of the congruence lattice*.

Proof. The general correspondence is clear by the previous lemma. By the lemma again, finitely generated congruences have finitely generated ideal kernels that must be principal. Conversely, a principal ideal $\langle a\rangle$ corresponds to the congruence generated from $(a, 0)$. □

Theorem 4.4.9. *The lattice* $\mathbf{Con}(\mathbf{S})$ *of all skew lattice congruences on a skew Boolean ∩-algebra* **S** *is the subdirect product of the interval* $[\Delta, \mathcal{D}]$ *and the sublattice* $\mathbf{Con}_{\cap}(\mathbf{S})$ *of skew Boolean ∩-algebra congruences on* **S** *under the map* $\mathbf{Con}(\mathbf{S}) \to [\Delta, \mathcal{D}] \times \mathbf{Con}_{\cap}(\mathbf{S})$ *given by the rule* $\theta \to (\theta\cap\mathcal{D}, \theta_{\theta[0]})$.

Proof. By Theorem 3.1.2, $\mathbf{Con}(\mathbf{S})$ is the subdirect product of $[\Delta, \mathcal{D}]$ and the interval $[\mathcal{D}, \nabla]$ under the map $\theta \to (\theta\cap\mathcal{D}, \theta\vee\mathcal{D})$. But $[\mathcal{D}, \nabla] \cong \mathbf{Con}(\mathbf{S}/\mathcal{D})$ which is isomorphic to the lattice of ideals of $\mathbf{S}/\mathcal{D}$ and in turn to the lattice of ideals of S, and thus to $\mathbf{Con}_{\cap}(\mathbf{S})$. The ∩-respecting congruence corresponding to $\theta\vee\mathcal{D}$ has kernel ideal $(\theta\vee\mathcal{D})(0) = \theta(0)$ so that $\theta_{(\theta\vee\mathcal{D})(0)} = \theta_{\theta(0)}$. □

Intersections and the canonical images of a skew lattice

Given a skew lattice **S**, the canonical skew lattice maps, $\mathbf{S} \to \mathbf{S}/\mathcal{L}$, $\mathbf{S} \to \mathbf{S}/\mathcal{R}$ and $\mathbf{S} \to \mathbf{S}/\mathcal{D}$, are all homomorphisms. How do intersections fit into this picture? We begin with:

Theorem 4.4.10. *Given a skew lattice* **S** *with finite intersections*:

i) *The canonical map* $\mathbf{S} \to \mathbf{S}/\mathcal{L}$ *preserves intersections iff* **S** *is right-handed, so that* $\mathbf{S}/\mathcal{L} \cong \mathbf{S}$.
ii) *The canonical map* $\mathbf{S} \to \mathbf{S}/\mathcal{R}$ *preserves intersections iff* **S** *is left-handed*, *so that* $\mathbf{S}/\mathcal{R} \cong \mathbf{S}$.
iii) *Both maps preserve finite intersections if and only if* **S** *is a lattice*, *in which case* ∩ *is* ∧.
iv) *The canonical map* $\mathbf{S} \to \mathbf{S}/\mathcal{D}$ *preserves intersections iff* **S** *is a lattice*, *so that* $\mathbf{S}/\mathcal{D} \cong \mathbf{S}$.

Proof. Suppose the map $\mathbf{S} \to \mathbf{S}/\mathcal{L}$ preserves finite intersections and let x and y be distinct in S, but $\mathcal{L}$-related. Then $x\cap y$ exists in a properly lower $\mathcal{D}$-class in S, while their images merge in $\mathbf{S}/\mathcal{L}$, giving them a trivial intersection in the $\mathcal{D}$-class of $\mathbf{S}/\mathcal{L}$ corresponding to that of both x and y in S. Thus (i) is seen. That cases for (ii) and (iv) are similar and (iii) follows immediately. □

Since neither $S/\mathcal{L}$ nor $S/\mathcal{R}$ need be isomorphic to S in general, they need not, in general, inherit intersections. This raises questions: *Under what added conditions must a skew lattice* S *having finite intersections imply that both* S/$\mathcal{L}$ *and* S/$\mathcal{R}$ *do also*? *Do examples exist where* **S** *has finite intersections but not* **S**/$\mathcal{L}$ *or* **S**/$\mathcal{R}$? *Conversely,* if S/$\mathcal{L}$ *and* S/$\mathcal{R}$ *do also:*

Theorem 4.4.11. *If a skew lattice* S *with finite intersections has a lattice section, then both* S/$\mathcal{L}$ *and* S/$\mathcal{R}$ *also have finite intersections*.

Proof. If T $\subseteq$ S is a lattice section in S, then a copy of S/$\mathcal{R}$ in S is given by T[$\mathcal{L}$] = $\bigcup_{t\in T}\mathcal{L}_t$. For all $x \in$ S let t_x be the unique element in $\mathcal{D}_x \cap$ T and set $x_\mathcal{L} = x \wedge t_x$ in T[$\mathcal{L}$]. Then $x_\mathcal{L}\ \mathcal{R}\ x$ and T[$\mathcal{L}$] = $\{x_\mathcal{L} | x \in S\}$. Given a, $b \in$ T[$\mathcal{L}$], we claim that its intersection $a \cap b$ in S is already in T[$\mathcal{L}$]. We use the fact that $x \geq y$ in S iff both $x_\mathcal{L} \geq y_\mathcal{L}$ in T[$\mathcal{L}$] and dually, $x_\mathcal{R} \geq y_\mathcal{R}$ in the dual subalgebra T[$\mathcal{L}$]. (In general, if x and y correspond to (x', x'') and (y', y'') in S/$\mathcal{R}$ $\times_{S/\mathcal{D}}$ S/$\mathcal{L}$, then $x \geq y$ iff both $x' \geq y'$ and $x'' \geq y''$.) Since $a = a_\mathcal{L}$ and $b = b_\mathcal{L}$, it follows that a, $b \geq (a \cap b)_\mathcal{L}$ in T[$\mathcal{L}$]. But since both $a \cap b\ \mathcal{D}\ (a \cap b)_\mathcal{L}$ and $a \cap b \geq (a \cap b)_\mathcal{L}$ by definition of $a \cap b$, $a \cap b$ *is* $(a \cap b)_\mathcal{L}$ in T[$\mathcal{L}$]. Being closed under $\cap$, T[$\mathcal{L}$] and also T[$\mathcal{R}$] have finite intersections and so do their copies, S/$\mathcal{R}$ and S/$\mathcal{L}$. □

Thus for a symmetric skew lattice S with finite intersections, if S/$\mathcal{D}$ is countable (so that S has a lattice section), then both S/$\mathcal{R}$ and S/$\mathcal{L}$ also have finite intersections. We can do better.

Theorem 4.4.12. *If a symmetric skew lattice* S *has finite intersections*, *so do* S/$\mathcal{R}$ *and* S/$\mathcal{L}$.

Proof. So let S be a symmetric skew lattice and let x' and y' in S/$\mathcal{R}$ be given with pre-images x and y in S. Let S_1 be the sub-algebra generated from x and y, let T be a lattice section of S_1 and let T[$\mathcal{L}$] be the maximal left-handed subalgebra T[$\mathcal{L}$] = $\bigcup_{t\in T}\mathcal{L}_t$ of S_1. If $x_\mathcal{L} = x \wedge t_x$ and $y_\mathcal{L} = y \wedge t_y$ in S_1 as in the previous proof, then $x_\mathcal{L}$ and $y_\mathcal{L}$ are also pre-images of x' and y' in S. Let S_2 be the generated result of adjoining $x_\mathcal{L} \cap y_\mathcal{L}$ in S to S_1 and let T_2 be a lattice section of S_2 extending T_1. Then $x_\mathcal{L} \cap y_\mathcal{L}$ must lie in $T_2[\mathcal{L}]$, as above.

Suppose next that x', $y' \geq z'$ in S/$\mathcal{R}$. If z is a pre-image of z' in S, let S_3 be the extension of S_2 generated from S_2 and z, and let T_3 be a lattice section of S_3 extending T_2. By what was seen in the above proof, $z_\mathcal{L}$ in $T_3[\mathcal{L}]$ is also a pre-image of z' such that both $x_\mathcal{L} \geq z_\mathcal{L}$ and $y_\mathcal{L} \geq z_\mathcal{L}$ hence $x_\mathcal{L} \cap y_\mathcal{L} \geq z_\mathcal{L}$. Clearly, if θ is the image of $x_\mathcal{L} \cap y_\mathcal{L}$ in S/$\mathcal{R}$, then x', $y' \geq \theta \geq z'$. Given that x' and y' are fixed and z' is arbitrary but subject to the constraint x', $y' \geq z'$, θ must be $x' \cap y'$ in S/$\mathcal{R}$. Since this must be true for all x', y' in S/$\mathcal{R}$, the latter has intersections. Likewise, so does S/$\mathcal{L}$. □

Indeed, the above arguments essentially show: *if a skew lattice* S *with finite intersections is such that for each countable sublattice* T *of* S/$\mathcal{D}$, *the inverse image* $S|_T$ *of* T *in* S *has a lattice section, then both* S/$\mathcal{R}$ and S/$\mathcal{L}$ *also have finite intersections*.

Theorem 4.4.13. *If a normal skew lattice* **S** *has finite intersections, then both* **S**/$\mathcal{L}$ *and* **S**/$\mathcal{R}$ *have finite intersections.*

Proof. A skew lattice **S** has finite intersections if and only if each principal ideal S∧x∧S has finite intersections. For a normal skew lattice S, each ideal S∧x∧S has a lattice section, namely x∧S∧x, with x∧S = (x∧S∧x)[$\mathcal{R}$] ≅ S/$\mathcal{L}$ and S∧x = (x∧S∧x)[$\mathcal{L}$] ≅ S/$\mathcal{R}$. Thus S/$\mathcal{L}$ and S/$\mathcal{R}$ must have finite intersections if S does. □

Conversely, one may ask: *if both* **S**/$\mathcal{L}$ *and* **S**/$\mathcal{R}$ *have finite intersections, must* S *also*? *Or do counterexamples exist*? Here is a case of the latter.

Example 4.4.8. Let S be the skew lattice defined on $\bigcup_{n\in\mathbb{Z}}\{a_n, b_n\}$, with the $\{a_n, b_n\}$ being a totally ordered family of $\mathcal{D}$-classes. For x, y in a common class $\{a_n, b_n\}$,

$$x \vee y = y \wedge x = \begin{cases} x, \text{ if } n \text{ is even.} \\ y, \text{ if } n \text{ is odd.} \end{cases}$$

Thus even classes are right-handed and odd classes are left-handed. If say $x \in \{a_m, b_m\}$ but $y \in \{a_n, b_n\}$ where $m < n$, then $x \vee y = y = y \vee x$ and $x \wedge y = x = y \wedge x$. Thus,

$$\text{both } a_{n+1}, b_{n+1} > \text{ both } a_n, b_n > \text{ both } a_{n-1}, b_{n-1} \quad \text{for all } n \in \mathbb{Z}.$$

Consequently neither $a_n \cup b_n$ nor $a_n \cap b_n$ exist, for all $n \in \mathbb{Z}$. On the other hand, both S/$\mathcal{L}$ and S/$\mathcal{R}$, have a $\mathcal{D}$-class structure

$$\ldots > \{a_{n+2}, b_{n+2}\} > \{c_{n+1}\} > \{a_n, b_n\} > \{c_{n-1}\} > \{a_{n-2}, b_{n-2}\} > \ldots$$

that is right-handed when n is even and left-handed when n is odd. Both $a_k \cap b_k = c_{k-1}$ and $a_k \cup b_k = c_{k+1}$. Cases of pairs of elements from distinct $\mathcal{D}$-classes are trivial, as they involve pairs x, y where $x > y$.

Observe that this example is trivially symmetric (since totally quasi-ordered), has many lattice sections, is distributive and thus is categorical. But for normal skew lattices, and in particular for skew Boolean algebras, we have the equivalence:

Theorem 4.4.14. *A normal skew lattice* **S** *has finite intersections if and only if both* **S**/$\mathcal{L}$ *and* **S**/$\mathcal{R}$ *have finite intersections. In particular, a skew Boolean algebra* **S** *has finite intersections iff both* **S**/$\mathcal{L}$ *and* **S**/$\mathcal{R}$ *do.*

Proof. We have already seen (⇒). As for (⇐), without loss in generality we represent **S** as the fibred product **S**/$\mathcal{L}$ $\times_{\mathbf{S}/\mathcal{D}}$ **S**/$\mathcal{R}$. So let both (x', x'') and (y', y'') in **S** be given where $x', y' \in$ **S**/$\mathcal{L}$ and

x'', $y'' \in \mathbf{S}/\mathcal{R}$. Since $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$ have finite intersections, $x'\cap y'$ and $x''\cap y''$ exist in $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$, respectively. If $x'\cap y'$ and $x''\cap y''$ share a common image in $\mathbf{S}/\mathcal{D}$, then $(x', x'') \cap (y', y'')$ is just $(x'\cap y', x''\cap y'')$. In general, let $u_0\wedge v_0$ be the meet in $\mathbf{S}/\mathcal{D}$ of the respective images u_0 of $x'\cap y'$ and v_0 of $x''\cap y''$ in $\mathbf{S}/\mathcal{D}$. In the respective $\mathcal{D}$ classes of $\mathbf{S}/\mathcal{L}$ and $\mathbf{S}/\mathcal{R}$ indexed by $u_0\wedge v_0$, unique elements w' and w'' exist (by normality) such that both $x'\cap y' \geq w'$ in $\mathbf{S}/\mathcal{L}$ and $x''\cap y'' \geq w''$ in $\mathbf{S}/\mathcal{R}$. The intersection $(x', x'') \cap (y', y'')$ is then precisely (w', w''). □

The practical consequence of all this as follows: just as all skew Boolean algebras can be constructed in principle from pairs of left- and right-handed skew Boolean algebras ($\mathbf{S}_L$, $\mathbf{S}_R$) having a common maximal commutative Boolean image $\mathbf{B}$ using the fibered product, $\mathbf{S}_L \times_{\mathbf{B}} \mathbf{S}_R$, so also all skew Boolean ∩-algebras can be constructed from pairs of left- and right-handed skew Boolean ∩-algebras ($\mathbf{S}_L$, $\mathbf{S}_R$) with a common maximal generalized Boolean algebra image $\mathbf{B}$ (with $\mathbf{S}_L$ and $\mathbf{S}_R$ viewed as skew Boolean algebras) by exactly the same process, $\mathbf{S}_L \times_{\mathbf{B}} \mathbf{S}_R$, even when their corresponding ∩-outcomes may have distinct locations relative to $\mathbf{B}$. It follows that the study of skew Boolean ∩-algebras can, in principle, be reduced to studying right-handed skew Boolean ∩-algebras, or their term-equivalent left-handed duals, since pairs sharing common maximal lattice images can be spliced together at will. All this is illustrated in the following case of infinite free skew Boolean algebras.

Atom splitting and the case of infinite free skew Boolean algebras

Consider the inclusion ${}_{\mathcal{L}}\mathbf{SBA}_n \subset {}_{\mathcal{L}}\mathbf{SBA}_{n+1}$ induced by $\{x_1, \ldots, x_n\} \subset \{x_1, \ldots, x_n, x_{n+1}\}$. The atoms of ${}_{\mathcal{L}}\mathbf{SBA}_n$ are no longer atomic in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$. The left-handed identity $x = (x\wedge y) + (x\backslash y)$ gives the following "subatomic" decomposition of the original atoms.

$$\begin{aligned}&(x_1\wedge x_2\wedge\ldots\wedge x_k) \backslash (x_{k+1}\vee\ldots\vee x_n)\\ &\qquad = (x_1\wedge x_2\wedge\ldots\wedge x_k\wedge x_{n+1}) \backslash (x_{k+1}\vee\ldots\vee x_n) + (x_1\wedge x_2\wedge\ldots\wedge x_k)\backslash (x_{k+1}\vee\ldots\vee x_{n+1})\end{aligned}$$

Both components of the new decomposition are atoms in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$. If say

$$(x_1\wedge x_2\wedge x_3)\backslash(x_4\vee\ldots\vee x_n) = (x_1\wedge x_3\wedge x_2)\backslash(x_4\vee\ldots\vee x_n)$$

in ${}_{\mathcal{L}}\mathbf{SBA}_n$, then their corresponding pairs of atomic components in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$ remain equal. But if say $(x_1\wedge x_2)\backslash(x_3\vee\ldots\vee x_n) \neq (x_2\wedge x_1)\backslash(x_3\vee x_4\vee\ldots\vee x_n)$ in ${}_{\mathcal{L}}\mathbf{SBA}_n$, then both corresponding pairs of components are likewise unequal in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$. One thus has *extended the decomposition* where, while a given element remains the same, its atomic decomposition doubles in length as each new generator is added. Thus given $u \in {}_{\mathcal{L}}\mathbf{SBA}_n$ with atomic decomposition $u = a_1 +\ldots+ a_r$ in ${}_{\mathcal{L}}\mathbf{SBA}_n$, each atom a_k splits as $b_k + c_k$ in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$, where $b_i = a_i\wedge x_{n+1}$ and $c_i = a_i\backslash x_{n+1}$, to give a revised atomic decomposition $u = b_1 + c_1 + \ldots + b_r + c_r$ in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$. Given the uniqueness of atomic decompositions (to within commutativity) of elements in ${}_{\mathcal{L}}\mathbf{SBA}_n$ or in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$, we have:

Lemma 4.4.15 *Given u be an atom of* $_{\mathcal{L}}\mathbf{SBA}_n$ *and let* $a = b + c$ *be the atomic decomposition* of a in $_{\mathcal{L}}\mathbf{SBA}_{n+1}$ *where* $b = a \wedge x_{n+1}$ *and* $c = a \backslash x_{n+1}$. *Then the following are equivalent*:

i) $u \geq a$ *in* $_{\mathcal{L}}\mathbf{SBA}_n$ (*and thus in* $_{\mathcal{L}}\mathbf{SBA}_{n+1}$).

ii) $u \geq b$ *in* $_{\mathcal{L}}\mathbf{SBA}_{n+1}$.

iii) $u \geq c$ *in* $_{\mathcal{L}}\mathbf{SBA}_{n+1}$. □

Comments. (1) Thus a is in the atomic decomposition of u in $_{\mathcal{L}}\mathbf{SBA}_n$ iff b [or c and hence both] is in the atomic decomposition of u in $_{\mathcal{L}}\mathbf{SBA}_{n+1}$.

(2) $_{\mathcal{L}}\mathbf{SBA}_{n+1}$ has n+1 "natural" copies of $_{\mathcal{L}}\mathbf{SBA}_n$ in it, each generated by one of n+1 subsets of $\{x_1, \ldots, x_{n+1}\}$ of size n. Likewise $\binom{n+1}{n-1}$ natural copies of $_{\mathcal{L}}\mathbf{SBA}_{n-1}$ lie in $_{\mathcal{L}}\mathbf{SBA}_{n+1}$, etc.

This leads us to infinite free algebras with necessarily infinite generating sets. If X is infinite, then $_{\mathcal{L}}\mathbf{SBA}_X$ is the upward directed union of its finite free subalgebras:

$$_{\mathcal{L}}\mathbf{SBA}_X = \bigcup\{_{\mathcal{L}}\mathbf{SBA}_Y \mid \varnothing \neq Y \subseteq X \;\&\; |Y| < \infty\}.$$

Given u and v of $_{\mathcal{L}}\mathbf{SBA}_X$, each occurs in some finite free subalgebra, say u in $_{\mathcal{L}}\mathbf{SBA}_Y$ and v in $_{\mathcal{L}}\mathbf{SBA}_Z$ for finite subsets Y and Z of X. Thus $u \wedge v$, $u \vee v$ and $u \backslash v$ are calculated in the larger finite subalgebra $_{\mathcal{L}}\mathbf{SBA}_{Y \cup Z}$ or in any finite $_{\mathcal{L}}\mathbf{SBA}_W$ where $Y \cup Z \subseteq W$. Of course calculations of $u \wedge v$, $u \vee v$ and $u \backslash v$ do not change in passing from $_{\mathcal{L}}\mathbf{SBA}_{Y \cup Z}$ to any properly larger $_{\mathcal{L}}\mathbf{SBA}_W$. What changes is their atomic decompositions; such changes, however, are derived from the original decompositions in $_{\mathcal{L}}\mathbf{SBA}_{Y \cup Z}$ by (possibly repeated) atomic splitting. *Ultimately in* $_{\mathcal{L}}\mathbf{SBA}_X$ *for X infinite, no atoms exist*. (If a is an atom, then it appears as such in $_{\mathcal{L}}\mathbf{SBA}_Y$ for some finite Y; but it immediately looses its atomic status in a properly larger free subalgebra.) Atoms are only relevant in its finite subalgebras. This is a fundamental difference between finite and infinite free algebras. Another fundamental difference is as follows.

Recall that the center of a skew lattice, consisting of elements that both $\wedge$-commute and $\vee$-commute with all elements, is the union of all singleton $\mathcal{D}$-classes. In $_{\mathcal{L}}\mathbf{SBA}_n$ (or $_{\mathcal{R}}\mathbf{SBA}_n$ or $\mathbf{SBA}_n$) it is the set of all n atoms of the form $x_1 \backslash (x_2 \vee \ldots \vee x_n)$ and the subalgebra they generate consisting of all orthosums of such atoms. But, except for 0, none of these orthosums remain central in $_{\mathcal{L}}\mathbf{SBA}_{n+1}$. For each atom,

$$\begin{aligned} x_{n+1} \wedge (x_1 \backslash (x_2 \vee \ldots \vee x_n)) &= ((x_{n+1} \wedge x_1) \backslash (x_2 \vee \ldots \vee x_n)) \\ &\neq ((x_1 \wedge x_{n+1}) \backslash (x_2 \vee \ldots \vee x_n)) = (x_1 \backslash (x_2 \vee \ldots \vee x_n)) \wedge x_{n+1} \end{aligned}$$

with the two new, unequal atoms being $\mathcal{D}$-related in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$. Thus, given any non-0 central element $c = a_1 + \ldots + a_k$ in ${}_{\mathcal{L}}\mathbf{SBA}_n$ with atoms a_i of the given form,

$$x_{n+1} \wedge c \;=\; (x_{n+1} \wedge a_1) + \ldots + (x_{n+1} \wedge a_k) \;\neq\; (a_1 \wedge x_{n+1}) + \ldots + (a_k \wedge x_{n+1}) \;=\; c \wedge x_{n+1}.$$

The case for ${}_{\mathcal{R}}\mathbf{SBA}_n$ and ${}_{\mathcal{R}}\mathbf{SBA}_{n+1}$, or $\mathbf{SBA}_n$ and $\mathbf{SBA}_{n+1}$, is similar. We thus have:

Theorem 4.4.16. *Given a finite free skew Boolean algebra on n generators, whether left-handed, right-handed or two sided, its center forms a Boolean algebra of order* 2^n. *In the case of an infinite free algebra, the center is just* $\{0\}$. □

We turn to a common property of all free algebras. Trivially, finitely generated free algebras have intersections since they are finite. It turns out that infinitely generated free algebras also have finite intersections. Thanks to our observations on adjoining free generators and their effects on atoms, *intersections are stable under the inclusion* ${}_{\mathcal{L}}\mathbf{SBA}_n \subset {}_{\mathcal{L}}\mathbf{SBA}_{n+1}$. Thus $\cap$ for pairs of elements in ${}_{\mathcal{L}}\mathbf{SBA}_3$ remains the same for *these* elements in the bigger, say ${}_{\mathcal{L}}\mathbf{SBA}_7$. What changes is the decomposition of all outcomes into atoms. The pool of atoms that two elements share in ${}_{\mathcal{L}}\mathbf{SBA}_n$, doubles by splitting to give rise to the new pool of atoms in ${}_{\mathcal{L}}\mathbf{SBA}_{n+1}$ that both share. As a result:

Theorem 4.4.17 *Given any set X, the free left-handed [right-handed] skew Boolean algebra* ${}_{\mathcal{L}}\mathbf{SBA}_X$ [${}_{\mathcal{R}}\mathbf{SBA}_X$] *on X has intersections. Given elements x and y in* ${}_{\mathcal{L}}\mathbf{SBA}_X$, $x \cap y$ *can be calculated in any subalgebra* ${}_{\mathcal{L}}\mathbf{SBA}_Y$, *where Y is any finite subset of X such that* ${}_{\mathcal{L}}\mathbf{SBA}_Y$ *contains both x and y. Similar remarks hold for* ${}_{\mathcal{R}}\mathbf{SBA}_X$.

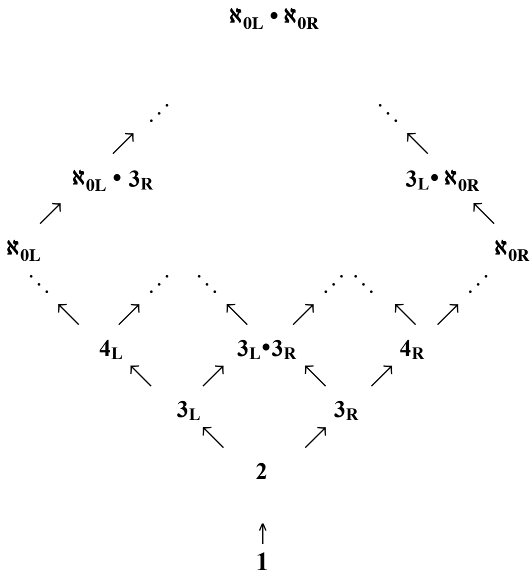
Proof. Suppose that x and y are encountered in ${}_{\mathcal{L}}\mathbf{SBA}_Y$ where Y is a finite subset of X and that u in ${}_{\mathcal{L}}\mathbf{SBA}_X$ is such that $u \leq$ both x and y. Then $x \cap y$ relative to ${}_{\mathcal{L}}\mathbf{SBA}_Y$ exists. By our remarks, this $x \cap y$ remains the intersection in any ${}_{\mathcal{L}}\mathbf{SBA}_Z$ where $Y \subset Z$ if Z is finite. Now u must be encountered in some finite subalgebra ${}_{\mathcal{L}}\mathbf{SBA}_U$ where $U \subseteq X$. Then both the current $x \cap y$ and u must lie in the larger subalgebra ${}_{\mathcal{L}}\mathbf{SBA}_{Y \cup U}$. Since $Y \cup U$ is finite, $x \cap y$ is the intersection here also, and $u \leq x \cap y$ follows. Thus $x \cap y$ remains the intersection of x and y throughout all ${}_{\mathcal{L}}\mathbf{SBA}_X$. The case for ${}_{\mathcal{R}}\mathbf{SBA}_X$ is similar. □

Since ${}_{\mathcal{R}}\mathbf{SBA}_X \cong \mathbf{SBA}_X/\mathcal{L}$ and ${}_{\mathcal{L}}\mathbf{SBA}_X \cong \mathbf{SBA}_X/\mathcal{R}$, by Theorem 4.4.14 we have:

Theorem 4.4.18. *Free skew Boolean algebras have finite intersections*. (Indeed, they have arbitrary intersections since a properly infinite subset must have 0-intersection.) □

The lattice of subvarieties of skew Boolean ∩-algebras

In what follows we observe the following notation. For all $n \le \aleph_0$, $\boldsymbol{n}_{\mathrm{L}}$ [respectively $\boldsymbol{n}_{\mathbf{R}}$] denotes the left-[right-]handed primitive skew Boolean ∩-algebra on $\boldsymbol{n} = \{0, 1, 2, \ldots , n-1\}$ and $\aleph_0 = \{0, 1, 2, \ldots \}$ with 0 the zero element. Given two primitive algebras **A** and **B**, we denote their fibred product $\mathbf{A} \times_{\mathbf{2}} \mathbf{B}$ by **A•B**. Finally, given any skew Boolean ∩-algebra **A**, $\langle \mathbf{A} \rangle^{\cap}$ denotes the principal subvariety of all skew Boolean ∩-algebras generated by **A** in that they satisfy all identities satisfied by **A**. Consider the following lattice of primitive subalgebras of the primitive algebra $\aleph_{0\mathbf{L}} \bullet \aleph_{0\mathbf{R}}$ viewed as a skew Boolean ∩-algebra



In this diagram, each $\boldsymbol{n}_{\mathrm{L}}$ is identified with the trivial fibered product, $\boldsymbol{n}_{\mathrm{L}}\bullet\mathbf{2}$, and each $\boldsymbol{n}_{\mathrm{R}}$ is identified with the trivial fibered product, $\mathbf{2}\bullet\boldsymbol{n}_{\mathrm{R}}$. The embeddings → are induced from the standard chain of inclusions: $\{0\} \subset \{0, 1\} \subset \{0, 1, 2\} \subset \{0, 1, 2, 3\} \subset \ldots$.

Proposition 4.4.19 *The map* $\mathbf{A} \to \langle \mathbf{A} \rangle^{\cap}$ *applied to the above diagram of inclusions induces a corresponding diagram of strict inclusions of the respective varieties, with*

i) $\langle \aleph_{0\mathbf{L}} \bullet \boldsymbol{n}_{\mathbf{R}} \rangle^{\cap} = \bigcup\{\langle \boldsymbol{m}_{\mathrm{L}}\bullet\boldsymbol{n}_{\mathbf{R}} \rangle^{\cap} \mid m < \aleph_{\mathbf{0}} \}$ *for all* $n < \aleph_{\mathbf{0}}$.

ii) $\langle \boldsymbol{m}_{\mathrm{L}} \bullet \aleph_{0\mathbf{R}} \rangle^{\cap} = \bigcup\{\langle \boldsymbol{m}_{\mathrm{L}}\bullet\boldsymbol{n}_{\mathbf{R}} \rangle^{\cap} \mid n < \aleph_{\mathbf{0}} \}$ *for all* $m < \aleph_{\mathbf{0}}$.

iii) $\langle \aleph_{0\mathrm{L}}\bullet \aleph_{0\mathbf{R}} \rangle^{\cap} = \bigcup\{\langle \boldsymbol{m}_{\mathrm{L}}\bullet\boldsymbol{n}_{\mathbf{R}} \rangle^{\cap} \mid m, n < \aleph_{\mathbf{0}} \}$.

Proof. Clearly we have a diagram of inclusions. Since any equation in the operations of the signature contains only finitely many variables, the final three assertions are also clear. Thus we need only show the inclusions of the induced subvarieties to be proper in the case of the finite primitive algebras. We first show $\langle \boldsymbol{n}_{\mathbf{L}}\rangle^{\cap} \subset \langle \boldsymbol{n}\mathbf{+1}_{\mathbf{L}}\rangle^{\cap}$ to be proper for all finite n. To begin, $x \approx y$ holds on $\langle \mathbf{1}\rangle^{\cap}$ but not on $\langle \mathbf{2}\rangle^{\cap}$. Next, for $n \geq 2$ we set

$$\Phi_n(x_1, x_2, \ldots, x_n) = (x_1 \setminus (x_1 \cap x_2)) \wedge (x_2 \setminus (x_2 \cap x_3)) \wedge \ldots \wedge (x_n \setminus (x_n \cap x_1))$$

and

$$\Psi_n(x_1, x_2, \ldots, x_n) = x_1 \setminus [(x_1 \cap x_2) \vee \ldots \vee (x_1 \cap x_n) \vee (x_2 \cap x_3) \vee \ldots \vee (x_2 \cap x_n) \vee \ldots \vee (x_{n-1} \cap x_n)].$$

Since $\Phi_n(a_1, a_2, \ldots, a_n) \neq 0$ only if none of $a_1, a_2, \ldots, a_n$ is 0.

$$\Phi_n(x_1, x_2, \ldots, x_n) \wedge \Psi_n(x_1, x_2, \ldots, x_n) \approx 0$$

holds on $\boldsymbol{n}_{\mathbf{L}}$ but not on $\boldsymbol{n}\mathbf{+1}_{\mathbf{L}}$ (respectively, on $\boldsymbol{n}_{\mathbf{R}}$ but not on $\boldsymbol{n}\mathbf{+1}_{\mathbf{R}}$) for all $n \geq 2$. Thus all inclusions at least along the two lower sides of the above diagram are proper. But this forces all links in the above diagram to be proper. For instance, $\langle \boldsymbol{m}_{\mathbf{L}}\bullet\boldsymbol{n}_{\mathbf{R}}\rangle^{\cap} \subset \langle \boldsymbol{m}\mathbf{+1}_{\mathbf{L}}\bullet\boldsymbol{n}_{\mathbf{R}}\rangle^{\cap}$ for $m < \aleph_0$ and $n \leq \aleph_0$ is indeed proper since $\Phi_m(x_1\wedge y, x_2\wedge y, \ldots, x_m\wedge y) \wedge \Psi_m(x_1\wedge y, x_2\wedge y, \ldots, x_m\wedge y) \approx 0$ must hold in $\langle \boldsymbol{m}_{\mathbf{L}}\bullet\boldsymbol{n}_{\mathbf{R}}\rangle^{\cap}$ but not in $\langle \boldsymbol{m}\mathbf{+1}_{\mathbf{L}}\bullet\boldsymbol{n}_{\mathbf{R}}\rangle^{\cap}$. □

This ascending array of principal varieties leads us to several results, beginning with:

Theorem 4.4.20. *Skew Boolean ∩-algebras are locally finite.*

Proof. We revise the argument from Theorem 4.1.10. Given a skew Boolean ∩-algebra S generated from a finite set X of size n, if φ: S → P is a nontrivial homomorphism from S to a primitive algebra, then φ[S] is a primitive subalgebra P′ of P that is isomorphic to a subalgebra of $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$. It follows that a homomorphism of φ': S → $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$ exists inducing the same congruence on S that φ has. Moreover, only finitely many distinct homomorphisms from S to $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$ are possible since S is generated from X. By the argument of Theorem 4.1.10, S can be embedded in a finite power of $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$ making S itself finite. □

Theorem 4.4.21. *A (quasi-)identity of signature* (∨, ∧, \, ∩, 0) *in n variables holds for all skew Boolean ∩-algebras if and only if it holds in* $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$. *Likewise the (quasi-)identity holds for all left-handed (right-handed) skew Boolean ∩-algebras if and only if it holds in* $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}$ *(or in* $\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$*). The question of when a given (quasi-)identity holds for all (left-handed or right handed) skew Boolean ∩-algebras is thus decidable.* □

Proof. If a (quasi-)identity in variables $x_1, \ldots, x_n$ holds for all skew Boolean ∩-algebras, in particular it holds for $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$. Conversely if it holds for $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$, it holds for all powers of $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}$. Thus, given skew Boolean ∩-algebra S with $a_1, a_2, \ldots, a_n \in$ S, the latter collectively generate a subalgebra S′ of S that (by previous arguments) can be embedded in some finite power of $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$. Since the given (quasi-)identity holds for all powers of $\boldsymbol{n}\mathbf{+1}_{\mathrm{L}}\bullet\boldsymbol{n}\mathbf{+1}_{\mathrm{R}}$, it holds for S′ and in particular for the assignments, x_1 to a_1, x_2 to a_2, … , and x_n to a_n. Since S and the a_i are arbitrary, the theorem follows in the two-sided case. The left-handed and right-handed cases are similar. ☐

This above ascending array of principal varieties is only part of the full lattice of all varieties of skew Boolean ∩-algebras.

Given a partially ordered set $\mathbf{P} = (P, \geq)$, an ***order ideal*** of $\mathbf{P}$ is any nonempty subset I of P satisfying the following implication: $x \in I$ and $x \geq y$ in P implies $y \in I$ also. The following result of Brian Davey is relevant. (See Theorems 3.3 and 3.5 in [Davey 79].)

Theorem 4.4.22. [Davey 79] *Let $\mathcal{V}$ be a locally finite, congruence distributive variety. Then its lattice of subvarieties of $\mathcal{V}$ is completely distributive and is isomorphic to the lattice of order ideals of its partially ordered set of principal subvarieties generated by finite, subdirectly irreducible algebras and ordered by subvariety inclusion.* ☐

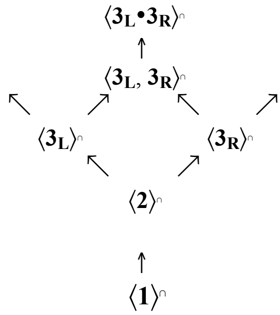
Theorem 4.4.23. *The lattice of subvarieties of skew Boolean ∩-algebras is isomorphic to the set of all order ideals of the following lattice of principal subvarieties generated by the finite primitive algebras.*

$$
\begin{array}{ccccccccccccc}
 & & & \ddots & & \iddots & & \ddots & & \iddots & & & \\
\ddots & & & & \langle \mathbf{4_L}\bullet\mathbf{3_R}\rangle^{\cap} & & & & \langle \mathbf{3_L}\bullet\mathbf{4_R}\rangle^{\cap} & & & & \iddots \\
 & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\
 & & \langle \mathbf{4_L}\rangle^{\cap} & & & & \langle \mathbf{3_L}\bullet\mathbf{3_R}\rangle^{\cap} & & & & \langle \mathbf{4_R}\rangle^{\cap} & & \\
 & & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & & \\
 & & & & \langle \mathbf{3_L}\rangle^{\cap} & & & & \langle \mathbf{3_R}\rangle^{\cap} & & & & \\
 & & & & & \nwarrow & & \nearrow & & & & & \\
 & & & & & & \langle \mathbf{2}\rangle^{\cap} & & & & & & \\
 & & & & & & \uparrow & & & & & & \\
 & & & & & & \langle \mathbf{1}\rangle^{\cap} & & & & & &
\end{array}
$$

The varieties of skew Boolean ∩-algebras, left- and right-handed skew Boolean ∩-algebras, and generalized Boolean algebras correspond to the order ideals given respectively by the entire lattice, the infinite ideal on the lower left $\{\langle \mathbf{1}\rangle^{\cap}, \langle \mathbf{2}\rangle^{\cap}, \langle \mathbf{3_L}\rangle^{\cap}, \langle \mathbf{4_L}\rangle^{\cap}, \ldots\}$, *the corresponding infinite ideal on the lower right, and the ideal* $\{\langle \mathbf{1}\rangle^{\cap}, \langle \mathbf{2}\rangle^{\cap}\}$.

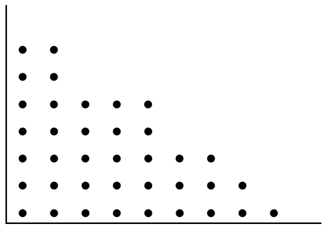
Proof. This follows from Proposition 4.4.19 and Theorems 4.4.20 and 4.4.22. ☐

The bottom of the lattice of subvarieties is as follows. We will look closely at a class of algebras in $\langle \mathbf{3_L}\rangle^{\cap}$ in the next section.



The sublattice of nontrivial varieties containing at least $\langle \mathbf{2} \rangle$, is described in Cartesian fashion with abbreviated notation in the following figure to the left. Here m,n stands for $(\boldsymbol{m}+\mathbf{1})_{\mathbf{L}}\bullet(\boldsymbol{n}+\mathbf{1})_{\mathbf{R}}$ with m and n counting the size of both non-0 $\mathcal{D}$-classes. One has $m,n \geq p,q$ when both $m \geq p$ and $n \geq q$, i.e., p,q lies non-strictly to the lower left of m,n.

⋮	⋮	⋮	⋮	⋮	⋮	⋰
5	5,1	5,2	5,3	5,4	5,5	⋯
4	4,1	4,2	4,3	4,4	4,5	⋯
3	3,1	3,2	3,3	3,4	3,5	⋯
2	2,1	2,2	2,3	2,4	2,4	⋯
1	1,1	1,2	1,3	1,4	1,5	⋯
	1	2	3	4	5	⋯



Within the quadrant, an ideal corresponds to a non-increasing array of the terraced form to the right and as such is described by a non-strictly decreasing function f from $\{1, 2, 3, \ldots\}$ to $\{0, 1, 2, 3, \ldots, \aleph_0\}$. The function f for the above array is thus

$$f = \begin{array}{c|cccccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\ \hline f(n) & 7 & 7 & 5 & 5 & 5 & 3 & 3 & 2 & 1 & 0 & \cdots \end{array}$$

corresponding to the variety $\langle \mathbf{8_L \bullet 2_R} \rangle^{\cap} \cup \langle \mathbf{6_L \bullet 4_R} \rangle^{\cap} \cup \langle \mathbf{4_L \bullet 7_R} \rangle^{\cap} \cup \langle \mathbf{3_L \bullet 9_R} \rangle^{\cap} \cup \langle \mathbf{2_L \bullet 10_R} \rangle^{\cap}$.

Since only finitely many strict decreases in the output are possible, the number of such functions [ideals] is countably infinite. Even the trivial ideal $\langle \mathbf{1} \rangle^{\cap}$ can be represented as the zero function: $z(n) = 0$ for all n. The lattice operations are evaluated in point-wise fashion: $(f \vee g)(n) = max\{f(n), g(n)\}$ and $(f \wedge g)(n) = min\{f(n), g(n)\}$. We thus have:

Theorem 4.4.24 *The lattice of varieties of skew Boolean algebras with intersections is isomorphic to the lattice of non-strictly decreasing functions from the set* $\{1, 2, 3, \ldots\}$ *to the set* $\{0, 1, 2, 3, \ldots \aleph_0\}$ *with the join and meet operations given pointwise. In either the right or left-handed cases, the lattice of varieties is isomorphic to the usual ordering on* $\{1, 2, \ldots, \aleph_0\}$. ☐

4.5 Omega algebras and skew Boolean covers

Given a lattice $\mathbf{L} = (L; \vee, \wedge)$, upon setting $\omega(\mathbf{L}) = \{(a, b) \in L \times L \mid a \geq b\}$ we obtain a sublattice of $\mathbf{L} \times \mathbf{L}$ that clearly is distributive when $\mathbf{L}$ is. But what if $\mathbf{L}$ is a (possibly generalized) Boolean lattice with zero 0, so that for each $a \in L$, the principal poset ideal $\lceil a \rceil = \{b \in L \mid a \geq b\}$ is a Boolean lattice?

Proposition 4.5.1. *Given a lattice* $\mathbf{L}$ *with minimal element* 0, $\omega(\mathbf{L})$ *forms a generalized Boolean lattice if and only if* $\mathbf{L}$, *and hence* $\omega(\mathbf{L})$, *is trivial.*

Proof. If $a > 0$ for some $a \in L$, then $(a, 0)$ has no complement in $\lceil (a, a) \rceil$. ☐

Given a generalized Boolean algebra $\mathbf{B}$, alternative *twisted* meet and join operations can be defined on the underlying set of $\omega(\mathbf{B})$ to give a Boolean skew lattice as follows:

$$(a, a') \wedge (b, b') = (a \wedge b, a' \wedge b) \quad \text{and} \quad (a, a') \vee (b, b') = (a \vee b, (a' \setminus b) \vee b').$$

Here all component operations are in $\mathbf{B}$ with $a \setminus b$ being the relative complement of $a \wedge b$ in $\lceil a \rceil$.

Lemma 4.5.2. *Under the operations above,* $\omega(\mathbf{B})$ *is left-handed skew lattice with a zero*.

Proof. That $\wedge$ is idempotent and associative is easily checked, as is the fact that $\vee$ is idempotent. Associativity of $\vee$ reduces to comparing the second coordinates of $[(a, a') \vee (b, b')] \vee (c, c')$ and $(a, a') \vee [(b, b') \vee (c, c')]$ to see if $\{[(a' \setminus b) \vee b'] \setminus c\} \vee c' = [a' \setminus (b \vee c)] \vee (b' \setminus c) \vee c'$. But

$$[(a' \setminus b) \vee b'] \setminus c = [a' \setminus (b \vee c)] \vee (b' \setminus c)$$

holds for generalized Boolean algebras. Next, expanding gives:

$$(a, a') \wedge [(a, a') \vee (b, b')] = (a, a') \wedge (a \vee b, (a' \setminus b) \vee b') = (a \wedge (a \vee b), a' \wedge (a \vee b)) = (a, a')$$

and

$$(a, a') \vee [(a, a') \wedge (b, b')] = (a, a') \vee (a \wedge b, a' \wedge b) = (a \vee (a \wedge b), [a' \setminus (a \wedge b)] \vee (a' \wedge b)) = (a, a').$$

In similar fashion, the remaining pair of absorption identities are verified. Clearly (0, 0) is the zero element. Finally, $(a, a') \wedge (b, b') \wedge (a, a') = (a, a') \wedge (b, b')$ since both reduce to $(a \wedge b, a' \wedge b)$. Thus the identity $x \wedge y \wedge x = x \wedge y$ holds, as well as its equivalent dual, $x \vee y \vee x = y \vee x$. ☐

Lemma 4.5.3. $(\omega(\mathbf{B}); \vee, \wedge)$ *is a symmetric skew lattice. If* (a, a') *and* (b, b') *commute, then* $(a, a') \wedge (b, b') = (a \wedge b, a' \wedge b')$ *and* $(a, a') \vee (b, b') = (a \vee b, a' \vee b')$.

Proof. Given $(a', a), (b', b) \in \omega(\mathbf{B})$, $(a, a') \wedge (b, b') = (b, b') \wedge (a, a')$ if and only if $a' \wedge b = a \wedge b'$ in the second coordinate of the outcomes, with both equaling $a' \wedge b'$. Likewise, $(a, a') \vee (b, b') = (b, b') \vee (a, a')$ if and only if $(a' \setminus b) \vee b' = (b' \setminus a) \vee a'$ in the second coordinate of the outcomes with both equaling $a' \vee b'$. Assuming $a' \wedge b = a \wedge b'$ we get

$$(a' \setminus b) \vee b' \;=\; (a' \setminus (a' \wedge b)) \vee b' \;=\; (a' \setminus a) \vee (a' \setminus b') \vee b' \;=\; (a' \setminus b') \vee b' \;=\; a' \vee b'.$$

Similarly $(b' \setminus a) \vee a' = a' \vee b'$ so that $(a' \setminus b) \vee b' \;=\; (b' \setminus a) \vee a'$ follows. Conversely given the latter, one has $a' \wedge [(a' \setminus b) \vee b'] \wedge b = a' \wedge [(b' \setminus a) \vee a'] \wedge b$ which yields $a' \wedge b' = a' \wedge b$. Similarly,

$$a \wedge [(a' \setminus b) \vee b'] \wedge b' = a \wedge [(b' \setminus a) \vee a'] \wedge b' \text{ yields } a \wedge b' = a' \wedge b. \;\square$$

Lemma 4.5.4. *In* $\omega(\mathbf{B})$:

i) $(a, a') \succeq (b, b')$ *if and only if* $a \geq b$ *in* B.
ii) $(a, a') \,\mathcal{D}\, (b, b')$ *if and only if* $a = b$ *in* B.
iii) $(a, a') \geq (b, b')$ *if and only if* $a \geq b$ *in* B *with* $b' = a' \wedge b$.

Proof. Since $(b, b') \wedge (a, a') \wedge (b, b') = (a \wedge b, b' \wedge a \wedge b)$, we have $(b, b') \wedge (a, a') \wedge (b, b') = (b, b')$ if and only if $a \wedge b = b$ in $\mathbf{B}$. Thus (i) holds and (ii) follows immediately. As for (iii),

$$(a, a') \wedge (b, b') \;=\; (b, b') \;=\; (b, b') \wedge (a, a')$$

becomes $a' \wedge b = b' = b' \wedge a$ in the second coordinate, and $b = a \wedge b$ in the first coordinate. But these conditions, plus $b \geq b'$ are equivalent to both $a \geq b$ and $b' = a' \wedge b$ holding in $\mathbf{B}$. $\square$

Theorem 4.5.5. *Upon letting* $(a, a') \setminus (b, b')$ *be the complement of* $(a, a') \wedge (b, b')$ *in the Boolean sublattice* $\lceil (a, a') \rceil$, $(\omega(\mathbf{B}); \vee, \wedge, \setminus, (0, 0))$ *becomes a skew Boolean algebra for which* $\omega(\mathbf{B})/\mathcal{D} \;\cong\; \mathbf{B}$. *Moreover,* $\omega(\mathbf{B})$ *has finite intersections. Thus, as a skew Boolean* $\cap$*-algebra* $\omega(\mathbf{B})$ *has a distributive congruence lattice.*

Proof. $\omega(\mathbf{B})$ is symmetric by Lemma 4.5.3. Since $\lceil (a, a') \rceil \;=\; \{(b, a' \wedge b) \mid a \geq b\} \cong \lceil a \rceil$ by Lemma 4.5.4, we see that $\lceil (a, a') \rceil$ is a Boolean lattice, making $\omega(\mathbf{B})$ also normal and hence a skew Boolean lattice. The first assertion is now clear.

Next, $(a, a') \cap (b, b')$ is given by $([a' \wedge b'] + [(a \wedge b) \setminus (a' \vee b')],\, a' \wedge b')$ where + is the symmetric difference in $\mathbf{B}$: $x + y = (x \setminus y) \vee (y \setminus x)$. That

$$(a, a'), (b, b') \;\geq\; ([a' \wedge b'] + [(a \wedge b) \setminus (a' \vee b')],\, a' \wedge b')$$

follows from Lemma 4.5.4 and basic Boolean algebra. So let $(a, a'), (b, b') \geq (c, c')$. Then $a \wedge b \geq c$ and $a' \wedge b' \geq c'$ with $a' \wedge c = c'$ and $b' \wedge c = c'$ imply

$$c \wedge \{[a' \wedge b'] + [a \wedge b \setminus (a' \vee b')]\} = c' + (c \setminus c') = c \;\text{ and }\; c \wedge [a' \wedge b'] = c' \wedge c' = c'$$

so that $([a' \wedge b'] + [a \wedge b \setminus (a' \vee b')],\, a' \wedge b') \;\geq\; (c, c')$. Thus $(a, a') \cap (b, b')$ is as stated. The final assertion follows from Theorem 4.4.3. $\square$

Algebras of the form $\omega(\mathbf{B})$ with operations as defined above are called ***omega algebras***. These algebras lie in the varieties of left-handed skew Boolean algebras and left-handed skew Boolean $\cap$-algebras, with the particular emphasis indicated by the context.

Characterizing omega algebras

Given a skew lattice $\mathbf{S}$, recall that a ***lattice section*** of $\mathbf{S}$ is any sublattice $\mathbf{S}_0$ that intersects every $\mathcal{D}$-class of $\mathbf{S}$. Any such lattice must be isomorphic to the maximal lattice image $\mathbf{S}/\mathcal{D}$ of $\mathbf{S}$.

Lemma 4.5.6. *Given an omega algebra* $\omega(\mathbf{B})$, $\mathbf{S}_0 = \{(a, a) \mid a \in B\}$ *is a lattice section.* ☐

Lemma 4.5.7. *Given* $(a, a') \in \omega(\mathbf{B})$, *the map* $(a, b') \to (a, a') \cap (a, b')$ *is a bijection of the* $\mathcal{D}$*-class of* (a, a') *with the principal ideal* $\lceil (a, a') \rceil$ *of* $(\omega(\mathbf{B}), \geq)$.

Proof. In $\omega(\mathbf{B})$ the ideal $\lceil (a, a') \rceil$ is just the set $\{(b, a' \wedge b) \mid a \geq b\}$. Given $a' \leq a$, then for all $x \leq a$, $(a, a') \cap (a, x) = ([a' \wedge x] + [a \setminus (a' \vee x)], a' \wedge x)$. Given any $b \leq a$, we seek some $x \leq a$ such that $([a' \wedge x] + [a \setminus (a' \vee x)], \; a' \wedge x) = (b, a' \wedge b)$ in $\lceil (a, a') \rceil$. Setting $x = a' \wedge b \vee [a \setminus (a' \vee b)]$ we get, $a' \wedge x = (a' \wedge b) \vee 0 = a' \wedge b$ and thus

$$
\begin{aligned}
[a' \wedge x] + [a \setminus (a' \vee x)] &= a' \wedge b + [a \setminus (a' \vee [a \setminus (a' \vee b)])] \\
&= a' \wedge b + [a \setminus (a' \vee [(a \setminus a') \wedge (a \setminus b)])] \\
&= a' \wedge b + [(a \setminus a') \wedge (a' \vee b)] = b.
\end{aligned}
$$

Is $x \leq a$ unique? Suppose that $y \leq a$ also satisfies the desired conditions. Then $a' \wedge x = a' \wedge b = a' \wedge y$ and $[a' \wedge x] + [a \setminus (a' \vee x)] = b = [a' \wedge y] + [a \setminus (a' \vee y)]$, from which also follows first $a \setminus (a' \vee x) = a \setminus (a' \vee y)$ and then $a' \vee x = a' \vee y$. Cancelling $a' \wedge x = a' \wedge y$ and $a' \vee x = a' \vee y$ in $\mathbf{B}$ gives $x = y$. ☐

We thus have the following characterization of omega algebras:

Theorem 4.5.8. *A left-handed skew Boolean* $\cap$*-algebra* $\mathbf{S}$ *is isomorphic to* $\omega(\mathbf{B})$ *for some generalized Boolean algebra* $\mathbf{B}$ *if and only if* $\mathbf{S}$ *has a lattice section* $\mathbf{S}_0$ *and for all* $e \in S$ *the map* $a \to e \cap a$ *is a bijection of the* $\mathcal{D}$*-class* $\mathcal{D}_e$ *with the principal poset ideal* $\lceil e \rceil$. *Under these conditions* $\mathbf{S}$ *is isomorphic to both* $\omega(\mathbf{S}_0)$ *and* $\omega(\mathbf{S}/\mathcal{D})$.

Proof. The conditions are clearly necessary. Conversely, given these conditions, for each $e \in \mathbf{S}_0$ let β_e: $\mathcal{D}_e \to \omega(\mathbf{S}_0)$ be given by $\beta_e(f) = (e, e \cap f)$. Next we define β: S $\to \omega(\mathbf{S}_0)$ to be the union $\bigcup_e \beta_e$. β is at least a bijection from S to $\omega(\mathbf{S}_0)$. It also preserves $\wedge$. Given e_1 and e_2 in $\mathbf{S}_0$, then for all $f_1 \, \mathcal{D} \, e_1$ and $f_2 \, \mathcal{D} \, e_2$ in S we have

$$\beta[f_1 \wedge f_2] = (e_1 \wedge e_2, \; e_1 \wedge e_2 \cap f_1 \wedge f_2) = (e_1 \wedge e_2, \; e_1 \wedge e_2 \cap f_1 \wedge f_2 \wedge e_2) = (e_1 \wedge e_2, \; e_1 \wedge e_2 \cap f_1 \wedge e_2)$$

and

$$\beta[f_1] \wedge \beta[f_2] = (e_1, \; e_1 \cap f_1) \wedge (e_2, \; e_2 \cap f_2) = (e_1 \wedge e_2, [e_1 \cap f_1] \wedge e_2) = (e_1 \wedge e_2, e_1 \cap f_1 \cap e_2).$$

Since $\wedge$ is a left normal operation, $e_1\wedge e_2 \cap (f_1\wedge e_2) = e_1 \cap e_2 \cap (f_1\wedge e_2) = e_1 \cap f_1 \cap e_2$. Indeed we have $e_1 \cap e_2 \cap (f_1\wedge e_2) \leq e_1 \cap e_2 \cap f_1 = e_1 \cap f_1 \cap e_2$. But on the other hand, $f_1 \cap e_2 \leq f_1\wedge e_2$ so that $e_1 \cap f_1 \cap e_2 \leq e_1 \cap (f_1\wedge e_2) \cap e_2 = e_1 \cap e_2 \cap (f_1\wedge e_2)$. β also preserves $\vee$. Given e_1 and e_2 in $\mathbf{S}_0$, then $\beta[f_1\vee f_2] = (e_1\vee e_2, (e_1\vee e_2) \cap (f_1\vee f_2))$ and

$$\beta[f_1]\vee\beta[f_2] = (e_1,\ e_1 \cap f_1) \vee (e_2,\ e_2 \cap f_2) = (e_1\vee e_2, [(e_1 \cap f_1) \setminus e_2] \vee [e_2 \cap f_2])$$

for all $f_1\ \mathcal{D}\ e_1$ and $f_2\ \mathcal{D}\ e_2$ in S. It remains to show that

$$(e_1\vee e_2) \cap (f_1\vee f_2) = ((e_1 \cap f_1) \setminus e_2) \vee (e_2 \cap f_2)$$

is an identity for left-handed skew Boolean $\cap$-algebras, subject to the conditions $f_1\ \mathcal{D}\ e_1$ and $f_2\ \mathcal{D}\ e_2$. This is indeed the case for left-handed primitive skew Boolean $\cap$-algebras. To see this, first assume $e_1 = f_1 = 0$. Here the equation reduces to the identity $e_2 \cap f_2 = 0 \vee (e_2 \cap f_2)$. Likewise, in the case $e_2 = f_2 = 0$ the equation reduces to the identity $e_1 \cap f_1 = ((e_1 \cap f_1) \setminus 0) \vee 0$. Otherwise all four elements lie in the unique nonzero class and we get $e_2 \cap f_2 = 0 \vee (e_2 \cap f_2)$. Thus the conditional equality holds on primitive algebras. But this explicit $\mathcal{D}$-condition can be removed upon replacing e_1, f_1, e_2 and f_2 by $e_1\wedge f_1$, $f_1\wedge e_1$, $e_2\wedge f_2$ and $f_2\wedge e_2$ respectively. Hence the above conditional identity holds for all left-handed skew Boolean $\cap$-algebras. We have shown that β also preserves joins and hence β: $\mathbf{S} \cong \omega(\mathbf{S}_0)$. $\square$

Example 4.5.1. In particular, *a primitive skew Boolean algebra* **S** *is isomorphic to an omega algebra if and only if it is left-handed of order* 3. In this case we are looking at a copy of $\mathbf{3}_{\mathrm{L}}$, which is what $\omega(\mathbf{2})$ is for the primitive Boolean algebra **2**: 1 > 0.

$$\mathbf{3}_{\mathrm{L}}:\quad \begin{array}{c} 1-2 \\ \vee \\ 0 \end{array}\ . \qquad\qquad \omega(\mathbf{2}):\quad \begin{array}{c} (1,1)-(1,0) \\ \vee \\ (0,0) \end{array}\ .$$

Theorem 4.5.9. $\omega(\mathbf{B})$ *is complete if and only if* **B** *is complete. It is both complete and atomic if and only if* **B** *is thus, in which case* $\omega(\mathbf{B}) \cong \mathbf{3}_{\mathrm{L}}{}^{|A(\mathbf{B})|}$ *where* $A(\mathbf{B})$ *is the set of atoms of* **B**. *Thus a complete, atomic skew Boolean algebra is isomorphic to some omega algebra if and only if it is left-handed and each of its atomic* $\mathcal{D}$*-classes has exactly two elements.*

Proof. If **B** is complete and $\mathcal{A} = \{(a_i, a'_i)\ |\ i \in \mathrm{I}\}$ is a pairwise commuting subset of $\omega(\mathbf{B})$, we claim that $sup(\mathcal{A}) = (sup a_j,\ sup a'_j)$ with both suprema taken in **B**. By Lemma 4.5.4, if $(b, b') \geq (a_i, a'_i)$, then $b \geq sup a_j$ and $b' \geq sup a'_j$. Thus, we need only show $(sup a_j, sup a'_j) \geq$ each (a_i, a'_i) in $\omega(\mathbf{B})$. Lemma 4.5.3 and the definition of $\wedge$ in $\omega(\mathbf{B})$ give

$$a_i \wedge sup_j\, a'_j \ =\ sup_j\, (a_i \wedge a'_j)\ =\ sup_j\, (a'_i \wedge a'_j)\ =\ a'_i$$

which is what Lemma 4.5.4 requires for $(sup a_j, sup a'_j) \geq (a_i, a'_i)$. Since $\omega(\mathbf{B})$ is normal with 0 and has commuting suprema, all nonempty subsets of $\omega(\mathbf{B})$ also have infima.

In general, for any **B** the atoms of $\omega(\mathbf{B})$ have either the form $(x, 0)$ or the form (x, x) where x is an atom of **B**. When **B** is both complete and atomic, then for all $(a, a') \in \omega(\mathbf{B})$,

$$(a, a') = sup\big[\{(x, x) \mid x \in A(\mathbf{B}), x \leq a'\} \cup \{(x, 0) \mid x \in A(\mathbf{B}), x \leq a \setminus a'\}\big].$$

Thus $\omega(\mathbf{B})$ is also atomic in which case $\omega(\mathbf{B})$ is isomorphic to a product of primitive algebras, all of which are copies of $\mathbf{3}_{\mathrm{L}}$ since each atomic $\mathcal{D}$-class of $\omega(\mathbf{B})$ has exactly two elements. The converse holds in general: if a skew Boolean algebra **S** is both complete and atomic, then so is its maximal (generalized) Boolean algebra image, $\mathbf{S}/\mathcal{D}$. ☐

Corollary 4.5.10. *Every left-handed skew Boolean algebra* **S** *can be embedded as a skew Boolean algebra into some omega algebra.*

Proof. By Corollary 4.1.7, every left-handed skew Boolean algebra can be embedded in a power of $\mathbf{3}_{\mathrm{L}}$ which in turn is isomorphic to the omega algebra of the same power of **2**. ☐

The ∩-version of Corollary 4.5.10 fails since, as seen in Section 4.4, as an ∩-algebra $\mathbf{3}_{\mathrm{L}}$ does not generate the variety $\langle \aleph_{\mathbf{0L}} \rangle^{\cap}$ of all left-handed skew Boolean ∩-algebras. But we have:

Theorem 4.5.11. *As algebras with* ∩, *all omega algebras lie in* $\langle \mathbf{3}_{\mathrm{L}} \rangle^{\cap}$. *A skew Boolean* ∩*-algebra* **S** *can be embedded in some omega algebra if and only if it lies in* $\langle \mathbf{3}_{\mathrm{L}} \rangle^{\cap}$.

Proof. In the variety $\langle \aleph_{\mathbf{0L}} \rangle^{\cap}$, primitive images of omega algebras are isomorphic to omega algebras (by Lemma 4.5.13 below) and thus are copies of $\mathbf{3}_{\mathrm{L}}$. Hence nontrivial omega algebras are subdirect products of copies of $\mathbf{3}_{\mathrm{L}}$ and must lie in $\langle \mathbf{3}_{\mathrm{L}} \rangle^{\cap}$. In general a skew Boolean ∩-algebra lies in $\langle \mathbf{3}_{\mathrm{L}} \rangle^{\cap}$ if and only if it can be embedded as an ∩-algebra in a power of $\mathbf{3}_{\mathrm{L}}$. But such powers are copies of omega algebras by Theorem 4.5.9. ☐

Skew Boolean covers of generalized Boolean algebras

Recall that the ***center*** of a skew lattice **S** is the set $\mathcal{Z}_{\mathbf{S}} = \{a \in S \mid a \wedge x = x \wedge a \text{ for all } x \in S\}$ or equivalently, the set $\{a \in S \mid a \vee x = x \vee a \text{ for all } b \in S\}$. $\mathcal{Z}_{\mathbf{S}}$ is also the union of all singleton $\mathcal{D}$-classes of **S**. (See Theorem 2.2.2.) For skew lattices $\mathcal{Z}_{\mathbf{S}}$ can be empty, but for a skew Boolean algebra **S**, at least $\{0\} \subseteq \mathcal{Z}_{\mathbf{S}}$. If $\mathcal{Z}_{\mathbf{S}} = \{0\}$, then **S** has a ***trivial center***. The center of a skew Boolean algebra **S** is always an ideal of **S**. A ***skew Boolean cover*** of a generalized Boolean algebra **B** is any skew Boolean algebra **S** with trivial center such that $\mathbf{S}/\mathcal{D} \cong \mathbf{B}$. In this case, **S** is a ***minimal cover*** if the center $\mathcal{Z}_{\mathbf{S}/\theta}$ of S/θ is nontrivial for all nontrivial congruences $\theta \subseteq \mathcal{D}$.

Theorem 4.5.12. *For any nontrivial generalized Boolean algebra* **B**, *the derived omega algebra* $\omega(\mathbf{B})$ *forms a minimal, skew Boolean cover of* **B.**

Proof. If $a > 0$ in $\mathbf{B}$, then (a, a) $\mathcal{D}$ $(a, 0)$ but $(a, a) \neq (a, 0)$ in $\omega(\mathbf{B})$. So let a nontrivial congruenec $\theta \subseteq \mathcal{D}$ be given and suppose that (a, a') θ (a, a'') with $a \geq a', a''$ in $\mathbf{B}$, but $a' \neq a''$. Applying $\wedge$ (b, b) we get $(a\wedge b, a'\wedge b)$ θ $(a\wedge b, a''\wedge b)$ for all $b \in \mathrm{B}$. Since $a' \neq a''$, one of these does not equal $a' \wedge a''$ in $\mathbf{B}$, say a'. Setting $b = a'$ we get (a', a') θ $(a', a''\wedge a')$ with $a' > a''\wedge a'$. Thus we may assume that (a, a) θ (a, a') with $a > a'$ in $\mathbf{B}$ from the outset. Next, setting $b = a \setminus a'$, we get that $a \geq b > 0$. Then $(a\wedge b, a\wedge b)$ θ $(a\wedge b, a'\wedge b)$, that is, (b, b) θ $(b, 0)$. We show that the θ-class of (b, b) is in fact its entire $\mathcal{D}$-class. Since $b > 0$, this forces $\omega(\mathbf{B})/\theta$ to have a singleton $\mathcal{D}$-class other than $\{0\}$, making its center nontrivial. Applying $\vee$ (c, c) to (b, b) θ $(b, 0)$ for any $c \leq b$, we get

$$(b, b) \vee (c, c) = (b\vee c, (b \setminus c) \vee c) = (b, b) \text{ and } (b, 0) \vee (c, c) = (b\vee c, (0 \setminus c) \vee c) = (b, c).$$

Thus (b, b) θ (b, c) for all $c \leq b$, so that the θ-class of (b, b) is precisely its $\mathcal{D}$-class in $\omega(\mathbf{B})$. □

Minimal skew Boolean covers can be created without the ω-construction. The latter is an elegant way of achieving this since (1) it gives us minimal covers and (2) it provides a systematic way of carrying this out.

While $\omega(\mathbf{B})/\theta$ has a nontrivial center for every nontrivial congruence $\theta \subseteq \mathcal{D}$, a second class of congruences exists for which the quotient algebra always has a trivial center, namely the class of ***∩-congruences*** β that *also* preserve intersections in that $a \,\beta\, b$ implies $a\cap c \,\beta\, b\cap c$, in which case $\mathbf{S}/\beta$ also has finite intersections. As we have seen, such congruences arise from ideals of $\mathbf{S}$. Given an ideal I, an ∩-congruence β_I is defined by: $a \,\beta_I\, b$ if $(a \setminus (a \cap b)) \vee (b \setminus (a \cap b)) \in I$. Conversely, every ∩-congruence β determines an ideal I_β given by the congruence class $\beta[0]$ of 0. The two assignment processes are reciprocal. (See Lemma 4.4.7 and Theorem 4.4.8.)

Lemma 4.5.13. *Given any ∩-congruence* β *on* $\omega(\mathbf{B})$, *its image* $\omega(\mathbf{B})/\beta$ *is isomorphic to an omega algebra and thus has trivial center. Put otherwise ∩-homomorphic images of omega algebras are copies of omega algebras.*

Proof. Let β be derived from ideal I. In turn let $I_0 \cong I/\mathcal{D}$ be the ideal of $\mathbf{B}$ consisting of all left-coordinates of elements in I so that $I = \omega(I_0)$. If β_0 is the congruence on $\mathbf{B}$ induced from I_0, then the map f: $\omega(\mathbf{B}) \to \omega(\mathbf{B}/\beta_0)$ defined by $f[(a, a')] = (\beta_0[a], \beta_0[a'])$ is a ∩-homomorphism for which $f^{-1}\{(0, 0)\}$ is precisely I. Thus β is the congruence induced by f and $\omega(\mathbf{B})/\beta \cong \omega(\mathbf{B}/\beta_0)$. □

We can now decompose the congruence lattice of $\omega(\mathbf{B})$ as a subdirect product of (i) the interval $[\Delta, \mathcal{D}]$ of all congruences (inclusively) between the trivial congruence Δ and $\mathcal{D}$ and (ii) the lattice $\mathbf{Con}_\cap(\omega(\mathbf{B}))$ of all ∩-congruences on $\omega(\mathbf{B})$. Indeed Theorem 4.4.9 gives us:

Theorem 4.5.14. *The full lattice of congruences* $\mathbf{Con}(\omega(\mathbf{B}))$ *is a subdirect product of the interval* $[\Delta, \mathcal{D}]$ *and* $\mathbf{Con}_\cap(\omega(\mathbf{B}))$ *under the embedding* $\mathbf{Con}(\omega(\mathbf{B})) \to [\Delta, \mathcal{D}] \times \mathbf{Con}_\cap(\omega(\mathbf{B}))$ *given by* $\theta \to (\theta\cap\mathcal{D}, \beta_{\theta[0]})$. *For all* $\theta \in [\Delta, \mathcal{D}]$ *except* Δ, *the image* $\omega(\mathbf{B})/\theta$ *has nontrivial center, while for all* $\beta \in \mathbf{Con}_\cap(\omega(\mathbf{B}))$ *the image* $\omega(\mathbf{B})/\beta$ *has a trivial center.*

Proof. This follows from Theorem 4.4.14 and the two previous results. □

Does being a minimal skew Boolean cover of its maximal lattice image characterize an ω-algebra? We briefly explore this question, beginning with a theorem. Its fairly straightforward proof is omitted.

Theorem 4.5.15. *Given an atomic skew Boolean algebra* **S** *with atomic* $\mathcal{D}$*-classes* D_i *for* i *in index set* I, *then* **S** *has a trivial center and hence is a skew Boolean cover of* **S**/$\mathcal{D}$ *if and only if* $|D_i| \geq 2$ *for all* i. **S** *is a minimal skew Boolean cover of* **S**/$\mathcal{D}$ *if and only if* $|D_i| = 2$ *for* $i \in I$. **S** ≅ ω(**S**/$\mathcal{D}$) *must occur whenever* **S** *is also left-handed and complete*. □

Example 4.5.2. A skew Boolean algebra **S** that is a minimal skew Boolean cover of **S**/$\mathcal{D}$ but not isomorphic to any ω(**S**/$\mathcal{D}$) is given by taking the subset lattice $\mathcal{P}$(X) of an infinite set X and letting **S** be the atomic subalgebra of ω($\mathcal{P}$(X)) consisting of all pairs (A, A′) for A′ finite. Given (A, A′) for A infinite and A′ finite, the $\mathcal{D}$-class of (A, A′) has |A| many elements in **S**, but |$\mathcal{P}$(A)| many elements in ω($\mathcal{P}$(X)). Since **S** and ω($\mathcal{P}$(X)) share $\mathcal{P}$(X) as a maximal lattice image, and |A| < |$\mathcal{P}$(A)| for all subsets of X, **S** cannot be isomorphic to any ω(**B**).

The functor ω *and its left adjoint*

In the following, ⟨**2**⟩ and ⟨$\mathbf{3}_{\mathrm{L}}$⟩ denote the respective varieties of generalized Boolean algebras and left-handed skew Boolean algebras viewed as categories. The ω-construction is the object stage of a functor ω: ⟨**2**⟩ → ⟨$\mathbf{3}_{\mathrm{L}}$⟩. Given homomorphism f: **B** → **B′** in ⟨**2**⟩, its ω-image in ⟨$\mathbf{3}_{\mathrm{L}}$⟩ is the homomorphism f^ω: ω(**B**) → ω(**B′**) in ⟨$\mathbf{3}_{\mathrm{L}}$⟩ given by $f^\omega(a, a') = (f(a), f(a'))$.

Theorem 4.5.16. ω: ⟨**2**⟩ → ⟨$\mathbf{3}_{\mathrm{L}}$⟩ *preserves limits. Thus it has a left adjoint,* Ω: ⟨$\mathbf{3}_{\mathrm{L}}$⟩ → ⟨**2**⟩.

Proof. Observe first that ω preserves products: indeed $\omega(\prod_{i\in I}\mathbf{B}_i) \cong \prod_{i\in I}\omega(\mathbf{B}_i)$ under the map $(\langle a_i\rangle, \langle a_i'\rangle) \to \langle(a_i), (a_i')\rangle$. Next, note that ω preserves equalizers: given f, g: **B** → **B′**, $equ(f^\omega, g^\omega) = (equ(f, g))^\omega$ where $equ(f, g)$ is the standard equalizer given by the inclusion of the maximal subalgebra of **B** on which f and g agree. Hence ω preserves limits.

Next we show that every skew Boolean algebra **S** has a solution set of homomorphisms $\mathcal{F}$ = $\{f_i$: **S** → ω($\mathbf{B}_i$) | $i \in \cap\}$ such that every homomorphism f: **S** → ω(**B**) in ⟨$\mathbf{3}_{\mathrm{L}}$⟩ can be written as a composite $f = h^\omega \circ f_i$ for some $f_i \in \mathcal{F}$ and some homomorphism h:$\mathbf{B}_i$ → **B** in ⟨**2**⟩. So given a homomorphism f: **S** → ω(**B**) of skew Boolean algebras, consider the subalgebra **B′** of **B** that is determined by the subset union

$$\bigcup\{\{a, a'\} \mid (a, a') = f(x) \in \omega(\mathbf{B}) \text{ for some } x \in \mathrm{S}\}.$$

Clearly f[S] ⊆ ω(**B′**). Moreover $|\mathbf{B}'| \leq max\{\aleph_0, |\mathbf{S}|\}$. Hence a solution set for **S** is given by the set of all homomorphisms f: **S** → ω($\mathbf{B}_j$) where $\{\mathbf{B}_j | j \in J\}$ is a maximal class of mutually non-isomorphic generalized Boolean algebras of order $|\mathbf{B}_j| \leq max\{\aleph_0, |\mathbf{S}/\mathcal{D}|\}$. Since ω is limit-preserving and each **S** in $\langle \mathbf{3}_L \rangle$ has a solution set, Freyd's Adjoint Functor Theorem implies that ω must have a left adjoint Ω. □

Thus given a skew Boolean algebra **S**, a homomorphism η_S: $\mathbf{S} \to \omega(\Omega(\mathbf{S}))$ exists, called the ***universal morphism*** from **S** to ω, such that for any homomorphism μ: $\mathbf{S} \to \omega(\mathbf{B}')$ in $\langle \mathbf{3}_L \rangle$, a unique homomorphism u: $\Omega(\mathbf{S}) \to \mathbf{B}'$ exists in $\langle \mathbf{2} \rangle$ such that μ factors as $u^\omega \circ \eta_S$. Due to Corollary 4.5.10, *each universal morphism* η_S *is an embedding*. Thus, given a homomorphism f: $\mathbf{S} \to \mathbf{S}'$, f^Ω: $\Omega(\mathbf{S}) \to \Omega(\mathbf{S}')$ is the unique homomorphism in $\langle \mathbf{2} \rangle$ making the following diagram commute.

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\eta_S} & \omega(\Omega(\mathbf{S})) \\ f \downarrow & & \downarrow \quad (f^\Omega)^\omega . \\ \mathbf{S}' & \xrightarrow{\eta_{S'}} & \omega(\Omega(\mathbf{S}')) \end{array}$$

What does η_S: $\mathbf{S} \to \Omega(\mathbf{S})$ look like for some typical left-handed skew Boolean algebras?

Theorem 4.5.17. *Let* **P** *be a primitive left-handed skew Boolean algebra with* P\{0} = X. *Then* $\Omega(\mathbf{P}) \cong \mathbf{FB}_X$, *the generalized Boolean algebra reduct of the free Boolean algebra on* X. *In particular,* $\Omega(\mathbf{P}) \cong \mathbf{2}^{2^{|X|}}$ *if* |X| *is finite*. (See [Leech and Spinks, 2008] Theorem 5.2.) □

In particular, $\Omega(\mathbf{2}) \cong \mathbf{2}^2$ and $\Omega(\mathbf{3_L}) \cong \mathbf{2}^4$. One can show that $\Omega(\mathbf{S}_1 \times \mathbf{S}_2) \cong \Omega(\mathbf{S}_1) \times \Omega(\mathbf{S}_2)$. (See [Leech and Spinks, 2008] Theorem 5.3.) Ω(**S**) is thus easily determined when $\mathbf{S}/\mathcal{D}$ is finite.

Twisted product constructions were introduced by Kalman [1958] as a means of building De Morgan algebras from distributive lattices. The ω-construction given here is a variation of a construction due to Pagliani [1998] for producing Nelson algebras from Heyting algebras. For applications of twisted product constructions to both algebra and logic see Pagliani [1997].

Historical remarks

Skew Boolean algebras in some form seem to have been studied first by Robert Bignall in his 1976 dissertation and then in a 1980 paper by his advisor, William Cornish. In 1990, a paper on skew Boolean algebras appeared among Jonathan Leech's early papers on skew lattices. Bignall and Leech then published a joint paper on skew Boolean algebras with intersections in 1995. Much in Sections 4.1 and 4.4 appeared in these two papers. In Section 4.4, the material on the lattice of varieties appeared in the 2017 paper by Leech and Spinks. The material in Section 4.2 on finite algebras and the free case in particular appeared in the 2016 paper by Ganya Kudryatseva and Leech, as did the material in Section 4.4 on infinite free algebras. The material in Section 4.3 appeared in a 2013 paper by Karin Cvetko-Vah, Matthew Spinks and Leech, but the initial information about strongly distributive skew lattices appeared in Leech's 1992 paper on normal skew lattices. The construction in Section 4.5 was due to Spinks and developed in a joint project with Leech published in 2007. Further developments of that material occur in a 2013 paper by Kudryatseva, who is also publishing a paper on free skew Boolean intersection algebras, set to appear in 2017.

Bignall, Cornish and Leech were not the only ones to initiate a study of noncommutative Boolean algebras in some form. As seen in Section 4.2, computer scientists (J. Berendsen *et al*) in studying the override and update operations introduced in 2010 a class of algebras that was shown by Cvetko-Vah, Leech and Spinks in their 2013 paper to be term equivalent to right-handed skew Boolean algebras. Indeed they are $(\vee, \setminus, 0)$-term reducts of the latter with a variant of $\wedge$ obtained as a defined operation. In a 2011 paper, Janis Cirulis, studied near lattices (meet semilattices with joins existing for pairs of elements with common upper bounds) that were supplied with an override operation (reducing to the conditional commuting join when it existed). Under special assumptions Cirulis obtained a class of skew Boolean algebras with intersections, with $\cap$ being the near lattice meet.

Skew Boolean algebras clearly form a natural class of objects to study. They play a significant role in the more general study of Boolean-like phenomena, with connections to discriminator varieties, iBCK algebras and their offspring which we meet in Chapter 7, and more recently, Church algebras. (See Spinks [2002] and Cvetko-Vah and Salibra [2015]) It is no coincidence that skew Boolean algebras have attracted the interest of some in computer science. Indeed, most of the individuals mentioned in these paragraphs have some degree of research interest in computer science. (In particular, see Bignall and Spinks [1996] – [1998] and Spinks and Veroff. [2006])

Topological representations of (generalized) Boolean algebras and distributive lattices have been studied since M. H. Stone's papers in the 1930s. More recently this has been extended to skew Boolean algebras (possibly with intersections) and strongly distributive skew lattices. Here one is given a skew Boolean algebra S with maximal (generalized) Boolean image $B = S/\mathcal{D}$. The latter is dual to a topological space $\mathcal{B}$ under standard Stone duality with S itself being dual to an étale covering $\pi: \mathcal{X} \to \mathcal{B}$. (See the papers below by Bauer and Cvetko-Vah (*et al*) and by Kudryatseva.)

In Chapter 6 we will look at the skew Boolean algebras of idempotents in rings, and in particular, the case where the idempotents in a ring are closed under multiplication and thus naturally form a skew Boolean algebra. In Chapter 7 we will look at algebraic structures that support a skew lattice reduct, in some cases with intersections; that is, we will look at algebras where skew Boolean operations are term-defined using the given operations of the algebra.

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V: FURTHER TOPICS IN SKEW LATTICES

Skew lattices were introduced in Section 1.3 and then studied in Chapter 2. Two main sources of initial examples were encountered:

i) Partial function algebras $\mathcal{P}(A, B)$ that form noncommutative variants of Boolean algebras defined on power sets $\mathcal{P}(A)$. (Example 2.6.1.)

ii) Sets of idempotents in rings that are closed under multiplication and the (often called) circle operation: $e \circ f = e + f - ef$. (Theorem 2.1.7.)

Both types of examples satisfy the primary conditions for an algebraic system $(S; \vee, \wedge)$ to be a skew lattice: $\vee$ and $\wedge$ are associative idempotent binary operations on S satisfying the absorption identities $x\wedge(x \vee y) = x = (y \vee x)\wedge x$ and $x\vee(x \wedge y) = x = (y \wedge x)\vee x$ that guarantee the basic dualities, $x\wedge y = y$ iff $x\vee y = x$ and $x\wedge y = x$ iff $x\vee y = y$. These examples are also a source of optional conditions that any skew lattice might satisfy.

In both cases commutation was *symmetric* in that $x\wedge y = y\wedge x$ iff $x\vee y = y\vee x$, thus making instances of commutation unambiguous. Also in both cases the skew lattices were *distributive* in that both $x\wedge(y \vee z)\wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x)$ and $x\vee(y \wedge z)\vee x = (x\vee y\vee x) \wedge (x\vee z\vee x)$ hold. For partial function algebras a stronger pair of identities hold: $x\wedge(y \vee z) = (x\wedge y) \vee (x\wedge z)$ and $(y \vee z)\wedge x = (y\wedge x) \vee (z\wedge x)$. In Section 2.3 we saw that a chain of implications holds for all skew lattices:

$$\forall x, y, z \in S,\ x\wedge(y \vee z) = (x\wedge y) \vee (x\wedge z)\ \&\ (y \vee z)\wedge x = (y\wedge x) \vee (z\wedge x)$$

$$\Rightarrow\ \forall x, y, z, w \in S,\ x\wedge(y \vee z)\wedge w = (x\wedge y\wedge w) \vee (x\wedge z\wedge w)$$

$$\Rightarrow\ \forall x, y, z \in S,\ x\wedge(y \vee z)\wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x)\ \&\ x\vee(y \wedge z)\vee x = (x\vee y\vee x) \wedge (x\vee z\vee x).$$

In general these implications are strict, but if the skew lattice is symmetric the converse of the first implication holds; likewise if the skew lattice is *normal* in that $x\wedge y\wedge z\wedge w = x\wedge z\wedge y\wedge w$, the converse of the second implication holds. We also observed that $x\wedge(y \vee z)\wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x)$ and its dual are not equivalent, but given the assumption of symmetry we will prove they are.

In addition, both types of algebras are *categorical* in that *nonempty* composites $\psi\circ\varphi$ of successive coset bijections from $\mathcal{D}$-classes to say lower $\mathcal{D}$-classes are also coset bijections. Indeed partial function algebras (and normal skew lattices in general) are *strictly categorical* in that all composites of coset bijections between successive $\mathcal{D}$-classes are also nonempty.

In this chapter we study these properties in greater detail.

In Section 5.1 we consider symmetric skew lattices where commutation is unambiguous as well as variations of this condition. Symmetry is first bisected into *lower symmetry* where commutation shifts "downward" ($x\vee y = y\vee x \Rightarrow x\wedge y = y\wedge x$) and its dual, *upper symmetry*, where commutation shifts "upward" ($x\wedge y = y\wedge x \Rightarrow x\vee y = y\vee x$). Characterizing identities for each are given in Theorem 5.1.1. Both types of partial symmetry are characterized in Theorem 5.1.2 by a pair of forbidden subalgebras (one right-handed and one left-handed). The absence of all four as subalgebras characterizes symmetric skew lattices.

In Section 5.2 we prove the equivalence of $x\wedge(y \vee z)\wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x)$ and its dual $x\vee(y \wedge z)\vee x = (x\vee y\vee x) \wedge (x\vee z\vee x)$ in the presence of symmetry. In particular, Theorem 5.2.3 states that given lower symmetry, $x\wedge(y \vee z)\wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x)$ implies

$$x\vee(y \wedge z)\vee x = (x\vee y\vee x) \wedge (x\vee z\vee x),$$

and dually, in the presence of upper symmetry, the converse implication holds. These results are due to Matthew Spinks, who had obtained in a very lengthy computer-generated proof, but eventually reduced the proof to one of a moderate size that was then "humanized" by Karin Cvetko-Vah. (All relevant references are given in the sections.)

For lattices, not only is $x\wedge(y \vee z) = (x\wedge y) \vee (x\wedge z)$ equivalent to $x\vee(y \wedge z) = (x\vee y) \wedge (x\vee z)$, both identities in turn are equivalent to a lattice not having copies of $\mathbf{M}_3$ or $\mathbf{N}_5$ as subalgebras. Another equivalent condition for lattices is this form of cancellation:

$$x\vee z = y\vee z \ \ \& \ \ x\wedge z = y\wedge z \ \Rightarrow \ x = y.$$

For skew lattices, however, these conditions are mutually nonequivalent in the absence of other qualifying assumptions. Either a distributive identity or the cancellative implication will by itself rule out $\mathbf{M}_3$ or $\mathbf{N}_5$ occurring as subalgebras, but not conversely. A skew lattice is called *cancellative* if both the above implication (cancelling on the right) and its dual,

$$x\vee y = x\vee z \ \& \ x\wedge y = x\wedge z \Rightarrow y = z \text{ (cancelling on the left)},$$

hold. Cancellative skew lattices are always strongly symmetric. Since every skew chain is cancellative (Proposition 5.3.2), but not necessarily distributive (Example 5.3.3), being cancellative does not imply being distributive. Conversely the four forbidden algebras of Section 5.1 are distributive, but not cancellative. Cancellative skew lattices, and their close variants, are studied in Section 5.3. They all form varieties that are characterized by small sets of forbidden subalgebras (Theorem 5.3.8). Their coset structure also produces some interesting counting features. (See Theorems 5.3.10 and 5.3.11.) Both partial function algebras and skew lattices of idempotents in rings are cancellative.

Like being symmetric, being categorical is desirable in a skew lattice as it makes cosets and their bijections well-behaved globally. Indeed, all distributive skew lattices are categorical (by Theorem 5.4.2). This condition is studied more closely in Section 5.4. Categorical skew lattices form a proper subvariety of skew lattices and in Theorem 5.4.4 we give a countable family of forbidden subalgebras that characterize this subvariety. Strictly categorical skew lattices are studied in the last part of the Section and Theorem 5.4.7 gives a number of equivalent conditions for a skew lattice to belong to this variety. Perhaps the most notable one is the *unique midpoint condition* on skew chains A > B > C in *S*: given $a > c$ with $a \in$A and $c \in$C, a unique *midpoint* $b \in$B exists such that $a > b > c$. This is equivalent to the condition: for each pair $e \leq f$ in a skew lattice, the interval subalgebra $[e, f] = \{x \in S \mid e \leq x \leq f\}$ is a sublattice. As a consequence, any strictly categorical skew lattice S for which the lattice image $S/\mathcal{D}$ is distributive is a distributive skew lattice. (Theorem 5.4.9.) All normal skew lattices belong to this variety as do their duals, *conormal* skew lattices (where $x\vee y\vee z\vee w = x\vee z\vee y\vee w$ holds). The subvariety of all skew lattices generated from these two classes of skew lattices turns out to be a proper subvariety of strictly categorical skew lattices. At the end of the section we ask if this generated subvariety is the variety of *paranormal* skew lattices (defined there). In general one has the following chain of subvarieties:

$$\langle \textit{Normal} \cup \textit{conormal} \rangle \subseteq \textit{Paranormal} \subset \textit{Strictly categorical.}$$

In Section 5.5 we take a closer look at distributivity. Every distributive skew lattice *S* must be *quasi-distributive* in that every lattice image of *S* is distributive. Quasi-distributivity is a necessary, but not sufficient condition for distributivity, thanks to Spinks' examples (Theorem 1.3.10). A possible complementary condition is for a skew lattice to be *linearly distributive* in that each subalgebra that is totally pre-ordered (by $\succeq$) is distributive. Linearly distributive skew lattices form a variety. (See Theorems 5.5.1 and 5.5.5.) For skew lattices one has the strict chain of subvarieties:

$$\textit{Strictly categorical} \subset \textit{Linearly distributive} \subset \textit{Categorical}.$$

In the strictly categorical case, quasi-distributivity suffices for the skew lattice to be distributive (Theorem 5.4.9). For linearly distributive algebras in general this does not occur (Spinks' examples again). However: *given symmetry*, *linearly distributive* + *quasi-distributive implies distributive.* (See Theorem 5.5.11.)

Being a vital part of distributivity, linear distributivity and distributive skew chains in particular are studied further in Section 5.6. Given a skew chain A > B > C with $a > c$ for $a \in$ A and $c \in$ C, consider the set $\mu(a, c) = \{b \in \mathrm{B} \mid a > b > c\}$ of midpoints b of a and c in B. One has $|\mu(a, c)| \geq 1$, and unless A > B > C is strictly categorical, $|\mu(a, c)| \geq 2$ for all such pairs. In general, B decomposes into a disjoint union of *AC-components* B_1, B_2, … that induce sub-skew chains A > B_i > C. (The definitions and relevant discussion occur prior to Example 5.4.3.) Each component B_i contains at least one midpoint b_i for $a > c$. In any case, A > B > C is distributive if and only if each A > B_i > C is strictly categorical, and in particular, each component B_i contains a unique midpoint b_i for each such $a > c$. $\mu(a, c)$ thus parameterizes in a natural way the AC-components of B. In so doing, distributivity minimizes the number of midpoints any $a > c$ can have in a skew chain A > B > C relative to the AC-component structure of B (Theorem 5.6.4). Here, and in the latter part of Section 5.4, the orthogonality of cosets from two $\mathcal{D}$-classes (A and C) in a third $\mathcal{D}$-class (B) occurs again. (Recall Lemma 2.4.8 and Theorem 2.4.9 in Section 2.4.)

In the final seventh section we present some combinatorial results about skew chains and skew diamonds. These counting theorems are a continuation of some results in Section 2.4, in particular Theorem 2.4.10 and its corollaries. Whereas the former are over 20 years old, the results in Section 7 were published in the past few years and are due primarily to Cvetko-Vah, Leech and Pita de Costa. The section concludes with some consequences for cancellative skew lattices. This is followed by a few remarks of a bibliographic nature.

5.1 Symmetric skew lattices

Recall from Theorem 2.2.4 that symmetric skew lattices are characterized as skew lattices by the following identities with both sides respectively equaling $x\wedge y\wedge x$ and $x\vee y\vee x$:

$$x\wedge y\wedge(x\vee y\vee x) \;=\; (x\vee y\vee x)\wedge y\wedge x \tag{5.1.1}$$

and

$$x\vee y\vee(x\wedge y\wedge x) \;=\; (x\wedge y\wedge x)\vee y\vee x. \tag{5.1.2}$$

Matthew Spinks in [Spinks 1998] observed that these identities can be trimmed to

$$x\wedge y\wedge(x\vee y) = (y\vee x)\wedge y\wedge x, \tag{5.1.3}$$

and

$$x\vee y\vee(x\wedge y) = (y\wedge x)\vee y\vee x \tag{5.1.4}$$

since $x\wedge y\wedge x \preceq$ both $x\wedge y$ and $x\vee y$, so that $x\vee y\vee(x\wedge y) = x\vee y\vee(x\wedge y\wedge x)\vee(x\wedge y) = x\vee y\vee(x\wedge y\wedge x)$ due to $x\wedge y\wedge x\ \mathcal{R}\ x\wedge y$ holding in any skew lattice. The three other terms are handled similarly.

Symmetry is parsed as follows. A skew lattice is ***lower symmetric*** if $x\vee y = y\vee x$ implies $x\wedge y = y\wedge x$. Dually, a skew lattice is ***upper symmetric*** if $x\wedge y = y\wedge x$ implies $x\vee y = y\vee x$.

Theorem 5.1.1. *Lower symmetry is characterized by identity* (5.1.3). *Dually, upper symmetry is characterized by identity* (5.1.4). *Hence both classes of skew lattices form varieties. A skew lattice* **S** *is thus upper* [*lower*] *symmetric if and only if both* $\mathbf{S}/\mathcal{L}$ *and* $\mathbf{S}/\mathcal{R}$ *are.*

Proof. Indeed, if $x\vee y = y\vee x$, then (5.1.3) plus absorption gives $x\wedge y = y\wedge x$ so that lower symmetry holds. Conversely, since $x \geq x\vee y\vee x$, occurrences of x and $(x\vee y\vee x)\wedge y\wedge(x\vee y\vee x)$ must $\vee$-commute. Hence if a skew lattice is lower symmetric, x and $(x\vee y\vee x)\wedge y\wedge(x\vee y\vee x)$ also $\wedge$-commute so that (5.1.1) and hence (5.1.3) follows. Similar remarks hold for (5.1.4) and upper symmetry. ☐

Consider the following pair of Hasse diagrams, each determining a right-handed skew diamond and its left-handed dual. (The dotted lines denote the natural partial order $\geq$.)

$$\begin{array}{ccc} & j_1 - j_2 & \\ \swarrow\swarrow & & \searrow\searrow \\ a_1 - a_2 & & b_1 - b_2 \\ \searrow & & \swarrow \\ & 0 & \end{array} \qquad\qquad \begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ a_1 - a_2 & & b_1 - b_2 \\ \searrow\searrow & & \swarrow\swarrow \\ & m_1 - m_2 & \end{array}$$

The induced right-handed algebras are denoted respectively by $\mathbf{NS}_7^{\mathcal{R},0}$ and $\mathbf{NS}_7^{\mathcal{R},1}$. Their left-handed duals by are denoted $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$. Cayley tables for $\mathbf{NS}_7^{\mathcal{R},0}$ are given by:

$\vee$	0	a_n	b_n	j_n
0	0	a_n	b_n	j_n
a_m	a_m	a_m	j_m	j_m
b_m	b_m	j_m	b_m	j_m
j_m	j_m	j_m	j_m	j_m

and

$\wedge$	0	a_n	b_n	j_n
0	0	0	0	0
a_m	0	a_n	0	a_n
b_m	0	0	b_n	b_n
j_m	0	a_n	b_n	j_n

.

All four algebras can be obtained from any one by $\vee$-$\wedge$ duality, $\mathcal{L}$-$\mathcal{R}$ duality or a combination of both. None is symmetric. Indeed, the tables above give $a_1 \wedge b_2 = 0 = b_2 \wedge a_1$ but $a_1 \vee b_2 \neq b_2 \vee a_1$. The following theorems appeared in 2011 in a paper of Cvetko-Vah, Kinyon, Leech and Spinks.

Theorem 5.1.2. *A skew lattice is upper symmetric if and only if it contains no copy of* $\mathbf{NS}_7^{\mathcal{R},0}$ *or* $\mathbf{NS}_7^{\mathcal{L},0}$. *It is lower symmetric if and only if it contains no copy of* $\mathbf{NS}_7^{\mathcal{R},1}$ *or* $\mathbf{NS}_7^{\mathcal{L},1}$. *Finally, it is symmetric if and only if it contains no copy of any of these skew lattices.*

Proof. Since neither $\mathbf{NS}_7^{\mathcal{R},0}$ nor $\mathbf{NS}_7^{\mathcal{L},0}$ is upper symmetric, any skew lattice containing a copy of one of them cannot be upper symmetric. Conversely, consider first the case where S is a non-upper symmetric, right-handed skew lattice. Thus a, $b \in$ S exist such that $a \wedge b = b \wedge a$ but $a \vee b \neq b \vee a$. Consider the subalgebra T generated by a and b. Since $a \wedge b \wedge x = a \wedge b = x \wedge a \wedge b$, for $x = a$ or b and hence for all $x \in$ T, $a \wedge b$ is the zero element of T. Moreover a and b must be in incomparable $\mathcal{R}$-classes for anti-symmetry to occur and thus one has the following configuration with seven distinct elements in four $\mathcal{D}$-classes within a right-handed skew diamond:

$$\begin{array}{ccc} & a \vee b - b \vee a & \\ \swarrow\swarrow & & \searrow\searrow \\ a - a \wedge (b \vee a) & & b \wedge (a \vee b) - b\,. \\ \searrow & & \swarrow \\ & a \wedge b & \end{array}$$

For if $a = a \wedge (b \vee a)$, then $b \vee a \geq a$, b forcing $a \vee b = b \vee a$ by Theorem 2.2.1, contradicting our assumption on a and b. Thus $a \neq a \wedge (b \vee a)$ and similarly $b \neq b \wedge (a \vee b)$ giving us at least the seven distinct displayed elements. We claim that this set is closed under ∨ and ∧. Each displayed $\mathcal{D}$-class is closed. We consider representatives of the less trivial remaining cases.

$$a\wedge(b\vee a) \vee (a\vee b) = [a\wedge(b\vee a) \vee a]\vee b = a\wedge(b\vee a) \vee b = b\vee a$$

since $a\wedge(b\vee a)\ \mathcal{R}\ a$ and both $a\wedge(b\vee a)$ and b are $\leq b\vee a$. Likewise,

$$a\vee[b\wedge(a\vee b)] = a\vee[b\wedge(a\vee b)]\vee(a\vee b) = a\vee(a\vee b) = a\vee b = [b\wedge(a\vee b)]\vee a\vee b = [b\wedge(a\vee b)]\vee a$$

and

$$[a\wedge(b\vee a)]\vee[b\wedge(a\vee b)] = [a\wedge(b\vee a)]\vee[b\wedge(a\vee b)]\vee a\vee b = [a\wedge(b\vee a)]\vee a\vee b = b\vee a$$

by a prior calculation. Thus, T is as in the diagram and it is a copy of $\mathbf{NS}_7^{\mathcal{R},0}$. Dual remarks involving $\mathbf{NS}_7^{\mathcal{L},0}$ arise in the left-handed case.

Suppose next that S is any non-upper symmetric skew lattice with a, $b \in$ S such that $a\wedge b = b\wedge a$, but $a\vee b \neq b\vee a$. Again, a and b generate a subalgebra S′ that forms a non-upper symmetric subalgebra for which $a\wedge b$ is the zero element. Moreover, one of S′/$\mathcal{R}$ or S′/$\mathcal{L}$ must be non-upper symmetric and thus a copy of $\mathbf{NS}_7^{\mathcal{L},0}$ or $\mathbf{NS}_7^{\mathcal{R},0}$. Whichever case it is, a copy also exists in S′ by Theorem 2.2.9. The lower symmetric case follows by duality. Both cases combine to characterize full symmetry. ☐

Upper and lower symmetry can each be parsed further in left-right fashion to give a four-fold partition of symmetry that will prove useful in Section 5.3.

Theorem 5.1.3. *Given a skew lattice* **S**:

i) **S**/$\mathcal{L}$ *being upper symmetric is equivalent to either of the following:*
   a) **S** *contains no copy of* $\mathbf{NS}_7^{\mathcal{R},0}$.
   b) **S** *satisfies* $x\vee y\vee x = (y\wedge x)\vee y\vee x$.

ii) **S**/$\mathcal{R}$ *being upper symmetric is equivalent to either of the following:*
   a) **S** *contains no copy of* $\mathbf{NS}_7^{\mathcal{L},0}$.
   b) **S** *satisfies* $x\vee y\vee x = x\vee y\vee(x\wedge y)$.

iii) **S**/$\mathcal{R}$ *being lower symmetric is equivalent to either of the following:*
   a) **S** *contains no copy of* $\mathbf{NS}_7^{\mathcal{L},1}$.
   b) **S** *satisfies* $x\wedge y\wedge x = (y\vee x)\wedge y\wedge x$.

iv) **S**/$\mathcal{L}$ *being lower symmetric is equivalent to either of the following:*
   a) **S** *contains no copy of* $\mathbf{NS}_7^{\mathcal{R},1}$.
   b) **S** *satisfies* $x\wedge y\wedge x = x\wedge y\wedge(x\vee y)$.

The above four conditions hence determine four varieties of skew lattices.

Proof. Notice that $\mathbf{S}/\mathcal{L}$ is upper symmetric if and only if $\mathbf{S}/\mathcal{L}$ itself contains no copy of $\mathbf{NS}_7^{\mathcal{R},0}$. Thus the equivalence of (i)(a) with the upper symmetry of $\mathbf{S}/\mathcal{L}$ follows from Theorem 2.2.9. To show the equivalence with (i)(b), we begin with $x, y \in \mathrm{S}$, set $u = u(x, y) = (y\wedge x)\vee y$. Since $x\wedge y\wedge x \;\mathcal{L}\; y\wedge x$, we have

$$(x\wedge y\wedge x) \vee y \vee (x\wedge y\wedge x) \;\mathcal{L}\; (y\wedge x) \vee y \vee (y\wedge x) = (y\wedge x)\vee y = u.$$

Setting $w = (x\wedge y\wedge x) \vee y \vee (x\wedge y\wedge x)$, since $x\wedge w = w\wedge x = x\wedge y\wedge x$ we get

$$(x\wedge u)\wedge(u\wedge x) = x\wedge u\wedge x = x\wedge u\wedge w\wedge x = (x\wedge u)\wedge(x\wedge w) = x\wedge u$$

and

$$(u\wedge x)\wedge(x\wedge u) = u\wedge x\wedge u = u\wedge w\wedge x\wedge u = u\wedge x\wedge w\wedge u = u\wedge x\wedge w = u\wedge w\wedge x = u\wedge x.$$

Thus, $x\wedge u \;\mathcal{L}\; u\wedge x$. Moreover, if $x\wedge y \;\mathcal{L}\; y\wedge x$, then $u(x, y)$ must reduce to y. $\mathrm{S}/\mathcal{L}$ is thus upper symmetric if and only if for all $x, y \in \mathrm{S}$, $x \vee u(x, y) \;\mathcal{L}\; u(x, y) \vee x$. Using the $x\wedge y = y\vee x$ identity on $\mathcal{D}$-classes we get, $(y\wedge x) \vee y \vee x \vee (y\wedge x) \vee y = x \vee (y\wedge x) \vee y$ and

$$x \vee (y\wedge x) \vee y \vee (y\wedge x) \vee y \vee x = (y\wedge x) \vee y \vee x$$

which reduce to $(y\wedge x) \vee y \vee x \vee y = x \vee y$ and $x \vee y \vee x = (y\wedge x) \vee y \vee x$ by Lemma 2.1.4, with the left identity being redundant in the presence of the right identity. (The lemma states that in any skew lattice, $a, c \succeq b \Rightarrow a\vee b\vee c = a\vee c$ while $a, c \preceq b \Rightarrow a\wedge b\wedge c = a\wedge c$.)

The three other cases are similar. The final assertion is now clear. ☐

5.2 Distributive identities in the symmetric case

We give Cvetko-Vah's 2006 proof of Spinks' Theorem (Spinks [1998]) stating that for symmetric skew lattices, the following identities are equivalent:

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x) \tag{5.2.1}$$

and

$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x). \tag{5.2.2}$$

Recall also the following results that are used in further computations often without reference

Lemma 5.2.1 *A band* S *is regular if and only if* $axb = ab$ *holds for all* $a, b \preceq x \in \mathrm{S}$. ☐

Lemma 5.2.2 *A skew lattice* S *satisfies any identity or equational implication satisfied by both its left factor* $\mathrm{S}/\mathcal{R}$ *and its right factor* $\mathrm{S}/\mathcal{L}$. ☐

If S is a right-handed skew lattice then the two symmetry identities of (5.1.1) and (5.1.2) above simplify as

$$x \wedge y \wedge (x \vee y) = y \wedge x \tag{5.2.3}$$

and

$$x \vee y = (y \wedge x) \vee y \vee x. \tag{5.2.4}$$

Likewise for right-handed skew lattices, identities (5.2.1) and (5.2.2) above simplify as

$$(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x) \tag{5.2.5}$$

and

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{5.2.6}$$

We are ready to prove Spinks' Theorem, doing so first for the right-handed case.

Theorem 5.2.3. *For any skew lattice* S *the identities* (5.2.1) *and* (5.1.1) *imply* (5.2.2), and *dually, the identities* (5.2.2) *and* (5.1.2) *imply* (5.2.1).

Proof. Assume S is a right-handed skew lattice satisfying (5.2.3) and (5.2.5). Set $\alpha = x \vee (y \wedge z)$ and $\beta = (x \vee y) \wedge (x \vee z)$. By (5.2.5) and absorption one also has

$$\beta = x \vee (y \wedge (x \vee z)). \tag{5.2.7}$$

Equality (5.2.3) yields $y \wedge (x \vee z) = (x \vee z) \wedge y \wedge (x \vee z \vee y)$. Thus by (5.2.7),

$$\begin{aligned} \beta &= x \vee ((x \vee z) \wedge y \wedge (x \vee z \vee y)) \\ &= x \vee (x \wedge y \wedge (x \vee z \vee y)) \vee (z \wedge y \wedge (x \vee z \vee y)) && \text{by (5.2.5)} \\ &= x \vee (z \wedge y \wedge (x \vee z \vee y)) && (5.2.8) \end{aligned}$$

by absorption. Apply $\wedge\ (x \vee y \vee z)$ on the right of (5.2.8). By (5.2.5) and absorption,

$$\beta \wedge (x \vee y \vee z) = x \vee (z \wedge y \wedge (x \vee y \vee z)) \tag{5.2.9}$$

On the other hand,

$$\alpha \vee (x \vee z \vee y) = x \vee (y \wedge z) \vee x \vee z \vee y = x \vee (y \wedge z) \vee z \vee y = x \vee z \vee y.$$

Hence $\alpha \leq x \vee z \vee y$ and thus $\alpha \wedge (x \vee z \vee y) = \alpha$. Thus by (5.2.5) and absorption again,

$$x \vee (y \wedge z) = x \vee ((y \wedge z) \wedge (x \vee z \vee y)). \tag{5.2.10}$$

Switch y and z in (5.2.10) to get $x \vee (z \wedge y) = x \vee ((z \wedge y) \wedge (x \vee y \vee z))$ and use (5.2.9) to obtain

$$\beta \wedge (x \vee y \vee z) = x \vee (z \wedge y). \tag{5.2.11}$$

Apply $\wedge\ (x \vee y)$ on the right to obtain

$$\beta \wedge (x \vee y \vee z) \wedge (x \vee y) = (x \vee (z \wedge y)) \wedge (x \vee y) = x \vee (z \wedge y \wedge (x \vee y)). \tag{5.2.12}$$

On the other hand regularity and right-handedness with (5.2.5) imply

$$\beta \wedge (x \vee y \vee z) \wedge (x \vee y) \;=\; \beta \wedge (x \vee y)$$

$$= \;(x \vee y) \wedge (x \vee z) \wedge (x \vee y) \;=\; (x \vee z) \;\wedge (x \vee y) \;=\; x \vee \;(z \wedge (x \vee y)). \qquad (5.2.13)$$

Together (5.2.12) and (5.2.13) give the identity

$$x \vee (z \wedge y \wedge (x \vee y)) \;=\; x \vee (z \wedge (x \vee y)). \qquad (5.2.14)$$

Notice that $\alpha \vee x \vee z = x \vee (y \wedge z) \vee z = x \vee z$. Hence $\alpha \leq x \vee z$ and thus $\alpha \wedge (x \vee z) = \alpha$. Finally,

$$\begin{aligned} \alpha = \alpha \wedge (x \vee z) \;&=\; (x \vee (y \wedge z)) \wedge (x \vee z) && \\ &=\; x \vee (y \wedge z \wedge (x \vee z)) && \text{by (5.2.5)} \\ &=\; x \vee (y \wedge (x \vee z)) && \text{by (5.5.16) with } y \text{ and } z \text{ switched} \\ &=\; \beta && \text{by (5.2.7).} \end{aligned}$$

A similar argument shows that a right-handed skew lattice satisfying (5.2.4) and (5.2.6) must also satisfy (5.2.5). Duality yields that the assertion follows for left handed-skew lattices. The theorem now follows from Lemma 5.2.2. ☐

Corollary 5.2.4 (Spinks [1998] and [2000]) *For symmetric skew lattices, the distributive identitie*s (5.2.1) *and* (5.2.2) *are equivalent*. ☐

5.3 Cancellation in skew lattices

A skew lattice is ***cancellative*** if both

$$x \vee z = y \vee z \text{ and } x \wedge z = y \wedge z \text{ imply } x = y, \qquad (5.3.1)\ 17$$

and

$$x \vee y = x \vee z \text{ and } x \wedge y = x \wedge z \text{ imply } y = z. \qquad (5.3.2)\ 18$$

A skew lattice satisfying (5.3.1) [or (5.3.2)] is ***right*** [***left***] ***cancellative***. Lattices are cancellative precisely when they are distributive. For skew lattices, we at least have:

Lemma 5.3.1. *Left [right, fully] cancellative skew lattices are quasi-distributive*.

Proof. All forms of cancellation prevent $\mathbf{M}_3$ or $\mathbf{N}_5$ from being subalgebras. ☐

Proposition 5.3.2 *Any skew chain is cancellative*.

Proof. Assume that S is a right-handed skew chain with $x, y, z \in$ S satisfying $x \vee z = y \vee z$ and $x \wedge z = y \wedge z$. Then $x\mathcal{D}y$ since S/$\mathcal{D}$ is totally ordered and thus cancellative. Assume that

$\mathcal{D}_x \leq \mathcal{D}_z$. From $x \wedge z = y \wedge z$ we get $x \wedge u = y \wedge u$ for all u in $\mathcal{D}_z$. Thus to prove that $x = y$ it thus suffices to find $u \in \mathcal{D}_z$ such that $x \leq u$ and $y \leq u$. Obviously, $u = x \vee z = y \vee z$ does the job. On the other hand, $\mathcal{D}_z \leq \mathcal{D}_x$ implies $x = x \vee z \vee x = x \vee z = y \vee z = y \vee z \vee y = y$. Hence S satisfies (5.3.1). A similar argument, but with $\vee$ and $\wedge$ interchanged, shows that (5.3.2) holds. The left-handed case follows by duality. Since any skew chain S for which S/$\mathcal{D}$ is finite is a subalgebra of S/$\mathcal{R} \times$ S/$\mathcal{L}$, the general case must follow. ☐

This leads us to the observation that *cancellative skew lattices need not be distributive*.

Example 5.3.3. Consider the following right-handed skew chain S on 8 elements.

$$\begin{array}{ccccccc} & & z' & -- & z & & \\ & \cdot^{\cdot^{\cdot}} & \vdots & & \vdots & \ddots & \\ y' & -- & y'' & -- & y & -- & y''' \\ & \ddots & \vdots & & \vdots & \cdot^{\cdot^{\cdot}} & \\ & & x & -- & x' & & \end{array}$$

$\wedge$	z'	z
y'	y'	y'''
y''	y''	y
y	y''	y
y'''	y'	y'''

$\vee$	y'	y''	y	y'''
x	y'	y''	y'	y''
x'	y	y'''	y	y'''

The $\mathcal{D}$-relation is denoted by $--$, while $\cdot^{\cdot^{\cdot}}$, $\vdots$ and $\ddots$ denote $\geq$. The partial tables indicate those operation outcomes not determined simply by S being right-handed or by $\geq$. By the previous proposition, S is cancellative. But S does not satisfy (5.2.1):

$$(x \vee y) \wedge z = y' \wedge z = y''' \text{ and } (x \wedge z) \vee (y \wedge z) = x' \vee y = y. \ \square$$

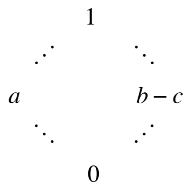
Theorem 5.3.4. *Cancellative skew lattices are strongly symmetric.*

Proof. An easy check shows that none of $\mathbf{NS}_7^{\mathcal{R},0}$, $\mathbf{NS}_7^{\mathcal{L},0}$, $\mathbf{NS}_7^{\mathcal{R},1}$ or $\mathbf{NS}_7^{\mathcal{L},1}$ is cancellative. Likewise it is easily verified that none of the $\mathbf{S}_{m,n}$ or $\mathbf{T}_{m,n}$ for $mn \geq 2$ can be cancellative. The theorem follows from Theorems 5.1.2 and 5.1.7. ☐

Corollary 5.3.5 (Cvetko-Vah [2006b]) *The distributive identities* (5.2.1) *and* (5.2.2) *are equivalent for all cancellative skew lattices*.

Proof. This is an immediate consequence of the theorem above and Corollary 5.2.4. ☐

Conversely, *distributivity does not imply cancellativity.* Minimal examples of distributive skew lattices that are neither left, right nor fully cancellative are given by a dual pair of skew lattices with the common Hasse diagram below. We denote the right-handed case by $\mathbf{NC}_5^{\mathcal{R}}$ and its left-handed dual by $\mathbf{NC}_5^{\mathcal{L}}$. The Cayley tables for $\mathbf{NC}_5^{\mathcal{R}}$ are also given. Transposing them gives the tables the left-handed variant.



$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	b	1
c	c	1	c	c	1
1	1	1	1	1	1

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	c	b
c	0	0	b	c	c
1	0	a	b	c	1

One might ask whether the absence of M_3, N_5 and the above pair of 5-element algebras as subalgebras of a skew lattice S will insure that S is cancellative. While this is not the case (the four 7-element algebras are counterexamples), the question does lead to a form of cancellation the underlies the three forms of cancellation above.

A skew lattice S is ***simply cancellative*** if for all $x, y, z \in$ S,

$$x\vee z\vee x = y\vee z\vee y \text{ and } x\wedge z\wedge x = y\wedge z\wedge y \text{ together imply } x = y. \tag{5.3.3}$$

This is equivalent to a second implication,

$$x\vee z = y\vee z,\ z\vee x = z\vee y,\ x\wedge z = y\wedge z \text{ and } z\wedge x = z\wedge y \text{ imply } x = y. \tag{5.3.4}$$

Indeed both sets of conditions imply each other. From $x\vee z\vee x = y\vee z\vee y$ we get

$$x\vee z\vee x\vee z = y\vee z\vee y\vee z,$$

that is $x\vee z = y\vee z$, and similarly, $z\vee x = z\vee y$. Conversely from $x\vee z = y\vee z$ and

$$z\vee x = z\vee y,\ x\vee z\vee x = y\vee z\vee y$$

must follow. Similar remarks hold for equalities involving $\wedge$. Thus, *simple cancellativity is implied separately by left, right and full cancellativity*.

Lemma 5.3.6. *A skew lattice* **S** *is simply cancellative if and only if it is quasi-distributive and all skew diamonds within* **S** *are simply cancellative. Given a skew diamond* **T** *with incomparable $\mathcal{D}$-classes* A *and* B, *join class* J *and meet class* M, *the following are equivalent:*

i) **T** *is simply cancellative*
ii) *Given $e > f$ with $e \in$ J and $f \in$ M, unique $a \in$A and $b \in$B exist such that both $e > a > f$ and $e > b > f$.*
iii) *Given $e > f$ with $e \in$ J and $f \in$ M, unique $a \in$ A and $b \in$ B exist such that $e = a\vee b = b\vee a$ and $f = a\wedge b = b\wedge a$.*
iv) **T** *contains no copy of* $\mathbf{NC}_5^{\mathcal{L}}$ *or* $\mathbf{NC}_5^{\mathcal{R}}$.

Proof. Simply cancellative skew lattices are quasi-distributive since M_3 and N_5 cannot occur as subalgebras. Assume **S** is at least quasi-distributive. If a, a', $b \in$ S exist such that $a \vee b \vee a = a' \vee b \vee a'$ and $a \wedge b \wedge a = a' \wedge b \wedge a'$, then $a \ \mathcal{D} \ a'$ since a and a' have the same image in the distributive lattice $\mathbf{S}/\mathcal{D}$. If a and a' are comparable to b (a, $a' \succeq b$ or $b \succeq a$, a'), then either $a \vee b \vee a = a' \vee b \vee a'$ or $a \wedge b \wedge a = a' \wedge b \wedge a'$ will reduce to $a = a'$. Thus we need only consider the case where b is incomparable to a and a'. In any case, the first assertion is by now clear. As for the equivalence of (i) through (iv) for a skew diamond **T**, (ii) and (iii) are trivially equivalent by Theorem 2.2.1. Clearly these conditions imply (iv). Conversely, suppose (iv) holds. Then given any $c \in$ A and $d \in$ B, setting $a = f \vee (e \wedge c \wedge e) \vee f$ and $b = f \vee (e \wedge d \wedge e) \vee f$ gives a pair a, b satisfying these inequalities. Suppose $e > a$, $a' > f$ where $a \ \mathcal{D} \ a'$. Then $a \ \mathcal{R} \ a \wedge a' \ \mathcal{L} \ a'$. Thus if $a \neq a'$ then either $a \neq a \wedge a'$ or $a \wedge a' \neq a'$. Moreover $e > a \wedge a' > f$ so that either $\{a, a \wedge a', b, e, f\}$ or $\{a', a \wedge a', b, e, f\}$ is a copy of $\mathrm{NC}_5^{\mathcal{L}}$ or $\mathrm{NC}_5^{\mathcal{R}}$ in T contrary to (iv). Thus a is unique and similarly so is b. Hence (iv) implies (ii).

Suppose that (i) holds and let $e \in$ J and $f \in$ M be as in (ii) and (iii). As we have seen, $a \in$ A and $b \in$ B exist such that $e > a$, $b > f$. If $a' \in$ A is such that $e > a' > f$ also, then $a \vee b \vee a = e = a' \vee b \vee a'$ and $a \wedge b \wedge a = f = a' \wedge b \wedge a'$. Simple cancellation yields $a = a'$. Similarly b is unique and (ii) follows. Conversely suppose that (ii) – (iv) hold and let $a \vee b \vee a = a' \vee b \vee a'$ and $a \wedge b \wedge a = a' \wedge b \wedge a'$ in T. Since **T** is quasi-distributive, at least $a \ \mathcal{D} \ a'$. Again, if a and b are comparable, then $a = a'$, as seen above. So we may assume they are incomparable and hence, say $a \in$ A and $b \in$ B. Setting $e = a \vee b \vee a = a' \vee b \vee a'$ in J and $f = a \wedge b \wedge a = a' \wedge b \wedge a'$ in M, we have $e \geq a$, $a' > f$. By (ii), $a = a'$ and (i) follows. ☐

Theorem 5.3.7. *A skew lattice* S *is simply cancellative if and only if it contains no copy of* $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{L}}$ *or* $\mathbf{NC}_5^{\mathcal{R}}$. *Collectively, these skew lattices form a variety.*

Proof. The first assertion is clear from the preceding discussion. An equational base for this class of skew lattices is given by the identity for quasi-distributivity and the identity

$$f \vee (e \wedge y \wedge z \wedge y \wedge e) \vee f = f \vee (e \wedge z \wedge y \wedge z \wedge e) \vee f$$

where $e = e(x, y, z) = x \vee y \vee z \vee x$ and $f = f(x, y, z) = x \wedge e \wedge y \wedge z \wedge y \wedge e \wedge x = x \wedge y \wedge z \wedge y \wedge x$. On either side are two typical $\mathcal{D}$-related elements having common commuting joins and meets with a third element (here x). The implication defining simple cancellativity (where x, y and z assume different roles!) equates these elements. ☐

Returning to left [right, full cancellativity] we have:

Theorem 5.3.8.

1) *The following are equivalent for a skew lattice* **S**.

   i) **S** *is left cancellative*.
   ii) *None of* $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{L}}$, $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NS}_7^{\mathcal{R},0}$ *nor* $\mathbf{NS}_7^{\mathcal{L},1}$ *are subalgebras of* S.
   iii) **S** *is simply cancellative*, $\mathbf{S}/\mathcal{L}$ *is upper symmetric, and* $\mathbf{S}/\mathcal{R}$ *is lower symmetric*.

2) *The following are equivalent for a skew lattice* **S**.

   i) **S** *is right cancellative*.
   ii) *None of* $\mathbf{M}_3$, $\mathbf{N}_5$, $\mathbf{NC}_5^{\mathcal{L}}$, $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NS}_7^{\mathcal{R},1}$ *nor* $\mathbf{NS}_7^{\mathcal{L},0}$ *are subalgebras of* S.
   iii) **S** *is simply cancellative*, $\mathbf{S}/\mathcal{L}$ *is lower symmetric, and* $\mathbf{S}/\mathcal{R}$ *is upper symmetric*.

3) *The following are equivalent for a skew lattice* **S**.

   i) **S** *is cancellative*.
   ii) *None of the above* 5 *or* 7*-element algebras occur as subalgebras of* S.
   iii) **S** *is simply cancellative and symmetric*.

4) *Left [right, fully] cancellative skew lattices form a variety*.

Proof. We begin with (1). If S is left cancellative, then S cannot contain copies of any of the algebras listed in (ii) since none of them are left cancellative. Thus (i) implies (ii). The equivalence of (ii) and (iii) follows from Theorems **5.3.7** and **5.1.2**. Now assume **S** satisfies (iii), and suppose a, b, $c \in$ S satisfy $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. Since $\mathbf{S}/\mathcal{L}$ is upper symmetric and $\mathbf{S}/\mathcal{R}$ is lower symmetric, Theorem 5.1.2 gives

$$b \vee a \vee b = (a \wedge b) \vee a \vee b = (a \wedge c) \vee a \vee c = c \vee a \vee c$$

and

$$b \wedge a \wedge b = (a \vee b) \wedge a \wedge b = (a \vee c) \wedge a \wedge c = c \wedge a \wedge c.$$

By simple cancellativity, $b = c$. Therefore S is left cancellative and (i) holds. The proof of (2) is similar. Indeed (2) follows from (1) via horizontal duality (new $x \vee y$ = old $y \vee x$; new $x \wedge y$ = old $y \wedge x$). (3) is a consequence of (1) and (2) combined, together with Theorems **5.3.7** and **5.1.1**. Finally (4) now follows from (1) – (3) and Theorems 5.1.1, 5.1.2 and 5.3.7. ☐

Equational bases. A basis for left cancellative skew lattices is thus as follows.

1) $[x\wedge(y\vee z)] \wedge [(x\wedge y) \vee (x\wedge z)] \wedge [x\wedge(y\vee z)] = x\wedge(y\vee z)$.

2) $f\vee(e\wedge y\wedge z\wedge y\wedge e)\vee f = f\vee(e\wedge z\wedge y\wedge z\wedge e)\vee f$ where $e = x\vee y\vee z\vee x$ and $f = x\wedge y\wedge z\wedge y\wedge x$.

3) $x\vee y\vee x = (y\wedge x)\vee y\vee x$ and $x\wedge y\wedge x = (y\vee x)\wedge y\wedge x$.

Identities (1) and (2) insure that a skew lattice S is simply cancellative (per Theorem 5.3.7) and (3) ensures that $S/\mathcal{L}$ is upper symmetric and $S/\mathcal{R}$ is lower symmetric (per Theorem 5.1.2). An equational base for right cancellative skew lattices is obtained by replacing (3) with the left-right dual identities. An equational base for cancellative skew lattices is given by replacing (3) with the two identities for symmetry.

Consider the lattice of varieties below.

$$
\begin{array}{ccc}
 & \textit{Simply cancellative skew lattices} & \\
\textit{Left cancellative skew lattices} & & \textit{Right cancellative skew lattices} \\
 & \textit{Cancellative skew lattices} &
\end{array}
$$

While the bottom variety is the inter-section of the middle varieties, we do not know if the top variety is generated from the middle varieties and thus is their join in the lattice of all skew lattice varieties. The bottom variety is, of course, the intersection of each variety with the variety of symmetric skew lattices. Using Mace4, four distinct minimal cases of order 12 exist that are simply cancellative, but are neither left nor right cancellative. They turn out to be the fibered product $\mathbf{NS}_7^{\mathcal{L},0} \times_{\mathbf{2}\times\mathbf{2}} \mathbf{NS}_7^{\mathcal{L},1}$ of $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$ over their maximal lattice image $\mathbf{2}\times\mathbf{2}$; the splice of $\mathbf{NS}_7^{\mathcal{L},0}$ with $\mathbf{NS}_7^{\mathcal{L},1}$ obtained by identifying the join class of $\mathbf{NS}_7^{\mathcal{L},0}$ with the meet class of $\mathbf{NS}_7^{\mathcal{L},1}$; their two right-handed duals.

Cancellation is often used to compare the sizes of sets. Indeed much of our discussion of simple cancellation in Lemma 5.3.6 can be recast as follows. Given a skew diamond with incomparable $\mathcal{D}$-classes A and B, join class J and meet class M, set:

$$\omega(J, M) \;=\; \{(j, m) \in J \times M \mid j > m\}$$

and

$$Comm^2(A, B) \;=\; \{(a, b) \in A \times B \mid a \vee b = b \vee a \;\&\; a \wedge b = b \wedge a\}.$$

$\omega(J, M)$ is the natural partial order between J and M, while $Comm^2(A, B)$, consists of all pairs, one from each class, that commute under both operations. Define $\xi\colon Comm^2(A, B) \to \omega(J, M)$ by $\xi(a, b) = (a \vee b, a \wedge b)$. ξ is at least well defined. In the proof of Lemma 5.3.6 we saw that ξ is surjective. The unique parallel commuting factorization of Lemma 5.3.6 (iii) gives us the first half of:

Theorem 5.3.10. *Given a simply cancellative skew diamond* {J > A, B > M}, *the function* ξ: *Comm*^{2}(A, B) → ω(J, M) *defined by* $\xi(a, b) = (a \vee b, a \wedge b)$ *is a bijection. Conversely, given any skew diamond* {J > A, B > M} *the map* ξ *as stated is always well-defined, but it is a bijection only if the skew diamond is simply cancellative.*

Proof. If {J > A, B > M} is not simply cancellative, then either a copy of $\mathbf{NC}_5^{\mathcal{L}}$ or $\mathbf{NC}_5^{\mathcal{R}}$ occurs which gives ξ a properly many-to-one instance. ☐

The above results in this section are from the 2011 paper of Cvetko-Vah, Kinyon, Leech and Spinks.

We conclude this section with several observations:

Theorem 5.3.11. *Skew lattices in rings are cancellative.*

Proof. So let $ab = ac$ and $a\nabla b = a\nabla c$. Expanding and making the obvious reductions reduces the latter to

$$b + ba - bab = c + ca - cac = c + ca - cab.$$

Thus $b(1 - a - ab) = c(1 - a - ab)$. But $(1 - a - ab)^2 = 1 - a + aba$ and so

$$b = b(1 - a + aba) = c(1 - a + aba).$$

Similarly $c = c(1 - a + aca) = c(1 - a + aba)$ and $b = c$ follows. Right cancellation is shown similarly. ☐

Theorem 5.3.12. *Strongly distributive skew lattices are cancellative.*

Proof. Strongly distributive skew lattices are clearly quasi-distributive, and by Theorem 2.3.4 also symmetric. By the same theorem they are also normal, so that neither $\mathbf{NC}_5^{\mathcal{L}}$ nor $\mathbf{NC}_5^{\mathcal{R}}$ can be a subalgebra. Hence none of the 5- or 7-element forbidden algebras can be a subalgebra, making any strongly distributive skew lattice cancellative. ☐

Primitive skew lattices and skew chains in general are trivially cancellative: they are clearly symmetric and quasi-distributive and it is impossible for either variant of $\mathbf{NC}_5$ to be a subalgebra. Thus we have:

Proposition 5.3.13. *Every skew lattice in the variety of skew lattices generated from the class of all primitive skew lattices is cancellative. More generally, every skew lattice in the variety of skew lattices generated from the class of all skew chains is cancellative.* ☐

Queries: *Are the varieties generated by these two classes of algebras the same? Is the second variety in fact the variety of all cancellative skew lattices?*

The answer to the first question is, *no*. Primitive skew lattices are trivially categorical, and thus all skew lattices in the first variety are categorical. We shall see in the next section that skew chains A > B > C of length 2 exist that are not categorical. The first variety is thus a proper subvariety of the second.

5.4 Categorical skew lattices

We continue our study begun in Section 2.4 of categorical skew lattices, where all nonempty composites of coset bijections are coset bijections. This reduces to the implication: if $a > b > c$ with $a' \mathcal{D} a$ and $c' \mathcal{D} c$ such that $c' = a' \wedge c \wedge a'$ in C and $a' = c' \vee a \vee c'$ in A (making both $a > c$ and $a' > c'$ part of a common coset bijection from A to C), then $a' \wedge b \wedge a' = c' \vee b \vee c'$ in B.

$$\begin{array}{lll} a & - & a' = c' \vee a \vee c' \\ \vdots & & \vdots \\ b & - & b' = a' \wedge b \wedge a' = c' \vee b \vee c' \quad \text{(where } \vdots \text{ denotes >)} \\ \vdots & & \vdots \\ c & - & c' = a' \wedge c \wedge a' \end{array}$$

For if χ, the unique coset bijection from A to C taking a to c, factors as $\psi o \varphi$, where φ and ψ are the unique coset bijections from A to B and from B to C taking a to b and b to c respectively, then one has $a' \wedge b \wedge a' = \varphi[a'] = \psi^{-1}[c'] = c' \vee b \vee c'$.

Contraposition gives the following *criterion for a skew lattice* S *to <u>not</u> be categorical*: given $a > b > c$ in S with $a' \mathcal{D} a$ and $c' \mathcal{D} c$ being such that $a' \wedge c \wedge a' = c'$ in $\mathcal{D}_c$ and $c' \vee a \vee c' = a'$ in $\mathcal{D}_a$, but $a' \wedge b \wedge a' \neq c' \vee b \vee c'$ in $\mathcal{D}_b$.

We have seen that *categorical skew lattices form a variety*. In the left-handed skew case:

Lemma 5.4.1. *A left-handed skew lattice is categorical if either of the following pair of dual identities hold:*

$$x \wedge [y \vee (x \wedge y \wedge z)] = x \wedge y \tag{5.4.1}$$

or

$$[(x \vee y \vee z) \wedge y] \vee z = y \vee z. \tag{5.4.2}$$

Proof. Let S be a left-handed *non*categorical skew lattice. Thus $a > b > c$ in S exist with $a' \mathcal{L} a$ and $c' \mathcal{L} c$ such that $a' \wedge c = c'$ in C, $a \vee c' = a'$ in A but $a' \wedge b \neq b \vee c'$ in B. This creates the following configuration

$$\begin{array}{lccccccc} \text{A:} & a & & - & & a' & & \\ & \vdots & & & \cdot^{\cdot^{\cdot}} & & \ddots & \\ \text{B:} & b & - & a' \wedge b & & - & & b \vee c'. \\ & \vdots & & & \ddots & & \cdot^{\cdot^{\cdot}} & \\ \text{C:} & c & & - & & c' & & \end{array}$$

Being left-handed, both $a' > a'\wedge b$ and $b\vee c' > c'$ are clear. Trivially $c'\wedge(a'\wedge b) = c'$ and $(b\vee c')\vee a' = a'$. On the other hand, $(a'\wedge b)\wedge c' = (a'\wedge b)\wedge(a'\wedge c) = a'\wedge(b\wedge c) = a'\wedge c = c'$ also so that $a'\wedge b > c'$. In dual fashion $a' > b\vee c'$. Notice that both

$$a' \wedge b \neq b \vee c' = a' \wedge (b \vee c') = a' \wedge [b \vee (a' \wedge b \wedge c')].$$

and

$$b \vee c' \neq a' \wedge b = (a' \wedge b) \vee c' = [(a' \vee b \vee c') \wedge b] \vee c'.$$

The lemma follows by contraposition. □

Comment: The converse (in the left-handed case) has been shown using Prover9. These identities do not imply left-handedness. As a consequence of this lemma we have:

Theorem 5.4.2. *Any skew lattice satisfying either distributive identity* (5.2.1) *or* (5.2.2) *is categorical. In particular, all distributive skew lattices are categorical.*

Proof. To begin, if a left-handed skew lattice satisfies (5.2.1), then (5.4.1) follows:

$$x \wedge [y \vee (x \wedge y \wedge z)] = (x\wedge y) \vee [x \wedge (x \wedge y \wedge z)] = (x\wedge y) \vee (x \wedge y \wedge z) = x\wedge y.$$

Likewise a left-handed skew lattice satisfying (5.2.2) must satisfy (5.4.2). Dually, a right-handed skew lattice satisfying either (5.2.1) or (5.2.2) are categorical. In general, if a skew lattice S satisfies either distributive identity, then so do its factors $S/\mathcal{R}$ and $S/\mathcal{L}$. Hence each factor is categorical and thus so is S, since it is isomorphic to a subalgebra of $S/\mathcal{R} \times S/\mathcal{L}$. □

Forbidden subalgebras

Given a comparable $\mathcal{D}$-classes, A > B, if $a, a' \in A$ lie in a common B-coset, we denote this by $a -_B a'$; likewise $b -_A b'$ in B if b, b' lie in a common A-coset. Of interest here are skew chains A > B > C, since a skew lattice is categorical if and only if all its skew chains are thus.

Two elements b and b' in the middle class B are **AC-*connected*** if a finite sequence $b = b_0, b_1, b_2, \ldots, b_n = b'$ exists such that $b_i -_A b_{i+1}$ or $b_i -_C b_{i+1}$ for all $i \leq n - 1$. Clearly this defines an equivalence relation on B. An **AC-*component*** of B (or just ***component*** when the context is clear) is an equivalence class for this relation, that is, a maximally AC-connected subset of B. (Connectedness is actually a congruence on the rectangular algebra B, making the components subalgebras of B. Indeed it is the join-equivalence of the congruence partitions given by the A-cosets and by the B-cosets.) In the examples below, B is connected. Given a component B_1 in the middle class B, a sub-skew chain is given by $A > B_1 > C$. Indeed if A_1 is a B-coset in A and C_1 is a B-coset in C, then $A_1 > B_1 > C_1$ is also a sub-skew chain; moreover, *a skew chain is categorical if and only if all such sub-skew chains are categorical.*

Our classification of forbidden subalgebras rests on the next lemma and its right-handed dual.

Lemma 5.4.3. *Given a left-handed skew chain* A > B > C, *let* $a > c$ *and* $a' > c'$ *be input-output pairs for a common coset bijection between* A *and* C *where* $a \neq a'$ *in* A *and* $c \neq c'$ *in* C. *Upon setting* A* = $\{a, a'\}$, B* = $\{x \in \mathrm{B} \mid a > x > c$ *or* $a' > x > c'\}$ *and* C* = $\{c, c'\}$, *one obtains a sub-skew chain:* A* > B* > C*. *In particular,*

i) $a' > x > c'$ *for* x *in* B* *implies*: $a >$ *both* $a \wedge x$ *and* $x \vee c > c$ *with* $a \wedge x -_{A^*} x -_{C^*} x \vee c$.
ii) $a > x > c$ *for* x *in* B* *implies*: $a' >$ *both* $a' \wedge x$ *and* $x \vee c' > c'$ *with* $a' \wedge x -_{A^*} x -_{C^*} x \vee c'$.

All A*-*cosets and all* C*-*cosets in* B* *are of order* 2. *An* A*C*-*component in* B* *is either a subset* $\{b, b'\}$ *that is simultaneously an* A* *and* C*-*coset in* B* *or else it is a larger subset having the alternating coset form:*

$$\dots -_{A^*} \bullet -_{C^*} \bullet -_{A^*} \bullet -_{C^*} \bullet -_{A^*} \bullet -_{C^*} \dots$$

Only the former can occur if the skew chain is categorical.

Proof. Being left-handed, we need only check the mixed outcomes, say $a \wedge x$, $x \wedge a$, $c \vee x$ and $x \vee c$ where $a' > x > c'$ for case (i). Trivially $x \wedge a = x = c \vee x$. As for $a \wedge x$, $a \wedge (a \wedge x) = a \wedge x = (a \wedge x) \wedge a$, due to left-handedness, so that $a > a \wedge x$; likewise $c \wedge (a \wedge x) = c$, while

$$(a \wedge x) \wedge c = a \wedge x \wedge a \wedge c' = a \wedge x \wedge c' = a \wedge c' = c$$

by left-handedness and the common coset bisection context. Hence $a \wedge x > c$ also, so that $a \wedge x$ is in B*. The dual argument gives $a > x \vee c > c$, so that $x \vee c \in$ B* also. Similarly (ii) holds and we have a sub-skew chain.

Clearly the A*-cosets in B* either all have order 1 or all have order 2. If they have order 1, then a, $a' >$ all elements in B*, and by transitivity, a, $a' >$ both c, c', so that $a > c$ belongs to a different coset bijection than $a' > c'$. Thus all A*-cosets in B* have order 2 and likewise all C*-cosets in B* have order 2. In an A*C*-component in B*, if the first case does not occur, a situation $x -_{C^*} y -_{A^*} z$ with x, y and z distinct develops. Since A* and C*-cosets have size 2, it extends in an alternating coset pattern in both directions, either doing so indefinitely or eventually connecting to form a cycle of even length. □

This leads to:

Example 5.4.4: Consider the class of skew chains A > B_n > C for $1 \leq n \leq \omega$, where

A = $\{a_1, a_2\}$,
$\mathrm{B}_n = \{a_1, a_2, a_3, \dots, a_{2n}\}$ or $\{\dots, b_{-2}, b_{-1}, b_0, b_1, b_2, \dots\}$ if $n = \omega$ and
C = $\{c_1, c_2\}$.

The natural partial ordering given by $a_1 > b_{odd} > c_1$ and $a_2 > b_{even} > c_2$. Both A and C are full B-cosets as well as full cosets of each other. A- and C-cosets in B are given respectively by the partitions:

$$\left\{b_1, b_2 \middle| b_3, b_4 \middle| \ldots \middle| b_{2n-1}, b_{2n}\right\} \text{ and } \left\{b_{2n}, b_1 \middle| b_2, b_3 \middle| \ldots \middle| b_{2n-2}, b_{2n-1}\right\}$$

and

$$\{b_{2k+1},\ b_{2k+2}\} \text{ and } \{b_{2k},\ b_{2k+1}\} \text{ when } n = \omega.$$

Clearly B_n is a single component. We denote the left-handed skew chain thus determined by $\mathbf{X}_n$ and its right-handed dual by $\mathbf{Y}_n$ for $n \leq \omega$. Their shared Hasse diagrams are as follows.

$$1: \quad \begin{array}{ccc} a_1 & - & a_2 \\ \vdots & & \vdots \\ b_1 & - & b_2 \\ \vdots & & \vdots \\ c_1 & - & c_2 \end{array} \qquad\qquad 2: \quad \begin{array}{c} a_1 - a_2 \\ \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \qquad \ddots \quad \ddots \\ b_1 \underset{\text{A}}{-} b_2 \underset{\text{C}}{-} b_3 \underset{\text{A}}{-} b_4 \underset{\text{C}}{-} b_1 \\ \ddots \quad \ddots \qquad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \\ c_1 - c_2 \end{array}$$

$$n: \quad \begin{array}{c} a_1 - a_2 \\ \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \quad \cdots \quad \ddots \quad \ddots \\ b_1 \underset{\text{A}}{-} b_2 \underset{\text{C}}{-} \cdots \underset{\text{C}}{-} b_{2n-1} \underset{\text{A}}{-} b_{2n} \underset{\text{C}}{-} (b_1) \\ \ddots \quad \ddots \quad \cdots \quad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \\ c_1 - c_2 \end{array} \qquad \omega: \quad \begin{array}{c} a_1 - a_2 \\ \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \quad \cdots \quad \ddots \quad \ddots \\ \cdots \underset{\text{A}}{-} b_{-2} \underset{\text{C}}{-} b_{-1} \underset{\text{A}}{-} b_0 \underset{\text{C}}{-} b_1 \underset{\text{A}}{-} b_2 \underset{\text{C}}{-} \cdots \\ \ddots \quad \ddots \quad \cdots \quad \cdot^{\cdot^{\cdot}} \quad \cdot^{\cdot^{\cdot}} \\ c_1 - c_2 \end{array}$$

In line with our remarks above, instances of left-handed operations on $\mathbf{X}_2$ are given by

$$a_1 \vee c_2 = a_1 \vee a_2 = a_2, \quad a_1 \wedge b_4 = b_3 \wedge b_4 = b_3 \quad \text{and} \quad b_1 \vee c_2 = b_1 \vee b_4 = b_4.$$

Except for $\mathbf{X}_1$ *and* $\mathbf{Y}_1$, *none of these skew lattices is categorical.* Indeed,

$$a_1 > b_1 > c_1, \qquad a_2 \wedge c_1 \wedge a_2 = c_2,\ c_2 \vee a_1 \vee c_2 = a_2,$$

but $a_2 \wedge b_1 \wedge a_2 = b_2$, while $c_2 \vee b_1 \vee c_2$ is either b_{2n} or b_0. *Thus, except for* $\mathbf{X}_1$ *and* $\mathbf{Y}_1$, *none of these is distributive.* Note also that each $\mathbf{X}_n$ is generated from a_1, c_2 and any b_i. Thus no $\mathbf{X}_n$ contains a copy of a lower $\mathbf{X}_m$ as a subalgebra. Similar remarks hold for the $\mathbf{Y}_n$.

Theorem 5.4.5. *A left-handed skew lattice is categorical if and only if it contains no copy of* $\mathbf{X}_n$ *for* $2 \leq n \leq \omega$. *Dually, a right-handed skew lattice is categorical if and only if it contains no copy of* $\mathbf{Y}_n$ *for* $2 \leq n \leq \omega$. *In general, a skew lattice is categorical if and only if it contains no copy of any of these algebras.*

Proof: We begin with a left-handed noncategorical skew chain S. Without loss of generality we may assume that A is a full B-coset in itself and that C is a full B-coset in itself. Let $a > b > c$ in S with $a' \mathcal{L}\, a$ and $c' \mathcal{L}\, c$ being such that $a' \wedge c = c'$ in C, $a \vee c' = a'$ in A so that a, a' correspond to c, c' under a coset bijection between A and C, but $a' \wedge b \neq b \vee c'$ in B where A, B, C are respective $\mathcal{L}$-classes. The first new elements formed are $a' \wedge b$ and $b \vee c'$ in B. Note that

$a\vee(a'\wedge b) = a'$ since both are images of $a'\wedge b$ in A, which is the unique B-coset within A. Likewise, $(b\vee c')\wedge c = c'$. No new elements are created thus far from $\{a, a', b, c, c'\}$ giving us:

$$
\begin{array}{llccccc}
\text{A:} & a & & - & a' & & \\
 & \vdots & & & \cdot^{\cdot^{\cdot}} & \ddots & \\
\text{B:} & b & - & a'\wedge b & - & & b\vee c' \\
 & \vdots & & & \ddots & \cdot^{\cdot^{\cdot}} & \\
\text{C:} & c & & - & c' & &
\end{array}
$$

At this stage it follows that the subalgebra formed from $\{a, a', b, c, c'\}$ only has $\{a, a'\}$ in the top $\mathcal{L}$-class and $\{c, c'\}$ in the bottom $\mathcal{L}$-class. Continuing, step-by-step, in both directions we get

$$
\begin{array}{ccccccccccccccccc}
- & a' & - & a & - & a' & - & a & - & a' & - & a & - & a' & - \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\underset{\text{A}}{-} & [a\wedge(b\wedge c')]\vee c' & \underset{\text{C}}{-} & a\wedge(b\wedge c') & \underset{\text{A}}{-} & b\vee c' & \underset{\text{C}}{-} & b & \underset{\text{A}}{-} & a'\wedge b & \underset{\text{C}}{-} & (a'\wedge b)\vee c & \underset{\text{A}}{-} & a\wedge[(a'\wedge b)\vee c] & \underset{\text{C}}{-} \\
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
.- & c' & - & c & - & c' & - & c & - & c' & - & c & - & c' & -
\end{array}
$$

Expansion thus continues in B. If repetition never occurs, we obtain a copy of X_ω. Otherwise, a cyclic closure arises and we have obtained a copy of some X_n. The left-handed case follows from this. The right-handed case is similar.

Clearly, a categorical skew lattice contains none of these algebras. Conversely, if a skew lattice S contains copies of none of them, then neither does S/$\mathcal{R}$ or S/$\mathcal{L}$ since every skew chain with finitely many $\mathcal{D}$-classes in either S/$\mathcal{R}$ or S/$\mathcal{L}$ can be lifted up into S. Thus both S/$\mathcal{L}$ and S/$\mathcal{R}$ are categorical, and hence so it S which is embedded in their product. ☐

Corollary 5.4.6. *Distributive skew lattices contain no copies of* $\mathbf{X}_n$ *or* $\mathbf{Y}_n$ *for* $n \geq 2$. ☐

Comment: If these skew chains seem familiar it is because they are precisely the maximal skew chains within the skew diamonds arising in the classification of all symmetric skew lattices. $\mathbf{X}_n$ and $\mathbf{Y}_n$ for all $n \geq 2$ in particular arise in the various non-categorical cases.

Strictly categorical skew lattices

We turn to categorical skew lattices that are ***strictly categorical*** in that for each skew chain of $\mathcal{D}$-classes A > B > C, each A-coset in B has nonempty intersection with each C-coset in B, making both B an entire AC-component and empty coset bijections unnecessary. Examples are:

a) *Normal* skew lattices characterized by the identity $x\wedge y\wedge z\wedge w = x\wedge z\wedge y\wedge w$; equivalently, each subset $[e]\downarrow = \{x \in S\,|\,e \geq x\} = \{e\wedge x\wedge e\,|\,x \in S\}$ is a sublattice.

b) *Conormal* skew lattices that satisfy $x\vee y\vee z\vee w = x\vee z\vee y\vee w$; equivalently, every subset $[a]\uparrow = \{x \in S\,|\,x \geq e\} = \{e\vee x\vee e\,|\,x \in S\}$ is a sublattice.

c) *Primitive* skew lattices consisting of two $\mathcal{D}$-classes: A > B and rectangular skew lattices. Any algebra in the variety that primitive skew lattices generate.

Their significance is due in part to skew Boolean algebras being normal as skew lattices.

Theorem 5.4.7. *Let* A > B > C *be a strictly categorical skew chain. Then*:

i) *For any* $a \in$A, *all images of* a *in* B *lie in a unique* C-*coset in* B.

ii) *For any* $c \in$C, *all images of* c *in* B *lie in a unique* A-*coset in* B.

iii) *Given* $a > c$ *with* $a \in$A *and* $c \in$C, *a unique* $b \in$B *exists such that* $a > b > c$. *This* b *lies jointly in the* C-*coset in* B *containing all images of* a *in* B *and in the* A-*coset in* B *containing all images of* c *in* B. (It is the ***midpoint*** of a and c in B.)

Proof. (i) Without loss of generality we assume that C is a full B-coset within itself. If $a\wedge C\wedge a = \{c \in C\,|\,a > c\}$ is the image set of a in C parameterizing the A-cosets in C and if $a > b$ in B, then the set $\{c\vee b\vee c\,|\,c \in a\wedge C\wedge a\}$ of all images of a in the C-coset $C\vee b\vee C$ in B, parameterizes the A-C cosets in B lying in $C\vee b\vee C$ (that is, the coset intersections $A\wedge b\wedge A \cap C\vee b\vee C$ in $C\vee b\vee C$) that are inverse images of the A-cosets in C relative to the coset bijection of $C\vee b\vee C$ onto C. (See Theorem 2.4.14 and its preceding discussion.) By assumption, all A-cosets X in B are in bijective correspondence with these A-C cosets under the map $X \to X \cap C\vee b\vee C$. Thus each x in $\{c\vee b\vee c\,|\,c \in a\wedge C\wedge a\}$ is the (necessarily) unique image of a in the A-coset in B to which x belongs and as we traverse through these x's, every such A-coset occurs as $A\wedge x\wedge A$. Thus all images of a in B lie in the C-coset $C\vee b\vee C$ in B.

In like fashion one verifies (ii).

Finally, given $a > c$ with $a \in$A and $c \in$C, a unique AC-coset U exists that is the intersection of the A-coset containing all images of c in B and the C-coset containing all images of a in B. In particular U contains any b in B such that $a > b > c$. Such a b exists in B, e.g., $b = a\wedge(c\vee u\vee c)\wedge a$ for any u in B. But being in a single AC-coset in B, at most one such b is in U. □

In the terminology of Section 2.4, the A-cosets in B are *orthogonal* to the C-cosets in B. All this leads to the following multiple characterization of strictly categorical skew lattices:

Theorem 5.4.8. *The following seven conditions on a skew lattice* S *are equivalent.*

i) S *is strictly categorical*.

ii) *Given both* $a > b > c$ *and* $a > b' > c$ *in* S *with* $b\ \mathcal{D}\ b'$, $b = b'$ *must follow*.

iii) *Given both* $a \geq b \geq c$ *and* $a \geq b' \geq c$ *in* S *with* $b\ \mathcal{D}\ b'$, $b = b'$ *must follow*.

iv) S *has no subalgebra isomorphic to either of the following* 4-*element skew chains*:

$$\begin{array}{ccc} & a & \\ \cdot^{\cdot^{\cdot}} & & \cdot_{\cdot_{\cdot}} \\ b & --_L & b'. \\ \cdot_{\cdot_{\cdot}} & & \cdot^{\cdot^{\cdot}} \\ & c & \end{array} \qquad\qquad \begin{array}{ccc} & a & \\ \cdot^{\cdot^{\cdot}} & & \cdot_{\cdot_{\cdot}} \\ b & --_R & b'. \\ \cdot_{\cdot_{\cdot}} & & \cdot^{\cdot^{\cdot}} \\ & c & \end{array}$$

v) *Given* $a > b$ *in* S, *the interval subalgebra* $[a, b] = \{x \in \mathrm{S} \mid a \geq x \geq b\}$ *is a sublattice*.

vi) *Given any* $a \in \mathrm{S}$, $[a]\!\uparrow\ = \{x \in \mathrm{S} \mid x \geq a\}$ *is a normal subalgebra of* S *and* $[a]\!\downarrow\ = \{x \in \mathrm{S} \mid a \geq x\}$ *is a conormal subalgebra of* S.

vii) S *is categorical and given any skew chain* $\mathrm{A} > \mathrm{B} > \mathrm{C}$ *of* $\mathcal{D}$*-classes in* S, *for each coset bijection* $\varphi: \mathrm{A} \to \mathrm{C}$, *unique coset bijections* $\psi: \mathrm{A} \to \mathrm{B}$ *and* $\chi: \mathrm{B} \to \mathrm{C}$ *exist such that* $\varphi = \chi\psi$.

Proof. Theorem 5.4.7(iii) gives us (i) ⇒ (ii). Conversely, if S satisfies (ii) then no subalgebra of S can be one of the forbidden algebras in Theorem 5.4.5, making S categorical. We next show that given $x, y \in \mathrm{B}$, there exist u and v in B such that $x -_{\mathrm{A}} u -_{\mathrm{C}} y$ and $x -_{\mathrm{C}} v -_{\mathrm{A}} y$. This guarantees that in B, every A-coset meets every C-coset. Indeed, pick $a \in \mathrm{A}$ and $c \in \mathrm{C}$ so that $a > x > c$. Note that $a > a \wedge (c \vee y \vee c) \wedge a$, $c \vee (a \wedge y \wedge a) \vee c > c$. But by assumption x is the unique element in B between a and c under $>$. Thus

$$a \wedge (c \vee y \vee c) \wedge a = x = c \vee (a \wedge y \wedge a) \vee c$$

so that both $x -_{\mathrm{A}} c \vee y \vee c -_{\mathrm{C}} y$ and $x -_{\mathrm{C}} a \wedge y \wedge a -_{\mathrm{A}} y$ in B, which gives (ii) ⇒ (i). Next let S be categorical with $\mathrm{A} > \mathrm{B} > \mathrm{C}$ as stated in (vii). The unique factorization in (vii) occurs precisely when (ii) holds, making (ii) and (vii) equivalent. Finally, (iii) – (vi) are easily seen to be equivalent variants of (ii) .☐

Corollary 5.4.9. *Strictly categorical skew lattices form a variety of skew lattices*.

Proof. Consider the following identity or its dual:

$$x \vee (y \wedge z \wedge u \wedge y) \vee x = x \vee (y \wedge u \wedge z \wedge y) \vee x. \tag{5.4.2}$$

Note that $x \vee y \vee x \geq x \vee (y \wedge z \wedge u \wedge y) \vee x$ & $x \vee (y \wedge u \wedge z \wedge y) \vee x \geq x$ by (1.1) and (1.6) with the middle expressions being $\mathcal{D}$-equivalent, since $z \wedge u \ \mathcal{D}\ u \wedge z$. Hence, if a skew lattice S is strictly categorical, then (5.4.2) holds by Theorem 5.4.8(iii). Conversely, let (5.4.2) hold in S and suppose that $a \geq$ both $b, b' \geq c$ in S with $b \ \mathcal{D}\ b'$. Assigning $x \to c$, $y \to a$, $z \to b \wedge b'$ and $u \to b' \wedge b$ reduces (5.4.2) to $b = b \wedge b' \wedge b = b' \wedge b \wedge b' = b'$ making S strictly categorical by Theorem 5.4.8(iii) again. ☐

While distributive skew lattices are categorical, they need not be strictly categorical; but:

Theorem 5.4.10. *A strictly categorical skew lattice S is distributive if and only if it is also quasi-distributive.*

Proof. Any distributive skew lattice is quasi-distributive. Conversely, in any strictly categorical skew lattice both $a \geq a \wedge (b \vee c) \wedge a$ and $a \geq (a \wedge b \wedge a) \vee (a \wedge c \wedge a)$. In turn, $a \wedge c \wedge b \wedge a \leq$ both $a \wedge (b \vee c) \wedge a$ and $(a \wedge b \wedge a) \vee (a \wedge c \wedge a)$. Indeed, regularity and absorption give, e.g.,

$$(a \wedge c \wedge b \wedge a) \wedge [a \wedge (b \vee c) \wedge a] = a \wedge c \wedge b \wedge (b \vee c) \wedge a = a \wedge c \wedge b \wedge a$$

and

$$\begin{aligned}(a \wedge c \wedge b \wedge a) \wedge [(a \wedge b \wedge a) \vee (a \wedge c \wedge a)] &= a \wedge c \wedge a \wedge b \wedge a \wedge [(a \wedge b \wedge a) \vee (a \wedge c \wedge a)] \\ &= a \wedge c \wedge a \wedge b \wedge a = a \wedge c \wedge b \wedge a\end{aligned}$$

In any quasi-distributive skew lattice, $a \wedge (b \vee c) \wedge a \ \mathcal{D}\ (a \wedge b \wedge a) \vee (a \wedge c \wedge a)$. Thus if S is quasi-distributive and strictly categorical, Theorem 5.4.8(iii) implies that both (5.2.1) and dually (5.2.2) must hold. The converse is clear. ☐

Theorems 5.4.8 and 5.4.10 can also be used to show that *a distributive, strictly categorical skew lattice S is simply cancellative*. It is (fully) cancellative when S is also symmetric.

Corollary 5.4.11. *A skew lattice is strictly categorical and distributive if and only if no subalgebra is a copy of lattices* $\mathbf{M}_3$ *or* $\mathbf{N}_5$ *or either of the skew chains in Theorem* 5.4.7(iv). ☐

Order-closure and paranormal skew lattice

Both normal skew lattices and conormal skew lattices are proper subvarieties of the variety of strictly categorical skew lattices. It is reasonable to ask if these subvarieties jointly generate the larger variety. This turns not to be the case since both types of algebras belong to another variety of skew lattices that excludes many primitive skew lattices, all of which must be strictly categorical.

A primitive skew lattice A > B is ***order-closed*** if for a, $a' \in$ A and b, $b' \in$ B, a, $a' > b$ and $a > b$, b' imply $a' > b'$.

$$\begin{array}{cccc} a & - & - & a' \\ \vdots & \ddots & \cdot^{\cdot^{\cdot}} & ? \\ \vdots & \cdot^{\cdot^{\cdot}} & \ddots & ? \\ b & - & - & b' \end{array}$$

A primitive skew lattice A > B is ***simply order-closed*** if $a > b$ holds for all $a \in$ A and all $b \in$ B. In this case the cosets of A and B in each other are precisely the singleton subsets. The following characterization of order-closed primitive skew lattices is easily verified: *a primitive skew lattice is order-closed if and only if it factors as the product* T × D *of a simply order-closed primitive skew lattice* T *with a rectangular skew lattice* D.

A skew lattice is ***order-closed*** if all its primitive subalgebras are order-closed. *Order-closed skew lattices form a subvariety of skew lattices*. Using variables x, y, u and v, with

$$w = x \wedge y \wedge \underline{u \wedge v} \wedge x \wedge y,$$

one has the following generic situation between two $\mathcal{D}$-classes ($\ddots$ denoting >):

$$\begin{array}{lll} x \wedge y & --- & w \vee (y \wedge x) \vee w \\ \vdots & \ddots \;\; \cdot^{\cdot^{\cdot}} & \\ \vdots & \cdot^{\cdot^{\cdot}} \;\; \ddots & \\ w & --- & x \wedge y \wedge \underline{v \wedge u} \wedge x \wedge y \end{array}$$

Thus $[w \vee (y \wedge x) \vee w] \wedge (x \wedge y \wedge v \wedge u \wedge x \wedge y) \wedge [w \vee (y \wedge x) \vee w] = x \wedge y \wedge v \wedge u \wedge x \wedge y$ characterizes these skew lattices and we have:

Proposition 5.4.12. *Order-closed skew lattices are a variety that includes both normal skew lattices and conormal skew lattices.*

Proof. That we have a variety is clear. That it includes all normal skew lattices is due to the fact that given a primitive normal skew lattice A > B, for any $a \in$ A only one $b \in$ B exists such that $a > b$, thus making A > B satisfy the defining condition for order-closure in a trivial manner. □

The variety join of the varieties of normal skew lattices and conormal skew lattices is thus included in the intersection of the varieties of strictly categorical and order-closed skew lattices. Moreover, primitive skew lattices not satisfying the order-closed criterion above are easily designed. See, e.g., the example of Theorem 2.4.3. Thus strictly categorical skew lattices are not the join of the normal and conormal skew lattice varieties.

A skew lattice is ***paranormal*** if is it is an order-closed, strictly categorical skew lattice. We can now pose the following question:

Is the variety of paranormal skew lattices jointly generated by the varieties of normal and conormal skew lattices? Otherwise put, *is it their join in the lattice of all skew lattice varieties*?

One can also consider conditioned versions of the question by asking if, say, the variety of symmetric paranormal skew lattices is the join of the varieties of symmetric normal and symmetric conormal skew lattices. "Symmetric," of course, could be replaced by "distributive" or by "distributive and symmetric." In the latter case, a related question is:

Is the variety of distributive, symmetric, paranormal skew lattices generated from the class of all order-closed primitive algebras?

The motivation for this is the fact that distributive, symmetric, normal skew lattices are generated from $\mathbf{3}_R$ and $\mathbf{3}_L$. (See Theorem 2.6.12.) Indeed, returning to strictly categorical skew lattices in general, one may ask:

Is the variety of distributive, symmetric, strictly categorical skew lattices generated from the class of all primitive algebras?

5.5 Distributive skew lattices

We begin with a broader class of skew lattices. A skew lattice S is ***linearly distributive*** if every subalgebra T that is totally preordered under $\succeq$ is distributive. Since totally preordered skew lattices are trivially symmetric, *a skew lattice* S *is linearly distributive if and only if each totally preordered subalgebra satisfies either* (5.2.1) *or equivalently* (5.2.2). Indeed, S is linearly distributive if it is distributive on each skew chain $A \geq B \geq C$ in S. Since skew chains need not be even categorical, they need not be distributive! However:

Theorem 5.5.1. *Linearly distributive skew lattices form a variety of skew lattices. Thus a skew lattice* S *is linearly distributive if and only if both* $S/\mathcal{R}$ *and* $S/\mathcal{L}$ *are*.

Proof. Consider the terms x, $y\wedge x\wedge y$ and $z\wedge y\wedge x\wedge y\wedge z$. Clearly $x \succeq y\wedge x\wedge y \succeq z\wedge y\wedge x\wedge y\wedge z$ holds for all skew lattices. Conversely given any instance $a \succeq b \succeq c$ in some skew lattice S, the assignment $x \to a, y \to b, z \to c$ will return this particular instance. Thus a characterizing set of identities for the class of all linearly distributive skew lattices is given by taking the basic identity

$$u\wedge(v \vee w)\wedge u = (u\wedge v\wedge u) \vee (u\wedge w\wedge u)$$

and forming all the identities possible in x, y, z by making bijective assignments from the variables $\{u, v, w\}$ to the terms $\{x, y\wedge x\wedge y, z\wedge y\wedge x\wedge y\wedge z\}$. □

We proceed with several lemmas, the first of which is evident.

Lemma 5.5.2. *Left-handed skew lattices that satisfy* (5.2.1) *are characterized by:*

$$x \wedge y \wedge x = x \wedge y,\ \ x \vee y \vee x = y \vee x \ \ \text{and}\ \ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z). \tag{5.5.1$\mathcal{L}$}$$

Dually, right-handed skew lattices that satisfy (5.2.1) *are characterized by:*

$$x \wedge y \wedge x = y \wedge x,\ \ x \vee y \vee x = x \vee y \ \ \text{and}\ \ (y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x). \tag{5.5.1$\mathcal{R}$}$$

Lemma 5.5.3. *In a left-handed totally preordered skew lattice, if* $a \wedge (b \vee c) \neq a \wedge b \vee a \wedge c$, *then* $a \succ b \succ c$. *Thus, for a left-handed skew lattice* S, *the following are equivalent:*

a) S *is linearly distributive*.

b) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *for all* $a \succ b \succ c$ *in* S.

c) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *for all* $a \succeq b \succeq c$ *in* S.

Left-handed linearly distributive skew lattices are thus characterized by:

$$x \wedge [(y \wedge x) \vee (z \wedge y \wedge x)] = (x \wedge y) \vee (x \wedge z \wedge y). \tag{5.5.2$\mathcal{L}$}$$

Dually, right-handed linearly distributive skew lattices are characterized by:

$$[(x \wedge y \wedge z) \vee (x \wedge y)] \wedge x = (y \wedge z \wedge x) \vee (y \wedge x). \tag{5.5.2$\mathcal{R}$}$$

Proof. If say $b \succeq a$, then $a \wedge (b \vee c) = a$ and $(a \wedge b) \vee (a \wedge c) = a \vee (a \wedge c) = a$. If $c \succeq a$, then $a \wedge (b \vee c) = a$ again, and $(a \wedge b) \vee (a \wedge c) = (a \wedge b) \vee a = (a \wedge b \wedge a) \vee a = a$. Thus inequality only occurs when $a \succeq b, c$. But even here, $a \succeq c \succeq b$ gives us $a \wedge (b \vee c) = a \wedge c$ and $(a \wedge c) \succeq (a \wedge b)$ so that $(a \wedge b) \vee (a \wedge c) = a \wedge c$ also. Thus, to completely avoid $a \wedge (b \vee c) = a \wedge b \vee a \wedge c$ we are only left with $a \succ b \succ c$. ☐

In particular *primitive skew lattices are distributive*. It turns out that linear distributivity is characterized succinctly by either of the dual pair of identities in the following lemma.

Lemma 5.5.4. *Identities* (5.2.1) *and* (5.2.2) *respectively imply*

$$x \wedge [(y\wedge x\wedge y) \vee (z\wedge x\wedge z)] \wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x) \qquad (5.5.3)$$

and

$$x \vee [(y\vee x\vee y) \wedge (z\vee x\vee z)] \vee x = (x\vee y\vee x) \wedge (x\vee z\vee x). \qquad (5.5.4)$$

For left-handed skew lattices, these identities simplify to:

$$x \wedge [(y\wedge x) \vee (z\wedge x)] = (x\wedge y) \vee (x\wedge z) \qquad (5.5.3\mathcal{L}).$$

and

$$[(x\vee y) \wedge (x\vee z)] \vee x = (y\vee x) \wedge (z\vee x). \qquad (5.5.4\mathcal{L}).$$

Dually, in the right-handed case they simplify to:

$$[(x\wedge y) \vee (x\wedge z)] \wedge x = (y\wedge x) \vee (z\wedge x) \qquad (5.5.3\mathcal{R}).$$

and

$$x \vee [(y\vee x) \wedge (z\vee x)] = (x\vee y) \wedge (x\vee z) . \qquad (5.5.4\mathcal{R}).$$

Proof. Since $x\wedge y\wedge x\wedge y\wedge x = x\wedge y\wedge x$ by regularity, (5.2.1) ⇒ (5.5.3) and also (5.2.2) ⇒ (5.5.4). □

Theorem 5.5.5. *For all skew lattices,* (5.5.3) *and* (5.5.4) *are equivalent. A skew lattice satisfies either and hence both if and only if it is linearly distributive.*

Proof. For left-handed skew lattices, (5.5.3$\mathcal{L}$) clearly implies (5.5.2$\mathcal{L}$), while (5.5.3$\mathcal{R}$) implies (5.5.2$\mathcal{R}$). Thus if a skew lattice S satisfies (5.5.3), so do both S/$\mathcal{R}$ and S/$\mathcal{L}$, making them linearly distributive, and hence S also by Theorem 5.5.1

Conversely, assume that S is linearly distributive. First, let S be left-handed also. Then left-handedness gives the first, third and sixth equalities below.

$$\begin{aligned}
x \wedge [(y\wedge x) \vee (z\wedge x)] &= x \wedge [(z\wedge x) \vee (y\wedge x) \vee (z\wedge x)] \\
&= \{x \wedge [(z\wedge x) \vee (y\wedge x)]\} \vee [x \wedge (z\wedge x)] \\
&= \{x \wedge [(y\wedge x) \vee (z\wedge x) \vee (y\wedge x)]\} \vee [x \wedge (z\wedge x)] \\
&= \{x \wedge [(y\wedge x) \vee (z\wedge x)]\} \vee [x \wedge (y\wedge x)] \vee [x \wedge (z\wedge x)] \\
&= [x \wedge (y\wedge x)] \vee [x \wedge (z\wedge x)] \\
&= (x \wedge y) \vee (x\wedge z).
\end{aligned}$$

Linear distributivity implies the second and fourth equalities, since e.g., $x \succeq (z\wedge x)\vee(y\wedge x) \succeq (z\wedge x)$. The fifth equality follows upon observing that $x \wedge [(y\wedge x)\vee(z\wedge x)]$ and $(x\wedge y\wedge x) \vee (x\wedge z\wedge x)$ are $\mathcal{L}$-related, since they are equal in any lattice, and in particular in S/$\mathcal{D}$ = S/$\mathcal{L}$. Thus (5.5.3$\mathcal{L}$) holds. Similarly (5.5.3$\mathcal{R}$) holds in the right-handed linearly distributive case. Again the embedding of S into S/$\mathcal{R}$ × S/$\mathcal{L}$ guarantees that all linearly distributive skew lattices satisfy (5.5.3). Thus linear distributivity is characterized by (5.5.3).

Since any totally pre-ordered context is symmetric, (5.2.1) and (5.2.2) are equivalent in such contexts. Thus linear distributivity is also characterized by (5.5.4) by the dual argument. ☐

There is more to linear distributivity than just what occurs in totally preordered contexts. Indeed (5.5.3) implies that for each a in a skew lattice S, the map $x \mapsto a \wedge x \wedge a$ defines a homomorphic retraction of the principal ideal $S \wedge a \wedge S$ onto the set $a \wedge S \wedge a$ of all $x \leq a$. The identity states directly that this map preserves joins, with meets preserved due to regularity. Likewise, (5.5.4) implies that $x \mapsto a \vee x \vee a$ defines a homomorphic retraction of the principal filter $S \vee a \vee S$ onto the set $a \vee S \vee a$ of all $x \geq a$. Thus all three aspects of distributivity are equivalent for any skew lattice. There is more. While skew diamonds need not be distributive, linearly distributive skew diamonds are.

Corollary 5.5.6. *Linearly distributive skew diamonds are distributive.*

Proof. Given a linearly distributive skew diamond T with $\mathcal{D}$-classes J > A, B > M, we check out (5.2.1) on T. Given $x \in J$, then $y \wedge x \wedge y = y$ and $z \wedge x \wedge z = z$ so that (5.5.3) reduces to (5.2.1). For x in M, (5.2.1) immediately reduces to $x = x \vee x$. So let x be in an intermediate $\mathcal{D}$-class, say say A. We consider several possible cases.

1) $y \,\mathcal{D}\, z$. Here $y \vee z = z \wedge y$ so that regularity gives

$$x \wedge (y \vee z) \wedge x \;=\; x \wedge (z \wedge y) \wedge x \;=\; (x \wedge z \wedge x) \wedge (x \wedge y \wedge x) \;=\; (x \wedge y \wedge x) \vee (x \wedge z \wedge x).$$

2) y or z is in A or J, say y. Here $y \vee z \succeq x$ and our equation reduces to the absorption identity $x = x \vee (x \wedge z \wedge x)$. Thus we may assume that y and z are in distinct $\mathcal{D}$-classes, but other than A and J.

3) So let, say $y \in M$ and $z \in B$. Then $(y \vee z \vee y) \,\mathcal{D}\, z$ and Case (1) gives

$$x \wedge (y \vee z) \wedge x = x \wedge [(y \vee z \vee y) \vee z] \wedge x = (x \wedge (y \vee z \vee y) \wedge x) \vee (x \vee z \wedge x).$$

The corollary will follow if we can show that $x \wedge (y \vee z \vee y) \wedge x = x \wedge y \wedge x$. Since $x \wedge (y \vee z \vee y)$, y and $(y \vee z \vee y) \wedge x$ are in M,

$$\begin{aligned} x \wedge (y \vee z \vee y) \wedge x \;&=\; x \wedge (y \vee z \vee y) \wedge (y \vee z \vee y) \wedge x \\ &=\; [x \wedge (y \vee z \vee y)] \wedge y \wedge [(y \vee z \vee y) \wedge x] \;=\; x \wedge y \wedge x, \end{aligned}$$

with the equalities due respectively to $\wedge$ being idempotenct, $uvw = uw$ holding in any rectangular band and absorption. (5.2.2) is seen in dual manner. ☐

Since distributive skew lattice are both linearly distributive and quasi-distributive, a natural question is: *Does linear distributivity plus quasi-distributivity imply distributivity?* While true for skew diamonds, it is not true in general. Spinks' examples (in Theorem 1.3.10) each satisfy just one of (5.2.1) or (5.2.2). Since totally pre-ordered subalgebras are trivially symmetric, these examples are linearly distributive. The following lattice of varieties is thus strictly ordered by inclusion.

$$\begin{array}{ccccc} & & \boldsymbol{LDist} \cap \boldsymbol{QDist} & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & \\ \wedge - \boldsymbol{Dist} & & & & \vee - \boldsymbol{Dist} \\ & \ddots & & \cdot^{\cdot^{\cdot}} & \\ & & \boldsymbol{Dist} & & \end{array}.$$

Dist is the variety of distributive skew lattices, **∧*-Dist*** and **∨*-Dist*** are the varieties of skew lattices satisfying (5.2.1) and (5.2.2) respectively, while ***LDist*** and ***QDist*** are the respective varieties of linearly distributive and quasi-distributive skew lattices. While ***Dist*** is the intersection of **∧*-Dist*** and **∨*-Dist***, it is unclear if ***LDist*** $\cap$ ***QDist*** is their join in the lattice of all skew lattice varieties.

All this leads us to modify our question and ask: *Does symmetry plus linear distributivity and quasi-distributivity imply distributivity?*

This is indeed the case. This was first shown using Prover 9, which has also shown that *linearly distributivity and distributivity are equivalent for simply cancellative skew lattices.* We will first justify the implication in the case of left-handed, symmetric skew lattices. Recall that *each* of the following identities characterizes left-handedness:

$$x \wedge y \wedge x = x \wedge y; \qquad x \vee y \vee x = y \vee x; \tag{5.5.5}$$

$$x \wedge (y \vee x) = x; \qquad (x \wedge y) \vee x = x. \tag{5.5.6}$$

We use them freely in what follows. Dual identities characterize the right-handed case.

Lemma 5.5.7. *For left handed skew lattices, the following identities hold*:

(1) $$[x \vee (y \wedge x)] \wedge x = x \vee (y \wedge x). \tag{5.5.7}$$

(2) $$[x \vee (y \wedge x)] \wedge y = y \wedge x. \tag{5.5.8}$$

Proof. Clearly $x \vee (y \wedge x)\ \mathcal{D}\ x$ holds for all skew lattices. Since $x \wedge [x \vee (y \wedge x)] = x$ by absorption, we get $x \vee (y \wedge x)\ \mathcal{L}\ x$ for all skew lattices, so that (5.5.7) follows in general. For (5.5.8), we have:

$$[x \vee (y \wedge x)] \wedge y \;=\; [x \vee (y \wedge x)] \wedge x \wedge y \wedge x \;=\; [x \vee (y \wedge x)] \wedge y \wedge x \;=\; y \wedge x.$$

The first equality follows from (5.5.7) and left-handedness, (5.5.1$\mathcal{L}$). The second and third equalities follow respectively from (5.5.7) again and absorption. ☐

We continue with a further characterization of quasi-distributivity in the left-handed case.

Lemma 5.5.8. *A left-handed skew lattice is quasi-distributive if and only if the following identity holds:*

$$x \wedge [(y\wedge x) \vee z] = x \wedge (y\vee z). \qquad (5.5.9)$$

Proof. Sufficiency is clear since neither $\mathbf{M}_3$ nor $\mathbf{N}_5$ can satisfy (5.5.9). For necessity, observe first that (5.5.9) holds in any distributive lattice. Hence in a quasi-distributive skew lattice one has at least $x \wedge [(y\wedge x) \vee z]\ \mathcal{D}\ x \wedge (y\vee z)$. If we can show that $x \wedge [(y\wedge x) \vee z] \leq x \wedge (y\vee z)$ in the left-handed case, then equality will hold, given the rectangular context of both expressions. Left-handedness clearly gives $y\wedge x \leq y$ since $y\wedge x = y\wedge x\wedge y$. Thus $(y\wedge x) \vee z \leq y\vee z$ follows since both

$$(y\wedge x) \vee z \vee y \vee z = (y\wedge x\wedge y) \vee y \vee z = y \vee z \;\text{ and }\; y \vee z \vee (y\wedge x) \vee z = y \vee (y\wedge x) \vee z = y \vee z$$

by left-handedness and absorption. Left-handedness plus the inequality gives

$$x\wedge[(y\wedge x)\vee z] \wedge x\wedge(y\vee z) = x\wedge[(y\wedge x)\vee z] \wedge (y\vee z) = x\wedge[(y\wedge x)\vee z]$$

and similarly $x\wedge(y\vee z) \wedge x\wedge[(y\wedge x)\vee z] = x\wedge[(y\wedge x)\vee z]$. Thus indeed $x\wedge[(y\wedge x)\vee z] \leq x\wedge(y\vee z)$ within a $\mathcal{D}$-class context and equality follows. ☐

Besides quasi-distributivity and linear distributivity, a third consequence of distributivity is the following pair of identities. Combined, all three consequences together guarantee that a skew lattice is distributive.

Lemma 5.5.9. *The following identities are respective consequence of* (5.2.1) *and* (5.2.2):

$$x \wedge [(y\wedge x\wedge y) \vee z \vee y \vee z \vee (y\wedge x\wedge y)] \wedge x = x \wedge (y \vee z \vee y) \wedge x. \qquad (5.5.10).$$

$$x \vee [(y\vee x\vee y) \wedge z \wedge y \wedge z \wedge (y\vee x\vee y)] \vee x = x \vee (y \wedge z \wedge y) \vee x. \qquad (5.5.11).$$

Their left are right-handed variants are:

$$x\wedge[y \vee z \vee (y \wedge x)] = x\wedge(z \vee y) \;\textit{ and }\; [(x \wedge y) \vee z \vee y]\wedge x = (y \vee z)\wedge x. \qquad (5.5.10\mathcal{L} \text{ and } \mathcal{R})$$

$$[(x \vee y) \wedge z \wedge y]\vee x = (y \wedge z)\vee x \;\textit{ and }\; x\vee [y \wedge z \wedge (y \wedge x)] = x\vee(z \wedge y). \qquad (5.5.11\mathcal{L} \text{ and } \mathcal{R})$$

Proof. (5.2.1) plus regularity first gives

$$x \wedge [(y\wedge x\wedge y)\vee z\vee y\vee z\vee(y\wedge x\wedge y)] \wedge x = (x\wedge y\wedge x) \vee (x\wedge z\wedge x) \vee (x\wedge y\wedge x) \vee (x\wedge z\wedge x) \vee (x\wedge y\wedge x)$$

which reduces to $(x\wedge y\wedge x) \vee (x\wedge z\wedge x) \vee (x\wedge y\wedge x)$ and then $x \wedge (y\vee z\vee y) \wedge x$ by (5.2.1) again. The derivation of (5.5.11) from (5.2.2) and regularity is similar. ☐

Both (5.5.10) and (5.5.11) can be given interpretations regarding the behavior of cosets and coset bijections within certain configurations of $\mathcal{D}$-classes. (See the final section of [Kinyon, Leech, Pita Costa].) For our purposes, however, their importance lies in the following "umbrella" result and in the fact that both are direct consequences of symmetry.

Theorem 5.5.10. *A quasi-distributive, linearly distributive skew lattice is* $\wedge$*-distributive if and only if it satisfies* (5.5.10). *Likewise, it is* $\vee$*-distributive if and only if it satisfies* (5.5.11). *In general, a skew lattice is distributive if and only if it is quasi-distributive, linearly distributive and satisfies both* (5.5.10) *and* (5.5.11).

Proof. Clearly, (5.2.1) and (5.2.2) imply respectively (5.5.10) and (5.5.11). Conversely, suppose that we have a skew lattice that is quasi-distributive and linearly distributive, and also satisfies (5.5.10). In the left-handed case we have:

$$
\begin{array}{rcll}
(x \wedge y) \vee (x \wedge z) & = & x \wedge [(y \wedge x) \vee (z \wedge x)] & \text{by } (5.5.3\mathcal{L}) \\
& = & x \wedge [y \vee (z \wedge x)] & \text{by } (5.5.9) \\
& = & x \wedge [(z \wedge x) \vee y \vee (z \wedge x)] & \text{by left-handedness} \\
& = & x \wedge [z \vee y \vee (z \wedge x)] & \text{by } (5.5.9) \\
& = & x \wedge (y \vee z) & \text{by } (5.5.10\mathcal{L}).
\end{array}
$$

Thus (5.2.1$\mathcal{L}$) holds. Likewise, the right-handed case must hold. The general case for (5.5.10) implying (5.2.1) now follows as usual. The argument equating (5.5.11) with $\vee$-distributivity, given that the skew lattice is both quasi-distributive and linearly distributive, is dual. □

A consequence of the above results is the following theorem. Recall that a skew lattice is lower symmetric if $x \vee y = y \vee x$ implies $x \wedge y = y \wedge x$. Dually, a skew lattice is upper symmetric if $x \wedge y = y \wedge x$ implies $x \vee y = y \vee x$. Recall that both types of partial symmetry were characterized respectively by (5.1.3) $x \wedge y \wedge (x \vee y) = (y \vee x) \wedge y \wedge x$ and by (5.1.4) $x \vee y \vee (x \wedge y) = (y \wedge x) \vee y \vee x$.

Theorem 5.5.11. *An upper symmetric skew lattice is lower distributive if and only if it is both quasi-distributive and linearly distributive; dually, a lower symmetric skew lattice is upper distributive if and only if it is both quasi-distributive and linearly distributive. Thus a symmetric skew lattice is distributive if and only if it is both quasi-distributive and linearly distributive.*

Proof. The first statement follows from Lemma 5.5.9 and the following proposition. The second statement follows by duality and the third follows in the usual way from the first two. □

Proposition 5.5.12. *Upper symmetric skew lattices satisfy* (5.5.10). *Dually, lower symmetric skew lattices satisfy* (5.5.11).

Proof. We need only prove the first assertion. We begin with the case of a *left-handed* skew lattice S, organizing the proof in this case in the following steps.

i) For all x, y, z in S, $x \vee y \geq [x \vee (y \wedge z)] \wedge [z \vee (y \wedge z)]$.

To begin, set $u = [x \vee (y{\wedge}z)] \wedge [z \vee (y{\wedge}z)]$. Since $[z \vee (y{\wedge}z)] \wedge y = y \wedge z$ by (5.5.8),

$$u{\wedge}y \;=\; [x \vee (y \wedge z)] \wedge (y \wedge z) \;=\; y \wedge z.$$

with the second identity due to absorption. But then,

$$y \wedge u = y \wedge u \wedge y = y \wedge y \wedge z = y \wedge z = u \wedge y$$

so that $y \vee u = u \vee y$ by upper symmetry. Thus:

$$\begin{aligned}
x \vee y \vee u &= x \vee (y \wedge z) \vee y \vee u\\
&= x \vee (y \wedge z) \vee u \vee y\\
&= x \vee (y \wedge z) \vee ([x \vee (y \wedge z)] \wedge [z \vee (y \wedge z)]) \vee y\\
&= x \vee (y \wedge z) \vee y\\
&= x \vee y,
\end{aligned}$$

where the first and fifth equalities are due to (5.5.6), the second equality is the established case of commutation, the third is replacing u by its full expression and the fourth is due to absorption. But since S is left-handed, $x \vee y \vee u = x \vee y$ establishes $x \vee y \geq u$.

ii) *For all* $x, y, z \in$ S, $z \wedge [x \vee (y \wedge z)] \;=\; z \wedge (x \vee y) \wedge [x \vee (y \wedge z)]$.

By duality $u = (x \vee y) \wedge u$, and so

$$\begin{aligned}
z \wedge [x \vee (y \wedge z)] &= z \wedge [z \vee (y \wedge z)] \wedge [x \vee (y \wedge z)]\\
&= z \wedge [z \vee (y \wedge z)] \wedge u\\
&= z \wedge [z \vee (y \wedge z)] \wedge (x \vee y) \wedge u\\
&= z \wedge [z \vee (y \wedge z)] \wedge (x \vee y) \wedge [x \vee (y \wedge z)]\\
&= z \wedge (x \vee y) \wedge [x \vee (y \wedge z)],
\end{aligned}$$

using reverse absorption in the first equality, left-handedness in the second and fourth equalities, part (i) in the middle equality and absorption in the final equality.

iii) *For all* $x, y, z \in$ S, $z \wedge [x \vee (y \wedge x \wedge z)] \;=\; z \wedge [x \vee (y \wedge x)]$.

Replace y with $y{\wedge}x$ in (ii) to get

$$\begin{aligned}
z \wedge [x \vee (y \wedge x \wedge z)] &= z \wedge [x \vee (y \wedge x)] \wedge [x \vee (y \wedge x \wedge z)]\\
&= z \wedge [x \vee (y \wedge x)] \wedge x \wedge [x \vee (y \wedge x \wedge \mathrm{z})]\\
&= z \wedge [x \vee (y \wedge x)] \wedge x\\
&= z \wedge [x \vee (y \wedge x)],
\end{aligned}$$

using (5.5.7) in the second and fourth equalities and absorption in the third.

iv) To conclude the left-handed case, replace x by $y \vee x$ in (*iii*). On the left side, absorption gives

$$z \wedge (y \vee x \vee [y \wedge (y \vee x) \wedge z]) \ = \ z \wedge [y \vee x \vee (y \wedge z)].$$

On the right side, absorption and (5.5.5) give

$$z \wedge (y \vee x \ \vee [y \wedge (y \vee x)]) = z \wedge (y \vee x \vee y) = z \wedge (x \vee y).$$

Therefore $z \wedge (x \vee y) = z \wedge [y \vee x \vee (y \wedge z)]$. But this is just (5.5.10$\mathcal{L}$) with permuted variables. The right-handed case for (5.5.10$\mathcal{R}$) follows by left-right duality, and the general implication of (5.5.10) thus holds. That lower symmetry implies (5.5.11) is seen in a ∨-∧ dual fashion. ☐

Theorem 5.5.13. *Strictly categorical skew lattices satisfy* (5.5.10) *and* (5.511).

Proof. Take (5.5.10). Both terms are $\mathcal{D}$-related in all skew lattices. But in all skew lattices we also have $x \geq$ both terms $\geq x \wedge y \wedge x$. The theorem follows by Theorem 5.4.8. ☐

Before proceeding to the next section, here are two consequences of Theorems 5.4.2 and 5.4.9. Both implications are strict.

Proposition 5.5.14. *All strictly categorical skew lattices are linearly distributive and all linearly distributive skew lattices are categorical*.

5.6 Midpoints and distributive skew chains

A skew lattice is linearly distributive if and only if each skew chain of $\mathcal{D}$-classes in it is distributive. In this section we characterize distributive skew chains in terms of the natural partial order. Given a skew chain A > B > C of comparable $\mathcal{D}$-classes, with $a \in$ A, $c \in$ C such that $a > c$, any element $b \in$ B such that $a > b > c$ is called a ***midpoint*** in B of a and c. We begin with several straightforward assertions.

Lemma 5.6.1. *Given a skew chain* A > B > C, *for all* $a \in$ A *and* $c \in$ C *with* $a > c$:

i) *For all* $b \in B$, *both* $a \wedge (c \vee b \vee c) \wedge a$ *and* $c \vee (a \wedge b \wedge a) \vee c$ *are midpoints in* B *of* a *and* c.

ii) *When* b *in* B *is already a midpoint of* a *and* c, *both midpoints in* (i) *reduce to* b.

iii) *When* A > B > C *is a distributive skew chain*, *both midpoints in* (i) *agree*:

$$a > a \wedge (c \vee b \vee c) \wedge a \ = \ c \vee (a \wedge b \wedge a) \vee c > c. \tag{5.6.1}$$

Midpoints provide a key to determining the effects of (5.2.1) and (5.2.2) in this context. To proceed further, we recall several concepts. Given a skew chain A > B > C, recall that elements b and b' in B are AC-*connected* if a finite sequence $b = b_0, b_1, b_2, \ldots, b_n = b'$ exists in B such that $b_i -_{\mathrm{A}} b_{i+1}$ or $b_i -_{\mathrm{C}} b_{i+1}$ for all $i \leq n - 1$. AC-connectedness is a congruence on B. Its congruence classes, the *components*, are thus subalgebras of B. Given a component B′ of B, a sub-skew chain is given by A > B′ > C. Since $a\wedge(c\vee b\vee c)\wedge a$ is the same for all b in a common C-coset and $c\vee(a\wedge b\wedge a)\vee c$ is the same for all b in a common A-coset, we can extend Lemma 5.6.1:

Lemma 5.6.2. *Given a distributive skew chain* A > B > C, *for any pair $a > c$ with $a \in$* A *and $c \in$* C, *each* AC-*component* B′ in B *has a unique midpoint b of a and c.*

We next sharpen Lemma 5.5.3 as follows.

Lemma 5.6.3. *Let S be a categorical skew chain consisting of* $\mathcal{D}$*-classes* A > B > C. *If* S *is left-handed, then* (5.2.1$\mathcal{L}$) *holds if and only if $a\wedge(b\vee c) = (a\wedge b)\vee c$ for all $a \succ b \succ c$ where $a > c$. Dually, if* S *is right-handed, then* (5.2.1$\mathcal{R}$) *holds if and only if $(c\vee b)\wedge a = c\vee(b\wedge a)$ for all $a \succ b \succ c$ where $a > c$.* (These identities are left and right-handed cases of (5.6.1) above.)

Proof. We consider the left-handed case. If S is indeed distributive with a, b, c as stated in the lemma, then $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c) = (a\wedge b)\vee c$, since $a > c$. Conversely, given just $a \succ b \succ c$ in the respective $\mathcal{D}$-classes A > B > C, set $c' = a\wedge c$. Then $a > c'$ and $(a\wedge b) \vee (a\wedge c) = (a\wedge b) \vee c'$. Next, since c and c' lie in the same A-coset in C and S is categorical, both $b\vee c$ and $b\vee c'$ lie both in a common C-coset in B and in a common A-coset in B so that $a\wedge(b\vee c) = a\wedge(b\vee c')$. Hence:

$$a\wedge(b\vee c) = a\wedge(b\vee c') = (a\wedge b) \vee c' = (a\wedge b) \vee (a\wedge c)$$

The lemma now follows from Lemma 5.5.3 and left-right duality. □

Theorem 5.6.4. *Given a skew chain* A > B > C, *the following condition are equivalent*:

i) A > B > C *is distributive*
ii) *For all $a \in$ A, $b \in$ B and $c \in$ C with $a > c$, $a\wedge(c \vee b \vee c)\wedge a = c\vee(a \wedge b \wedge a)\vee c$.*
iii) *Given $a \in$ A and $c \in$ C with $a > c$, each component* B′ *of* B *contains a unique midpoint b of a and c.*
iv) *For each component* B′ *of* B, A > B′ > C *is strictly categorical*.

When these conditions hold, each coset bijection φ: A → C *uniquely factors through each component* B′ *of* B *in that unique coset bijections* ψ: A → B′ *and* χ: B′ → C *exist such that* $\varphi = \chi\psi$ *under the usual composition of partial bijections*.

Proof. Clearly (i) implies (ii). Given $a > c$ in (ii), for each element x in B, both $b_1 = a\wedge(c\vee x\vee c)\wedge a$ and $b_2 = c\vee(a\wedge x\wedge a)\vee c$ are midpoints of a and c in B Replacing x by any element in its C-coset, does not change the b_1-outcome. Likewise, replacing x by any element in its A-coset, does not change the b_2-outcome. Hence (ii) is equivalent to asserting that given $a > c$ fixed, for all x in a common AC-component B′ of B, both $a\wedge(c\vee x\vee c)\wedge a$ and $c\vee(a\wedge x\wedge a)\vee c$ produce the same output b in B′ such that $a > b > c$. Conversely, for any b in B′ such that $a > b > c$ we must have

$$a\wedge(c\vee b\vee c)\wedge a = b = c\vee(a\wedge b\wedge a)\vee c.$$

Thus (ii) and (iii) are equivalent. Their equivalence with (iv) follows from Theorem 5.4.7 above. Given (ii) – (iv), (iv) forces A > B > C to categorical, since for each component B′ in B, A > B′ > C is categorical. Denoting the skew chain by S, (ii) forces $S/\mathcal{R}$ and $S/\mathcal{L}$ to be distributive by Lemma 5.6.3 and thus $S \subseteq S/\mathcal{R} \times S/\mathcal{L}$ to be distributive. In light of Theorem 5.4.7, the final comment is clear. ☐

Given $a > c$ as above, their midpoint b in component B′ depends on the interplay of the A-cosets and C-cosets within B′. Indeed, given any $a \in$ A, the set of ***images*** in B′ of a is the set $a \wedge B' \wedge a = \{a \wedge b \wedge a \mid b \in B'\} = \{b \in B' \mid a > b\}$. This set parameterizes the A-cosets in B′ since each possesses exactly one b such that $a > b$. Likewise, for $c \in$ C the image set

$$c \vee B' \vee c = \{c \vee b \vee c \mid b \in B'\} = \{b \in B' \mid b > c\}$$

parameterizes all cosets of C in B′. (See Theorem 2.4.1.) Both images sets are ***orthogonal*** in B′ in the following sense. *For any $a \in$ A, all images of a in B′ lie in a unique C-coset in B′. Likewise for any $c \in$ C, all images of c in B′ lie in a unique A-coset in B′. Finally, given $a > c$ with $a \in$ A and $c \in$ C, their unique midpoint $b \in$ B′ lies jointly in the C-coset in B′ containing all images of a in B′ and in the A-coset in B′ containing all images of c in B′.* (See Theorem 5.4.6.) Of course, every b in B′ is the midpoint of some pair $a > c$. For a fixed pair $a > c$, the set $\mu(a, c)$ of all midpoints in B is a rectangular subalgebra that parameterizes the class of all AC-components in B: let b in $\mu(a, c)$ correspond to the component B′ containing b.

		$*$	$\cdots$	
A-coset →	$\bullet$	b	$\bullet \cdots \bullet$	$\bullet$
		$*$		
	$\vdots$	$\vdots$	$\ddots$	$\vdots$
		$*$		
		$*$	$\cdots$	

C-coset ↑

The A-coset of b contains all images ($\bullet$'s) of c in B′. The C-coset of b has all images ($*$'s) of a in B′. Element b is the unique image of both a and c.

Example 5.6.5. Using Mace 4, two minimal 12-element categorical skew chains have been found that are not linearly distributive, one left-handed and the other its right-handed dual. Below are the Cayley Tables in the left-handed case. Here i and j assume the values 1 and 2, and k assumes the values 3 and 4.

$\wedge$	a_i	b_j	b_k	d_j	d_k	c_j
a_1	a_1	b_1	b_3	d_1	d_3	c_j
a_2	a_2	b_2	b_4	d_2	d_4	c_j
b_1	b_1	b_1	b_1	b_1	b_1	c_1
b_2	b_2	b_2	b_2	b_2	b_2	c_1
b_3	b_3	b_3	b_3	b_3	b_3	c_1
b_4	b_4	b_4	b_4	b_4	b_4	c_1
d_1	d_1	d_1	d_1	d_1	d_1	c_2
d_2	d_2	d_2	d_2	d_2	d_2	c_2
d_3	d_3	d_3	d_3	d_3	d_3	c_2
d_4	d_4	d_4	d_4	d_4	d_4	c_2
c_1	c_1	c_1	c_1	c_1	c_1	c_1
c_2	c_2	c_2	c_2	c_2	c_2	c_2

$\vee$	a_i	b_j	b_k	d_j	d_k	c_1	c_2
a_1	a_i	a_j	a_{k-2}	a_j	d_{k-2}	a_1	a_1
a_2	a_i	a_j	a_{k-2}	a_j	d_{k-2}	a_2	a_2
b_1	a_i	b_j	b_k	d_j	d_k	b_1	d_3
b_2	a_i	b_j	b_k	d_j	d_k	b_2	d_2
b_3	a_i	b_j	b_k	d_j	d_k	b_3	d_1
b_4	a_i	b_j	b_k	d_j	d_k	b_4	d_4
d_1	a_i	b_j	b_k	d_j	d_k	b_3	d_1
d_2	a_i	b_j	b_k	d_j	d_k	b_2	d_2
d_3	a_i	b_j	b_k	d_j	d_k	b_1	d_3
d_4	a_i	b_j	b_k	d_j	d_k	b_4	d_4
c_1	a_i	b_j	b_k	d_j	d_k	c_1	c_2
c_2	a_i	b_j	b_k	d_j	d_k	c_1	c_2

.

Their common Hasse diagram is as follows, with b_i –C d_j iff $i + j \equiv 0 \pmod 4$.

$$\begin{array}{cc}
A & a_1 - a_2 \\
\vdots & \\
B & b_1 \underset{A}{-} b_2 - b_3 \underset{A}{-} b_4 - d_1 \underset{A}{-} d_2 - d_3 \underset{A}{-} d_4 \\
\vdots & \\
C & c_1 - c_2
\end{array}$$

In both cases, $a_1 > b_{odd}, d_{odd}$ and $a_2 > b_{even}, d_{even}$, all $b_i > c_1$, all $d_i > c_2$, and $a_1, a_2 >$ both c_1, c_2. Thus both skew chains are categorical since all cosets involving just A and C are trivial. We denote the left-handed skew lattice thus determined by **U** and its right-handed dual by **V**. Both **U** and **V** are not distributive. Indeed, given the coset structure on B, we get $a_1 \wedge (b_2 \vee c_2) = a_1 \wedge d_2 = d_1$, while $(a_1 \wedge b_2) \vee (a_1 \wedge c_2) = b_1 \vee c_2 = d_3 \neq d_1$ in **U**. **V** is handled similarly. Note that in both **U** and **V**, B is a AC-connected, but $a_1 > b_1$, $b_3 > c_1$, and also $a_2 > b_2$, $b_4 > c_1$, etc.

This example is the $n = 2$ case of a sequence of similar skew chains A > B$_n$ > C, where A $= \{a_1, a_2\}$ and C $= \{c_1, c_2\}$ as above, but B$_n = \{b_1, b_2, b_3, \ldots, b_{2n}\} \cup \{d_1, d_2, d_3, \ldots, d_{2n}\}$ for n finite or $\{\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots\} \cup \{\ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots\}$ for $n = \omega$. In all cases,

$$a_1 > b_{\text{odd}}, d_{\text{odd}};\; a_2 > b_{\text{even}}, d_{\text{even}};\; \text{all } b_i > c_1; \text{ all } d_i > c_2; \text{ and } a_1, a_2 > \text{both } c_1, c_2.$$

(That $a_1, a_2 >$ both c_1, c_2 insures that these skew chains are categorical since all "outer" cosets involving just A and C are trivial.)

An important aspect of these examples is the fact that all A-cosets and C-cosets in B$_n$ are of size 2 and are interconnected in the following cyclic fashion in the finite case,

$$b_1 \underset{A}{-} b_2 \underset{C}{-} d_2 \underset{A}{-} d_1 \underset{C}{-} b_3 \underset{A}{-} b_4 \underset{C}{-} d_4 \underset{A}{-} d_3 \underset{C}{-} b_5 \cdots b_{2n-1} \underset{A}{-} b_{2n} \underset{C}{-} d_{2n} \underset{A}{-} d_{2n-1} \underset{C}{-} (b_1)$$

or for $n = \omega$,

$$\cdots d_{-3} \underset{C}{-} b_{-1} \underset{A}{-} b_0 \underset{C}{-} d_0 \underset{A}{-} d_{-1} \underset{C}{-} b_1 \underset{A}{-} b_2 \underset{C}{-} d_2 \underset{A}{-} d_1 \underset{C}{-} b_3 \underset{A}{-} b_4 \underset{C}{-} d_4 \cdots .$$

We denote the left-handed skew lattice thus determined by $\mathbf{U}_n$ and its right-handed dual by $\mathbf{V}_n$. Our example is $\mathbf{U}_2$. All $\mathbf{U}_n$ and $\mathbf{V}_n$ for $n \geq 2$ are nondistributive since A-cosets and C-cosets in the single component B_n for $n \geq 2$ need not have nonempty intersection. *Must a categorical skew lattice containing no copies of* $\mathbf{U}_n$ *or* $\mathbf{V}_n$, *be linearly distributive*? Here is a counterexample.

Example 5.6.6: The underlying set of S is $\mathbf{U}_2 \times \mathbb{Z}_2$.

$$\begin{array}{c}
(a_1,0) - (a_1,1) - (a_2,0) - (a_2,1) \\
\therefore \quad \ldots \quad \ddots \\
(b_1,0) \underset{A}{-} (b_1,1) \underset{A}{-} (b_2,0) \underset{A}{-} (b_2,1) - \ldots - (d_3,0) \underset{A}{-} (d_3,1) \underset{A}{-} (d_4,0) \underset{A}{-} (d_4,1) \\
\ddots \quad \ldots \quad \therefore \\
(c_1,0) - (c_1,1) - (c_2,0) - (c_2,1)
\end{array}$$

All cosets are just cartesian products of $\mathbf{U}_2$-cosets with $\mathbb{Z}_2$. The coset bijections are as follows.

1) $\boxed{(a_1,0) - (a_1,1) - (a_2,0) - (a_2,1)} \to \boxed{(b_1,0) - (b_1,1) - (b_2,0) - (b_2,1)}$

2) $\boxed{(a_1,0) - (a_1,1) - (a_2,0) - (a_2,1)} \to \boxed{(b_3,0) - (b_3,1) - (b_4,0) - (b_4,1)}$

3) $\boxed{(a_1,0) - (a_1,1) - (a_2,0) - (a_2,1)} \to \boxed{(d_1,0) - (d_1,1) - (d_2,0) - (d_2,1)}$

4) $\boxed{(a_1,0) - (a_1,1) - (a_2,0) - (a_2,1)} \to \boxed{(d_3,0) - (d_3,1) - (d_4,0) - (d_4,1)}$

5) $\boxed{(b_1,0) - (b_1,1) - (d_3,0) - (d_3,1)} \to \boxed{(c_1,0) - (c_1,1) - (c_2,0) - (c_2,1)}$

6) $\boxed{(b_2,0) - (b_2,1) - (d_2,0) - (d_2,1)} \to \boxed{(c_1,0) - (c_1,1) - (c_2,\mathbf{1}) - (c_2,\mathbf{0})}$

7) $\boxed{(b_3,0) - (b_3,1) - (d_1,0) - (d_1,1)} \to \boxed{(c_1,0) - (c_1,1) - (c_2,0) - (c_2,1)}$

8) $\boxed{(b_4,0) - (b_4,1) - (d_4,0) - (d_4,1)} \to \boxed{(c_1,0) - (c_1,1) - (c_2,\mathbf{1}) - (c_2,\mathbf{0})}$

For each bijection, the outcomes occur in the same order as the inputs are listed. These coset bijections collectively determine operations $\vee$ and $\wedge$ on S making S a left-handed categorical skew chain that is not distributive since its homomorphic image $\mathbf{U}_2$ is not distributive. The twisted outcomes for (6) and (8) prevent it from containing copies of any of the $\mathbf{U}_n$ for $n \geq 2$.

5.7 Counting theorems and cancellative skew lattices

In this chapter we have thus far revisited properties that skew lattices might possess. We began by further studying symmetric skew lattices and then introducing strongly symmetric skew lattices. We then gave Karin Cvetko-Vah's proof of Spinks' Theorem on the equivalence of the distributive identities (5.2.1) and (5.2.2) for symmetric skew lattices. Symmetry played an important role in our analysis of cancellation in skew lattices. We saw that simply cancellative skew diamonds {J > A. B > M} are characterized by a naturally defined bijection

$$\xi\colon Comm^2(\mathrm{A}, \mathrm{B}) \to \omega(\mathrm{J}, \mathrm{M}),$$

given by the function $\xi(a, b) = (a \vee b, a \wedge b)$, between the set $Comm^2(\mathrm{A}, \mathrm{B})$ of all pairs (a, b) in A×B for which both $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$, and the set $\omega(\mathrm{J}, \mathrm{M})$ of all pairs (j, m) in J×M for which $j > m$. This is, of course, an instance of the type of counting theorem, where two related but distinct sets necessarily have the same size. Other counting theorems exist for (fully) cancellative – and hence symmetric – skew lattices. But first we recall some definitions:

Given a primitive skew lattice A > B. Recall that for any $b \in \mathrm{B}$, its image set in A is the set $b \vee \mathrm{A} \vee b = \{b \vee a \vee b \mid a \in \mathrm{A}\}$ which also coincides with $\{a \in \mathrm{A} \mid a \geq b\}$. Dually, for any $a \in \mathrm{A}$, its image set in B is $a \wedge \mathrm{B} \wedge a = \{a \wedge b \wedge a \mid b \in \mathrm{B}\}$ which also coincides with $\{b \in \mathrm{B} \mid a \geq b\}$. Both image sets are rectangular subalgebras. Recall from Section 2.4 that:

i) Image sets of all elements from one $\mathcal{D}$-class have the same size in its opposite $\mathcal{D}$-class. E.g., given b, b' in B, $|b \vee \mathrm{A} \vee b| = |b' \vee \mathrm{A} \vee b'|$.
ii) Image sets naturally parameterize the cosets of either class in the other. Thus $b \vee \mathrm{A} \vee b$ is a cross-section of all B-cosets in A while $a \wedge \mathrm{B} \wedge a$ is a cross-section of all A-cosets in B.

The ***index*** of B in A, denoted [A: B], is the size $|b \vee \mathrm{A} \vee b|$ that counts the number of B-cosets in A. Likewise, the index of A in B, denoted [B: A], is the size $|a \wedge \mathrm{B} \wedge a|$ that counts counts the number of A-cosets in B. In general, no direct relationship need exist between [A: B] and [B: A]. The common size of all A-cosets in B and all B-cosets in A is denoted by ω_{AB}, or equivalently, ω_{BA}. Clearly the following basic equalities hold:

$$|\mathrm{A}| = [\mathrm{A}: \mathrm{B}]\omega_{\mathrm{AB}} \text{ and } |\mathrm{B}| = [\mathrm{B}: \mathrm{A}]\omega_{\mathrm{AB}}. \qquad (5.7.1)$$

Given a skew diamond {J > A, B > M} one has five pairs of indices. If the skew diamond is cancellative, one has the following "Index Laws" that connect opposite pairs of indices. They are from Karin Cvetko-Vah's Dissertation [Ref.]. (See also CKLS.)

Theorem 5.7.1. *Let* **S** *be a cancellative skew diamond* {J > A, B > M}. *Then*:

$$[M: A] = [B: J] \textit{ and } [A: M] = [J: B]. \tag{5.7.2}$$

$$[M: B] = [A: J] \textit{ and } [B: M] = [J: A]. \tag{5.7.3}$$

In detail, given $j \in J$, $a \in A$, $b \in B$ *and* $m \in M$ *such that* $j > a$, $b > m$, *isomorphisms*

$$\alpha\colon \{x \in A \mid x > m\} \cong \{y \in J \mid y > b\} \quad \textit{and} \quad \beta\colon \{u \in B \mid u < j\} \cong \{v \in M \mid v < a\}$$

are defined by $\alpha(x) = x \vee b \vee x$ *and* $\beta(u) = u \wedge a \wedge u$. *Isomorphisms between other pairs of image sets are defined similarly.*

Conversely, if all such maps in a skew diamond are isomorphisms, then it is cancellative.

Proof. Since $b > m$ and **S** is symmetric, b commutes under both $\wedge$ and $\vee$ with all $x \in A$ such that $x > m$ so that α is well-defined. Since S is cancellative, α is also one-to-one. Lemma 2.1.4 implies

$$\alpha(x \vee x') = (x \vee x') \vee b \vee (x \vee x') = (x \vee b \vee x) \vee (x' \vee b \vee x') = \alpha(x) \vee \alpha(x').$$

Thus α is a $\vee$-homomorphism that must be a $\wedge$-homomorphism since index sets are rectangular subalgebras. To show it is onto and thus an isomorphism, let $y > b$ be given in J and set $c = y \wedge a \wedge y$ in A. Then $m < c < y$ and $y = \alpha(c)$ by Theorem 2.2.1. The verification that β is an isomorphism is similar.

Conversely, given a skew diamond, if all such maps are at least bijections, copies of $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NC}_5^{\mathcal{L}}$ and the algebras $\mathbf{NS}_7^{\mathcal{R},0}$, $\mathbf{NS}_7^{\mathcal{R},1}$, $\mathbf{NS}_7^{\mathcal{L},0}$ and $\mathbf{NS}_7^{\mathcal{L},1}$ cannot occur as subalgebras. Since any skew diamond is trivially quasi-distributive, it must be cancellative. ☐

What else can be said? First, observe:

Theorem 5.7.2. *A skew diamond is simply cancellative if and only if it is strictly categorical, in which case it is also distributive. It is cancellative if and only if it is symmetric and strictly categorical.*

Proof. A skew diamond is already quasi-distributive. To be simply cancellative it needs to exclude both $\mathbf{NC}_5$ subalgebras, and to be strictly categorical it needs to exclude both 4-element skew chains in Theorem 5.4.8(iv). But both constraints are equivalent for skew diamonds. Theorem 5.4.10 insures the addendum of distributivity. ☐

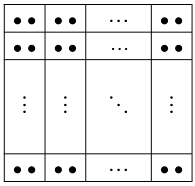
The following two results from João Pita da Costa's dissertation are relevant.

Theorem 5.7.3 *Given a strictly categorical skew chain* $A > B > C$, *if* A *and* C *are finite, then so is* B *and*

$$|B| = \frac{\omega_{AB}\omega_{BC}}{\omega_{AC}} = [B : A]\, \omega_{AC}\, [B : C].$$

In general, a strictly categorical skew lattice S *is finite if and only if its maximal lattice image* $S/\mathcal{D}$ *is finite and both the maximal and minimal* $\mathcal{D}$*-classes in* S *are finite.*

Proof. B is partitioned by A-cosets, each of size ω_{AB}, and also by C-cosets, each of size ω_{BC}. Thus there is a double partition by coset intersections of the form X∩Y where X is an A-coset in B and Y is a C-coset in B. Since A > B > C is strictly categorical, each $|X \cap Y| = \omega_{AC}$.



Thus if A and C are finite, then so are ω_{AB}, ω_{BC} and ω_{AC} giving ω_{BC}/ω_{AC} many A-cosets in B, ω_{AB}/ω_{AC} many C-cosets in B and thus $(\omega_{BC}/\omega_{AC})\omega_{AC}(\omega_{AB}/\omega_{AC}) = \omega_{AB}\omega_{BC}/\omega_{AC} < \infty$ elements in B. One has $|B| = [B{:}A]\omega_{AC}[B{:}C]$ also since this double partition has [B:A][B:C] coset intersections, all of size ω_{AC}.

Given a strictly categorical skew lattice S, the condition is clearly necessary for S to be finite. Conversely given that its maximal and minimal $\mathcal{D}$-classes are finite, so are all intermediate $\mathcal{D}$-classes. If there are only finitely many of them, then S is finite. ☐

Theorem 5.7.4. *Given any skew chain* A > B > C, [C: A] ≤ [C: B][B: A]. *If the skew chain is strictly categorical and both* A *and* C *are finite, then*

$$[C{:}\,A] = [C{:}\,B]\,[B{:}\,A].$$

In general, given any skew chain $A_1 > A_2 > \dots > A_n$ *in a in a strictly categorical skew lattice S, if* A_1 *and* A_n *are finite then so are all intermediate* $\mathcal{D}$*-classes and*

$$[A_1{:}\,A_n] = [A_1{:}\,A_2]\,[A_2{:}\,A_3]\,\dots\,[A_{n-1}{:}\,A_n].$$

Proof. The general inequality is a consequence of the fact that given $a > c$ with $a \in$ A and $c \in$ C, there exists a $b \in$ B such that $a > b > c$. Hence given a has [B: A] images in B, each of which has [C: B] images in C, so that a has at most [C: B][B: A] images in C.

Assuming S is also strictly categorical, then B is finite. The following equalities thus hold and with them the first half of the theorem:

$$[A:C] = \frac{|A|}{\omega_{AC}} = [A:B]\frac{\omega_{AB}}{\omega_{AC}} \quad \text{and} \quad [B:C] = \frac{|B|}{\omega_{BC}} = \frac{\omega_{AB}\omega_{BC}}{\omega_{AC}}\frac{1}{\omega_{BC}} = \frac{\omega_{AB}}{\omega_{AC}}.$$

Given the chain $A_1 > A_2 > \dots > A_n$, the factorization of $[A_1{:}A_n]$ proceeds from the special case:

$[A_1: A_n] = [A_1: A_2][A_2: A_n] = [A_1: A_2][A_2: A_3][A_3: A_n] = \ldots = [A_1: A_2][A_2: A_3]\ldots[A_{n-1}: A_n]$. ☐

Returning to skew diamonds, the following restatement of Theorem 2.4.10 parallels to some extent the theorem above:

Theorem 5.7.5. *In a symmetric skew diamond* {J > A, B > M}*,* [M: J] = [M: A] [M: B] *and* [J: M] = [J: A][J : B] . ☐

Lemma 5.7.6. *Given finite $\mathcal{D}$-classes* X > Y *in a skew lattice*, |X|[Y: X] = |Y|[X: Y], *or put otherwise*, $[Y: X] = \frac{|Y|}{|X|}[X: Y]$.

Proof. |X|[Y: X] and |Y|[X: Y] expand to [X: Y] ω_{XY} [Y: X] and to [Y: X] ω_{XY} [X: Y]. ☐

This next result of relevance is from Pita da Costa's dissertation.

Theorem 5.7.7. *Given a finite cancellative skew diamond* {J > A, B > M}, |A| |B| = |J| |M|.

Proof. One has $\frac{|A|}{|J|} = \frac{[A : J]}{[J : A]} = \frac{[M : B]}{[B : M]} = \frac{|M|}{|B|}$. The first and third equalities come from the previous lemma. The middle equality is from [B: M] = [J: A] and [M: B] = [A: J] in Theorem 5.7.1. The equality now follows by cross-multiplying. ☐

This outcome fails in the four simply cancellative $\mathbf{NS}_7$ variants and both symmetric $\mathbf{NC}_5$ variants. (But see Corollary 2.4.11.) One thus has:

Corollary 5.7.8. *A skew lattice* S *is cancellative if and only if it is quasi-distributive and all of its skew diamonds are cancellative. The latter occurs if and only if* |A| |B| = |J| |M| *holds in all finite skew diamonds* {J > A, B > M} *of* S. ☐

This situation is sharpened if the skew diamond {J > A, B > M} is ***pointed*** in that |J| = 1 or |M| = 1. If J has a unique element, it is often denoted by 1; if it is M, the single point is often denoted by 0. $\mathbf{NS}_7^{\mathcal{L},0}$, $\mathbf{NS}_7^{\mathcal{R},0}$, $\mathbf{NS}_7^{\mathcal{L},1}$ and $\mathbf{NS}_7^{\mathcal{R},1}$ are pointed, while $\mathbf{NC}_5^{\mathcal{L}}$ and $\mathbf{NC}_5^{\mathcal{R}}$ are doubly pointed. Indeed, *a skew diamond is simply cancellative if and only if all doubly pointed skew diamond subalgebras are sublattices* (thus eliminating any possible $\mathbf{NC}_5$ subalgebras). What about full cancellation? The next two Theorems are from the 2011 paper of Cvetko-Vah, Kinyon, Leech and Spinks.

Theorem 5.7.9. *A quasi-distributive skew lattice is cancellative if and only if all pointed skew diamonds in it factor as products of primitive skew lattices*.

Proof. Given a skew lattice S, the condition on pointed skew diamonds in it excludes copies of $\mathbf{NC}_5^{\mathcal{L}}$, $\mathbf{NC}_5^{\mathcal{R}}$, $\mathbf{NS}_7^{\mathcal{L},0}$, $\mathbf{NS}_7^{\mathcal{R},0}$, $\mathbf{NS}_7^{\mathcal{L},1}$ and $\mathbf{NS}_7^{\mathcal{R},1}$ from being subalgebras since in each of these six cases the order of the join or meet classes is inconsistent with such a factorization. This insures that S is cancellative. Conversely, given a cancellative skew lattice, we must show that all pointed skew diamonds in it factor as stated. Our task quickly reduces to showing that a cancellative pointed skew lattice S factors as stated. So let S = {J > A, B > M} be such a skew diamond with say M = {0}. A pair of primitive subalgebras are $A^0 = A \cup \{0\}$ and $B^0 = B \cup \{0\}$. Claim: An isomorphism σ: $A^0 \times B^0 \cong S$ is given by $\sigma[(x, y)] = x\vee y$ for all $x \in A^0$ and $y \in B^0$.

Thanks to Theorem 2.2.1, σ is easily seen to be surjective. It is clearly bijective from {0}×{0} to {0}, from A×{0} to A and from {0}×B to B. Thus σ is bijective overall if it is bijective from A×B to J. Since S is symmetric, $a\vee b = b\vee a$ for each pair (a, b) in A × B since $a\wedge b = 0 = b\wedge a$. Thus the bijectivity of A×B with J is given by Theorem 5.3.10.

Finally for all $x_1, x_2 \in A^0$ and $y_1, y_2 \in B^0$,

$$\begin{aligned}\sigma[(x_1, y_1)\vee(x_2, y_2)] &= \sigma[(x_1\vee x_2, y_1\vee y_2)] &&= x_1\vee x_2 \vee y_1\vee y_2\\ &= x_1\vee y_1 \vee x_2\vee y_2 &&= \sigma[(x_1, y_1)] \vee \sigma[(x_2, y_2)],\end{aligned}$$

since elements from A^0 commute with elements from B^0. Expanding $\sigma[(x_1, y_1)\wedge(x_2, y_2)]$ and $\sigma[(x_1, y_1)] \wedge \sigma[(x_2, y_2)]$, we get respectively: $(x_1\wedge x_2) \vee (y_1\wedge y_2)$ and $(x_1\vee y_1) \wedge (x_2\vee y_2)$.

Case 1) One of the x_i and one of the y_j is 0. Here both expressions above reduce to 0.

Case 2) Neither x_i is 0 but one of the y_j is 0. We get $x_1\wedge x_2$ on the left and on the right either $(x_1\vee y_1) \wedge x_2$ or $x_1 \wedge (x_2\vee y_2)$. Since the x_i and y_j commute, normality plus absorption gives,

$$(x_1\vee y_1)\wedge x_2 = (y_1\vee x_1)\wedge x_2\wedge x_1\wedge x_2 = (y_1\vee x_1)\wedge x_1\wedge x_2 = x_1 \wedge x_2$$

and

$$x_1\wedge(x_2\vee y_2) = x_1\wedge x_2\wedge x_1\wedge(x_2\vee y_2) = x_1\wedge x_2\wedge(x_2\vee y_2) = x_1 \wedge x_2.$$

Case 3) One of the x_i is 0, but neither of the y_j is 0. This is similar to Case 2.

Case 4) None of the x_i or y_j are 0. By x-y commutation plus $u\wedge v = v\vee u$ on $\mathcal{D}$-classes, both $(x_1\wedge x_2)\vee(y_1\wedge y_2)$ and $(x_1\vee y_1)\wedge(x_2\vee y_2)$ are easily seen to reduce to $x_2\vee y_2\vee x_1\vee y_1$.

Thus σ is an isomorphism. The case where J = {1} follows by the dual argument. □

There is more. In general, every skew diamond S (cancellative or not) is a union of maximal pointed subalgebras in two ways. $S = \bigcup_{m\in M} m\vee S\vee m = \bigcup_{j\in J} j\wedge S\wedge j$ where

$m\vee S\vee m = \{m\vee x\vee m \mid x \in S\} = \{x \in S \mid x \geq m\}$ is pointed below with zero m, and
$j\wedge S\wedge j = \{j\wedge x\wedge j \mid x \in S\} = \{x \in S \mid j \geq x\}$ is pointed above with identity 1,

Given $m, m' \in M$, define f: $m\vee S\vee m \to m'\vee S\vee m'$ by $f(x) = m'\vee x\vee m'$ and g: $m'\vee S\vee m' \to m\vee S\vee m$ by $g(x) = m\vee x\vee m$. Regularity and the fact $x\vee y\vee x = x$ holds on M imply that f and g are a reciprocal bijections. Again regularity gives $m'\vee(x\vee y)\vee m' = (m'\vee x\vee m')\vee(m'\vee y\vee m')$. Thus f and

g are reciprocal $\vee$-isomorphisms that must restrict to isomorphisms between corresponding $\mathcal{D}$-classes (since $x \wedge y = y \vee x$ on $\mathcal{D}$-classes), giving, e.g., $m \vee \mathrm{A} \vee m \cong m' \vee \mathrm{A} \vee m'$. Thus f induces isomorphisms between pointed primitive algebras,

$$m \vee \mathrm{A} \vee m \cup \{m\} \cong m' \vee \mathrm{A} \vee m' \cup \{m'\} \text{ and } m \vee \mathrm{B} \vee m \cup \{m\} \cong m' \vee \mathrm{B} \vee m' \cup \{m'\}.$$

If S *is also cancellative*, then the previous theorem gives $m \vee \mathrm{S} \vee m \cong m' \vee \mathrm{S} \vee m'$.

Theorem 5.7.10. *If* $\{\mathrm{J} > \mathrm{A}, \mathrm{B} > \mathrm{M}\}$ *is a cancellative skew diamond*, *then all pointed skew diamonds* $m \vee \mathrm{S} \vee m \subseteq \mathrm{S}$ *for* $m \in \mathrm{M}$, *are isomorphic. Dually, all pointed skew diamonds* $j \wedge \mathrm{S} \wedge j \subseteq \mathrm{S}$ *for* $j \in \mathrm{J}$ *are isomorphic*. □

Summing up much of the discussion about cancallative skew lattices we have:

Theorem 5.7.11. *For a skew lattice* S *the following are equivalent*:

i) S *is cancellative*.
ii) S *is quasi-distributive and all [finite]skew diamonds in* S *are cancellative*.
iii) S *is quasi-distributive and symmetric, with all [finite]skew diamonds in it being strictly categorical*.
iv) S *is quasi-distributive and all [finite]pointed skew diamonds in* S *factor as a product of two primitive skew lattices*.
v) S *is quasi-distributive and* $|\mathrm{A}||\mathrm{B}| = |\mathrm{J}||\mathrm{M}|$ *in any finite skew diamond* $\{\mathrm{J} > \mathrm{A}, \mathrm{B} > \mathrm{M}\}$ *in* S.

While neither the classes of distributive skew lattices or cancellative skew lattices includes the other, all skew diamonds in cancellative skew lattices are distributive (being strictly categorical), and all skew chains in distributive skew lattices are cancellative (being true in general).

Historical remarks

The results in Section 5.1, on symmetry come from [Cvetko-Vah, Kinyon, Leech and Spinks, 2011]. The results on comparing distributive identities in Section 2 are due to [Spinks, 1998 and 2000] and [Cvetko-Vah, 2006]. The material in Section 3 on cancellation is mostly from [Cvetko-Vah, Kinyon *et al*, 2011] again, while the results in Section 4 on categorical behavior are from [Kinyon and Leech, 2013]. The material in Sections 5 and 6 on distributivity and its consquences comes from [Kinyon, Leech and Pita Costa, 2014?]. The various counting results in the final section are from the dissertations of Pita Costa [2012] and Cvetko-Vah [2005] as well as [Cvetko-Vah, Kinyon *et al*, 2011].

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VI: SKEW LATTICES IN RINGS

From the initial research into skew lattices in the 1980s, skew lattices of idempotents in rings and their particular examples have provided fundamental ideas about the subject. The absorption identities came from observing that nonempty sets of idempotents in a ring that were closed under both multiplication and the circle operation ($x \bigcirc y = x + y - xy$) satisfied them. The significance of the distributive identities $a \wedge (b \vee c) \wedge a = (a \wedge b \wedge a) \vee (a \wedge c \wedge a)$ and its dual was due to the fact they hold for all skew lattices in rings, whereas say $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ need not hold. Occurrences of being symmetric, cancellative or categorical were observed first in the ring context. Maximal left-regular (right-regular) multiplicative bands turned out to be maximal left-handed (right-handed) skew lattices under $\bullet$ and $\bigcirc$; and maximal normal bands formed skew Boolean algebras in their host rings (with $\bigcirc$ often replaced by ∇). In this chapter we look more closely at skew lattices (of idempotents) in rings.

In Section 6.1 left- and right-handed *skew lattice extensions* of a lattice of idempotents S_0 in a ring are introduced. These are the uniquely largest left-handed and right-handed skew lattices containing S_0 as a lattice section. More generally, *quadratic* skew lattices (with join $\bigcirc$) are studied. Theorems about the center $\mathcal{Z}(S)$ of a quadratic skew lattice S and decompositions of S related to its center are given. Attention is given to what occurs in matrix rings over fields.

Section 6.2 looks at ∇-bands, that is, multiplicative bands of idempotents that are closed also under the cubic join ∇ where $x \nabla y = (x \bigcirc y)^2 = x + y + yx - xyx - yxy$. Even when ∇ is not associative, ∇-bands share many properties of quadratic skew lattices. (Theorem 6.2.2.) ∇ is thus a nocommutative join. When it is associative, the ∇-band is a *cubic* skew lattice that is necessarily distributive and cancellative, and hence categorical and symmetric. (Quadratic skew lattices are seen as being trivially cubic.) We consider various criteria given by Cvetko-Vah and Leech [2007] for the ∇-operation in ∇-bands to be associative. These include the following: the $\mathcal{L}$ and $\mathcal{R}$ congruences relative to multiplication are also ∇-congruences; ∇ is associative on every primitive subalgebra of comparable $\mathcal{D}$-classes A > B in the band. In particular, normal ∇-bands where multiplication is normal, are seen (again) to be normal skew lattices. For many ∇-bands S where ∇ is not associative, a closely related associative join $\vee$ exists making (S; $\vee$, $\bullet$) a skew lattice. (See Theorems 6.2.12 and 6.2.13 and the preceding discussion.) This holds for ∇-bands in finite dimensional algebras over fields (in the ring-theoretic sense of "algebra").

While maximal left [right] regular bands and normal bands in a ring form skew lattices under $\bullet$ and ∇, maximal regular bands in rings need not form skew lattices or even ∇-bands. Equivalently, a regular band in a ring need not generate a ∇-band under $\bullet$ and ∇. In Section 6.3 we consider ∇-inductive conditions. These are conditions which guarantee that a regular band satisfying them will generate at least a ∇-band in the host ring. (See Theorems 6.3.1 and 6.3.2.) Being normal or being left [right] regular are cases of ∇-inductive conditions. We conclude with Cvetko-Vah's nice theorem (6.3.5) stating that a regular band in a ring with totally ordered $\mathcal{D}$-

classes must generate a ∇-band. In particular, maximal totally pre-ordered regular bands in rings are ∇-bands.

Not only does a maximal normal band S form a skew Boolean algebra in its host ring (with $e \setminus f = e - efe$), it is also the full set of idempotents in the subring R' that it generates: $S = \mathbf{E}(R')$. Conversely, if $\mathbf{E}(R)$ is multiplicatively closed in a given ring R, then $\mathbf{E}(R)$ forms a skew Boolean algebra. (See Theorem 2.3.7 or Theorem 6.4.2 below.) In Section 4 we begin our study of such *idempotent-closed* rings. Much of the focus is on the case of *idempotent-dominated* rings where $R = Q(R)$, the ideal generated from $\mathbf{E}(R)$. In this case, if $\mathcal{K}_R$ is the canonical nilpotent ideal $\{k \in R \mid xky = 0 \text{ for all } x, y \in R\}$, then $R/\mathcal{K}_R$ is the maximal abelian image of R. (See Theorem 6.4.10. Recall that R is *abelian* if its idempotents commute.) When $\mathbf{E}(R)$ also has a lattice section, R is a semidirect sum $\mathcal{A} \oplus \mathcal{K}_R$, that is direct under addition with $\mathcal{A}$ being a maximal abelian subring of R that is necessarily isomorphic to $R/\mathcal{K}_R$. (Theorem 6.4.11.) This is the case for all idempotent-closed and dominated rings of $n\times n$ matrices. When R is not idempotent-dominated, these facts apply directly to $Q(R)$; but upon setting $\mathcal{K} = \mathcal{K}_{Q(R)}$, then $\mathcal{K}$ is a nilpotent ideal of R also, with R being idempotent-closed if and only if $R/\mathcal{K}$ is abelian. (Theorem 6.4.15.) These and related facts are studied in the fourth section.

Like Boolean algebras, skew Boolean algebras decompose almost at will. (See Theorem 4.1.4 or Theorem 6.5.1 below.) To what extent does this extend to idempotent-closed rings, especially if they are idempotent-dominated? In particular, given certain finiteness conditions (e.g. the ACC or DCC on idempotents), must ring decompose as a direct sum of subrings that in some sense are "atomic"? These questions are pursued for idempotent-dominated rings in the fifth section. The "atomic" rings turn out to be *rectangular* rings – idempotent-dominated rings whose non-0 idempotents form a rectangular band under multiplication. Theorem 6.5.6 states that each idempotent-closed and dominated ring R satisfying the DCC on idempotents is an *orthosum* of ideals Q_i (that is, $R = \sum Q_i$ with $Q_iQ_j = \{0\}$ for $i \neq j$) where each Q_i is a rectangular ring. While the orthosum condition is a weakening of the direct sum condition, if the annihilator ideal of R reduces to $\{0\}$, the sum must be direct. Rectangular rings are characterized in Theorems 6.5.11 and 6.5.14.

The results of Sections 6.4 and 6.5 are then "tested" in the context of matrix rings over fields in Sections 6.6 and 6.7. The former studies upper triangular representations of normal skew lattices and skew Boolean algebras in matrix rings, and Section 6.7 studies upper triangular representations of (maximal) idempotent-closed and dominated subrings of matrix rings. (See Theorems 6.7.4 – 6.7.6.)

The chapter ends with historical comments and relevant references.

6.1 Quadratic skew lattices in rings

Recall that a ***quadratic skew lattice*** in a ring R is any multiplicative band in R that is also closed under the circle operation: $x \bigcirc y = x + y - xy$. Letting • denote multiplication, by Theorem 2.1.7 such a band S satisfies the following absorption identities that guarantee that (S, ○, •) is indeed a skew lattice:

$$x \bullet (x \bigcirc y) = x = (y \bigcirc x) \bullet x.$$

$$x \bigcirc (x \bullet y) = x = (y \bullet x) \bigcirc x.$$

In particular both (S, ○) and (S, •) are regular bands. We typically identify ○ as the join ∨ and • as the meet ∧. Joins are generally higher than their constituent elements and this is certainly the case here. In particular, in matrix rings

$$rank(e \bullet f) \leq rank(e), rank(f) \leq rank(e \bigcirc f).$$

By Theorem 2.1.9 every maximal right [left] regular multiplicative band S in a ring is closed under ○, making (S, ○, •) is a maximal right-handed quadratic skew lattice in the ring. Our first result is the dual of this theorem.

Theorem 6.1.1. *Let* R *be a ring and let* S *be a subset of* R *forming a left regular band in* R *under the circle operation. If* S *is a maximal such* ○*-band in* S, *then* S *is also closed under multiplication and forms a maximal right-handed quadratic skew lattice in* S.

Proof. First suppose that R has an identity 1. Then $\gamma(x) = 1 - x$ induces a bijection on **E**(R) such that $\gamma(xy) = \gamma(x) \bigcirc \gamma(y)$ and $\gamma(x \bigcirc y) = \gamma(x)\gamma(y)$ regardless of the outcomes also being idempotent. Hence, if S is a maximal left regular band in R under ○, γ[S] must be a maximal left regular multiplicative band in R and thus along with ○ forms a maximal left-handed skew lattice in R. Clearly S = γγ[S] is indeed a maximal right-handed skew lattice in R.

If R does not have an identity, then it can be embedded in a ring R′ with identity 1. In R′ we extend S to a maximal left regular ○-band S′ that forms a maximal right-handed skew lattice in R′. Hence S itself must generate a right-handed skew lattice in R. But given maximal status of S in R, it is this skew lattice. ☐

Our main emphasis in this section is with classes of quadratic skew lattices. To begin, recall that for any $e \in \mathbf{E}(R)$, its ***R-set*** $\mathcal{R}_e = e + eR(1 - e)$ is the maximal right-zero semigroup in R containing e. Since $x \bigcirc y = yx$ on $\mathcal{R}_e$, $\mathcal{R}_e$ is a maximal right-rectangular skew lattice in R. Recall also that if $e > f$ in **E**(R), then $\mathcal{R}_e \cup \mathcal{R}_f$ forms a maximal right-primitive skew lattice in R. This has an immediate generalization.

Theorem 6.1.2. *Given any naturally totally ordered set of idempotents* T *in a ring* R, $S = \bigcup_{e\in T} \mathcal{R}_e$ *is a band that together with* $\circ$ *forms a right-handed skew lattice in* R *that is maximal with respect to being right-handed and containing* T *as a lattice section*. ☐

This result is illustrated by the following chain T of length 4 and its induced right-handed skew lattice S in $\mathcal{M}_4(\mathbb{Q})$. The asterisks denote free variable positions in the matrices.

$$
\begin{array}{ccccc}
 & \mathbf{I} & & & \mathbf{I} \\[2ex]
 & \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} & & & \begin{bmatrix} 1&0&0&*\\0&1&0&*\\0&0&1&*\\0&0&0&0 \end{bmatrix} \\[2ex]
\mathrm{T}: & \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} & \rightarrow & \mathrm{S}: & \begin{bmatrix} 1&0&*&*\\0&1&*&*\\0&0&0&0\\0&0&0&0 \end{bmatrix} \\[2ex]
 & \begin{bmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} & & & \begin{bmatrix} 1&*&*&*\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \\[2ex]
 & \mathbf{0} & & & \mathbf{0}
\end{array}
$$

Finding right-handed ***skew chains*** (skew lattices whose $\mathcal{D}$-classes are totally ordered) is thus comparatively easy. We turn our attention to the general case where the $\mathcal{D}$-classes need not be totally ordered. A simple strategy for finding right-handed skew lattices in rings is as follows:

(1) *Find a lattice* T *in a ring with* $\vee = \circ$ *and* $\wedge = \bullet$.

(2) *Consider the union* $\bigcup_{e\in T} \mathcal{R}_e = \bigcup_{e\in T}\, e + e\mathrm{R}(1 - e)$.

(3) *Search for skew lattices* $S \subseteq \bigcup_{e\,\in\,T} \mathcal{R}_e$ *containing* T *as a lattice section*.

This leads us to the following fundamental result:

Theorem 6.1.3. *Given the lattice* T *as above, for each* $e \in \mathrm{T}$ *set*

$$\mathrm{S}_e \;=\; \{e' \in \mathcal{R}_e \,\big|\; \forall f \in \mathrm{T},\, e'f \in \mathcal{R}_{ef}\,\}.$$

Then $\mathrm{S} = \bigcup_{e\in\mathrm{T}} \mathrm{S}_e$ *is the unique maximal right-handed skew lattice in* R *having section* T.

(S is called the ***right extension*** of T and each S_e is called the ***S-set*** of e relative to T.)

Proof. Given a right-handed skew lattice S′ in **E**(R) having T as a lattice section, clearly $\mathrm{S}' \subseteq \mathrm{S}$. We need only show that S is closed under both skew lattice operations and thus is a right-handed skew lattice. To begin, given $e' \in \mathrm{S}_e$ and $f' \in \mathrm{S}_f$,

$$e'f' \;=\; e'ff' \;\in \mathcal{R}_{ef}$$

since $e'f \in \mathcal{R}_{ef}$ by definition of S_e and thus $(e'f)f' \in \mathcal{R}_{ef}$ also. Next, given $g \in \mathrm{T}$, $f'g \in \mathcal{R}_{fg}$ with $f'g = fgf'g$. Thus

$$(e'f')g \;=\; e'(f'g) \;=\; e'(fgf'g) \;=\; (e'fg)f'g \;\in \mathcal{R}_{efg}$$

because $e'fg \in \mathcal{R}_{efg}$ and $\mathcal{R}_{fg} \cup \mathcal{R}_{efg}$ is a right primitive skew lattice. Hence $e'f' \in \mathrm{S}_{ef}$ and S is at least a right regular band under multiplication. As such S generates a right-handed skew lattice S′. But since $\mathcal{R}$ is the $\mathcal{D}$-congruence on S′, T must be a lattice section on S′ forcing $\mathrm{S}' \subseteq \mathrm{S}$, that is $\mathrm{S}' = \mathrm{S}$. □

Examples 6.1.1 from $\mathcal{M}_4(\mathcal{F})$ for any field $\mathcal{F}$. The lattice section in each case is the lattice T of all diagonal matrices in the skew lattice, one from each $\mathcal{D}$-class. Notice that *any $\mathcal{D}$-class that is comparable to all other classes is a full $\mathcal{R}$-set. This is true for all right extensions of lattices in rings*. Incomparable pairs of $\mathcal{D}$-classes, however, in some way create “interference” with each other. Both classes are properly less than full $\mathcal{R}$-sets. *That this is always the situation for matrix rings over fields is justified later*.

(a)

$$
\begin{tikzcd}[column sep=small]
& \mathbf{I} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & * \\ 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dr, dash, dotted] & & \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ * & \mathbf{1} & 0 & 0 \\ * & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \arrow[dl, dash, dotted] \\
& \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ * & \mathbf{1} & 0 & * \\ * & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
\begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ * & \mathbf{1} & 0 & * \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dr, dash, dotted] & & \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{0} & 0 & 0 \\ * & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dl, dash, dotted] \\
& \mathbf{0} &
\end{tikzcd}
$$

(b)

$$
\begin{tikzcd}[column sep=small]
& \mathbf{I} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
\begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & * \\ 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dr, dash, dotted] & & \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ * & \mathbf{1} & 0 & 0 \\ * & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix} \arrow[dl, dash, dotted] \\
& \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ * & \mathbf{1} & 0 & * \\ * & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[d, dash, dotted] & \\
& \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ * & \mathbf{1} & * & * \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[d, dash, dotted] & \\
& \mathbf{0} &
\end{tikzcd}
$$

In the next case something else of note occurs. First: ***complementary*** pairs of $\mathcal{D}$-classes A and B exist where A ∨ B = {I} and A ∧ B = {0}. Here **I** and **0** denote respectively the ***unique maximum*** and ***unique minimum*** of S, although in these cases they also denote the identity and zero matrices of the matrix ring. In the case below, complementary classes are in bold type.

(c)

$$
\begin{tikzcd}[column sep=tiny]
& & & \mathbf{I} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & \\
& & \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & * \\ 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & & \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \arrow[dl, dash, dotted] \\
& \begin{bmatrix} \mathbf{1} & 0 & 0 & 0 \\ 0 & \mathbf{1} & * & * \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dl, dash, dotted] \arrow[dr, dash, dotted] & & \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & * \\ 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dl, dash, dotted] & \\
\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \arrow[dr, dash, dotted] & & \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{1} & * & * \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \arrow[dl, dash, dotted] & & \\
& \mathbf{0} & & &
\end{tikzcd}
$$

Second: these complementary pairs are trivial $\mathcal{D}$-classes and thus form part of the center of the skew lattice.

Third: this case is ***centrally complemented*** in that each central (trivial) $\mathcal{D}$-class has a complementary class.

Finally, we seemingly have an internal direct product. Indeed, setting

$$S_1 = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \cup \{\mathbf{0}\} \text{ and } S_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \cup \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & * \\ 0 & 0 & \mathbf{1} & * \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \cup \begin{bmatrix} \mathbf{0} & 0 & 0 & 0 \\ 0 & \mathbf{1} & * & * \\ 0 & 0 & \mathbf{0} & 0 \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \cup \{\mathbf{0}\}$$

an isomorphism of μ: $S_1 \times S_2 \cong S$ is given by setting $\mu(x, y) = x \vee y = x + y$ (since $xy = 0$). This leads us to the following three results:

Theorem 6.1.4. *If* S *is a cancellative skew lattice with a unique maximum* 1 *and a unique minimum* 0, *then all pairs of complementary $\mathcal{D}$-classes in* S *are trivial and thus lie in the center of* S. *In particular, this occurs for all skew lattices in rings with unity* 1 *that contain both* 1 *and* 0.

Proof. Let A, B be a complementary $\mathcal{D}$-classes with say a, $a' \in$ A and $b \in$ B. From $a \vee b = a' \vee b = 1$ and $a \wedge b = a' \wedge b = 0$, cancellation implies $a = a'$. □

Theorem 6.1.5. *Every maximal [right-handed] skew lattice* (*under* $\circ$ *and* $\bullet$) *in a ring with identity* 1 *is* (a) *centrally complemented and* (b) *contains all central idempotents of the ring.*

Proof. (a) If $e \in \mathcal{Z}(S)$ but not $1 - e$, then $1 - e \notin S$ so that $S' = eS + (1 - e)S$ is a larger [right-handed] skew lattice in R. Indeed given $x + x', y + y' \in S'$, we get

$$(x + x')(y + y') = xy + x'y' \text{ and } (x + x') \circ (y + y') = x \circ y + x' \circ y'.$$

Thus S′ is a skew lattice in R containing both S and $1 - e$. Given the maximal status of S, $1 - e \in S$ and hence $1 - e \in \mathcal{Z}(S)$.

(b) If e is a central idempotent that is not in S, then clearly $S' = eS + (1 - e)S$ contains both e and S. As in (a), S′ must be a skew lattice, which by the maximal status of S equals the latter, and $e \in S$ follows. □

Theorem 6.1.6. *Let* S *be a distributive, symmetric skew lattice with both maximal and minimal elements,* 1 *and* 0. *Let* B *be a finite Boolean lattice such that* $\{1, 0\} \subseteq \mathrm{B} \subseteq \mathcal{Z}(\mathrm{S})$. *Then*:

1) *For each atom* α *of* B, $\alpha\wedge \mathrm{S} = \{\alpha\wedge x \mid x \in \mathrm{S}\} = \{y \in \mathrm{S} \mid 0 \leq y \leq \alpha\}$ *is a skew lattice with maximal element* α.
2) *For atoms* $\alpha \neq \beta$, *and all* $x \leq \alpha$ *and* $y \leq \beta$, $x\wedge y = 0 = y\wedge x$ *and* $x\vee y = y\vee x$.
3) *Given atoms* $\alpha_1, \dots ,\alpha_n$ *of* B, *an isomorphism* φ: $\mathrm{S} \cong \prod_1^n(\alpha_i \wedge \mathrm{S})$ *is defined by* $\varphi(x) = (\alpha_1\wedge x, \dots, \alpha_n\wedge x)$ *with* $\varphi^{-1}(x_1, \dots, x_n) = x_1 \vee \dots \vee x_n$.

Proof. Since $\mathrm{B} \subseteq \mathcal{Z}(\mathrm{S})$, $\alpha\wedge x\wedge\alpha\wedge y = \alpha\wedge x\wedge y$ and

$$(\alpha\wedge x)\vee(\alpha\wedge y) = (\alpha\wedge x\wedge\alpha)\vee(\alpha\wedge y\wedge\alpha) = \alpha\wedge(x\vee y)\wedge\alpha = \alpha\wedge(x\vee y).$$

Thus assertion (1) follows. Next, since $\alpha\wedge\beta = 0$ for x and y as stated in (2),

$$x\wedge y = x\wedge\alpha\wedge y\wedge\beta = x\wedge y\wedge\alpha\wedge\beta \quad x\wedge y\wedge 0 = 0.$$

Similarly $y\wedge x = 0$, so that $x\vee y = y\vee x$ by symmetry and (2) follows. Moreover, φ is tat least a homomorphism. For all $x \in \mathrm{S}$,

$$\begin{aligned} x &= x \wedge 1 \wedge x = x \wedge (\alpha_1 \vee \dots \vee \alpha_n) \wedge x \\ &= (x \wedge \alpha_1 \wedge x) \vee\dots\vee (x \wedge \alpha_n \wedge x) = (\alpha_1 \wedge x) \vee\dots\vee (\alpha_n \wedge x), \end{aligned}$$

which guarantees that φ is also one-to-one. Finally, given any $(x_1, \dots, x_n)$ in $\prod_1^n(\alpha_i \wedge \mathrm{S})$, each $x_i = \alpha_i\wedge x_i$ so that

$$\alpha_j\wedge(x_1 \vee \dots \vee x_n) = (\alpha_j\wedge x_1) \vee \dots \vee (\alpha_j\wedge x_n) = (\alpha_j\wedge x_j) = x_j.$$

Hence $\varphi(x_1 \vee \dots \vee x_n) = (x_1, \dots, x_n)$, making the map φ surjective and hence an isomorphism. ☐

Our next goal is to refine our description of the right extension S of a lattice T is a ring. In this regard, given commuting idempotents e and f in a ring R, $\mathrm{S}(e \mid f)$ will denote the $\mathcal{D}$-class of e in the skew diamond that is the right extension S of the lattice $\mathrm{T} = \{e, f, e\vee f, e\wedge f\}$ in **E**(R).

Theorem 6.1.7. *Let* T *be a lattice in a ring* R *with identity* 1 *and let S be the right extension of* T. *If* $e \in \mathrm{T}$, *then* $\mathrm{S}_e = \bigcap\{\mathrm{S}(e \mid f) \mid f \in \mathrm{T}\}$. *For* $e, f \in \mathrm{T}$, *moreover*,

$$\mathrm{S}(e \mid f) = e + ef\mathrm{R}f(1- e) + e\mathrm{R}(1- e)(1-f).$$

In particular, given e and f in T:

i) *if e and f are comparable, then* $\mathrm{S}(e \mid f) = \mathcal{R}_e$.
ii) *if e and f are disjoint* ($ef = 0$), *then* $\mathrm{S}(e \mid f) = e + e\mathrm{R}(1- e)(1-f)$.

Proof. In light of Theorem 6.1.3, the assertion about S_e is obvious. In general, for all $x \in \mathcal{R}_e$,

$$x = e + ea(1-e) = e + ea(1-e)f + ea(1-e)(1-f)$$

for some a in R (x itself will do). Thus $xf = ef + ea(1-e)f$, so that $xf \in \mathcal{R}_{ef}$ only when $ea(1-e)f = efb(1-ef)$ for some b in R which occurs precisely when $ea(1-e)f = efb(1-e)f$ for some b since $(1-e)f = (1-ef)f$. Thus,

$$S(e\,|\,f) \subseteq e + ef\mathrm{R}f(1-e) + e\mathrm{R}(1-e)(1-f) \subseteq \mathcal{R}_e.$$

Since

$$\{e + ef\mathrm{R}f(1-e) + e\mathrm{R}(1-e)(1-f)\}f = ef + ef\mathrm{R}f(1-e) = ef + ef\mathrm{R}f(1-ef) \subseteq \mathcal{R}_{ef},$$

the reverse inclusion holds. Case (ii) follows immediately from the general case. For case (i), if $f > e$, then $S(e\,|\,f) = e + e\mathrm{R}f(1-e) + e\mathrm{R}(1-e)(1-f)$ which reduces to $e + e\mathrm{R}(1-e) = \mathcal{R}_e$. If $e > f$, then both $f(1-e) = 0$ and $(1-e)(1-f) = 1 - \mathrm{e}$, so that $S(e\,|\,f) = e + e\mathrm{R}(1-e) = \mathcal{R}_e$ again. □

The general case and the two special cases are illustrated in the following block matrix diagrams where u, x, y and z hold arbitrary values.

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S(e\,|\,f) = \text{all} \begin{bmatrix} 1 & 0 & 0 & y \\ 0 & 1 & x & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S(e\,|\,f) = \text{all} \begin{bmatrix} 1 & 0 & u & y \\ 0 & 1 & x & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S(e\,|\,f) = \text{all} \begin{bmatrix} 1 & 0 & 0 & y \\ 0 & 1 & 0 & z \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The case of matrix rings over fields

Given a matrix ring $\mathrm{R} = \mathcal{M}_n(\mathcal{F})$ over a (possibly skew) field $\mathcal{F}$, let Δ denote the maximal lattice of 0-1 diagonal matrices and let $e_1, \ldots, e_n$ denote the atoms of Δ. Thus e_i is the matrix with 1 in the ii-position and 0 elsewhere. The ***support*** of any $e \in \Delta$ is $supp(e) = \{i \in \mathbb{N} \mid e_{ii} = 1\}$.

In what follows, without loss of generality *we assume that* T *is a sublattice of* Δ *such that both matrices* **0** *and* **I** *lie in* T. With this in mind, for each $e \in \Delta$,

the **T-*cover*** of e is s $\overline{e} = \bigwedge \{f \in T \mid f \geq e\}$,

and

the **T-*interior*** of e is $e^{o} = \bigvee \{f \in T \mid f \leq e\}$.

Clearly an idempotent $e \in \Delta$ belongs to T if and only if it equals either, and hence both, its T-cover and its T-interior. We next state a pair of elementary lemmas, the first of which describes an R-set in matrix form.

Lemma 6.1.8. *Given* $e \in \Delta$ *and* $x \in \mathcal{R}_e = e + e\mathrm{R}(1 - e)$, *then* e *and* x *have the same diagonal entries. Moreover, the nonzero diagonal entries of* x *occur only in those rows indexed by* $supp(e)$ *and in those columns with support indexed by* $supp(1 - e)$. ☐

Lemma 6.1.9. *Given* $e \in \mathrm{T}$ *and* $j \in supp(1 - e)$, $e \wedge \overline{e_j} = 0$ *iff* $j \in supp(1 - e)^{o}$. ☐

We now state our principal result for right-handed skew lattices in matrix rings.

Theorem 6.1.10. *Let* R *be the ring of all n×n matrices over a (skew) field* $\mathcal{F}$, *let* Δ_n *be the lattice of all* 0-1 *diagonal matrices and let* T *be any sublattice of* Δ_n *containing at least the zero and identity matrices*, **0** *and* **I**. *If the skew lattice* S *is the right extension of* T *in* R, *then for each* $e \in \mathrm{S}$ *its* $\mathcal{D}$-*class* S_e *in* S *is described as follows*:

i) *All matrices in* S_e *have the same diagonal as e.*
ii) *The nonzero, non-diagonal entries of any matrix in* S_e *occur only in those columns indexed by* $supp(1 - e)$.
iii) *For j in* $supp(1 - e)$, *the only positions in the* j^{th} *column of a matrix in* S_e *that admit nonzero entries are given by* $supp(e \wedge \overline{e_j})$.
iv) *For any* $j \in supp(1 - e)$, *the* j^{th} *column only has* 0*s when* $j \in supp(1 - e)^{o}$.
v) *No further restrictions are imposed on the matrices in* S_e.

Proof. Letting $e \in \mathrm{T}$ as stated, the class of matrices in S_e is the intersection of classes of matrices themselves obtained by appropriate juggling of the block designs given after Theorem 6.1.7. Thus to determine the matrices in S_e, one need only discover which non-diagonal positions can hold nonzero entries in these matrices since these positions can hold any member of $\mathcal{F}$. Since $\mathrm{S}_e \subseteq \mathcal{R}_e$, these "free" positions can only occur in rows indexed by $supp(e)$ and columns indexed by $supp(1 - e)$. Thus (i) and (ii) are seen. Next let $j \in supp(1 - e)$ and let $\Gamma(e, j) \subseteq \mathrm{supp}(e)$ consist of those j^{th} column positions admitting non-0 entries.

Claim: $\Gamma(e, j) = supp(e \wedge \overline{e_j})$ for all $j \in supp(1 - e)$.

If $j \in supp(1-e)^{o}$ then $\Gamma(e, j)$ is empty by Theorem 6.1.7(ii), while $e \wedge \overline{e_j} = 0$ by Lemma 5.1.9 and assertion (iv) is seen. Otherwise, if $j \notin supp(1-e)^{o}$ then by the previous theorem

$$\begin{aligned}\Gamma(e, j) &= \bigcap\{supp(e \wedge f) \mid f \in \mathrm{T} \text{ such that } j \in supp(f) \text{ and } e \wedge f \neq 0\}\\ &= supp(e \wedge \bigwedge\{f \in \mathrm{T} \mid j \in supp(f) \text{ and } e \wedge f \neq 0\}).\end{aligned}$$

But by Lemma 6.1.9, the latter case is equivalent to asserting that $e \wedge \overline{e_j} \neq 0$. Thus in this case,

$$\overline{e_j} = \bigwedge\{f \in \mathrm{T} \mid e \wedge f \neq 0 \text{ and } j \in supp(f)\})$$

so that $\Gamma(e, j) = supp(e \wedge \overline{e_j})$ here also. Thus $\Gamma(e, j) = supp(e \wedge \overline{e_j})$ indeed holds for all $j \in supp(1-e)$ and assertion (iii) is finally seen. □

Given the above definitions in the context of a diagonal lattice T and the matrix ring R, the ***class space*** Γ of T in R is the vector subspace consisting of all matrices A $\in$R such that for $1 \leq i \leq n$, in the i^{th} column of A the only nonzero entries occur in the supp($\overline{e_j}$) positions. The class space of T allows use to express S_e for any $e \in \mathrm{T}$ in the following succinct manner.

Theorem 6.1.11. *Given* T *and* R *as above with right extension* S *of* T, *if* Γ *is the class space of* T *in* R, *then for each* $e \in \mathrm{T}$, $\mathrm{S}_e = e + e\Gamma(1-e)$.

Examples 6.1.2 The class space for each of the skew lattices of Example 6.1.1 are, in the same order, as follows:

a) $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ b) $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$` c) $\begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$. □

Corollary 6.1.12. *If* R *is a matrix ring over a skew field,* T *is a lattice in* R *containing both* 0, 1 *and* S *is the right extension of* T, *then* $\mathcal{Z}$(S) *is the Boolean lattice of all complemented elements in* T.

Proof. By Theorem 6.1.4 we need only show that every central element in S comes from a complemented element in T. So let $\mathrm{S}_e = \{e\}$. Thus $e \wedge \overline{e_j} = 0$ for all $j \in$supp$(1-e)$ so that

$$1 - e \geq \bigvee\{\overline{e_j} \mid j \in \mathrm{supp}(1-e)\}.$$

Since $1 - e \leq \bigvee\{\overline{e_j} \mid j \in \mathrm{supp}(1-e)\}$ always holds, e is complemented in T. □

A lattice T in a ring R with unity is ***centrally closed*** if the Boolean lattice of all central idempotents lies in T. The previous result can be generalized as follows.

Corollary 6.1.13. *Given a semisimple, Artinian ring* R *and a centrally closed lattice* T *in* R*, the center* $\mathcal{Z}$(S) *of the right extension* S *of* T *in* R *is the Boolean lattice of all complemented elements in* T.

Proof. This follows from the Wedderburn structure theorem for semisimple, Artinian ring and the previous theorem. ☐

We conclude this section with a further consequence of the above results. Recall that a skew chain is any skew lattice whose $\mathcal{D}$-classes are totally ordered.

Theorem 6.1.14. *Given* R = $\mathcal{F}^{n\times n}$:

i) *Every maximal right zero semigroup in* R *is a maximal rectangular band in* R.

ii). *Every maximal right-handed skew chain in* R *is a maximal skew lattice in* R.

Proof. Consider a maximal right zero semigroup S given by all matrices of block form $\begin{bmatrix} I^{j\times j} & X \\ 0 & 0 \end{bmatrix}$. If S is not a maximal rectangular semigroup in R, then some $\beta = \begin{bmatrix} I^{j\times j} & 0 \\ B & 0 \end{bmatrix}$ exists in **E**(R) \ S such that β together with S generates a properly larger rectangular band. But $\begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 \\ B & 0 \end{bmatrix} = \begin{bmatrix} I+XB & 0 \\ 0 & 0 \end{bmatrix}$ is idempotent only if XB = 0 for all possible X. But this is not possible since β ∈ S and thus B ≠ 0. Hence (i) must follow.

Suppose next that S is a maximal chain of right-rectangular skew lattices in R. Clearly S is a quadratic skew lattice. Suppose that S lies in a larger skew lattice S′. The inclusion S ⊆ S′ cannot increase any $\mathcal{D}$-class already in S. Hence a new $\mathcal{D}$-class B exists in S′ that is incomparable to some $\mathcal{D}$-class A in S. If T is a lattice section of S′, then Theorem 6.1.7 guarantees that the right extension of T will not include the full R-set A. (Put succinctly, increasing the lattice shape of S decreases some of its $\mathcal{D}$-class sizes.). Hence it is impossible to join new $\mathcal{D}$-class to S. ☐

Query: *Given a lattice* T *in a matrix ring* R *of maximal height, is its right extension* S *a maximal* (*right-handed*) *skew lattice in* R? *Conversely, are all maximal (right handed) skew lattices in* R *of maximal height* = *n*? Settling these questions even for matrix rings would be of interest.

6.2 ∇-bands and cubic skew lattices

Given a ring R and e, f, in **E**(R) recall that $e\nabla f = (e \circ f)^2 = (e + f - ef)^2$. The latter expands to $e + f + fe - ef - efe - fef + efef$ in general, but reduces to $e + f + fe - efe - fef$ when ef is idempotent. Recall also that ∇ extends $\circ$ in that:

i) Every skew lattice (S; $\circ$, $\bullet$) in a ring is also a skew lattice under ∇ and $\bullet$ since in this case $(e \circ f)^2 = e \circ f$ so that $e\nabla f$ reduces to $e \circ f$.

ii) Whenever $e, f, ef, fe \in$ **E**(R), then so are efe, fef and $e\nabla f$ by Theorem 2.7.5.

iii) Situations occur where $e\nabla f$ is idempotent, but not $e \circ f$; but ∇ need not be associative, even if idempotent closure occurs. (See Examples 2.3.1 and 2.3.2.)

iv). Due to (*i*), maximal right [left] regular bands in any ring form skew lattices under ∇ and $\bullet$. Every right [left] regular band in a ring generates a skew lattice under ∇ and $\bullet$.

v). Every maximal normal band in a ring R is a normal skew lattice under ∇ and $\bullet$; indeed it forms a skew Boolean algebra. Thus a normal band in R generates a strongly distributive skew lattice under ∇ and $\bullet$. (Theorem 2.3.6)

vi) Maximal regular bands in rings, however, need not be closed under ∇, much less be skew lattices under ∇. (See Example 2.3.5.)

Recall that every band is naturally partially ordered by $e \le f$ if $ef = e = fe$ which refines the natural preorder given by $e \preceq f$ if $efe = e$. The equivalence induced from $\preceq$ is the Green's relation $\mathcal{D}$. Turning to $\triangledown$-bands proper, a band congruence θ on a $\triangledown$-band is a ***$\triangledown$-congruence*** if θ is also a congruence under $\triangledown$. The assertions in the following lemma, coming from a 2004 paper of Cvetko-Vah, are easily verified.

Lemma 6.2.1. *Given a $\triangledown$-band* S *in a ring, for all $a, b \in$* S:

i) *Given $b \preceq a$ in* S, *$a\nabla b = a + ba - aba$ and $b\nabla a = a + ab - aba$.*

ii) *$a\nabla b\nabla a$ is unambiguous: $(a\nabla b)\nabla a = a\nabla(b\nabla a) = a + b - bab$.* ☐

Even when $\triangledown$ is not associative, $\triangledown$-bands are very much like skew lattices in rings.

Theorem 6.2.2. *For any $\triangledown$- band* S *in a ring the following hold.*

i) *As a band,* S *is regular. That is, $abaca = abca$ holds on* S.

ii) *$\mathcal{D}$ is a $\triangledown$-congruence and* S/$\mathcal{D}$ *is a lattice with $\mathcal{D}_x \vee \mathcal{D}_y = \mathcal{D}_{x\triangledown y}$.*

iii) *$a(a\triangledown b) = a = (b\triangledown a)a$ and $a\triangledown(ab) = a = (ba)\triangledown a$.*

iv) *$ab = ba$* iff *$a\triangledown b = b\triangledown a$.*

v) *$a(b\triangledown c)a = aba \triangledown aca$ and $a\triangledown(bc)\triangledown a = (a\triangledown b\triangledown a)(a\triangledown c\triangledown a)$.*

vi) *$a\nabla c\nabla a = b\nabla c\nabla b$ and $aca = bcb$ implies $a = b$.*

(S; $\triangledown$, $\bullet$) *is thus a distributive, symmetric, cancellative skew lattice when $\triangledown$ is also associative.*

Proof. (i) Suppose A and B are $\mathcal{D}$-classes in S with A > B. Let $a \in$ A and $b \in$ B so that $a \succeq b$. Consider $b \triangledown a \triangledown b = a + b - aba$. From

$$(a + b - aba)a(a + b - aba) = a + b - aba \quad \text{and} \quad a(a + b - aba)a = a,$$

$b \triangledown a \triangledown b \in$A follows. On the other hand, clearly $b \triangledown a \triangledown b \geq b$. Thus given $\mathcal{D}$-classes A > B in S, for all $b \in$ B there exists $a \in$ A such that $a > b$. Hence S is regular by Theorem 1.2.18.

(ii) Let $a \succeq b$ and $a \succeq c$. Since S is regular, $bab = b$, $bac = bc$, $cab = cb$ and $cac = c$. Hence $(b \triangledown c)a(b \triangledown c) = b \triangledown c$ so that $a \succeq b \triangledown c$. Thus the $\mathcal{D}$-class of $b \triangledown c$ is the join-class of $\mathcal{D}_b$ and $\mathcal{D}_c$ and $\mathcal{D}$ is a $\triangledown$-congruence.

(iii) follows from routine calculations such as $a \triangledown (ab) = a + ab + aba - aaba - abaab$ wich immediately reduces to $a + ab + aba - aba - ab = a$.

(iv) Just observe that $a \triangledown b$ and $b \triangledown a$ differ only by their third terms, ba and ab.

(v) For the first part, regularity gives

$$a(b \triangledown c)a = a(b + c + bc - bcb - cbc)a$$

$$= aba + aca + abaca - abacaba - acabaca = aba \triangledown aca.$$

For the other identity, first observe that $[b(c \triangledown a)]^2 = (bc + ba + bac - bcac - baca)^2$ expands as

$$(bc + bcba - bca) + (babc + ba - baca) + (babc + bacba - baca)$$

$$- (bcabc + bcaba - bca) - (babc + bacba - baca)$$

$$= \qquad bc + ba - baca + babc - bcabc$$

which in turn must equal $b(c \triangledown a) = bc + ba + bac - bcac - baca$. Equating and canceling common terms gives $babc - bcabc = bac - bcac$, that is, the identity

$$babc + bcac = bac + bcabc. \tag{6.2.2}$$

Thus

$$(a \triangledown b \triangledown a)(a \triangledown c \triangledown a) = (a + b - bab)(a + c - cac) = a + bc - bcac - babc + bac$$

$$= a + bc - bac - bcabc + bac = a + bc - bcabc = a \triangledown (bc) \triangledown a.$$

(vi) Given $aca = bcb$, the regularity of $\bullet$ implies $cac = c(aca)c = c(bcb)c = cbc$. Cancelling in $a + c - cac = b + c - cbc$ ($a \nabla c \nabla a = b \nabla c \nabla b$) in turn gives $a = b$. ☐

We turn to other properties observed in skew lattices. The next result demonstrates the important role played by instances of commutativity, especially on the algebraic reducts (S, •) and (S, $\triangledown$) of any $\triangledown$-band S.

Theorem 6.2.3. *Join classes and meet classes are given by commuting joins and meets. Thus given $\mathcal{D}$-classes* A *and* B *in a* $\triangledown$*-band* S*, their join* $\mathcal{D}$*-class* J *and meet* $\mathcal{D}$*-class* M *are*

$$\mathrm{J} = \{a\triangledown b \mid a \in \mathrm{A}, b \in \mathrm{B} \ \&\ a\triangledown b = b\triangledown a\} \ \ \textit{and} \ \ \mathrm{M} = \{ab \mid a \in \mathrm{A}, b \in \mathrm{B} \ \&\ ab = ba\}.$$

Moreover, for every $a \in \mathrm{A}$ *there exists* $b \in \mathrm{B}$ *such that* $a\triangledown b = b\triangledown a$ *in* J *and* $ab = ba$ *in* M.

Proof. Given $v \in \mathrm{J}$, there exist $a \in \mathrm{A}$ and $b \in \mathrm{B}$ such that $v \geq a, b$. (Set $a = va'v$ and $b = vb'v$ for any $a' \in \mathrm{A}$ and $b' \in \mathrm{B}$.) For such a and b we have $a\triangledown b \in \mathrm{J}$ and

$$v \ = \ v(a\triangledown b)v \ = \ v(a + b + ba - aba - bab)v \ = \ a + b + ba - aba - bab \ = \ a\triangledown b.$$

Similarly, $b\triangledown a$ equals v also and the assertion about J is seen. The case for M is similar. For the final assertion, pick a in A and let $v \in \mathrm{J}$ be such that $v \geq a$. That $b \in \mathrm{B}$ exists such that $a\triangledown b = v = b\triangledown a$ is now clear. The rest follows from symmetry. □

Corollary 6.2.4. *Given a* $\triangledown$*-band* S and $e \in \mathrm{S}$, *the following are equivalent*:

i) $\mathcal{D}_e = \{e\}$.

ii) *For all* $x \in \mathrm{S}$, $e\triangledown x \ = \ x\triangledown e$ *and* $ex \ = \ xe$.

Proof. Clearly (*ii*) implies (*i*); and (*ii*) implies (*i*) due to the final assertion of Theorem 6.2.3. □

Corollary 6.2.5. *A set of commuting elements in a* $\triangledown$*-band* S *generates a sublattice*. □

We next turn to the question when $\triangledown$ is associative, giving various criteria. When it is associative on a particular $\triangledown$-band, the latter is called a ***cubic skew lattice*** in the ring. Clearly quadratic skew lattices in a ring (studied in the previous section) are trivially cubic.

The associativity of $\triangledown$*: the role of the commutator* $[x, y]$

Recall that the *commutator* of elements x, y in a ring R is $[x, y] = xy - yx$. Clearly x and y commute if and only if $[x, y] = 0$. For any pair of idempotents e and f in a band S in a ring R

$$e\triangledown f - e\circ f \ = \ ef + fe - efe - fef \ = \ [e, f]^2.$$

Thus $[e, f]^2 = 0$ on a $\triangledown$-band S if and only if $\triangledown = \circ$ as binary operations on S. In this section we show that a $\triangledown$-band S is associative if and only if for all $e, f \in \mathrm{S}$, $[e, f]^2$ lies in the center of the subring of R generated from S. That is, $\triangledown$ is associative if and only if $g[e, f]^2 = [e, f]^2g$ for all $e, f, g \in \mathrm{S}$. We begin with a pair of somewhat technical lemmas.

Lemma 6.2.6. $(a\triangledown b)c(a\triangledown b) = -(abca - aca - bca - bcb + bcab)$.

Proof. Multiplying out $(a + b + ba - aba - bab)c(a + b + ba - aba - bab)$ and cancelling yields

$(aca + \underline{acb} - \underline{acab}) + (bca + bcb - bcab) - (abca + \underline{abcb} - \underline{abcab}) = aca + bca + bcb - bcab - abca$

where the underlined terms vanish collectively by identity (6.2.2). □

Lemma 6.2.7. *Given a, b, c in a* $\triangledown$*-band* S, $a\triangledown(b\triangledown c) = (a\triangledown b)\triangledown c$ *if and only if*

$$[b, c]^2a - a[b, c]^2a = c[a, b]^2 - c[a, b]^2c. \tag{6.2.7}$$

Proof. $a\triangledown(b\triangledown c) = a + (b + c + cb - bcb - cbc) + (ba + ca + cba - bcba - cbca) - (aba + aca + acba - abcba - acbca) - (b\triangledown c)a(b\triangledown c)$

while $(a\triangledown b)\triangledown c = (a + b + ba - aba - bab) + c + (ca + cb + cba - caba - cbab) - (cac + cbc + cbac - cabac - cbabc) - (a\triangledown b)c(a\triangledown b)$.

Equating $a\triangledown(b\triangledown c)$ with $(a\triangledown b)\triangledown c$ and then canceling common terms yields

$$\begin{aligned} &- bcb - bcba - cbca - aca - acba + abcba + acbca - (b\triangledown c)a(b\triangledown c) \\ &\quad = - bab - caba - cbab - cac - cbac + cabac + cbabc - (a\triangledown b)c(a\triangledown b). \end{aligned}$$

Applying the previous lemma gives

$$\begin{aligned} &- bcb - bcba - cbca - aca - acba + abcba + acbca + (bcab - bab - cab - cac + cabc) \\ &\quad = - bab - caba - cbab - cac - cbc - cbac + cabac + cbabc + (abca - aca - bca - bcb + bcab) \end{aligned}$$

which reduces to

$$\begin{aligned} &- bcba - cbca - acba + abcba + acbca - cab + cabc \\ &\quad = - caba - cbab - cbac + cabac + cbabc + abca - bca. \end{aligned}$$

Adding $cab + bca$ to both sides, grouping the aXa terms on the left and the cYc terms on the right and then factoring gives,

$$bca - bcba - cbca - a(bc - cb)^2a = cab - caba - cbab - c(ab - ba)^2c,$$

Adding cba to both sides, then grouping and factoring once again gives

$$(bc - cb)^2a - a(bc - cb)^2a = c(ab - ba)^2 - c(ab - ba)^2c$$

which is the statement of the lemma. □

Theorem 6.2.8. *A* $\triangledown$*-band* S *is associative if and only if for all a, b, c* $\in$ S,

$$a[b, c]^2 = [b, c]^2a.$$

Proof. This identity implies that of Lemma 6.2.7, making $\triangledown$ associative. On the other hand, replacing a by aba and b by bab in the latter gives

$$[babc - cbab]^2 aba - aba[bab - cbab]^2 aba \;=\; c[ab - ba]^2 - c[ab - ba]^2 c.$$

Regularity first gives

$$aba[babc - cbab]^2 aba = aba(c - c)aba = 0$$

and then

$$[babc - cbab]^2 aba = (babc - cbab)(c - c)aba = 0.$$

Thus $c(ab - ba)^2 c = c(ab - ba)^2$ and by (6.2.7), $a(bc - cb)^2 a = (bc - cb)^2 a$. Permuting variables in $c(ab - ba)^2 c = c(ab - ba)^2$ gives $a(bc - cb)^2 a = a(bc - cb)^2$ from which $a[b, c]^2 = [b, c]^2 a$ follows. ☐

The associativity of $\triangledown$ thus reduces to cases of possible commutation: does $[x, y]^2$ always produce elements lying in the center of the subring $S^{\pm}$ generated from S?

The associativity of $\triangledown$: the Green's relations $\mathcal{L}$ and $\mathcal{R}$

We next consider connections between the associativity of $\triangledown$ and the equivalences $\mathcal{L}$ and $\mathcal{R}$ that refine $\mathcal{D}$. Recall that $\mathcal{L}$ is a ***right congruence*** on S in that $a\mathcal{L}b$ implies $ac\mathcal{L}bc$ for all $c \in$S while $\mathcal{R}$ is a ***left congruence*** on S. If S is regular, then both $\mathcal{L}$ and $\mathcal{R}$ are full congruences. In particular, $\mathcal{L}$ and $\mathcal{R}$ are multiplicative congruences on all $\triangledown$-bands. We turn to the status of $\mathcal{L}$ and $\mathcal{R}$ as $\triangledown$-congruences on a $\triangledown$-band. But first let $a\mathcal{R}_{\triangledown}b$ denote the conjunction, $a\triangledown b = b$ and $b\triangledown a = a$, and similarly let $a\mathcal{L}_{\triangledown}b$ denote $a\triangledown b = a$ and $b\triangledown a = b$.

Lemma 6.2.9. *In a* $\triangledown$-band, *$a\mathcal{L}b$ if and only if $a\mathcal{R}_{\triangledown}b$ and similarly $a\mathcal{R}b$ if and only if $a\mathcal{L}_{\triangledown}b$. In general, $a\mathcal{L}b$ implies $(c\triangledown a)(c\triangledown b) = c\triangledown a$ for all c and $a\mathcal{R}b$ implies*

$$(a\triangledown c)(b\triangledown c) = b\triangledown c \quad \textit{for all } c.$$

Proof. Expanding, $a\triangledown b = b$ and $b\triangledown a = a$ reduce to $a = aba + bab - ba$ and $b = aba + bab - ab$. Multiplying on the left by a and b respectively, yields $a = ab$ and $b = ba$, that is $a\mathcal{L}b$. Conversely, if $a\mathcal{L}b$ under the ring multiplication, then $a\triangledown b$ reduces to b and $b\triangledown a$ reduces to a. In general, for all $c \in$S, $(c\triangledown a)(c\triangledown b) = (c + a + ac - aca - cac)(c + b + bc - bcb - cbc)$ which with the assistance of $\mathcal{L}$ as a congruence on the multiplicative band expands as

$$c + (ac + a - acb) + (ac) - (ac + aca - acb) - (cac) \;=\; c + a + ac - aca - cac \;=\; c\triangledown a$$

Similarly, $(c\triangledown b)(c\triangledown a) = (c\triangledown b)$ so that $c\triangledown a \;\mathcal{L}\; c\triangledown b$. The case for $\mathcal{R}$ is similar. ☐

In general $\mathcal{L}$ and $\mathcal{R}$ need not be $\triangledown$-congruences. However:

Lemma 6.2.10. *$\mathcal{L}$ and $\mathcal{R}$ are both $\triangledown$-congruences on a $\triangledown$-band* S *if and only if for all a, b, c* $\in$S,

$$i)\quad a(bc - cbc)a = a(bc - cbc).$$
$$ii)\quad a(bc - bcb)a = (bc - bcb)a.$$

Proof. Given $u\mathcal{L}v$, $(u\triangledown x)(v\triangledown x) = (u + x + xu - uxu - xux)(v + x + xv - vxv - xvx)$. Distributing each term of the left factor over the right factor, then adding and simplifying gives

$$(u) + (xv + x - xvx) + (xu) - (uxu) - (xu) = u + x + xv - uxu - xvx.$$

(Here we use various consequences of regularity. E.g.,

$$u(v + x + xv - vxv - xvx) = u + ux + uxv - uxv - uxvx = u + ux - ux = u.)$$

Thus, $(u\triangledown x)(v\triangledown x) = u\triangledown x$ holds for $u\mathcal{L}v$ if and only if $xv - xvx = xu - xux$. Replacing u and v by generic values, uvu and vu, we see that $\mathcal{L}$ is a $\triangledown$-congruence if $xuvu - xuvux = xvu - xvux$ holds for all u, x, v in S. Rearranging the terms gives $xvux - xuvux = xvu - xuvu$ which is equivalent to (i) upon switching variables. In similar fashion, $\mathcal{R}$ being a $\triangledown$-congruence is equivalent to (ii). □

Theorem 6.2.11. *Given a $\triangledown$-band* S *in a ring, both $\mathcal{L}$ and $\mathcal{R}$ are $\triangledown$-conguences if and only if $\triangledown$ is associative and thus* S *is a skew lattice.*

Proof. If $\triangledown$ is associative so that S is a skew lattice, then both $\mathcal{L}$ and $\mathcal{R}$ must be full skew lattice congruences. On the other hand, given that $\mathcal{L}$ and $\mathcal{R}$ are $\triangledown$-congruences, the identities of the above lemma imply that

$$\begin{aligned} a[b, c]^2 &= a(bc - cbc + cb - bcb) = a(bc - cbc + cb - bcb)a \\ &= a(bc - bcb + cb - cbc)a = (bc - bcb + cb - cbc)a = [b, c]^2 a. \end{aligned}$$

Thus, the criterion of Theorem 3.2.8 is satisfied and hence $\triangledown$ is associative. □

The associativity of $\triangledown$: the role of Kimura factorizations

A closely related criterion involves a canonical factorization that can occur in $\triangledown$-bands. We begin with bands. Every subsemilattice T of any band S meets each $\mathcal{D}$-class in S in at most once element. A ***semilattice section*** in a band S is a subsemilattice T of S that meets each $\mathcal{D}$-class exactly once, thus making T is an internal copy of S/$\mathcal{D}$, the maximal semilattice image of S. If S is also regular, then $T^{\mathcal{R}} = \bigcup_{t\in T}\mathcal{R}_t$ is a maximal right regular band in S and $T^{\mathcal{L}} = \bigcup_{t\in T}\mathcal{L}_t$ is a maximal left regular band in S with $T^{\mathcal{R}}$ and $T^{\mathcal{L}}$ being copies of the maximal right and left regular images S/$\mathcal{L}$ and S/$\mathcal{R}$ of S. Each $e \in$S factors as $e_L e_R$ where $e_L = et_e \in T^{\mathcal{L}}$ and $e_R = t_e e \in T^{\mathcal{R}}$ where t_e is the unique element in $T \cap \mathcal{D}_e$. Put otherwise, e_L and e_R are the unique elements in $\mathcal{D}_e$ related to e and t_e in the following $\mathcal{D}$-class picture.

$$\begin{array}{ccccc} & e & \mathcal{R} & e_L & = et_e \\ & \mathcal{L} & & \mathcal{L} & \\ t_e e = & e_R & \mathcal{R} & t_e & \end{array}$$

Due to S being regular

$$ef = (e_L e_R)(f_L f_R) = (e_L f_L)(e_R f_R)$$

holds for all $e, f \in$ S. Indeed,

$$ef = efefef = (e_L e_R)(f_L f_R)ef(e_L e_R)(f_L f_R) = e_L f_L efe_R f_R = (e_L f_L)(e_R f_R)$$

with both latter reductions due to regularly. Multiplication on S thus decomposes internally into separate products on $T^{\mathcal{L}}$ and $T^{\mathcal{R}}$ so that S is isomorphic to a sub-band of $T^{\mathcal{L}} \times T^{\mathcal{R}}$. We call the factorization $e = e_L e_R$ the *internal Kimura factorization* of *e relative to* T.

Even more is true if S is a ∇-band in a ring. In this case T is a ***lattice section*** in that for all e, f in T, $e \nabla f = e \circ f$ is in T also making (T; $\circ$, $\bullet$) a lattice that meets each $\mathcal{D}$-class exactly once, so that T $\cong$ S/$\mathcal{D}$. (If $u \in$ T is in the join $\mathcal{D}$-class of e and f, then $u \geq$ both e, f and so $u \geq e \circ f$ follows. But Since u and $e \circ f$ lie in a common $\mathcal{D}$-class, $u = e \circ f$.) Given a ∇-band S with a lattice section T, a modified join operation $\vee_T$ can be defined such that (S, $\vee_T$, $\bullet$) is a distributive, symmetric skew lattice. To begin, $T^{\mathcal{L}}$ and $T^{\mathcal{R}}$ are in fact skew lattices. Indeed, suppose that $a, b \in T^{\mathcal{L}}$. Certainly $a \nabla b \in$ S, but since a and b lie in the left regular band $T^{\mathcal{L}}$, we also have $a \nabla b = a \circ b$. Let t_a, $t_b \in$ T be such that $a \;\mathcal{L}\; t_a$ and $b \;\mathcal{L}\; t_b$. By left regularity, we have $abt_a = ab$, $at_b t_a = at_b$, $bat_b = ba$ and $bt_a t_b = bt_a$. Hence

$$(a \circ b)(t_a \circ t_b) = (a + b - ab)(t_a + t_b - t_a t_b) = a + b - ab = a \circ b$$

Similarly, $(t_a \circ t_b)(a \circ b) = t_a \circ t_b$. Thus $a \circ b \in T^{\mathcal{L}}$, and $T^{\mathcal{L}}$ is a skew lattice in S as claimed. Likewise, $T^{\mathcal{R}}$ is a skew lattice in S. The internal Kimura decomposition of the band S with respect to T enables us to define an operation $\vee_T$ on S by setting

$$e \vee_T f = (e_L \circ f_L)(e_R \circ f_R).$$

Clearly $(e \vee_T f)_L = e_L \circ f_L$ and $(e \vee_T f)_R = e_R \circ f_R$. It follows that (S, $\vee_T$, $\bullet$) is an "internal" fibered product of the skew lattices ($T^{\mathcal{L}}$, $\circ$, $\bullet$) and ($T^{\mathcal{R}}$, $\circ$, $\bullet$) over their common sublattice T and thus is a skew lattice. We will call $\vee_T$ the ***associative join*** on S induced from T. Cleary $\vee_T$ is dependent on T and is somewhat more complex in design than $\triangledown$. Indeed we will soon see that distinct lattice sections of S can produce different associative joins. But first we have:

Theorem 6.2.12. *Given a* $\triangledown$*-band* S *with a lattice section* T, *the binary operation* $\triangledown$ *is associative if and only if* $e \triangledown f = e \vee_T f$ *for all* e, f *in* S, *where* $\vee_T$ *is the* T*-induced associative join on* S. *Thus if* $\triangledown$ *is associative, all lattice sections* T *of* S *induce a common associative join, namely* $\triangledown$.

Proof. If $\triangledown$ equals $\vee_T$ for some lattice section T, then clearly $\triangledown$ is associative. Conversely, let $\triangledown$ be associative and thus S a skew lattice. The regularity of $\triangledown$ plus the fact the $x \triangledown y = yx$ in any $\mathcal{D}$-class of S gives,

$$\begin{aligned} e \triangledown f &= (e \triangledown t_e \triangledown f \triangledown t_f) \triangledown (t_e \triangledown e \triangledown t_f \triangledown f) = (t_e \triangledown e \triangledown t_f \triangledown f)(e \triangledown t_e \triangledown f \triangledown t_f) \\ &= (et_e \triangledown f t_f)(t_e e \triangledown t_f f) = (e_L \circ f_L)(e_R \circ f_R) = e \vee_T f. \end{aligned}$$

The next-to-last equality is because $\triangledown$ reduces to $\circ$ on any left or right regular band in S. The final assertion of the theorem is clear. □

Consider the following pair of lattice sections for lattice sections for Example 2.3.2.

$$\text{Let T} = \left\{ \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} > \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \right\} \text{ and T}' = \left\{ \begin{bmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} > \begin{bmatrix} 0&0&0&0\\0&1&1&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \right\}.$$

If A and B as chosen as in that example, then

$$\text{A} \vee_T \text{B} = \begin{bmatrix} 0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix} \text{ while A} \vee_{T'} \text{B} = \begin{bmatrix} 0&1&1&0\\0&1&0&0\\0&0&1&0\\0&0&0&0 \end{bmatrix}.$$

Hence the choice of T affects the outcome of the induced join when $\triangledown$ is not associative.

The preceding theorem and example raise the question of whether having a unique associative join $\vee$ implies $\triangledown$ is associative. A related question is that of when lattice sections must exist and in what abundance to make this question meaningful. Both issues are treated in the following theorem.

Theorem 6.2.13. *If* S *is a* $\triangledown$*-band such that* S/$\mathcal{D}$ *is at most countable*, *then* S *has a lattice section* T. *Such a lattice section can always be found to include any given finite subset* T_0 *of pairwise commuting elements in* S. *Finally*, *if all lattice sections* T *of* S *induce a common associative join on* S *then* $\triangledown$ *is this join and as such is associative.*

Proof. Let $D_1, D_2, \ldots$ be a complete listing of the $\mathcal{D}$-classes of S. Pick $a_1 \in D_1$ and consider the set $X = \{x \in S \mid a_1x = xa_1\}$. X is a band containing a_1 and meeting each $\mathcal{D}$-class of S. Indeed, given any other $\mathcal{D}$-class D′ with $y \in D'$ we have $a_1 \geq a_1ya_1 \leq a_1ya_1 \triangledown y \triangledown a_1ya_1 = b$. It follows that $a_1b = a_1ya_1 = ba_1$ so that $b \in X$. Moreover, X is a $\triangledown$-band. Indeed, given $x, y \in X$, then a_1 and $x\triangledown y = x + y + yx - xyx - yxy$ clearly commute, so that $x\triangledown y \in X$ also. Note that $D_1 \cap X = \{a_1\}$. Next pick $a_2 \in D_2 \cap X$, set $Y = \{y \in X \mid a_2y = ya_2\}$. Again Y is a $\triangledown$-band that meets each $\mathcal{D}$-class of S and moreover $D_i \cap X = \{a_i\}$ for $i = 1, 2$. The process continues through the countable set of $\mathcal{D}$-classes to produce the lattice section $T = \{a_1, a_2, a_3, \ldots\}$. By placing the $\mathcal{D}$-classes of the elements in T_0 at the front of the list of $\mathcal{D}$-classes, the second assertion follows. Finally, suppose that all lattice sections T induce a common associative join $\vee$ on S. Let $e, f \in S$ be given. Then e and f generate a skew lattice S_0 in S with at most 16 elements. (See Theorem 2.7.5 and the surrounding discussion.) Take a lattice section T_0 of S_0 and extend it to a full lattice section T of S, using the first part of this theorem. Then $e \vee_T f = e \vee_{T_0} f$ and the latter equals $e\triangledown f$ in S_0 by Theorem 6.2.12. Thus $\triangledown$ is the common associative join induced by all lattice sections. ☐

Thus while "really big" $\triangledown$-bands need not have lattice sections, thanks to Theorem 6.2.13 all $\triangledown$-bands in finite dimensional matrix rings over fields have them so that all of the theorem applies.

Since $\triangledown$ is associative if and only if it is thus on all finitely generated subalgebras of a given $\triangledown$-band, an alternative to Theorem 6.2.12 is given in the following corollary to the above theorems and its own following corollary in turn.

Corollary 6.2.14. *Given a $\triangledown$-band* S, $\triangledown$ *is associative if and only if for all $e\mathcal{D}u$ and $f\mathcal{D}v$ situations in* S *where $uv = vu$, $e\triangledown f$ is calculated as* $(eu \circ fv)(ue \circ vf)$. ☐

Corollary 6.2.15. $\triangledown$ *is associative on a $\triangledown$-band* S *if and only if it is associative on all sub-algebras generated from at most* 2 *$\mathcal{D}$-classes*. ☐

The associativity of $\triangledown$: the role of primitive algebras

As with skew lattices, a $\triangledown$-band S is ***primitive*** if it has exactly two $\mathcal{D}$-classes, A > B. In general, a band S is ***totally pre-ordered*** if either $e \succeq f$ or $f \succeq e$ for all $e, f \in S$. (Again, $e \succeq f$ if $fef = f$, or equivalently for $\triangledown$-bands, $e\triangledown f\triangledown e = e$.) These bands are of interest because maximal, totally pre-ordered regular bands in a ring form $\triangledown$-bands. (See Corollary 6.3.5 below.) For such $\triangledown$-bands, the previous results imply that to check for the associativity of $\triangledown$ one need only check for associativity on its primitive subalgebras. It is thus natural to ask if the "totally pre-ordered" condition can be removed from this observation? Put otherwise, *if $\triangledown$ is associative on all primitive subalgebras of a $\triangledown$-band, must it be associative on that band*? Or does a $\triangledown$-band S exist with exactly four $\mathcal{D}$-classes, two of which are mutually incomparable, such that $\triangledown$ is associative on all five maximal primitive subalgebras but not on all of S? This leads us to:

Theorem 6.2.16 *If $\triangledown$ is associative on all primitive subalgebras of a $\triangledown$-band* S, *then $\triangledown$ is associative on* S, *and conversely*.

Proof. Granted the assumption, we need only show that $\triangledown$ is associative when S is generated from two incomparable $\mathcal{D}$-classes, say A and B. Let their meet class be M and join class be J. Let a lattice section of S be given by T = $\{a_0, b_0, m_0, j_0\}$. Take $a \in$ A, $b \in$ B. We show that $a \triangledown b = a \vee_T b$. To begin

$$\begin{aligned} a \vee_T b &= (aa_0 + bb_0 - aa_0bb_0)(a_0a + b_0b - a_0ab_0b) \\ &= a + \underline{am_0b} - \underline{ab_0b} + bm_0a + b - bm_0ab - abm_0a - \underline{aa_0b} + \underline{ab} \\ &= a + bm_0a + b - bm_0ab - abm_0a \end{aligned}$$

since $aa_0b + ab_0b - am_0b = aj_0b = ab$ by regularity. Of course, $a \triangledown b = a + b + ba - aba - bab$. Denoting the difference $(a \vee_T b) - (a \triangledown b)$ by $\Delta_{(a,\, b)}$, we have

$$\Delta_{(a,\, b)} = bm_0a - bm_0ab - abm_0a - ba + aba + bab.$$

The associativity of $\triangledown$ on the primitive $\triangledown$-band M∪B implies that

$$\begin{aligned} \Delta_{(abm_0a,\, b)} &= bm_0a - bm_0ab - abm_0a - bam_0a + abm_0a + bab \\ &= bm_0a - bm_0ab - bam_0a + bab = 0. \end{aligned}$$

Subtracting from $\Delta_{(a,\, b)}$ yields the refinement $\Delta_{(a,\, b)} = bam_0a + aba - abm_0a - ba$. Using the latter to calculate $\Delta_{(a,\, bam_0b)}$ in the associative context of M∪A gives:

$$0 = \Delta_{(a,\, bam_0b)} = bam_0a + aba - abm_0a - ba = \Delta_{(a,\, b)}.$$

The converse is trivial. □

Since $\triangledown$ is associative on a $\triangledown$-band only if it is thus on all primitive subalgebras, two of our earlier associativity criteria can be fine-tuned for the primitive case as follows.

Corollary 6.2.17. *Given a primitive $\triangledown$-band* P *with $\mathcal{D}$-classes* A > B, *the following are equivalent*:

i) *$\triangledown$ is associative on* S.

ii) *For all $a \in$ A and $b, c \in$ B, $a[b, c]^2 = [b, c]^2a$.*

iii) *For all $a \in$ A and $b, c \in$ B, both*

$$a(bc - cbc)a = a(bc - cbc) \quad \text{and} \quad a(bc - bcb)a = (bc - bcb)a.$$

Proof. That (i) implies (ii) and (iii) follows from Theorem 6.2.8, Lemma 6.2.10 and Theorem 6.2.11. The identities of (ii) and (iii), in their unconditional form, conversely imply (i). Given a, b, $c \in$ P, the only case where the identities need not hold, is the case where $a \in$ A and b, $c \in$ B. In all alternative cases these identities hold. We check the case where $b \in$ A and $c \in$ B. Here we have $b \succeq a, c$ so that $xby = xy$ whenever x and y are either a or c. Hence

$$a[b, c]^2 = a[bc + cb - bcb - cbc] = ac + acb - acb - ac = 0,$$
$$a(bc - cbc) = abc - acbc = ac - ac = 0 = a(bc - cbc)a,$$

and similarly $[b, c]^2 a = 0$ and $(bc - bcb)a = 0 = a(bc - bcb)a$. The cases b, $c \in$ A, or $b \in$ B but $c \in$ A are similarly verified, as is the case where a, b, $c \in$ B. ☐

Given the matrices A $\succ$ B, C from Example 2.3.2, we have:

$$\mathrm{A[B, C]}^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{[B, C]}^2\mathrm{A}, \text{ and}$$

$$\mathrm{A(BC - BCB)A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{(BC - BCB)A}.$$

The case of normal ∇-bands

That ∇ is associative for any normal ∇-band has been seen already. The various criteria in this section provide essentially new proofs:

Corollary 6.2.18. *A normal ∇-band* S *is a skew lattice*. (That is, ∇ must be associative.)

Proof 1. The identity $xyzw = xzyw$ implies $a[b, c]^2 = 0 = [b, c]^2 a$. Hence ∇ is associative by Theorem 6.3.8.

Proof 2. Again, $xyzw = xzyw$ implies that the criteria of Lemma 6.3.10 is satisfied:

$$a(bc - cbc)a = 0 = a(bc - cbc) \text{ and } a(bc - bcb)a = 0 = (bc - bcb)a.$$

Proof 3. Finally, $xyzw = xzyw$ implies $xyz = xyxz = xzyz$. Thus given any $e\mathcal{D}u$ and $f\mathcal{D}v$ in S with $uv = vu$, the product $(eu\circ fv)(ue\circ vf) = (eu + fv - eufv)(ue + vf - uevf)$ must reduce to $(e\circ f)^2 = e\triangledown f$, so that $\triangledown$ is associative by the criterion of Corollary 6.2.14. ☐

6.3 The question of $\triangledown$-closure

While every $\triangledown$-band in a ring is regular as a band, not every regular band in a ring need generate a $\triangledown$-band in that ring. Thus far no simple necessary and sufficient criteria for a regular band in a ring to generate a $\triangledown$-band are known, though cases exist where successful generation is guaranteed. (By contrast, *every regular band* S *is isomorphic to a regular band* S′ *in* <u>*some*</u> *ring such that* S′ *generates a cubic skew lattice in that ring*.) When a $\triangledown$-band is generated from a band S, it is called the ***$\triangledown$-closure*** of S. Our next theorem describes what must occur at any stage in a successful generation of a $\triangledown$-band from a regular band in a ring.

Theorem 6.3.1. *Given a regular band* S *in a ring* R *and elements* $e, f \in$ S, S $\cup \{e\triangledown f\}$ *generates a (possibly larger) regular band* S′ *in* R *if and only if for all a, b, c* $\in$ S,

I) $\quad ea(e \circ f)bf = eabf.$

II) $\quad a(e - ef)abc(f - ef)c = a(e - ef)b(f - ef)c.$

Proof. Observe first that since S is regular, $ea(ef + fe - efe - fef)bf = 0$ making $ea(-ef)bf = ea(fe - efe - fef)bf$, so that $ea(e\triangledown f)bf = ea(e\circ f)bf$. Thus if the semigroup S′ generated from S and $e\triangledown f$ is a regular band, then since ea, $bf \preceq e\triangledown f$ we have $ea(e\triangledown f)bf = eabf$ so that (I) must follow.

In general, elements of the semigroup S′ generated from S and $e\triangledown f$ look like

$$a_0(e\triangledown f)a_1(e\triangledown f)a_2(e\triangledown f)\ldots a_{n-1}(e\triangledown f)a_n$$

with $a_0, a_1, \ldots, a_n \in \mathrm{S}^1$ for $n \geq 1$. (The $n = 0$ case is just a_0) For S′ to be a regular band, both

$$(e\triangledown f)a(e\triangledown f)b(e\triangledown f) = (e\triangledown f)ab(e\triangledown f) \qquad (6.3.1)$$

and

$$[a(e\triangledown f)b(e\triangledown f)c]^2 = a(e\triangledown f)b(e\triangledown f)c \qquad (6.3.2)$$

or by (6.3.1)

$$a(e\triangledown f)bcab(e\triangledown f)c = a(e\triangledown f)b(e\triangledown f)c \qquad (6.3.2')$$

must hold for all a, b $c \in$ S. Conversely, (6.3.1) guarantees that all elements in S′ have the form $a(e\triangledown f)b(e\triangledown f)c$, (6.3.2) then guarantees that S′ is a band, and together with the regularity of S they guarantee that S′ is a regular band. Note in passing that the regularity of S implies that (6.3.2′) can be replaced by

$$a(e\triangledown f)abc(e\triangledown f)c = a(e\triangledown f)b(e\triangledown f)c. \qquad (6.3.2'')$$

Returning to (6.3.1), upon setting $\Lambda = \{e, f, fe, -efe, -fef\}$, we get

$$(e\triangledown f)a(e\triangledown f)b(e\triangledown f) = \textstyle\sum_{u,\, v\, \in\, \Lambda} ua(e\triangledown f)bv.$$

Except for the two cases, $ea(e\triangledown f)bf$ and $fa(e\triangledown f)be,$ all of the $ua(e\triangledown f)bv$ terms reduce to $uabv$ terms due to the regularity of S. For example,

$$efa(e\triangledown f)bfef = efae(e\triangledown f)bfef = efaebfef = efabfef.$$

Thus (6.3.1) reduces first to $ea(e\triangledown f)bf + fa(e\triangledown f)be = eabf + fabe$ and then to

$$ea(e \circ f)bf + fa(f \circ e)be = eabf + fabe. \qquad (6.3.1')$$

Clearly (I) implies (6.3.1′) and when S′ is regular, (I) holds so that (6.3.1′) follows. Thus (6.3.1′) can be replaced by the stronger assertion, (I). Next, expanding (6.3.2″) gives

$$a(e + f + fe - efe - fef)abc(e + f + fe - efe - fef)c$$

$$= a(e + f + fe - efe - fef)b(e + f + fe - efe - fef)c.$$

Using the identity $aeabcec = aebec$ holding for all regular bands, multiplying out the left side of (6.3.2″) creates a number of terms that reduce immediately to the corresponding terms on the right. Of course we have, $aeabcec = aeaebecec = aebec$ and $afabcfc = afafbfcfc = afbfec$, but also cases such as

$$afe(abc)fefc = afef(abc)fefc = afef(b)fefc = afe(b)fefc,$$

and

$$aef(abc)fefc = aef(abc)efc = aef(b)efc = aef(b)fefc.$$

Where this matching fails, (6.3.2″) first reduces to

$$a(e)abc(f)c + a(e)abc(\underline{fe - efe} - fef)c + a(\underline{fe} - efe - \underline{fef})abc(f)c$$

$$= a(e)b(f)c + a(e)b(\underline{fe - efe} - fef)c + a(\underline{fe} - efe - \underline{fef})b(f)c.$$

Using regularity again on products involving the underlined terms, the above reduces to

$$a(e)abc(f - ef)c - a(ef)abc(f)c = a(e)b(f - ef)c - a(ef)b(f)c$$

or

$$a(e)abc(f)c - a(e)abc(ef)c - a(ef)abc(f)c = a(e)b(f)c - a(e)b(ef)c - a(ef)b(f)c.$$

Regularity gives $a(ef)abc(ef)c = a(ef)a(ef)bc(ef)c = a(ef)b(ef)c$. (6.3.2) is thus reduced to (II). □

A successful generation of a $\triangledown$-band from a regular band requires that at each stage of generation the new regular band S′ also satisfies (I) and (II) in Theorem 6.3.1 for all $a, b, c, e, f \in$ S′. While (I) and (II) generally need not be passed on to later bands, here is a strategy for generating $\triangledown$-bands. To this end, a condition *C* potentially satisfied by regular bands in rings is ***$\triangledown$-inductive*** if

(i) any regular band S in a ring that satisfies C must satisfy (I) and (II), and

(ii) for all $e, f \in$ S, the regular band S′ generated from S $\cup \{e \triangledown f\}$ also satisfies C.

Clearly:

Theorem 6.3.2. *Any regular band of idempotents in a ring that satisfies a $\triangledown$-inductive condition C will generate a $\triangledown$-band (its $\triangledown$-closure) in that ring.* □

We consider three conditions that are $\triangledown$-inductive (in this parlance). We give new proofs based on the concept of $\triangledown$-induction. Since the first two cases are already known, we omit their elementary proofs based on $\triangledown$-induction.

Corollary 6.3.3. *Left [right] regularity is a $\triangledown$-inductive condition*. □

Corollary 6.3.4. *Normality is a $\triangledown$-inductive condition*. □

Recall that a band S is totally pre-ordered if for all $e, f \in$S, either $e \preceq f$ or $f \preceq e$, that is, either $efe = e$ or $fef = f$. The following nice result is due to Karin Cvetko-Vah [2005a].

Theorem 6.3.5. *Being totally pre-ordered is a $\triangledown$-inductive condition. Thus every maximal totally pre-ordered regular band in a ring is a $\triangledown$-band.*

Proof. In verifying (I), we use the fact that $ea(e \circ f)bf$ multiplies out to $eaebf + eafbf - eaefbf$. There are two cases to consider.

Case 1: the $\preceq$-maximum is e or f. Say e. Then $eaebf + eafbf - eaefbf$ reduces to $eabf + eafbf - eafbf = eabf$. The subcase for f is similar.

Case 2: the $\preceq$-maximum is a or b. Say a. Here $eaebf + eafbf - eaefbf$ reduces to $ebf + efbf - efbf = ebf = eabf$. The subcase for b is similar.

For (II), first suppose that $e \preceq f$, so that $e = efe$. Then both sides of (II) reduce to 0:

$$a(e - ef)abc(f - ef)c = a(e - ef)fabc(f - ef)c = a0abc(f - ef)c = 0$$

and

$$a(e - ef)b(f - ef)c = a(e - ef)fb(f - ef)c = a0b(f - ef)c = 0.$$

Likewise, this occurs when $f \preceq e$, so that (II) follows.

Hence S $\cup \{e \triangledown f\}$ generates a regular band S′ in R. Is S′ also totally pre-ordered? Suppose that $e \preceq f$ in S. (The $f \preceq e$ case is similar.) Here $e \triangledown f = f + fe - fef$. Thus $f(e \triangledown f)f = f$ while $(e \triangledown f) f (e \triangledown f) = f(e \triangledown f) = f(f + fe - fef) = e \triangledown f$. Since $e \triangledown f \,\mathcal{D}\, f$ in S′, every element u' in S′ must be $\mathcal{D}$-equivalent to some element u in S. Since S is totally pre-ordered, so is S′. □

∇-bands open the door to cubic skew lattices that are distinct from any possible quadratic skew lattice a ring. This observation, however, is tempered by the following result:

Theorem 6.3.6. *Every cubic skew lattice* (S: ∇, $\bullet$) *having a lattice section* T *in a ring* R *is isomorphic to a quadratic skew lattice* (S′: $\circ$, $\bullet$) *in some ring* R′. *This always occurs when* S/$\mathcal{D}$ *is countable and in particular for any* (S: ∇, $\bullet$) *in a matrix ring over a (skew) field, in which case* R′ *can also be a matrix ring.*

Proof. Given a lattice section T with left and right extensions S_L and S_R, the internal Kimura decomposition gives an embedding of S into a quadratic skew lattice $S \to S_R \times S_L \subseteq R \times R$. ☐

6.4 Idempotent-closed rings

A ring *R* is ***idempotent-closed*** if its set of idempotents $\mathbf{E}(R)$ is multiplicatively closed. We begin with a result in basic ring theory.

Proposition 6.4.1 *Let ring R be idempotent-closed. If R has an identity, then its idempotents commute. In general, if the idempotents of R commute, then* $\mathbf{E}(R)$ *is in the center of R and forms a generalized Boolean algebra* ($\mathbf{E}(R)$; $\vee$, $\wedge$, \, 0) *upon setting*

$$e \wedge f = ef,\ \ e \vee f = e \circ f = e + f - ef \text{ and } e \setminus f = e - efe.$$

In this case, given $e \in \mathbf{E}(R)$, *both eR and* $ann(e) = \{x \in R \mid ex = 0\}$ *are ideals and R decomposes internally as* $eR \oplus ann(e)$ *under the map* $x \to ex + (x - ex)$.

Proof. If $1 \in R$, then $e(1-e) = 0$ for all $e \in E(R)$. Hence $ef(1-e) = [ef(1-e)]^2 = 0$ and thus $ef = efe$ for all e, f in $\mathbf{E}(R)$. Similarly, $fe = efe$ and thus $ef = fe$ for all e, f in $\mathbf{E}(R)$. Next, assuming all idempotents commute, choose $e \in \mathbf{E}(R)$ and $x \in R$. Observe first that if $ex = 0$, then $e + xe$ is idempotent; likewise, if $xe = 0$, then $e + ex$ is idempotent. Thus in general, since $e(x - ex) = 0$ one has that $e + xe - exe$ is idempotent. Multiplying each expression with e and using commutation, gives $e + xe - exe = e = e + ex - exe$, from which $xe = exe = ex$ follows. The remaining assertions are easy consequences of this. ☐

A ring whose idempotents commute, and thus lie in the center of the ring, is called ***abelian***. Such rings are clearly idempotent-closed. We are less concerned here with the internal structure of these rings, than in their role in the class of idempotent-closed rings. Suffice it to say, under mild hypotheses abelian rings decompose into direct sums of rings with identity whose only idempotents are 0 and 1. (See Section 6.5.)

Moving beyond the abelian case, first recall that a band *S* is ***normal*** if it satisfies any and hence all of the following equivalent identities:

(a) $efge = egfe$. (b) $efgh = egfh$. (c) $efege = egefe$.

Theorem 6.4.2 *If a ring R is idempotent-closed, then:*

i) $\mathbf{E}(R)$ *is a normal band under multiplication.*
ii) $\mathbf{E}(R)$ *is also closed under* ∇ *which is idempotent and associative on* $\mathbf{E}(R)$.
iii) $(\mathbf{E}(R); \nabla, \bullet)$ *is a strongly distributive (distributive, normal and symmetric) skew lattice.*
iv) *Upon setting* $e \setminus f = e - efe$, $(\mathbf{E}(R); \nabla, \bullet, \setminus, 0)$ *is a skew Boolean algebra.*

Proof. (i) Given $e \in \mathbf{E}(R)$, the principal subring eRe has an identity e and thus $e\mathbf{E}(R)e = \mathbf{E}(eRe)$ is commutative by the above proposition. Hence $\mathbf{E}(R)$ satisfies (c) and is normal. (ii) and (iii) follow from Theorems 2.3.6 and 2.3.7. Finally, let $e\setminus f$ as $e - efe$ in R. As such, $e\setminus f$ is the complement of $e\wedge f\wedge e$ in $\mathbf{E}(eRe)$ and satisfies the characterizing identities in Section 4.1: $e = (e\wedge f\wedge e) \vee (e \setminus f)$ and $(e\wedge f\wedge e) \wedge (e \setminus f) = 0$. □

Recall that $\mathbf{E}(R)$ is partially ordered by $e \geq f$ if $ef = f = fe$ with $e > f$ denoting $e \geq f \neq e$. Recall also that an idempotent $e > 0$ is ***primitive*** if no idempotent f exists such that $e > f > 0$. $\mathbf{E}(R)$ is ***0-primitive*** if all of its non-0 elements are primitive. In this case, $\mathbf{M}(R)$ denotes $\mathbf{E}(R)\setminus\{0\}$. Recall that a band is rectangular if $efe = e$, or equivalently, $efg = eg$ holds. A ***0-rectangular band*** M^0 consist of a rectangular band M and a distinct element $0 \notin M$, so that $x0 = 0 = 0x$. Abstractly viewed, every primitive skew Boolean algebra has operations induced from a 0-rectangular band M^0: given $x, y \in M$, $0\wedge x = 0 = x\wedge 0$, $0\vee x = x = x\vee 0$, $x\wedge y = xy = y\vee x$, $x\setminus 0 = x$ and $0\setminus x = 0 = x\setminus y$. *In this simple manner, 0-rectangular bands are in 1-1 correspondence to the class of primitive skew Boolean algebras.*

Theorem 6.4.2 *If* $\mathbf{E}(R)$ *is 0-primitive and closed under multiplication, then under multiplication,* $\mathbf{E}(R)$ *is a 0-rectangular band with* $e\nabla f = fe$ *on* $\mathbf{M}(R)$.

Proof. Given the assumptions on $\mathbf{E}(R)$, let $e \neq f$ in $\mathbf{M}(R)$. Then $e \geq efe \geq 0$ in $\mathbf{E}(R)$ so that efe is either 0 or e. If $efe = 0$, then so are fef, ef and fe (since, *e.g.*, $ef = efef = 0$). It follows that $e + f$ is an idempotent greater than either e or f and hence not primitive. Thus $efe = e$ for all e, f in $\mathbf{M}(R)$ making $\mathbf{M}(R)$ a rectangular band with $e\nabla f = e + f + fe - efe - fef$ reducing to fe. □

Idempotent-dominated rings

How does $\mathbf{E}(R)$ being multiplicatively closed affect the behavior of R? To answer this one is pressed to find reasonable assumptions connecting $\mathbf{E}(R)$ to all of R in order to obtain consequences for all of R. To do so we begin with the following lemma.

Theorem 6.4.3 *Given a ring R, set* $\mathbf{\Gamma}(R) = \{x \mid ex = x = xf, \text{ for some } e, f \in \mathbf{E}(R)\}$. *Then:*
i) $\mathbf{\Gamma}(R)$ *is a multiplicatively closed set containing* $\mathbf{E}(R)$ *that also has negatives*:
$$x \in \mathbf{\Gamma}(R) \text{ implies } -x \in \mathbf{\Gamma}(R).$$
ii) *When* $\mathbf{E}(R)$ *is idempotent-closed,* $\mathbf{\Gamma}(R) = \bigcup\{eRe \mid e \in \mathbf{E}(R)\}$.
iii) *In general, the set* $\boldsymbol{Q}(R)$ *of all finite sums of elements in* $\mathbf{\Gamma}(R)$ *is the smallest ideal in R containing* $\mathbf{E}(R)$.

Proof. (ii) is clear. If $\mathbf{E}(R)$ is multiplicative, $x \in \mathbf{\Gamma}(R)$ and $e, f \in \mathbf{E}(R)$ are such that $ex = x = xf$, then set $g = f\nabla e$. By absorption, $ge = e$ and $fg = f$. Hence $gx = gex = ex = x$ and likewise $xg = x$ so that $gxg = x$ and (ii) holds. For (iii), first, let Q' be the least ideal of R containing $\mathbf{E}(R)$. Clearly $\boldsymbol{Q}(R)$ is a subring of Q'. Conversely, Q' consists of sums of terms of the form xe, ey and xey where $e \in \mathbf{E}(R)$ and $x, y \in R$. For such e, x, y, observe that $f = e + xe - exe$ and $h = e + ey - eye$ are both idempotent, so that xe, ey and thus xey must lie in $\boldsymbol{Q}(R)$. Indeed as defined, $ef = e$ and $fe = f$ so that $ff = (fe)f = f(ef) = fe = f$. Similar observations hold for h. □

Elements in $\mathbf{\Gamma}(R)$ are said to be ***idempotent-covered***. R as a whole is idempotent-covered if its elements are thus. Rings with identity and von Neumann regular rings are trivially idempotent-covered. For such rings we have an extension of Proposition 6.4.1.

Theorem 6.4.4 *Given an idempotent-covered ring R,* $\mathbf{E}(R)$ *is multiplicative if and only if R is abelian.*

Proof. Let R be idempotent-covered and idempotent-dominated. Given e, f in $\mathbf{E}(R)$, by Theorem 6.4.3 $g \in \mathbf{E}(R)$ exists such that $g(ef - fe) = ef - fe = (ef - fe)g$. By normality,

$$ef - fe \;=\; g(ef - fe)g \;=\; gefg - gfeg \;=\; gefg - gefg \;=\; 0.$$

Thus $\mathbf{E}(R)$ is commutative and R is abelian. The converse is clear. □

A ring R is ***idempotent-dominated*** if it is generated from the set $\mathbf{\Gamma}(R)$ of all idempotent-covered elements, or put otherwise, if $R = Q(R)$. Clearly idempotent-covered rings are idempotent-dominated, but not conversely.

Idempotent-covered abelian rings enter into the general case of idempotent-closed rings in at least two ways. First, for any idempotent e in an idempotent-closed ring R, the principal subring eRe is abelian. Secondly, we will see that each idempotent-closed and dominated ring has a maximal abelian image that in many cases arises as a major subring of the given ring.

Being idempotent-dominated can have a side effect. For any idempotent-covered ring R, the annihilator ideal vanishes, $ann(R) = \{0\}$. If R is just idempotent-dominated, this need not be the case, even when $\mathbf{E}(R)$ is also multiplicative. Indeed, given e, f in a multiplicatively closed $\mathbf{E}(R)$, the following small rectangular band arises

$$\begin{array}{ccc} efe & \mathcal{R} & ef \\ \mathcal{L} & & \mathcal{L} \\ fe & \mathcal{R} & fef \end{array}$$

and the combination $\alpha(e, f) = ef + fe - efe - fef$ can lie in $ann(R)$. This is of interest since for all idempotents e and f we have $e\nabla f = e \circ f + \alpha(e, f)$. Indeed we have:

Lemma 6.4.5 *When R is idempotent-closed and dominated, then:*

i) *For all $x, y \in R$ and all $e, f \in \mathbf{E}(R)$, $xefy = xfey$.*
ii) *In particular, if $e\,\mathcal{D}\,f$ then $xey = xfy$.*

Consequently, $\alpha(e, f) \in ann(R)$ *for all* $e, f \in \mathbf{E}(R)$ *so that* $e\nabla f = e \circ f$ *when* $ann(R) = \{0\}$.

Proof. Given that R is idempotent-dominated and $\mathbf{E}(R)$ is normal, $xefy = xfey$ must hold as stated. If $e\,\mathcal{D}\,f$ then $xey = xeffey = xfeefy = xfy$. Thus

$$x\alpha(e, f) = x(ef + fe - efe - fef) = x(ef + fe - fe - ef) = 0$$

for all $x \in$ R, and similarly, $\alpha(e, f)x = 0$. ☐

Given the above assumptions it is easy to see that $ann(R) = \{0\}$ when $\mathbf{E}(R)$ is either left-handed ($efe = ef$) or right-handed ($efe = fe$). While $ann(R)$ need not vanish, this is not the full story.

Lemma 6.4.6 *If R is idempotent-dominated, then*

i) $ann(R) = \{x \in R \mid xe = 0 = ex \text{ for all } e \in \mathbf{E}(R)\}$
ii) $ann[R/ann(R)] = \{0\}$.

Proof. In general, $ann(R) = \{x \in R \mid xy = 0 = yx$ for all $y \in R\}$ and if $\pi: R \to R/ann(R)$ is the induced homomorphism, then $ann(R) \subseteq \pi^{-1}\{ann[R/ann(R)]\}$ with

$$\pi^{-1}\{ann[R/ann(R)]\} = \{x \in R \mid xyz = 0 = yzx = yxz \text{ for all } y, z \in R\}.$$

But if R is idempotent-dominated, then $ann(\mathrm{R}) = \{x \in R \mid xe = 0 = ex$ for all $e \in \mathbf{E}(R)\}$. Hence $\pi^{-1}\{ann[R/ann(R)]\} \subseteq ann(R)$ and equality follows and $ann[R/ann(R)]$ vanishes in $R/ann(R)$. Thus, if R is idempotent-dominated, the description of $ann(R)$ in the lemma insures that $\pi^{-1}\{ann[R/ann(R)]\} \subseteq ann(R)$. Equality follows, as does the lemma. ☐

Lemma 6.4.7 *If R is a ring with ideal $I \subseteq ann(R)$, then the induced epimorphism $\pi: R \to R/I$ restricts to a bijection $\pi_E: \mathbf{E}(R) \to \mathbf{E}(R/I)$. If $\mathbf{E}(R)$ is also multiplicative, so is $\mathbf{E}(R/I)$ and π_E is an isomorphism of skew Boolean algebras.*

Proof. π_E is a well-defined map between the stated sets. If $\pi(e) = \pi(f)$ for $e, f \in \mathbf{E}(R)$, then $e = f + a$ for some $a \in ann(\mathrm{R})$ and squaring gives $e = f$. Thus π_E is at least injective. Given $x + \mathrm{I} \in \mathbf{E}(R/I)$, $x^2 = x+a$ for some $a \in ann(R)$ and hence $x^4 = x^2$ so that $x^2 \in \mathbf{E}(R)$ and π_E is bijective. Since π is a ring homomorphism, the rest of the lemma follows. ☐

Since $eRe \cap ann(R) = \{0\}$ for all idempotents e in any ring R, we have:

Theorem 6.4.8 *If R is both idempotent-dominated and idempotent-closed, then $R/ann(R)$ has both properties with $ann(R/ann(R)) = \{0\}$ so that $e\nabla f = e\bigcirc f$ in $R/ann(R)$. The natural epimorphism π: $R \to R/ann(R)$ induces a skew Boolean algebra isomorphism π_E: $\mathbf{E}(R) \cong \mathbf{E}(R/ann(R))$ and ring isomorphisms π_E: $eRe \to \pi(e)(R/ann(R))\pi(e)$ between corresponding principal subrings.* □

When $ann(R) \neq \{0\}$, $R/ann(R)$ provides a cleaner, trimmer version of R, sharing many of its characteristics, but without a non-vanishing annihilator ideal.

The canonical ideal $\mathcal{K}_R$

The annihilator ideal is a part of generally larger canonical nilpotent ideal $\mathcal{K}_R$ that all rings possess. We begin with an example.

Example 1 For $n \geq 1$, $\mathcal{M}_n(F)$ is the ring of $n \times n$ matrices over a field F. $\mathcal{M}_n(F)$ is trivially idempotent-covered since it has an identity, but $\mathbf{E}(\mathcal{M}_n(F))$ is never multiplicative unless $n = 1$. For $n \geq 2$, given fixed integer parameters $i, j, k \geq 0$ subject to $i + j + k = n$ and $1 \leq j \leq n$, consider the subring $R^n_{i,j,k}$ of all matrices with the following block design satisfying the added restriction that D be a diagonal matrix:

$$D_{ijk} = \begin{bmatrix} 0^{i\times i} & A^{i\times j} & C^{i\times k} \\ 0^{j\times i} & D^{j\times j} & B^{j\times k} \\ 0^{k\times i} & 0^{k\times j} & 0^{k\times i}k \end{bmatrix}.$$

$R^n_{i,j,k}$ is idempotent-dominated and idempotent-closed. $R^n_{i,j,k}$ and $\mathbf{E}(R^n_{i,j,k})$ are noncommutative when $j < n$ and commutative when $j = n$. The idempotents of $R^n_{i,j,k}$ are the matrices where D has only 0-1 entries in the diagonal, $AD = A$, $DB = B$ and $AB = C$ (for cases where A, B or C do not disappear.) $\mathbf{E}(R^n_{i,j,k})$ is right-handed when $i = 0$ and left-handed when $k = 0$, with C vanishing in either case along with A or B, respectively. For $i, k > 0$ so that $j \leq n - 2$, $ann(R^n_{i,j,k})$ is nontrivial consisting of all matrices for which blocks D, A and B are 0-submatrices. The strictly upper triangular matrices (where $D = 0$) form a nilpotent ideal, K. Considering just addition, the additive group of the ring is the direct sum of the group of all diagonal matrices (a maximal idempotent-covered subring with central idempotents) and the group of strictly upper triangular matrices (the nilpotent ideal K). To what extent does such a direct decomposition characterize idempotent-dominated rings with multiplicative sets of idempotents? This leads us to the following considerations. □

Given a ring R, $\mathcal{K}_R$ denotes the ideal $\{x \in R \,|\, uxv = 0$ for all $u, v \in R\}$. Clearly $\mathcal{K}_R\mathcal{K}_R\mathcal{K}_R = \{0\}$ so that $\mathcal{K}_R$ is a nilpotent ideal of index 3.

Lemma 6.4.9 *If R is idempotent-dominated, then*

$$\mathcal{K}_R = \{x \in R \,|\, exf = 0 \textit{ for all } e, f \in \mathbf{E}(R)\}.$$

If R is also idempotent-closed, then $\mathcal{K}_R = \{x \in \mathrm{R} \mid exe = 0$ *for all* $e \in \mathbf{E}(R)\}$.

Proof. The first statement is clear. If R is also idempotent-closed and $exe = 0$ for all $e \in \mathbf{E}(R)$, then for all $e, f \in \mathbf{E}(R)$, $exf = e(e\nabla f)x(e\nabla f)f = e0f = 0$ and the second statement follows. ☐

Theorem 6.4.10 *If R is an idempotent-dominated and idempotent-closed ring, then the ring* $R/\mathcal{K}_R$ *is the maximal abelian image of R. It is also idempotent-covered with* $\mathbf{E}(R/\mathcal{K}_R) \cong \mathbf{E}(R)/\mathcal{D}$. *Conversely, if R is an idempotent-dominated ring for which* $R/\mathcal{K}_R$ *is abelian, then R is idempotent-closed.*

(In general, if R is any ring with an ideal K such that $RKR = \{0\}$, then R is idempotent-closed if and only if R/K is idempotent-closed, in which case $\mathbf{E}(R/K)$ is a homomorphic image of $\mathbf{E}(R)$ with both sharing a common maximal lattice image.)

Proof. The quotient ring R/K is automatically idempotent-dominated. Suppose $x + K \in \mathbf{E}(R/K)$ so that $x + K = x^n + K$ for all $n \geq 1$. Then $x^2 = x + k$ for some $k \in K$. From this we get $x^4 = x^3 + xkx = x^3$, and hence $x^6 = x^3$ so that $x + K = e + K$ for some $e \in \mathbf{E}(R)$. Thus all idempotents in R/K come from idempotents in R, making $\mathbf{E}(R/K)$ multiplicative. By Lemma 6.4.5, $ef - fe \in K$ for all $e, f \in \mathbf{E}(R)$, making $\mathbf{E}(R/K)$ commutative. Being idempotent-dominated, this forces R/K to be idempotent-covered. Indeed, given any idempotent-dominated abelian ring S, for any $x = e_1x_1e_1 + \cdots + e_nx_ne_n \in S$, upon setting $f = e_1\nabla e_2\nabla \ldots \nabla e_n$ we have $f \geq$ all e_i so that $fxf = f$.

Next, let α: $R \to A$ be a homomorphism onto a ring A having only central idempotents. A is automatically idempotent-dominated and hence idempotent-covered so that $\mathcal{K}_\mathrm{A} = \{0\}$. Given $k \in \mathcal{K}$, since α is surjective, $\alpha(k)$ is in $\mathcal{K}_A$ so that $\alpha(k) = 0$. Thus $\mathcal{K} \subseteq \ker(\alpha)$ and the maximality of the abelian image $R/\mathcal{K}$ follows.

The converse follows from the remark following the theorem. So let R be a ring with an ideal K such that $RKR = \{0\}$, and let $e, f \in \mathbf{E}(R)$ be given. If R/K is idempotent-closed, then at least $efef = ef + k$ for some $k \in K$. But then

$$efef = e(efef)f = e(ef + k)f = ef + ekf = ef,$$

so that R is idempotent-closed. Likewise, if R is idempotent-closed, then so is R/K by the above argument for R/K with $\mathbf{E}(R/K)$ again a homomorphic image of $\mathbf{E}(R)$. Given idempotent closure, if $e > f$ in $\mathbf{E}(R)$, then $e + K \neq f + K$ in $\mathbf{E}(R/K)$, for otherwise $e = f + k$ for some $k \in K$ so that $e = e(f + k)e = efe = f$ in R. This forces images of distinct $\mathcal{D}$-classes in $\mathbf{E}(R)$ to remain distinct in $\mathbf{E}(R/K)$, so that $\mathbf{E}(R)/\mathcal{D} \cong \mathbf{E}(R/K)/\mathcal{D}$. □

In general, $ann(R) \subseteq \mathcal{K}_R$ with the inclusion often proper; but thanks to Theorems 6.4.8 and 6.4.10, for an idempotent-closed and dominated ring R, $ann(R) = \mathcal{K}_R$ precisely when both ideals vanish and R idempotent-covered with central idempotents.

When $\mathbf{E}(R)$ has a lattice section, the ideal $\mathcal{K}_R$ has a natural near-complement in the ring R.

Theorem 6.4.11 *Let R be idempotent-dominated and idempotent-closed. If $\mathbf{E}(R)$ has a lattice section E_0, then setting $\mathcal{A} = \{x \in S \mid \exists e \in E_0, x = exe\}$, we have the following*:

i) *$\mathcal{A}$ is an idempotent-covered abelian subring of R.*

ii) *As additive groups, $R = \mathcal{A} \oplus \mathcal{K}_R$.*

iii) *The natural epimorphism $R \to R/\mathcal{K}_R$ induces a ring isomorphism $\mathcal{A} \cong R/\mathcal{K}_R$.*

Conversely, let S be any ring with an abelian subring A and an ideal K such that $SKS = \{0\}$ and as additive groups, $S = A \oplus K$ so that as rings, $S/K \cong A$. Then S is idempotent-closed and $\mathbf{E}(A)$ is a lattice section of $\mathbf{E}(S)$; moreover, for all $a \in \mathbf{E}(A)$, the $\mathcal{D}$-class $\mathcal{D}_a$ in $\mathbf{E}(S)$ consists of all elements of the form $(a + ka)(a + ak)$ with $k \in K$.

Proof. Suppose $exe = x$ and $fyf = y$ for $e, f \in E_0$. Setting $g = e\nabla f = e\circ f$ in E_0, $g \geq e, f$ so that $gxg = x$ and $gyg = y$ also. This gives $g(x \pm y)g = gxg \pm gyg = x \pm y$ and $g(xy)g = (gx)(yg) = xy$. Thus $\mathcal{A}$ must be an idempotent-covered subring for which $\mathbf{E}(\mathcal{A})$ is multiplicative. By Theorem 6.4.4, (i) follows. Suppose that we are given $x = e_1x_1e_1 + \cdots + e_nx_ne_n \in R$. For each e_i in $\mathbf{E}(R)$ let $f_i \in E_0$ be such that $e_i \mathcal{D} f_i$ and set $y = f_1x_1f_1 + \cdots + f_nx_nf_n \in \mathcal{A}$. We claim that $x - y$ is in $\mathcal{K}_R$. We need only show that each $e_ix_ie_i - f_ix_if_i \in \mathcal{K}_R$. But due to Lemma 6.4.5(ii), for all u, v in R:

$$u(e_ix_ie_i - f_ix_if_i)v = ue_ix_ie_iv - uf_ix_if_iv = ue_ix_ie_iv - ue_ix_ie_iv = 0.$$

Hence $x - y \in \mathcal{K}_R$ so that $x = y + (x - y) \in \mathcal{A} + \mathcal{K}_R$. Thus $R = \mathcal{A} + \mathcal{K}_R$ and clearly $\mathcal{A} \cap \mathcal{K}_R = \{0\}$ so that (ii) follows. From (i) and (ii), (iii) follows.

Conversely, that S is idempotent-closed follows from the remark after Theorem 6.4.10. Given $a \in A$ and $k \in K$, $(a + k)^2 = a^2 + (ka + ak + k^2) \in A \oplus K$. Thus $a + k$ is idempotent if and only if $a^2 = a$ in A and $k = ka + ak + k^2$ in K. The latter gives $k^2 = kak$ so that when idempotent,

$$a + k = (a + k)^2 = (a + ka)(a + ak).$$

Here $a(a + k)a = a + aka = a$ and $(a + k)a(a + k) = (a + ka)(a + ak) = a + k$. Thus $a\,\mathcal{D}\,(a + k)$ making $\mathbf{E}(A)$ is a lattice section of $\mathbf{E}(S)$. It is easily seen that any element $(a + ka)(a + ak)$ is idempotent whenever $a \in \mathbf{E}(A)$ and $k \in K$. □

Thus the observations of Example 1 generalize to all idempotent-closed and dominated rings R such that $\mathbf{E}(R)$ has a lattice section. For rings satisfying the chain conditions of the next section, this is the case. (The latter include all finite-dimension matrix ring examples.) $\mathbf{E}(R)$ also has a lattice section if it has a maximal $\mathcal{D}$-class M, since for any $e \in M$, $\{f \in \mathbf{E}(R) \mid f \leq e\}$ is a lattice section.

While $\mathcal{K}_R$ generally exceeds $ann(R)$, this is not the whole story. $\mathcal{K}_R$ also contains two related canonical ideals, the left and right annihilator ideals of R:

$$ann_L(R) = \{x \in R \mid xy = 0 \text{ for all } y \in R\} \text{ and } ann_R(R) = \{y \in R \mid xy = 0 \text{ for all } x \in R\}.$$

It happens that $ann_L(R) + ann_R(R) = \mathcal{K}_R$ for all idempotent-dominated and idempotent-closed rings. We prove this when $\mathbf{E}(R)$ is bounded with a maximal $\mathcal{D}$-class. The general case follows from the next proposition and Theorems 6.5.1 and 6.5.3 below.

An idempotent-dominated and closed ring R is ***bounded*** if $\mathbf{E}(R)$ has a maximal $\mathcal{D}$-class, M, consisting of all m in $\mathbf{E}(R)$ such that $m\nabla e\nabla m = m$ and $eme = e$ for all $e \in \mathbf{E}(R)$. Such an m is called ***maximal*** in $\mathbf{E}(R)$ and $xmy = xy$ for all x, y in R. Every element $x \in R$ is a sum of elements $m_i x_i m_i$ where each m_i is in M

Proposition 6.4.12 *Let R be a bounded idempotent-dominated and closed ring and let m be maximal in* $\mathbf{E}(R)$. *Then as an additive group,* $\mathcal{K}_R = \mathcal{K}_R m \oplus ann(R) \oplus m\mathcal{K}_R$ *where* $\mathcal{K}_R m = \{km \mid k \in \mathcal{K}_R\}$ *and* $m\mathcal{K}_R = \{mk \mid k \in \mathcal{K}_R\}$. *In particular,*

$$ann_R(R) = \mathcal{K}_R m \oplus ann(R) \text{ and } ann_L(R) = ann(R) \oplus m\mathcal{K}_R.$$

Finally, μ: $R \to mRm$ *defined by* $\mu(x) = mxm$ *is a ring homomorphism onto the abelian subring* mRm, *with kernel* $\mathcal{K}_R$.

Proof. Setting $\mathcal{K} = \mathcal{K}_R$, the identity $k = km + (k - km - mk) + mk$ gives $\mathcal{K} = \mathcal{K}m + ann(R) + m\mathcal{K}$. Let that $k, k' \in \mathcal{K}$ and $a \in ann(R)$ be such that $km + a + mk' = 0$. Then

$$mk' = m(km + a + mk') = m0 = 0$$

and similarly $km = 0$, leaving $a = 0$ also. Thus the sum is direct. Next let $x \in ann_L(R)$. Being in $\mathcal{K}$, x has the form $km + a + mk'$. Thus $0 = xm = km$, so that $x = a + mk'$, giving $ann_L(R) \subseteq ann(R) \oplus m\mathcal{K}$. The reverse inclusion is trivial. That $ann_R(R) = \mathcal{K}m \oplus ann(R)$ is seen in similar fashion. Since $xmy = xy$ in R, the final statement is clear. Thus as additive subgroups,

$$R = mRm \oplus \mathcal{K}m \oplus ann(R) \oplus m\mathcal{K}. \quad \square$$

Returning to Example 1, blocks D, A, C and B correspond respectively to mRm, $\mathcal{K}m$, $ann(R)$ and $m\mathcal{K}$.

We briefly consider a class of rings that are always idempotent-closed. A ring is ***weakly commutative*** if the identity $xyzw = xzyw$ holds. Such a ring R has a nil radical $\mathcal{N}_R$ consisting of all nilpotent elements in R. $\mathcal{N}_R$ is indeed an ideal and $R/\mathcal{N}_R$ is commutative with a vanishing nil radical. Given a commutative ring A and a normal band S, the semigroup ring $A[S]$ is weakly commutative. This makes idempotent-closed rings easy to find. Indeed, *all examples in this section are weakly commutative*. In any idempotent-closed ring, $\mathbf{E}(R)$ generates a weakly commutative subring, denoted *if* $Q_0(R)$.

Theorem 6.4.13 *If R is a weakly commutative ring, then* $\mathbf{E}(R)$ *is multiplicative and the subring eRe is commutative for each $e \in$* $\mathbf{E}$(R). *The converse also holds if R is idempotent-dominated. Finally, for any ring R,* $\mathbf{E}(R)$ *is multiplicative if and only if $Q_0(R)$ is weakly commutative.*

Proof. Given $e, f \in \mathbf{E}(R)$, $(ef)^2 = efef = eeff = ef$. Also, given exe, eye in eRe, we have

$$(exe)(eye) = e(exe)(eye)e = e(eye)(exe)e = (eye)(exe).$$

Conversely let R be idempotent-dominated with $\mathbf{E}(R)$ being multiplicative and each subring eRe, for $e \in \mathbf{E}(R)$, being commutative. Let eae, fbf, gcg, hdh in $\mathbf{\Gamma}(R)$ be given with $e, f, g, h \in \mathbf{E}(R)$. Since $\mathbf{E}(R)$ is multiplicative, as in the proof of Lemma 6.4.7, e', f', g', h' in $\mathbf{E}(R)$ exist such that $e' \geq e, f' \geq f, g' \geq g, h' \geq h$ with e', f', g' and h' being $\mathcal{D}$-related. Thus we may assume at the outset that e, f, g and h are $\mathcal{D}$-related. This plus the assumption that each eRe be commutative gives

$$\begin{aligned}(eae)(fbf)(gcg)(hdh) &= (eae)e(fbf)e(gcg)e(hdh)\\ &= (eae)e(gcg)e(fbf)e(hdh) = (eae)(gcg)(fbf)(hdh)\end{aligned}$$

holding in $\mathbf{\Gamma}(R)$. Distribution extends the identity $xyzw = xzyw$ from $\mathbf{\Gamma}(R)$ to all of R. If we just assume $\mathbf{E}(R)$ is normal, then this property extends via distribution to weak commutativity of the generated subring $Q_0(R)$. The converse is clear. □

Idempotent-closed rings in general

What can be said about an arbitrary idempotent-closed ring R where $Q(R)$ could be a proper ideal? We begin by quoting a standard result in ring theory.

Lemma 6.4.14 *Given a ring R with a nil ideal N, every idempotent in R/N is of the form $e + N$ for some idempotent e in R.* □

As an immediate consequence we have:

Theorem 6.4.15 *In any ring R,* $\mathcal{K} = \mathcal{K}_{Q(\mathrm{R})}$ *is a nil ideal and* $Q(R/\mathcal{K}) = Q(R)/\mathcal{K}$. *Thus R is idempotent-closed if and only if* $R/\mathcal{K}$ *is abelian*. (Note that $R\mathcal{K}R$ need not be $\{0\}$ here.)

Proof. Indeed, for any ideal I in R, $\mathcal{K}_I$ is a nil ideal of R. Thus for $\mathcal{K} = \mathcal{K}_{Q(\mathrm{R})}$, each idempotent in $R/\mathcal{K}$ has the form $e + \mathcal{K}$ for some idempotent e in R, so that $Q(R/\mathcal{K}) = Q(R)/\mathcal{K}$. Suppose $R/\mathcal{K}$ is abelian. Then $Q(R)/\mathcal{K} = Q(R/\mathcal{K})$ is also, in which case $Q(R)$ is idempotent-closed by Theorem 6.4.10, and so is R. Conversely, if R is idempotent-closed, then $Q(R)/\mathcal{K}$ is abelian, and since $Q(R)$ contains all the idempotents of R, Lemma 6.4.14 above assures that all idempotents in $R/\mathcal{K}$ commute, making it abelian. ☐

What can one say about $R/Q(R)$ in general? In particular, is $\mathbf{E}(R/Q(R)) = \{0\}$? This would be an extreme case of idempotent-closure. The answer is affirmative when R is idempotent-closed. We begin with a special case:

Lemma 6.4.16 *If a ring R is abelian, then* $\mathbf{E}(R/Q(R)) = \{0\}$.

Proof. Letting $Q = Q(R)$, suppose that $x + Q$ is idempotent in R/Q. Thus $x^2 - x \in Q$. It follows that e in E(R) exists such that $x^2 - x = e(x^2 - x) = (ex)^2 - ex$. Moreover, x decomposes as $ex + y$ where $y = x - ex$ so that $ey = 0$. Then

$$x^2 - x = (ex + y)^2 - (ex + y) = (ex)^2 + y^2 - (ex + y) = [(ex)^2 - ex] + [y^2 - y].$$

But since $x^2 - x = (ex)^2 - ex$, $y^2 - y = 0$ in R, that is, y is in $\mathbf{E}(R)$, so that $x \in Q$ and $x + Q$ is the zero element in R/Q. ☐

Theorem 6.4.17 *If R is idempotent-closed, then* $\mathbf{E}(R/Q(R)) = \{0\} = Q(R/Q(R))$.

Proof. Again, $Q = Q(R)$ is an ideal and $\mathcal{K} = \mathcal{K}_Q$ is a nil ideal of R. From $R/Q \cong (R/\mathcal{K})/(Q/\mathcal{K})$ and the preceding two results, $\mathbf{E}(R/Q(R)) = \{0\}$, and hence $Q(R/Q(R)) = \{0\}$ follows. ☐

Thus squeezing $Q(R)$ to a point eliminates any nonzero idempotents in $R/Q(R)$ provided R is idempotent-closed. In general, this need not be so. In any case, every idempotent-closed ring R is a ring extension of an idempotent-dominated subring Q by a ring $T = R/Q$ for which $\mathbf{E}(T) = \{0\}$. From the standpoint of the idempotents, Q and T are polar opposites: the idempotents are maximally engaged in Q, while they are minimally engaged in T. In turn Q is an extension of a nilpotent ring $\mathcal{K}$ for which $\mathbf{E}(\mathcal{K}) = \{0\}$ by an idempotent-covered abelian ring A. Indeed, in many cases Q is a "subdirect sum" of $\mathcal{K}$ and an internal copy of A. We consider other decompositions in the following section.

6.5 Decomposing E(R) and R

Recall that, every skew Boolean algebra is a subdirect product of primitive algebras, thanks largely to the following restatement of Theorem 4.1.4.

Theorem 6.5.1 *Given a $\mathcal{D}$-class A of a skew Boolean algebra S, set*

$$S_1 = \{e \in S \mid e\wedge a\wedge e = e \text{ for some (and hence all) } a \in A\},$$

and

$$S_2 = \{f \in S \mid f\wedge a = a\wedge f = 0 \text{ for some (and hence all) } a \in A\}.$$

Then both S_1 and S_2 are subalgebras of S, all elements of S_1 commute with all elements of S_2 and the map μ: $S_1 \times S_2 \to S$ defined by $\mu(e_1, e_2) = e_1 \vee e_2$ is an isomorphism of skew Boolean algebras. The inverse isomorphism is given by $\mu^{-1}(e) = (e\wedge a\wedge e,\ e \setminus e\wedge a\wedge e)$. ☐

Described otherwise, S_1 is the union of the $\mathcal{D}$-class A and all lower $\mathcal{D}$-classes in the generalized Boolean lattice $S/\mathcal{D}$, while S_2 consists of all $\mathcal{D}$-classes B that meet A and its lower $\mathcal{D}$-classes at {0} in $S/\mathcal{D}$. In fact S_1 and S_2 are ideals of S where by an ***ideal*** of a skew lattice S is meant any subset I such that $e\vee f \in I$ for all $e, f \in I$, and both $e\wedge g$, $g\wedge e \in I$ for all $e \in I$ and all $g \in S$. S_1 corresponds to the principal ideal in $S/\mathcal{D}$ determined by the element A of $S/\mathcal{D}$ while S_2 corresponds to the ideal in $S/\mathcal{D}$ consisting of all elements of $S/\mathcal{D}$ that meet A at 0. If $S = \mathbf{E}(R)$, how does this play out in the full context of the host ring, R? We begin by passing from skew lattice ideals I in $\mathbf{E}(R)$ to their induced ring ideals Q_I in R.

Lemma 6.5.2 *Let R be idempotent-closed and let I be an ideal of* $\mathbf{E}(R)$. *Then the least ideal Q_I of R containing I consists of all sums $\sum e_i x_i f_i$ where both $e_i, f_i \in I$ and $x_i \in R$.* (As with Theorem 6.4.3, all elements in Q_I also have the form $\sum e_i x_i e_i$ where $e_i \in I$.)

Proof. Since $-(xy) = (-x)y = x(-y)$, the set of all such sums is at least a subring R' of R. Clearly $I \subseteq R' \subseteq Q_I$. On the other hand, Q_I consists of sums of the form $\sum x_i e_i y_i$ where $e_i \in I$ and $x_i, y_i \in R$. But each such sum lies in R' since its terms must. Indeed, given $e \in I$, $f = e + xe - exe$ satisfies $fe = f$ and $ef = e$, forcing f to be an idempotent in I, thus ensuring $xe = f - e + exe \in R'$. Similarly ey and thus xey also lie in R'. ☐

Theorem 6.5.3 *Let ring R be idempotent-closed and dominated, and let I and J be ideals of* $\mathbf{E}(R)$ *such that each $e \in \mathbf{E}(R)$ is uniquely $f + g$ for some $f \in I$ and $g \in J$. Then:*

i) *$R = Q_I + Q_J$ and $Q_I Q_J = \{qq' \mid q \in Q_I \;\&\; q' \in Q_J\} = \{0\} = Q_J Q_I$.*

ii) *σ: $Q_I \oplus Q_J \to$ R given by $\sigma(q, q') = q + q'$ is a ring homomorphism onto R, that restricts to isomorphisms, $\sigma_I : Q_I \oplus \{0\} \cong Q_I$ and $\sigma_J : \{0\} \oplus Q_J \cong Q_J$.*

iii) *In general, $Q_I \cap Q_J \subseteq ann(R)$ with $Q_I \cap Q_J$ possibly exceeding $\{0\}$;*

iv) *σ is an isomorphism when $Q_I \cap Q_J = \{0\}$, and in particular when $ann(R) = \{0\}$.*

Proof. The given decomposition of $\mathbf{E}(R)$ implies first that $I \cap J = \{0\}$ and thus $IJ = JI = \{0\}$. Lemma 6.5.2 now gives $Q_IQ_J = \{0\} = Q_JQ_I$. Since $\mathbf{E}(R) = I + J$ and R is both idempotent-closed and dominated, R consists of sums of elements of the form $(e+f)x(e+f)$ where $e \in I$ and $f \in J$. But

$$(e+f)x(e+f) = exe + exf + fxe + fxf = exe + fxf \in Q_I + Q_J.$$

Indeed $exf \in Q_I \cap Q_J$ and so $exf = e(exf) = 0$ since $Q_IQ_J = \{0\}$. Likewise $fxe = 0$ and (i) is seen. (ii) is a consequence of (i), as is the inclusion of (iii). That $Q_I \cap Q_J$ can exceed $\{0\}$ is seen in the next example. Since $ker(\sigma) = \{(x, -x) \mid x \in Q_I \cap Q_J\}$, (iv) follows. □

Example 6.5.1 R is the matrix ring $\left\{ \begin{bmatrix} 0 & a & b & p \\ 0 & u & 0 & c \\ 0 & 0 & v & d \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| a, b, c, d, u, v, p \in F \right\}$ where F is any field. $\mathbf{E}(R)$ is multiplicatively closed with four $\mathcal{D}$-classes described as follows:

$$\left\{\begin{bmatrix} 0 & m & n & mp+nq \\ 0 & 1 & 0 & p \\ 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} > \left\{\begin{bmatrix} 0 & a & 0 & ac \\ 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} \;\&\; \left\{\begin{bmatrix} 0 & 0 & b & bd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 \end{bmatrix}\right\} > \left\{\mathbf{0}^{4\times 4}\right\}$$

where $a, b, c, \ldots, q$ vary freely over F. If I and J are the primitive skew lattice ideals determined by the middle left and middle right $\mathcal{D}$-classes, then Q_I, Q_J and $ann(R) = ann(Q_I)$ are represented respectively by

$$\begin{bmatrix} 0 & a & 0 & v \\ 0 & u & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & b & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & u & d \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & v \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \;\square$$

Theorem 6.5.3 can be applied repeatedly. Doing so leads to a modification of direct sum decompositions of rings, which we describe this in full generality, independent of any special assumption on R and $\mathbf{E}(R)$. We begin with a ring R and a set of subrings of R, $\{Q_i \mid i \in I\}$. Let $\sum^{\oplus}_{i\in I} Q_i$ be the direct sum of the Q_i, the subring of the direct product $\prod Q_i$ consisting of all I-tuples with only finitely many non-0 components. Define σ: $\sum^{\oplus}_{i\in I} Q_i \to R$ by $\sigma(\langle x_i \mid i \in I\rangle) = \sum x_i$. σ preserves addition. It preserves multiplication precisely when $\sum x_i \sum y_i = \sum x_iy_i$ holds, which it does if $x_iy_j = 0$ for all $i \neq j$. If in addition the image $\sum Q_i = R$, then each Q_i must be an ideal of R.

When R is a sum of ideals Q_i where $Q_iQ_j = \{0\}$ for all $i \neq j$, we say that R is the ***orthosum*** of the Q_i. With σ as above, $\sigma([\sum^{\oplus}_{i\in I} ann(Q_i)] = ann(R)$ and $\sigma^{-1}[ann(R)] = \sum^{\oplus}_{i\in I} ann(Q_i)$. From this we have:

Proposition 6.5.4 *If R is an orthosum of ideals Q_i and σ: $\sum^{\oplus}_{i\in I} Q_i \to R$ is the sum epimorphism, then σ: $\sum^{\oplus}_{i\in I} Q_i \to R$ is an isomorphism if and only if $\sum^{\oplus}_{i\in I} ann(Q_i)$ is isomorphic with $ann(R)$ under the restricted map. An isomorphism τ: $\sum^{\oplus}_{i\in I} \left(Q_i \big/ ann(Q_i)\right) \to R/ann(R)$ is defined by*

$$\tau(\langle x_i + ann(Q_i) \mid i \in I\rangle) = \sum x_i + ann(R)\ .$$

Proof. Since $ann(\sum^{\oplus}_{i\in I} Q_i) = \sum^{\oplus}_{i\in I} ann(Q_i)$, if σ is an isomorphism of $\sum^{\oplus}_{i\in I} Q_i$ with R, then σ restricts to an isomorphism, $\sum^{\oplus}_{i\in I} ann(Q_i) \cong ann(R)$. Conversely, σ is at least surjective. Suppose $\sigma(\langle x_i \mid i \in I\rangle) = 0$ in R. If some $x_i \neq 0$, then for some finite set of non-0 elements from distinct Q_i we have $x_1 + x_2 + \ldots + x_n = 0$. Since 0 is in $ann(R)$, each x_i lies in the annihilator of it respective ideal. Thus if σ restricts to an isomorphism, $\sum^{\oplus}_{i\in I} ann(Q_i) \cong ann(R)$, then $\langle x_i \mid i \in I\rangle = \langle 0 \mid i \in I\rangle$, making σ an isomorphism. The final isomorphism is the one induced from the combined epimorphism $\sum^{\oplus}_{i\in I} Q_i \xrightarrow[\sigma]{} R \xrightarrow[\pi]{} R/ann(R)$ where π is the canonical map. □

Regarding the idempotents, we also have:

Proposition 6.5.5 *If R is the orthosum of ideals Q_i and σ : $\sum^{\oplus}_{i\in I} Q_i \to R$ is the sum epimorphism, then the restriction σ_{E}: $\sum^{\oplus}_{i\in I} \mathbf{E}(Q_i) \to \mathbf{E}(R)$ is a bijection of sets that is an isomorphism of skew Boolean algebras, whenever $\mathbf{E}(R)$ is multiplicative. $\sum^{\oplus}_{i\in I} \mathbf{E}(Q_i)$ is the subset of $\sum^{\oplus}_{i\in I} Q_i$ where all $x_i \in \mathbf{E}(Q_i)$. It equals $\mathbf{E}(\sum^{\oplus}_{i\in I} Q_i)$ and is multiplicative iff each $\mathbf{E}(Q_i)$ is.)*

Proof. Since any sum of mutually orthogonal idempotents is also idempotent, σ_{E} is at least well-defined. Let $e \in \mathbf{E}(R)$ equal $x_1 + ... + x_r$, with each $x_i \in Q_i$. Then $e = e^2 = x_1^2 + ... + x_r^2$ and thus for each $i \leq r$, $x_i^2 = x_i + a_i$ where $a_i \in ann(R)$. Again one has $x_i^4 = x_i^2$ so that each $x_i^2 \in \mathbf{E}(Q_i)$ and σ_{E} is at least surjective. Let $e \in \mathbf{E}(R)$ be represented as both $e_1 + ... + e_r$ and $f_1 + ... + f_r$ where $e_i, f_i \in \mathbf{E}(Q_i)$. (By letting some values be 0 we may assume a common indexing.) But then $e_i = f_i + a_i$ where $a_i \in ann(R)$ for each index i. Squaring both sides gives $e_i = f_i$. Thus σ_{E} is a bijection. Finally, since σ is a ring homomorphism, the final assertions are clear. □

Given a ring R, $\mathbf{E}(R)$ satisfies the ***descending chain condition*** (the DCC) if any sequence

$$e_1 \geq e_2 \geq e_3 \geq \ldots$$

in $\mathbf{E}(R)$ eventually stabilizes: $e_n = e_{n+1} = \ldots$.

The ***ascending chain condition on*** $\mathbf{E}(R)$ (the ACC) is defined in dual fashion. The latter implies the former since a descending chain $e_1 \geq e_2 \geq e_3 \geq \ldots$ in $\mathbf{E}(R)$ induces a corresponding ascending chain of idempotents $e_1 - e_2 \leq e_1 - e_3 \leq \ldots$ with both stabilizing, if they do, simultaneously.

In what follows by a ***rectangular ring*** we mean an idempotent-dominated ring R for which $\mathbf{E}(R)$ is a 0-rectangular band.

Theorem 6.5.6 *Given an idempotent-closed and dominated ring R for which $\mathbf{E}(R)$ satisfies the descending chain condition:*

i) $\mathbf{E}(R) = \sum_{i\in I}^{\oplus} P_i$ *where the P_i are the primitive bands given by the union the minimal nonzero $\mathcal{D}$-classes of* $\mathbf{E}(R)$ *with* $\{0\}$.

ii) *R is an orthosum ideals $\sum_{i\in I} Q_i$ where each ideal Q_i is a rectangular subring for which* $\mathbf{E}(Q_i) = P_i$.

iii) $R/ann(R) \cong \sum_{i\in I}^{\oplus} Q_i \big/ ann(Q_i)$ *where annihilators of all quotient rings vanish.*

iv) *In particular,* $R \cong \sum_{i\in I}^{\oplus} Q_i$ *when* $ann(R)$ *vanishes.*

v) *As skew Boolean algebras,* $\mathbf{E}(R) \cong \mathbf{E}[R/ann(\mathrm{R})]$ *and* $P_i \cong \mathbf{E}[Q_i/ann(Q_i)]$.

Proof. (iii) through (v) follow from (i) and (ii) and Results 6.4.6 and 6.5.3 - 6.5.5. The DCC on $\mathbf{E}(R)$ plus the normality of $\mathbf{E}(R)$ guarantee that each idempotent $e > 0$ is a unique sum of primitive idempotents, $e = p_1 + \cdots + p_n$, where each p_i covers 0 in $(\mathbf{E}(R), \leq)$ and $p_1, \cdots, p_n$ are mutually orthogonal, coming from different primitive subalgebras of $\mathbf{E}(R)$. Assertion (i) follows from this. Next let $x \in \mathbf{\Gamma}(\mathrm{R})$ be given. By Theorem 6.4.3.

$$x = exe = (p_1 + \cdots + p_n)x(p_1 + \cdots + p_n) = \sum_{i,j} p_i x p_j = \sum_i p_i x p_i$$

for the appropriate primitive idempotents, where $p_i x p_j = 0$ for $i \neq j$ thanks to Theorem 6.5.3. Thus $x = exe = \sum_i p_i x p_i$ where $p_i x p_i \in Q_i$ and $(p_i x p_i)(p_j x p_j) = 0$ for $i \neq j$. Since every element in R is a finite sum of elements in $\mathbf{\Gamma}(R)$, (ii) follows. □

Corollary 6.5.7 *The conclusions of Theorem* 6.5.6 *hold if we assume the ascending chain condition on* $\mathbf{E}(R)$. *In this case the number of summands Q_i is finite, equaling the number of atoms in* $\mathbf{E}(R)/\mathcal{D}$. *Conversely, when only finitely many summands Q_i exist,* $\mathbf{E}(R)$ *satisfies the ascending chain condition*.

Proof. The DCC must hold on $\mathbf{E}(R)$ also. The ACC also prevents $\mathbf{E}(R)$ from having an infinite number of 0-minimal $\mathcal{D}$-classes and thus R from having an infinite number of ortho-summands Q_i. The converse is clear. □

Corollary 6.5.8 *If R satisfies the conditions of Theorem* 6.5.6, *then* $\mathcal{K}_R = \sum \mathcal{K}_{Q_i}$. *Upon choosing a nonzero idempotent $e_i \in Q_i$ for each i,* $\mathcal{A} = \sum_i e_i Q_i e_i$ *is an idempotent-covered abelian ring such that* $R = \mathcal{A} \oplus \mathcal{K}_R$ *as additive groups and* $R/\mathcal{K}_R \cong \mathcal{A}$ *as rings*. □

When the DCC holds on $\mathbf{E}(R)$, the question of $\mathbf{E}(R)$ being multiplicative can be reduced as follows. To begin, let $\mathbf{M}(R)$ denote the set of primitive idempotents of $\mathbf{E}(R)$ and let $\mathbf{M}_0(R)$

denote $\mathbf{M}(R) \cup \{0\}$. If $\mathbf{E}(R)$ satisfies this chain condition, then for any $e > 0$ in $\mathbf{E}(R)$ an $m \in \mathbf{M}(R)$ exists such that $e \geq m$. A result of Dolˇzan [8] for a case when R is abelian can be extended:

Theorem 6.5.9 *If* $\mathbf{E}(R)$ *satisfies the descending chain condition, then* $\mathbf{E}(R)$ is *multiplicative if and only if* $\mathbf{M}_0(R)$ *is multiplicative.*

Proof. Let $\mathbf{M}_0(R)$ be multiplicative and let S consist of all possible finite sums $\sum e_i$ of elements from distinct $\mathcal{D}$-classes in $\mathbf{M}_0(R)$. Since all products ef from distinct $\mathcal{D}$-classes in $\mathbf{M}_0(R)$ equal 0, S is also a set of idempotents that is closed under multiplication. Given $e > 0$ in $\mathbf{E}(R)$, let $m_1 \in \mathbf{M}(R)$ be such that $e \geq m_1$. If $e = m_1$, we stop. Otherwise we have $e > e - m_1 \geq m_2$ in $\mathbf{M}(R)$ with m_2 orthogonal to m_1 in $\mathbf{E}(R)$, since $m_1 \mathcal{D} m_2$ implies $m_2 = m_2(e - m_1)m_2 = m_2 - m_2 = 0$. If $e - m_1 = m_2$, then $e = m_1 + m_2$ with $m_1 \perp m_2$ in $\mathbf{E}(R)$. Otherwise, $e - m_1 - m_2 \geq$ some m_3 in $\mathbf{M}(R)$. The DCC insures this process eventually halts to give $e = m_1 + \cdot \cdot \cdot + m_n$ with the m_i mutually orthogonal and thus $\mathbf{E}(R) = S$. The converse is trivial. ☐

Although we do not use this, it can be proved that *if an idempotent-closed and dominated ring R satisfies the DCC [ACC] on (left, right) ideals, then it must satisfy the DCC [ACC] on idempotents*.

Rectangular rings

Thus to within isomorphism, the rings of the last section are direct sums of rectangular rings $\sum^{\oplus}_{i \in I} R_i$ or quotient rings $(\sum^{\oplus}_{i \in I} R_i)/I$ for some ideal $I \subseteq ann(\sum^{\oplus}_{i \in I} R_i)$. We study these "atomic" rings R_i with the goal of describing them in terms of rectangular bands S and rings A with identity 1 for which $\mathbf{E}(A) = \{0, 1\}$. Our main concern is not the precise structure of the latter "subatomic" ring A, but rather their role in the larger "atomic" picture.

We begin with a special case that is suggestive of what occurs generally. Given a ring A such that $\mathbf{E}(A) = \{0, 1\}$ and a rectangular band S, we form a ring $A[S]$. Under addition $A[S]$ is the free A-module on generating set S. Thus it consists of formal sums $\sum a_s s$ with $a_s \in$ A and $a_s \neq 0$ for only finitely many s. Addition is given by: $\sum a_s s + \sum b_s s = \sum(a_s + b_s)s$; multiplication is given by distributivity subject to the constraints: $(as)(bt) = (ab)(st)$ and $0s = 0 = \sum 0s$. If $s \in S$ is identified with $1s \in A[S]$, then S is a multiplicative band inside $\mathbf{E}(A[S])$. But is it a maximal rectangular band in $A[S]$? In what follows, at times we use just finite expressions $a_1s_1 +...+ a_ns_n$ with a_i in A and s_i in S, assuming that $a_j = 0$ for all a_js_j terms not showing.

Lemma 6.5.10 *Given A and S as above and $s \in S$, then in $A[S]$*

$$\mathcal{L}_S = \{\sum a_i s_i \mid \sum a_i = 1 \textit{ in } A \ \& \ a_i \neq 0 \Rightarrow s_i \,\mathcal{L}\, s \textit{ in } S\}$$

and

$$\mathcal{R}_S = \{\sum b_j t_j \mid \sum b_j = 1 \textit{ in } A \ \& \ b_j \neq 0 \Rightarrow t_j \,\mathcal{R}\, s \textit{ in } S\}$$

are the sets of idempotents respectively $\mathcal{L}$-related or $\mathcal{R}$-related to s in $A[S]$. Moreover,

$$\mathcal{M}_S = \mathcal{L}_S\mathcal{R}_S = \{\sum(a_ib_j)(s_it_j) \mid \sum a_i = 1 = \sum b_j \textit{ in } A \textit{ with } a_ib_j \neq 0 \Rightarrow s_i \,\mathcal{L}\, s \,\mathcal{R}\, t_j \textit{ in } S\}$$

is the maximal rectangular band in $A[S]$ containing s and hence all of S.

Proof. Indeed, given $x = \sum a_i s_i$ where $\sum a_i = 1$ in A and $s_i \,\mathcal{L}\, s$ in S if $a_i \neq 0$, one easily sees that $xs = x$, $sx = s$ and thus $x^2 = xsx = x$. Conversely, if $\sum a_i s_i$ is an idempotent that is $\mathcal{L}$-related to s, then

$$s = s(\sum a_i s_i)s = \sum a_i(ss_is) = \sum a_i s = (\sum a_i)s \text{ so that } \sum a_i = 1.$$

Moreover,

$$\sum a_i s_i = (\sum a_i s_i)s = \sum a_i(s_i s).$$

Since all s_is are $\mathcal{L}$-related to s in S, by the uniqueness of the representation, all s_i with nonzero coefficients in $\sum a_i s_i$ are $\mathcal{L}$-related to s in S. Thus $\mathcal{L}_S$ and likewise $\mathcal{R}_S$ are indeed as described. Finally, for any rectangular band M, given $e \in M$ one has $\mathcal{R}_e = eM$, $\mathcal{L}_e = Me$ in M so that $M = MeM = \mathcal{L}_e\mathcal{R}_e$. Thus we need only show that under multiplication $\mathcal{M}_S$ is a rectangular band. This follows from the easily verified identity in $A[S]$:

$$(\sum a_i s_i)(\sum b_j t_j)(\sum c_k u_k)(\sum d_m v_m) = (\sum a_i s_i)(\sum d_m v_m)$$

given $\sum a_i = \sum b_j = 1 = \sum c_k = \sum d_m$ in A and s_i, t_j u_k and v_m are in S. ☐

One can use the above "Inflation Lemma" to show that for nontrivial A and S, $\mathcal{M}_S$ will properly include S except in three cases: $\mathbb{Z}_2[S]$ for $|S| = 2$ (two cases) and $\mathbb{Z}_2[S]$ where S is the 4-element rectangular band of Example 3 below. On the other hand, if S has an $\mathcal{L}$- or $\mathcal{R}$-class with ≥ 3 distinct elements a, b, c, then a new element in the inflated class is given by $a - b + c$. Or if $\alpha \in A \setminus \{0, 1\}$, then given $a \neq b$ in an $\mathcal{L}$- or $\mathcal{R}$-class of S, $\alpha a + (1 - \alpha)b$ is a new element in the inflated class. Now, that the status of S and $\mathcal{M}_S$ in $A[S]$ is decided, we turn to all of $\mathbf{E}(A[S])$.

Theorem 6.5.11 *$A[S]$ is idempotent-dominated and $\mathbf{E}(A[S])$ is the 0-rectangular band $\mathcal{M}_S \cup \{0\}$ and $\mathcal{K}_{A[S]} = \{(\sum a_i s_i \mid \sum a_i = 0 \textit{ in } A\}$.*

Proof. Since $A[S] = \sum_{s\in S} As$ and $As = sA[S]s$, the first assertion holds. We show that any $f = f^2 \neq 0$ in $A[S]$ lies in $\mathcal{M}_S$. Let $f = \sum a_i s_i$. Given s in S, with i and j representing the same index values we have

$$fsf = (\sum a_i s_i)s(\sum a_j s_j) = \sum a_i a_j s_i s s_j = \sum a_i a_j s_i s_j = (\sum a_i s_i)(\sum a_j s_j) = ff = f$$

and

$$sfs = s(\sum a_i s_i)s = \sum a_i s s_i s = \sum a_i s = (\sum a_i)s.$$

Since $(sfs)^2 = sfsfs = sfs$, $\sum a_i \in \mathbf{E}(A)$. $\mathbf{E}(A) = \{0, 1\}$ by assumption. If $\sum a_i = 0$, so that $sfs = 0$, then also $f = fsfsf = 0$, contradicting $f \neq 0$. This leaves $\sum a_i = 1$, so that

$$f = fsf = (\sum a_i s_i)s(\sum a_i s_i) = (\sum a_i s_i s)(\sum a_i s s_i) \in \mathcal{L}_S\mathcal{R}_S = \mathcal{M}_S.$$

Next, since for all $a, b \in A$ and all $s, t \in S$, $(as)(\sum a_i s_i)(bt) = [a(\sum a_i)b](st)$, the condition $\sum a_i = 0$ is necessary for $\sum a_i s_i$ to be in $\mathcal{K}_{A[S]}$. Since $A[S]$ is additively generated from all $a\bullet s$ terms, it is also sufficient. □

In the general rectangular case, letting $\mathbf{M}(R)$ denote $\mathbf{E}(R) \setminus \{0\}$, we have:

Theorem 6.5.12 *If ring R is a rectangular ring, then*

i) $\boldsymbol{\Gamma}(R) = \cup_{e\in\mathbf{M}(R)} eRe$ *with* $(eRe)(fRf) = efRef$ *for all* $e, f \in \mathbf{M}(R)$.

ii) *Given* $e, f \in \mathbf{M}(R)$, *the map* $\chi: x \to fxf$ *defines a ring isomorphism of eRe with fRf. Thus every element y in fRf is uniquely expressed as fxf for some x in eRe.*

iii) *Given* $y = fxf \in fRf$ *and* $z = gx'g \in gRg$, $yz = (fg)(xx')(fg) \in fgRfg$.

iv) $R = \sum_{e\in\mathbf{M}(R)} eRe$ *with all summands being isomorphic subrings.*

Proof. Lemma 6.4.7 gives the first equality in (i). The second equality in (i) follows from $(eRe)(fRf) = efeRefRfef \subseteq efRef = (efRe)f \subseteq (eRe)(fRf)$. To see (ii), note that χ is at least an additive homomorphism from eRe to fRf. Let $x, y \in eRe$ be given. Then

$$f(xy)f = f(xey)f = f(xefey)f = (fxef)(feyf) = (fxf)(fyf)$$

so that χ is a homomorphism of rings. Indeed it is an isomorphism with inverse isomorphism given by $y \to eye$ from fRf back to eRe. (iii) follows from

$$yz = (fxf)(gx'g) = (fexef)(gex'eg) = fexex'eg = fgexx'efg = fgxx'fg.$$

The final assertion is now clear. □

Example 6.5.1 continued for n =3, $i = j = k = 1$, $e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 0 & a & ab \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$.

Here $eRe = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} \middle| d \in F \right\}$, $fRf = \left\{ \begin{bmatrix} 0 & ad & abb \\ 0 & d & db \\ 0 & 0 & 0 \end{bmatrix} \middle| d \in F \right\}$ and the isomorphism $x \to fxf$

of Theorem 6.5.12 (ii), sends $\begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in eRe to $\begin{bmatrix} 0 & ad & abb \\ 0 & d & db \\ 0 & 0 & 0 \end{bmatrix}$ in fRf. ☐

How close is this class of A[S]-rings to the class of all rectangular rings? We begin our answer with the following canonical co-representation:

Proposition 6.5.13 *Let R be a rectangular ring. If $A = fRf$ for a fixed $f \in M = \mathbf{M}(R)$, then A is a ring with identity $1 = f$ such that $\mathbf{E}(A) = \{0, f\}$; moreover the map β: $A[M] \to R$ defined by $\beta(\sum a_i e_i) = \sum e_i a_i e_i$ in R is a homomorphism onto R that is bijective between the copy $\{1e \mid e \in M\}$ of M in $A[M]$ and M in R, and also between subrings Ae in $A[M]$ and their images eAe in R.*

Proof. By Theorem 6.5.12 (ii) and (iv), $R = \sum_{e \in M} eRe$ with all summands being isomorphic to A. Thus β is an additive epimorphism that is bijective where stated. That it preserves multiplication follows from Theorem 6.5.12 (iii) and distribution. ☐

Thus, all rectangular rings essentially arise as rings of the form $A[S]$ as above and certain homomorphic images of these rings. The entire situation is put more precisely as follows.

Theorem 6.5.14 *Given a ring A with identity such that $\mathbf{E}(A) = \{0, 1\}$ and a rectangular band S, then $A[S]$ is a rectangular ring. Moreover given any ideal K of $A[S]$ such that $K \subseteq \mathcal{K}_{A[S]}$, the quotient ring $A[S]/K$ is also rectangular. Conversely, to within isomorphism every rectangular ring is obtained in this fashion.*

Proof. The first assertion is Theorem 6.5.11. The second comes from Theorem 6.4.14. To see the converse note that for the map β above, if $\beta(\sum a_i e_i) = \sum e_i a_i e_i = 0$ in R, then in $fRf = A$, $\sum a_i = \sum f a_i f = f(\sum a_i) f = f0f = 0$. Thus ker(β) $\subseteq \mathcal{K}_{A[M]}$ by Theorem 6.5.11. The converse now follows by Theorem 6.4.14. ☐

Example 6.5.2 Let $A = \mathbb{Z}_2$ and let S be the rectangular band determined by the array:

$$\begin{array}{ccc} a & \mathcal{R} & b \\ \mathcal{L} & & \mathcal{L} \\ c & \mathcal{R} & d \end{array} \qquad \text{e.g. } a \wedge d = b \text{ and } a \wedge d = c.$$

Setting $s = a+b+c+d$, the sixteen elements of $\mathbb{Z}_2[S]$ are arrayed in the following diagram.

$$\begin{array}{c|c|cccc|ccc} x & 0 & a & b & c & d & a+b & a+c & a+d \\ \hline s-x & s & b+c+d & a+c+d & a+b+d & a+b+c & c+d & b+d & b+c \end{array}.$$

Again $\mathbf{E}(\mathbb{Z}_2[S]) = \{0\} \cup S$, $\mathcal{K} = \{0, s, a+b, c+d, a+c, b+d, a+d, b+c\}$ so that $A[S]/\mathcal{K} \cong \mathbb{Z}_2$ and $ann(\mathbb{Z}_2[S]) = \{0, s\}$. $\mathbb{Z}_2[S]/ann(\mathbf{Z}_2[S])$ has 8 elements that are parameterized by the x-row entries. $\mathbb{Z}_2[S]$ is weakly commutative with $\mathcal{K}$ equaling the nil radical $\mathcal{N}$. □

6.6 Idempotent-closed rings of matrices

In this section F is again a field and n is a positive natural number. We characterize those idempotent-closed subrings of $\mathcal{M}_n(F)$ that are maximal subject to certain constrains. This is done modulo maximal idempotent-closed matrix subrings R for which $\mathbf{E}(R) \subseteq \{0, I\}$. While we are unaware of any general characterization of the latter, if F is the field of complex numbers $\mathbb{C}$, a result of Livshits *et al* [2003] implies that if A is an *algebra* in $\mathcal{M}_n(\mathbb{C})$ such that $\mathbf{E}(A) \subseteq \{0, \mathrm{I},\}$, then either $A = N$ or $A = \mathbb{C}I + N$ for some nil algebra N. A maximal such algebra would be simultaneously similar to the algebra of all upper-triangular matrices with constant diagonals, since any subring of nilpotent matrices in $\mathcal{M}_n(\mathbb{C})$ is triangularizable. (See Example 6.7.2 below. See also Okninski [1997] or Radjavi and Rosenthal [2000].)

The abelian case

We begin with maximal idempotent-covered abelian subrings of $\mathcal{M}_n(F)$. Such subrings necessarily have and identity E and the "maximal" constraint insures that E = I. The following pair of extreme examples are suggestive of the general case.

Example 6.6.1. R is the abelian subring of all diagonal matrices in $\mathcal{M}_n(F)$ and $\mathbf{E}(R)$ is the set of all diagonal matrices with only 0 or 1 entries. *R is a maximal idempotent-closed ring in* $\mathcal{M}_n(F)$. For suppose that ring R' in $\mathcal{M}_n(F)$ properly contains R. Let A ∈ R' be a non-diagonal matrix and let $i \neq j$ be such that $\mathrm{E}_i\mathrm{A}\mathrm{E}_j \neq 0$. ($\mathrm{E}_i$ is the matrix with the *i-th* diagonal entry 1, and 0 elsewhere.) Then $(\mathrm{E}_i + \mathrm{E}_i\mathrm{A}\mathrm{E}_j)^2 = \mathrm{E}_i + \mathrm{E}_i\mathrm{A}\mathrm{E}_j \in \mathbf{E}(R)$, but $(\mathrm{E}_i + \mathrm{E}_i\mathrm{A}\mathrm{E}_j)\mathrm{E}_j = \mathrm{E}_i\mathrm{A}\mathrm{E}_j$ is nilpotent. Thus R' is not idempotent-closed.

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

Example 6.6.2. Let R consist of all upper triangular matrices in $\mathcal{M}_n(F)$ with constant diagonals. R is trivially idempotent-closed since $\mathbf{E}(R) = \{0, \mathrm{I}\}$.

$$\begin{bmatrix} d & a_{1,2} & \cdots & a_{1,n} \\ 0 & d & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d \end{bmatrix}$$

For any field, R is maximal subject to $\mathbf{E}(R) = \{0, \mathrm{I}\}$. In fact, *$R$ is a maximal idempotent-closed subring of* $\mathcal{M}_n(F)$. For its verification, see the discussion in Example 9 in Cvetko-Vah and Leech [2011]. That a subring has only idempotents 0 and I is unremarkable. But that a maximal idempotent-closed subring of $\mathcal{M}_n(F)$ can be thus is interesting. □

Theorem 6.6.1. *For each $n \geq 1$, $\mathcal{M}_n(F)$ has a maximal idempotent-closed subring R that is an algebra over F and for which* $\mathbf{E}(R) = \{0, \mathrm{I}\}$. □

Conversely, one may ask: are all maximal idempotent-closed subalgebras R of $\mathcal{M}_n(F)$ for which $\mathbf{E}(R) = \{0, I\}$ simultaneously similar to such an example, as happens when $F = \mathbb{C}$? In any case, a general way of constructing maximal idempotent-closed subrings of $\mathcal{M}_n(F)$ with identity I (and hence abelian) follows from the next result.

Proposition 6.6.2. *An idempotent-closed ring R in $\mathcal{M}_n(F)$ containing I is similar simultaneously to a ring in block form* (1) *below where for each index i all blocks D_i form a subring R_i in $\mathcal{M}_{n(i)}(F)$ with* $\mathbf{E}(R_i) = \{0_{n(i)}, I_{n(i)}\}$. *$R$ is maximally idempotent-closed in $\mathcal{M}_n(F)$ if and only if each R_i is maximally idempotent-closed in the matrix ring $\mathcal{M}_{n(i)}(F)$.*

$$\text{(1)} \qquad \mathrm{U} = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_n \end{bmatrix}$$

Proof. By Proposition 6.4.1, $\mathbf{E}(R)$ is a Boolean algebra. Let $\mathrm{E}_1, \ldots, \mathrm{E}_k$ be the atoms of $\mathbf{E}(R)$. Then $\mathrm{E}_1 + \ldots + \mathrm{E}_k = \mathrm{I}$ and we can choose a basis for F^n such that in this basis each E_i is similar to the diagonal matrix that has $I_{n(i)}$ (of the appropriate dimension) on the *i-th* diagonal block entry and 0s elsewhere. All elements of $\mathbf{E}(R)$ are sums of atoms and thus diagonal with the block form (1) where each $D_i = I_{n(i)}$ or $0_{n(i)}$. Given

$$V = \begin{bmatrix} V_{11} & V_{12} & \cdots & V_{1n} \\ V_{21} & V_{22} & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_{1n} & V_{2n} & \cdots & V_{nn} \end{bmatrix}$$

in R. If $V_{ij} \neq 0$ for some $i \neq j$, then $(E_i + E_iVE_j) \in \mathbf{E}(R)$ but $(E_i + E_iVE_j)E_j = E_iVE_j \notin \mathbf{E}(R)$ and R is not idempotent-closed. Thus under this basis R is indeed similar to a ring in the stated block from with each block ring $R_i \cong \{E_iVE \mid V \in R\}$. The atomic nature of the E_i insures that each $\mathbf{E}(R_i)$ is as stated. Given the diagonal block design, the final assertion is clear. ☐

R is isomorphically a direct sum $\oplus_i R_i$. Internally,

$$R = E_1RE_1 \oplus \ldots \oplus E_kRE_k.$$

What is more,

$$\mathbf{E}(R) = \mathbf{E}(E_1RE_1) \oplus \cdots \oplus \mathbf{E}(E_kRE_k) = \{E_1, 0\} \oplus \cdots \oplus \{E_k, 0\}$$

in that each idempotent in R decomposes uniquely as a sum of idempotents in each E_iRE_i.

The general case: the idempotents

To pass to idempotent-closed subrings of $\mathcal{M}_n(F)$ in general, we first look at bands and skew lattices in matrix rings. As it turns out: *all bands in $\mathcal{M}_n(F)$ are simultaneously similar to a band of upper-triangular matrices*. Consequently, *each skew lattice in $\mathcal{M}_n(F)$ is simultaneously similar to a skew lattice of upper-triangular matrices.* The result for bands was proved for algebraically closed fields by Radjavi [1997]. The arbitrary case for bands follows from Okninski's results [1997]. For maximal normal bands, i.e. maximal skew Boolean algebras, results of Cvetko-Vah ([2005b] and [2007]) are relevant. We summarize her results for convenience.

Theorem 6.6.3. *In any matrix ring $\mathcal{M}_n(F)$, the following are true:*

1. *Given a normal skew lattice S in $\mathcal{M}_n(F)$, all matrices in S can be simultaneously triangularized to form an isomorphic skew lattice of matrices S^Δ of the following fixed block form where $0'$ and $0''$ are fixed square 0-blocks, each D_i is a 0 or I square block of fixed dimensions, $A_iD_i = A_i$, $D_iB_i = B_i$ and $C = \sum A_iB_i$:*

(2)
$$U = \begin{bmatrix} 0' & A_1 & \cdots & A_k & C \\ 0 & D_1 & \cdots & 0 & B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & B_k \\ 0 & 0 & \cdots & 0 & 0'' \end{bmatrix}$$

2. *This triangularization can be chosen so that for all such U, the following diagonal matrix E_U also lies in S^Δ with $U \mathcal{D} E_U$, with the E_U collectively form a lattice section S_0 of S^Δ.*

$$E_U = \begin{bmatrix} 0' & 0 & \cdots & 0 & 0 \\ 0 & D_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & 0 \\ 0 & 0 & \cdots & 0 & 0'' \end{bmatrix}$$

3. *When S is a maximal normal skew lattice, and the elements of S^Δ in block form look like*

$$U = \begin{bmatrix} 0 & A & C \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix} \quad \textit{with} \quad E_U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where D is any possible 0-1 diagonal matrix of a fixed size and A, B and C are all possible submatrices of appropriate dimensions such that $AD = A$, $DB = B$ and $AB = C$. The lattice section S_0 of all E_U forms a Boolean algebra.

(The pattern allows for the possibility that either $0'$ or $0''$ vanishes or both. In the first case the $0'$-row and $0'$-column disappear so that the main diagonal begins with D. Similar remarks hold when $0''$ or both vanish. The maximal case of "both" is the idempotent part of Example 6.6.1.)

The general case: the idempotent-closed subrings

If R is an idempotent-closed ring in $\mathcal{M}_n(F)$ and E is in the maximal $\mathcal{D}$-class of $\mathbf{E}(R)$ then ERE is a maximal abelian subring of R with the identity E and $\mathcal{B} = \mathbf{E}(\mathrm{E}R\mathrm{E})$ is a Boolean algebra**.**

Let $E_1, ..., E_k$ be the atoms of $\mathcal{B}$. Then for all $i \le k$, $R_i = E_iRE_i$ is a ring with identity E_i and no non-trivial idempotents, and $\mathcal{A} \cong R_1 \oplus \cdots \oplus R_k$ is an idempotent-covered ring with identity E and $\mathbf{E}(\mathcal{A}) \cong E_1\mathbf{E}(R)E_1 \oplus \cdots \oplus E_k\mathbf{E}(R)E_k$ as above. We are now ready to state the first of the two main results in this section.

Theorem 6.6.4. *If R is a maximal idempotent-closed and idempotent-dominated ring in $\mathcal{M}_n(F)$, then R is simultaneously similar to the ring of all matrices of block form*

(3)
$$\begin{bmatrix} 0' & A_1 & \cdots & A_k & C \\ 0 & D_1 & \cdots & 0 & B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & B_k \\ 0 & 0 & \cdots & 0 & 0'' \end{bmatrix}$$

where each D_i comes from a maximal idempotent-closed matrix subring R_i of $\mathcal{M}_{n(i)}(F)$ for which $\mathbf{E}(R) = \{I_{n(i)}, 0_{n(i)}\}$. *Idempotent matrices of this form have only $0_{n(i)}$s or $I_{n(i)}$s on the diagonal, and satisfy $A_iD_i = A_i$, $D_iB_i = B_i$ and $C = \sum A_iB_i$.*

(Remarks similar to those given for Theorem 6.6.1 apply when $0'$ or $0''$ vanish. When both vanish one has a maximal idempotent covered subring with central idempotents.)

Proof. We first choose a basis $\mathcal{B}$ for F^n such that relative to $\mathcal{B}$ all idempotents in R have matrix form (1). Assuming that all elements in R are matrix-represented relative to $\mathcal{B}$, the assumption that R is idempotent-dominated implies the leftmost column of blocks and the bottom row of blocks of these representations consist only of zero blocks, as in (3). What can arise in the central block of all R-matrices, the say $m \times m$ blocks that exclude the two outermost rows of blocks and the two outermost columns of blocks relative to (1).

$$\begin{bmatrix} D_1 & \cdots & ? \\ \vdots & \ddots & \vdots \\ ? & \cdots & D_k \end{bmatrix} \quad D_i \in \mathcal{M}_{n(i)}(F)$$

Given $i \in \{1, . . . , k\}$ let E_i be the matrix in $\mathcal{M}_{n(i)}(F)$ with $D_i = I_{n(i)}$ and 0 elsewhere. By Theorem 6.6.1 we may assume that all E_i lie in $\mathbf{E}(R)$. If $E = E_1 + \cdot \cdot \cdot + E_k$, then $\mathcal{A} = ERE$ is an idempotent-closed subring of R with identity E as are each E_iRE_i where $\mathbf{E}(E_iRE_i) = \{E_i, 0\}$. Note that for $\mathcal{A}$-matrices, all blocks in the first row and last column are also 0-blocks. Thus $\mathcal{A}$ is isomorphic to an idempotent-closed ring in $M_m(\mathcal{F})$ with identity I_m under the map sending each matrix P in $\mathcal{A}$ to its central $m \times m$ block. $\mathcal{A}$ is further block diagonalized with each set of D_i-blocks forming a ring R_i in $\mathcal{M}_{n(i)}(F)$ that is isomorphic to the subring E_iRE_i. Since $ERE = E_1RE_1$

$\oplus \ldots \oplus E_kRE_k$ by the remark after Proposition 6..7.2, given any matrix in $\mathcal{A}$, all nondiagonal blocks in its central submatrix are 0-blocks. Indeed, since P and EPE share the same central submatrix, every matrix P in R has this pattern in its central submatrix. Hence, given our assumptions on $\mathbf{E}(R)$, every matrix in R is at least of the block form (3).

Using simplified block-of-blocks format, $\begin{bmatrix} 0 & A & C \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & A' & C' \\ 0 & D' & B' \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & AD' & AB' \\ 0 & DD' & DB' \\ 0 & 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & A & C \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & A & C \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix}$ iff $\begin{cases} AD = A, \ \ DB = B, \\ AB = C \text{ and } D^2 = D. \end{cases}$ It follows that any such ring of matrices is idempotent-closed if and only if its central ring of D-matrices is idempotent-closed. Thus the status of a subring R as being idempotent-closed in $\mathcal{M}_n(\mathcal{F})$ remains unchanged if its design is extended by first allowing arbitrary A_i, B_i and C blocks and then enlarging each ring R_i of D_i-blocks to a maximal idempotent-closed subring of $\mathcal{M}_{n(i)}(F)$ that includes $0_{n(i)}$ and $I_{n(i)}$. If such a subring extension is idempotent-covered, then the A_i, B_i, C blocks and the R_i rings already had this maximal status as stated in the theorem.

It remains to see that the ring R of *all* matrices of form (3) is idempotent-dominated. We again use this simplified block form. To begin, the identities

$$\begin{bmatrix} 0 & A & C \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & D & B \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

reduce this verification to showing any $\begin{bmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ lies in Q_R. Such a matrix is a sum of matrices of this type having only a single nonzero entry; but any such summand easily factors into a product of the form $\begin{bmatrix} 0 & A & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}$ and thus lies in Q_R. □

$\mathcal{A}$ is the maximal abelian subring of R and the subring of all matrices in R for which all D_i-blocks are 0-blocks is the nilpotent ideal $\mathcal{K}_R$ encountered in Section 4. $R = \mathcal{A} \oplus \mathcal{K}$ as additive groups and as rings $R/\mathcal{K} \cong \mathcal{A}$ making $\mathcal{A}$ the maximal abelian image of R. (See Section 6.4.) Again in our simplified block notation:

$\mathcal{A}$ is all $\begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}$; $\mathcal{K}_R$ is all $\begin{bmatrix} 0 & A & C \\ 0 & 0 & B \\ 0 & 0 & 0 \end{bmatrix}$; max $\mathcal{D}$-class in $\mathbf{E}(R)$ is all $\begin{bmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{bmatrix}$.

Also, in the situation above, in the decomposition of R into an orthosum of rectangular ideals, the particular ideals involved are the k distinct rectangular subrings obtained by letting all D_i, A_i and B_i blocks be 0-matrices except for a particular index j where the D_j, A_j and B_j blocks are subject to just the constraints of Theorem 6.6.2 and the C block is generated by the A_jBj outcomes.

When $F = \mathbb{C}$ we have the following crisp result:

Corollary 6.6.5. *If R is a maximal idempotent-closed and idempotent-dominated ring in* $\mathcal{M}_n(\mathbb{C})$*, thenR is simultaneously similar to the ring of all matrices with the block form* (3) *above, where the* D_i *lie in the subring of all upper triangular matrices in* $\mathcal{M}_{n(i)}(\mathbb{C})$ *with constant diagonals*. □

Can such a ring be extended to a larger subring of $\mathcal{M}_n$(F), which although no longer idempotent-dominated, has no new idempotents and thus is still idempotent-closed? (A maximal idempotent-dominated and idempotent-closed subring R of $\mathcal{M}_n(F)$ cannot be contained in a properly larger idempotent-closed subring R' of $\mathcal{M}_n(F)$ unless R' has no new idempotents.)

Suppose that $0'$ in the upper left corner of the above block design is a $p\times p$ 0-matrix and that $0''$ in the lower right is a $q\times q$ 0-matrix. Let T' be a subring of $\mathcal{M}_p(F)$ that is maximal with respect to only having 0 as an idempotent. Likewise, let T'' be a subring of $\mathcal{M}_q(F)$ that is maximal with respect to only having 0 as an idempotent. Such "fringe" subrings are trivially idempotent-closed. Consider the following design

$$(4) \qquad \begin{bmatrix} G & A_1 & \cdots & A_k & C \\ 0 & D_1 & \cdots & 0 & B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & B_k \\ 0 & 0 & \cdots & 0 & H \end{bmatrix}$$

where the As, Bs, Cs and Ds are as in Theorem 6.6.4 above but G ∈ T′ and H ∈ T″. When G and H are 0 we are in the previous context. The collection R' of all matrices with this design forms a subring of $\mathcal{M}_n(F)$. But in this larger ring no new idempotents are created since

$$\begin{bmatrix} G & A_1 & \cdots & A_k & C \\ 0 & D_1 & \cdots & 0 & B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & B_k \\ 0 & 0 & \cdots & 0 & H \end{bmatrix}^2 = \begin{bmatrix} G & A_1 & \cdots & A_k & C \\ 0 & D_1 & \cdots & 0 & B_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_k & B_k \\ 0 & 0 & \cdots & 0 & H \end{bmatrix}$$

precisely when first $G^2 = G = 0$ and $H^2 = H = 0$ and the As, Bs, Cs and Ds behave as described in Theorem 6.6.4. We have proved the following theorem.

Theorem 6.6.6. *Let R be the set of all matrices of a Type* (4) *design in* $\mathcal{M}_n(F)$ *where*:

1) *The A_i, B_j and C blocks can be any matrix of the prescribed dimensions*.

2) *The D_i-blocks are subject to the constraints of Theorem* 6.6.4.

3) *G and H belong to subrings T′ and T″ of* $\mathcal{M}_p(F)$ and $\mathcal{M}_q(F)$ *respectively, that are maximal with respect to having no nonzero idempotent*.

Then R is a maximal idempotent-closed subring of $\mathcal{M}_n(F)$*. Its Type* (3) *matrix subring is a maximal idempotent-closed and dominated subring of* $\mathcal{M}_n(F)$*.*

The converse (*a maximal idempotent-closed subring of* $\mathcal{M}_n(F)$ *containing a maximal idempotent-closed and dominated subring of* $\mathcal{M}_n(F)$*, is simultaneously similar to a ring of the above type*) is also true. Its proof is given in Cvetko-Vah and Leech [2011].

Historical remarks

The results in Section 6.1 are from Leech's initial paper on skew lattices [1989]. Those in Sections 6.2 and 6.3 are from Cvetko-Vah and Leech's 2008 paper, some of which generalized results in two earlier papers of Cvetko-Vah ([2004] and ([2005a]. All results in Sections 6.4 and 6.5 are from the 2012 paper of Cvetko-Vah and Leech. Section 6.6 is based on earlier results of Cvetko-Vah ([2005b] and [2007]).except for those mentioned at the end which are from their 2011 paper.

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VII: FURTHER TOPICS IN SKEW BOOLEAN ALGEBRAS

In Chapter IV the implicit perspective of a skew Boolean algebra is that of a skew lattice with added structure, namely a constant 0 and a difference operation \, which with $\wedge$ and $\vee$ satisfy certain identities. There is an alternative approach: to consider algebras with a reduced signature ($\wedge$, \, 0) or even (\, 0) that are subject to a set of identities, and view skew Boolean algebras as instances where these simpler algebras acquire added structure. These reduced algebras can, however, be of independent interest, as was the case for the iBCK algebras encountered below.

This chapter begins by looking at algebras of signature (\, 0) that satisfy "subtractive" identities such as $x \backslash 0 = x$ and $x \backslash x = 0 = 0 \backslash x$. Indeed Section 7.1 considers seven such identities. The first six (indeed the first four) characterize implicative BCS-algebras (or just *iBCS-algebras*). The significance of these algebras lies in the fact that (\, 0)-*reducts of skew Boolean algebras are iBCS-algebras*. The seventh identity, when joined to the rest, characterizes implicative BCK-algebras (or just *iBCK-algebras*). *A skew Boolean algebra is a generalized Boolean algebra if and only if its* (\, 0)-*reduct is an iBCK-algebra*. A skew Boolean algebra is simultaneously both a strongly distributive skew lattice and an iBCS-algebra, having both types of algebras as reducts. The *Signature Bisection Theorem* (Theorem 7.1.1) tells how a strongly distributive skew lattice and an iBCS-algebra, if defined on a common set, must interact to form a skew Boolean algebra having the initial pair of algebras as reducts. We next define an alternative iBCK-difference / on skew Boolean $\cap$-algebras by $x/y = x \backslash x \cap y$, in which case the reduct (S; /, 0) is an iBCK-algebra. Theorem 7.1.2 characterizes skew Boolean $\cap$-algebras as algebras of signature ($\vee$, $\wedge$, /, 0).

The section continues by looking at the role of discriminator terms and discriminator varieties. Skew Boolean algebras form a binary discriminator variety (Theorem 7.1.4) and all binary discriminator algebras have an iBCS-algebra reduct; what is more, all iBCS-algebras also have left-normal-band-with-0 reducts. Thus much, but not all, of the structure of a left-handed skew Boolean algebra is encoded in its iBCS reduct (S; /, 0). The main result (Theorem 7.1.9) states: *If* $\mathbf{V}$ *is a binary discriminator variety with constant term* $\mathbf{0}$ *and additive term* $\boldsymbol{x+y}$, *then every algebra* $\mathbf{A}$ *of* $\mathbf{V}$ *has a left-handed skew Boolean algebra term reduct* $\mathbf{A_S}$. (Here $\boldsymbol{x+y}$ is a binary term satisfying $x+0 = x = 0+x$.) Clearly a right-handed version also holds. The section concludes by looking at ternary discriminator varieties. Theorems 7.1.10 and 7.1.11 reveal a close connection between skew Boolean $\cap$-algebras and pointed ternary discriminator varieties.

Most of Section 7.1 is based on the work of Robert Bignall and his student Matthew Spinks, with some input from Jonathan Leech.

7.1 Differences, discriminators and connections with other algebras

Besides a strongly distributive skew lattice reduct (S; ∨, ∧), a skew Boolean algebra also has a complementary reduct, (S; \, 0). To understand the behavior of the latter, consider the following identities:

(*a*) $x \setminus x \approx 0$.
(*b*) $x \setminus (y \setminus x) \approx x$.
(*c*) $(x \setminus y) \setminus z \approx (x \setminus z) \setminus y$.
(*d*) $(x \setminus y) \setminus z \approx (x \setminus z) \setminus (y \setminus z)$.
(*e*) $x \setminus 0 \approx x$
(*f*) $0 \setminus x \approx 0$,
(*g*) $x \setminus (x \setminus y) \approx y \setminus (y \setminus x)$.

An algebra (S; \, 0) of type (2, 0) satisfying (*a*) – (*d*) is called an ***implicative BCS-algebra*** (***iBCS-algebra*** for short), in which case it also satisfies both (*e*) and (*f*) making it a 0-subtractive algebra. Indeed (*e*) follows by putting $y = x$ in (*b*) and then using (*a*), while (*f*) follows by setting $x = 0$ in (*b*) and then using (*e*). If in addition (S; \, 0) satisfies the "commutative" identity (*g*), it is called an ***implicative BCK-algebra*** (***iBCK-algebra*** for short).

iBCK-algebras were introduced in Lyndon [**1951**]. They have been studied by various authors including Abbott [**1967**], Cornish [**1982**], Iseki and Tanaka [**1978**] and Kalman [**1960**]. iBCS-algebras were introduced and studied by Bignall and Spinks in [**2003**] and [**2007**]. For skew Boolean algebras we have the following *Signature Bisection Theorem*.

Theorem 7.1.1. *An algebra* (S; ∨, ∧, \, 0) *of type* ⟨2, 2, 2, 0⟩ *forms a skew Boolean algebra if and only if*:

i) (S; ∨, ∧) *is a strongly distributive skew lattice*.
ii) (S; \, 0) *is an iBCS-algebra*.
iii) *The identity* $e \setminus (e \setminus f) \approx e \wedge f \wedge e$ *holds.*

(S; ∨, ∧, \, 0) *is a generalized Boolean algebra if and only if* (ii) *can be strengthened to*:

ii′) (S; \, 0) *is an iBCK-algebra*.

Proof Given skew Boolean algebra, (S; ∨, ∧, \, 0), (i) holds and it is easily seen that the reduct (S; \, 0) is an iBCS algebra and that $e \setminus (e \setminus f) \approx e \wedge f \wedge e$. Just check the situation for 0-primitive algebras. Thus (i) - (iii) follow. The converse will be proved after Theorem 1.11 below. Given (i)-(iii), (S; \, 0) is an iBCK-algebra if and only if identity (g) holds, which in this context is equivalent to $e \wedge f \wedge e \approx f \wedge e \wedge f$ holding. But the latter is equivalent to $e \wedge f \approx f \wedge e$, making (S; ∨, ∧) a lattice and (S; ∨, ∧, \, 0) a generalized Boolean algebra. ☐

To distinguish an iBCK operation from the more general iBCS operation, we use the symbol / when referring to the former. *Given an iBCK algebra* (S; /, 0), *set* $x \cap y = x/(x/y)$. Then:

$$x\cap x = x/(x/x) = x/0 = x; \quad x\cap 0 = x/(x/0) = x/x = 0; \quad \text{and } x\cap y = y\cap x \text{ by } (g).$$

With a bit more work one sees that (S; $\cap$, 0) *is a meet semilattice with a minimum* 0 *such that each principal ideal* $\lceil x\rceil = \{y\,|\,y \le x\}$ *forms a Boolean lattice, with* x/y *being the complement of* $x\cap y$ *in* $\lceil x\rceil$. For any pair $u, v \in \lceil x\rceil$, $u \vee v$ is given as $x / [(x/u) \cap (x/v)]$. (See the references above.) The algebra (S; $\cap$, /, 0) is sometimes called a ***Boolean semilattice***.

This is relevant to skew Boolean $\cap$-algebras. Given such an algebra (S: $\vee$, $\wedge$, \, $\cap$, 0), define the ***iBCK difference*** e/f on S by

$$e/f \;=\; e \setminus e\cap f.$$

For all e and f both differences agree, that is $e/f = e\setminus f$, if and only if e and f commute. Indeed one has $e \setminus e\wedge f\wedge e = e\setminus e\cap f$ if and only if $e\wedge f\wedge e = e\cap f$. But then $e\wedge f = e\wedge f\wedge e\wedge f = (e\cap f) \wedge f = e\cap f$ and likewise, $f\wedge e = e\cap f$. The converse is clear. In particular, $e/f = e\setminus f$, if $e \ge f$. Both $\cap$ and the skew Boolean difference \ can be recovered from the iBCK difference / by

$$e\cap f \;=\; e/(e/f) \quad \text{and} \quad e\setminus f \;=\; e/(e\wedge f\wedge e).$$

Thus skew Boolean $\cap$-algebras can be viewed as algebras with three binary operations: $\vee$, $\wedge$, and /, plus a constant, 0. What identities involving $\{\vee, \wedge, /, 0\}$ characterize such algebras? Our remarks above, together with an examination of what occurs in the primitive case, yield a *Second Signature Bisection Theorem* due to Bignall and Leech [1995].

Theorem 7.1.2. *Every skew Boolean* $\cap$*-algebra* (S: $\vee$, $\wedge$, \, $\cap$, 0) *is term equivalent to an algebra*, (S; $\vee$, $\wedge$, /, 0) *of type* $\langle 2, 2, 2, 0\rangle$ *where*:

i) (S; $\vee$, $\wedge$, 0) *is a symmetric, normal skew lattice with* 0.
ii) (S; /, 0) *is an iBCK-algebra.*
iii) *The induced meet,* $e\cap f = e/(e/f)$, *of* (S; /, 0) *satisfies the identities of Lemma* 4.3.2:

$$e \cap (e\wedge f\wedge e) = e\wedge f\wedge e \;\; \textit{and} \;\; e \wedge (e\cap f) = e\cap f = (e\cap f) \wedge e.$$

Proof. Given a skew Boolean $\cap$-algebra, (i) is clear. The reduct (S; /, 0), where $e/f = e \setminus e\cap f$, satisfies the conditions for an iBCK algebra on primitive skew Boolean algebras and hence on all skew Boolean $\cap$-algebras so that (ii) follows. Finally, the iBCK meet $e/(e/f)$ reduces to the natural intersection $\cap$. Indeed since both e/f, $e\cap f \le e$, we get

$$e/(e/f) = e\setminus(e/f) = e\setminus(e\setminus e\cap f) = e\cap f,$$

and (iii) follows. Conversely, given (i), (S; $\vee$, $\wedge$, 0) is a symmetric, normal skew lattice with zero 0. By (iii) it shares a common natural partial order $\le$ with the /-induced meet $\cap$. This forces both algebras to share a common natural meet $\cap$. Each common principal ideal $\lceil e\rceil = \{y\,|\,y \le e\}$ is a Boolean lattice by (ii), which forces $(S/\mathcal{D}; \vee, \wedge)$ to be a generalized Boolean lattice. It follows that (S; $\vee$, $\wedge$, $\cap$, 0) is an implicit skew Boolean $\cap$-algebra with iBCK difference /. ☐

(Dual) binary discriminator varieties and iBCS algebras

Let A be a set with distinguished element 0. The ***binary 0-discriminator*** and the ***dual binary 0-discriminator*** on A are defined respectively by

$$x \setminus y = \begin{cases} x \text{ } if \text{ } y = 0 \\ 0 \text{ } otherwise \end{cases} \quad \text{and} \quad x \wedge y = \begin{cases} 0 \text{ } if \text{ } y = 0 \\ x \text{ } otherwise \end{cases}.$$

Clearly $x \setminus y$ and $x \wedge y$ are left-handed skew Boolean operations on A as a primitive skew Boolean algebra. When no ambiguity exists about the constant 0 we simply use the term (***dual***) ***binary discriminator***. The details of the following lemma are easily verified.

Lemma 7.1.3. *Given a set* A *with constant* 0, *the functions* $x \setminus y$ *and* $x \wedge y$ *satisfy identities*:

B1.	$a \wedge a = a$.	B2.	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
B3.	$a \wedge (b \wedge c) = a \wedge (c \wedge b)$.	B4.	$a \wedge 0 = 0 \wedge a = 0$.
B5.	$a \wedge b = a \setminus (a \setminus b)$.	B6.	$a \setminus a = 0$.
B7.	$(a \setminus b) \setminus c = (a \setminus c) \setminus b$.	B8.	$(a \setminus b) \setminus c = (a \setminus c) \setminus (b \setminus c)$.
B9.	$a \setminus (b \setminus a) = a$.	B10.	$a \setminus 0 = a$.
B11.	$0 \setminus a = a$.	B12.	$(a \setminus b) \setminus b = a \setminus b$. ☐

Identities B1 to B4 characterize a left normal band with zero. Identities B6 through B9 reprise identities (*a*) – (*d*) at the onset of this section, with any algebra (A; \, 0) satisfying them being an iBCS-algebra. *For such algebras* B10 – B12 *also hold*. B10 and B11 are just (*e*) and (*f*) above, while

$$(a \setminus b) \setminus b =_{\mathrm{B8}} (a \setminus b) \setminus (b \setminus b) =_{\mathrm{B6}} (a \setminus b) \setminus 0 =_{\mathrm{B10}} a \setminus b.$$

A variety $\mathcal{V}$ with a constant term **0** is a ***[dual] binary discriminator variety*** if a binary term $\boldsymbol{x \setminus y}$ [$\boldsymbol{x \wedge y}$] exits such that $\mathcal{V}$ is generated by a class $\mathcal{K}$ of algebras on which that binary term induces the [dual] binary 0-discriminator.

Binary discriminator varieties are widespread. Examples include Stone algebras, pseudo-complemented semilattices, implicative BCK-algebras and, as we will shortly show, any ternary discriminator variety with a constant term.

Any binary discriminator variety is also a dual binary discriminator variety, due to B5, but not conversely. Indeed the variety of left normal bands with zero is a dual binary 0-discriminator variety, but it cannot be a binary 0-discriminator variety because any term $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{y})$ in the language of left normal bands with zero in which both variables appear explicitly must satisfy the implication: if $y = 0$ then $t(x, y) = 0$. Hence a binary term $\boldsymbol{x \setminus y}$ satisfying the identity $\boldsymbol{x \setminus 0 \approx x}$ cannot be defined in the band.

These definitions generalize the concepts of pointed ternary discriminator and pointed ternary discriminator variety. Indeed from a ternary discriminator $d(x, y, z)$ and 0 one defines the binary discriminator by setting $x \setminus y = d(0, y, x)$. Thus pointed ternary discriminator varieties are binary discriminator varieties and all the later are dual binary discriminator varieties. These relationships cannot be reversed in general. Allowing for a bit of repetition we have:

Theorem 7.1.4. *Skew Boolean algebras form a binary discriminator variety with* \ *being a binary 0-discriminator on primitive skew Boolean algebras. Thus if* (S: ∨, ∧, \, 0) *is a skew Boolean algebra, then the reduct* (S: \, 0) *is an iBCS algebra.* □

It is well known that the variety of left normal bands with zero is generated by the three-element band ({0, 1, 2}; ∧, 0), where ∧ is the dual binary discriminator on the base set {0, 1, 2}. Since any algebra of the form (A; ∧, 0) is a left normal band when ∧ is the dual binary discriminator on A, the class of left normal bands with zero is called the ***generic*** dual binary discriminator variety. Strongly distributive skew lattices with zero provide another example of a dual binary discriminator variety.

Before giving a semigroup characterization of iBCS-algebras we will need some further properties of iBCS-algebras. We follow Matthew Spinks' Monash University dissertation [2002].

Lemma 7.1.5. *Upon setting* $x \wedge y = x \setminus (x \setminus y)$, *an iBCS-algebra also satisfies identities*:

B13. $(a \setminus b) \setminus (c \setminus a) \approx a \setminus b$.
B14. $a \setminus (b \setminus (c \setminus a)) \approx a \setminus b$.
B15. $a \setminus (a \setminus (a \setminus b)) \approx a \setminus b$.
B16. $(a \wedge b) \setminus c = (a \setminus c) \setminus (a \setminus b)$.
B17. $(a \setminus c) \setminus b = (a \setminus c) \setminus (a \setminus b)$.
B18. $(a \setminus c) \wedge (b \setminus c) = (a \setminus c) \setminus (a \setminus b)$.
B19. $a \wedge (b \setminus c) = (a \setminus c) \setminus (a \setminus b)$.

Proof.

B13. $(a \setminus b) \setminus (c \setminus a) =_{B7} [a \setminus (c \setminus a)] \setminus b =_{B9} a \setminus b$.

B14. $a \setminus (b \setminus (c \setminus a)) =_{B9} [a \setminus (c \setminus a)] \setminus [b \setminus (c \setminus a)] =_{B8} (a \setminus b) \setminus (c \setminus a) =_{B13} a \setminus b$.

B15. $a \setminus b =_{B9} (a \setminus b) \setminus [a \setminus (a \setminus b)] =_{B8} \{a \setminus [a \setminus (a \setminus b)]\} \setminus \{b \setminus [a \setminus (a \setminus b)]\}$
$=_{B13} \{a \setminus [a \setminus (a \setminus b)]\} \setminus (b \setminus a) =_{B7} [a \setminus (b \setminus a)] \setminus [a \setminus (a \setminus b)]$
$=_{B9} a \setminus (a \setminus (a \setminus b))$.

B16. $(a \wedge b) \setminus c =_{B5} [a \setminus (a \setminus b)] \setminus c =_{B7} (a \setminus c) \setminus (a \setminus b)$.

B17. $(a \setminus c) \wedge b =_{B5} (a \setminus c) \setminus [(a \setminus c) \setminus b] =_{B7} (a \setminus c) \setminus [(a \setminus b) \setminus c]$
$=_{B8} [a \setminus (a \setminus b)] \setminus c =_{B5} (a \wedge b) \setminus c =_{B16} (a \setminus c) \setminus (a \setminus b)$.

B18. $(a \setminus c) \wedge (b \setminus c) =_{B5} (a \setminus c) \setminus [(a \setminus c) \setminus (b \setminus c)] =_{B8} (a \setminus c) \setminus [(a \setminus b) \setminus c]$
$=_{B7} (a \setminus c) \setminus [((a \setminus c) \setminus b] =_{B5} (a \setminus c) \wedge b =_{B17} (a \setminus c) \setminus (a \setminus b)$.

B19. $a \wedge (b \setminus c) =_{B5} a \setminus [(a \setminus (b \setminus c)] =_{B9} [a \setminus (c \setminus a)] \setminus [a \setminus (b \setminus c)]$
$=_{B7} (a \setminus [a \setminus (b \setminus c)]) \setminus (c \setminus a)$

$$=_{B13}\ (a \setminus [(a \setminus (b \setminus c)]) \setminus (c \setminus [a \setminus (b \setminus c)])\ =_{B8}\ (a \setminus c) \setminus (a \setminus (b \setminus c))$$
$$=_{B7}\ (a \setminus [a \setminus (b \setminus c)]) \setminus \mathrm{c}\ =_{B8}\ (a \setminus c) \setminus ([(a \setminus (b \setminus c)] \setminus \mathrm{c})$$
$$=_{B7}\ (a \setminus c) \setminus\ [(a \setminus c) \setminus (b \setminus c)]\ =_{B5}\ (a \setminus c) \wedge (b \setminus c)\ =_{B18}\ (a \setminus c) \setminus (a \setminus b).\ \square$$

This leads us to:

Theorem 7.1.6. *Given an iBCS-algebra* (A; \, 0), *upon setting* $x \wedge y = x \setminus (x \setminus y)$ *the derived algebra* (A; ∧, 0) *is a left normal band with zero*.

Proof. B1 follows from $a \wedge a = a \setminus (a \setminus a) = a \setminus 0 = a$. B4 follows from

$$a \wedge 0 = a \setminus (a \setminus 0) = a \setminus a = 0 \text{ and } 0 \wedge a = 0 \setminus (0 \setminus a) = 0$$

provided we know that $0 \setminus a = 0$ holds in general. But the latter follows from $0 \setminus a = (a \setminus a) \setminus a = (a \setminus a) \setminus (a \setminus a) = 0 \setminus 0 = 0$. To verify B2 and B3 we again follow Spinks [2002]. To begin, observe that

$$(a \wedge b) \wedge c\ =_{B5}\ (a \wedge b) \setminus [(a \wedge b) \setminus c]\ =_{B5,\,16}\ [a \setminus (a \setminus b)] \setminus [(a \setminus c) \setminus (a \setminus b)]$$
$$=_{B8}\ [a \setminus (a \setminus c)] \setminus (a \setminus b)\ =_{B7}\ [a \setminus (a \setminus b)] \setminus (a \setminus c)$$

where the latter expression must also equal $(a \wedge c) \wedge b$. Hence establishing B2 will also establish B3. But

$$(a \wedge b) \wedge c\ =_{\text{again}}\ [a \setminus (a \setminus c)] \setminus (a \setminus b)\ =_{B8}\ [a \setminus (a \setminus b)] \setminus [(a \setminus c) \setminus (a \setminus b)]$$
$$=_{B7}\ \{a \setminus [(a \setminus c) \setminus (a \setminus b)]\} \setminus (a \setminus b)$$
$$=_{B19}\ \{a \setminus [a \wedge (b \setminus c)]\} \setminus (a \setminus b)\ =_{B5}\ [a \setminus (a \setminus (a \setminus (b \setminus c)))] \setminus (a \setminus b)$$
$$=_{B15}\ [a \setminus (b \setminus c)] \setminus (a \setminus b)\ =_{B19}\ \mathrm{a} \wedge [b \setminus ((b \setminus c))]$$
$$=_{B5}\ a \wedge (b \wedge c).\ \square$$

Theorem 7.1.6 holds trivially for any iBCS-algebra arising as a reduct of an algebra in a binary discriminator variety, for one begins with a generating class $\mathcal{K}$ of binary discriminator algebras that implicitly satisfy B1-B5, where B5 defines ∧. From these algebras B1-B5 are passed to all algebras in the variety. Although we do not show this here, the variety of iBCS-algebras is in fact a binary discriminator variety (Bignall and Spinks [2007]). Theorem 7.1.6 is thus a corollary to this fact. This theorem, however, takes us only halfway to characterizing iBCS-algebras in terms of left normal bands with zero. Indeed:

Theorem 7.1.7. *If* (S; ∧, 0) *is the derived left normal band with zero of an iBCS-algebra* (S; \, 0), *then for each* $a \in S$ *the set* $\lceil a \rceil = \{b \in \mathrm{S} \mid b \leq a\}$ *is both a subalgebra of* S *and a Boolean lattice under the natural partial ordering of* S.

Conversely, given a left normal band with zero (S; ∧, 0) *such that for each* $a \in S$ *the set* $\lceil a \rceil$ *is a Boolean lattice under* ≥, *a derived iBCS-structure* (S; \, 0) *is given by letting* $a \setminus b$ *be the relative complement of* $a \wedge b$ *in* $\lceil a \rceil$ *for all a, b in* S.

Finally, both derivations (S; \, 0) → (S; ∧, 0) *and* (S; ∧, 0) → (S; \, 0) *are reciprocal. In particular,* $a \setminus b$ *is the complement of* $a \wedge b$ *in* $\lceil a \rceil$.

Proof. First, note that for left normal bands $\leq$ is described by $b \leq a$ if $a \wedge b = b$ and $\lceil a \rceil$ is the set $\{a \wedge s \mid s \in S\}$. Let b, c lie in $\lceil a \rceil$. By Lemma 4.4.9, $b \setminus c = (a \wedge b) \setminus (a \wedge c) = a \wedge (b \setminus c) \in \lceil a \rceil$ also. In addition $0 = a \wedge 0 \in \lceil a \rceil$ so that $\lceil a \rceil$ is seen to be an iBCS-subalgebra of S. Since $\wedge$ is commutative on $(\lceil a \rceil; \setminus, 0)$, the latter satisfies the iBCK identities (i) – (iv) above and in particular (ii) which implies that $\wedge$ is commutative on $\lceil a \rceil$. Since $(\lceil a \rceil; \setminus, 0)$ is an iBCK-algebra with maximal element a, it forms a Boolean lattice $(\lceil a \rceil; \vee, \wedge, \setminus, a, 0)$ with $\wedge$ and $\setminus$ as already given, and $x \vee y$ defined as $a \setminus [(a \setminus x) \wedge (a \setminus y)]$ for all $x, y \leq a$.

Conversely, let $(S; \wedge, 0)$ be a left normal band with zero such that for each $a \in S$ the set $\lceil a \rceil$ is a Boolean lattice under $\leq$. Let $a \setminus b$ denote the usual relative complement $a \setminus a{\wedge}b$ in $\lceil a \rceil$. The clearly B6 holds. B7 reduces to

$$(a \setminus a{\wedge}b) \setminus [(a \setminus a{\wedge}b) \wedge c] \approx (a \setminus a{\wedge}c) \setminus [(a \setminus a{\wedge}c) \wedge b].$$

This holds in the Boolean case of $\lceil a \rceil$ where both sides reduce to $[a \setminus (a{\wedge}b \vee a{\wedge}c)] \vee (a{\wedge}b{\wedge}c)$. In similar fashion B8 and B9 are seen to hold. That both processes are reciprocal follows from B14 which implies $a \setminus (a \wedge b) = a \setminus b$ and the Boolean identity $a \setminus (a \setminus a{\wedge}b) = a{\wedge}b$. ☐

Observe that this theorem implies the converse of the main statement of Theorem 7.1.1 in the case of left-handed skew Boolean algebras. That is, we have:

Theorem 7.1.1$_\mathbf{L}$. *An algebra* $(S; \vee, \wedge, \setminus, 0)$ *of type* $\langle 2, 2, 2, 0\rangle$ *forms a left-handed skew Boolean algebra if and only if*:

i) $(S; \vee, \wedge)$ *is a left-handed strongly distributive skew lattice.*
ii) $(S; \setminus, 0)$ *is an iBCS-algebra.*
iii) *The identity* $e \setminus (e \setminus f) \approx e{\wedge}f$ *holds.*

Proof. $(\Rightarrow)$ is clear. For $(\Leftarrow)$, (ii) and (iii) along with the previous theorem imply that for all a in S, $\lceil a \rceil = \{b \in S \mid b \leq a\}$ is a Boolean lattice on which $\setminus$ is the relative complement. $(S; \vee, \wedge, 0)$ is thus at least a skew Boolean algebra reduct. The $\setminus$ of its skew Boolean structure agrees with the given $\setminus$ within all $\lceil a \rceil$. But then they agree in general since $a \setminus b = a \setminus (a{\wedge}b)$ holds for the algebra. ☐

Given a skew lattice $(S; \vee, \wedge)$, set $x \wedge_L y = x{\wedge}y{\wedge}x$ and $x \vee_R y = y{\vee}x{\vee}y$. Then $(S; \wedge_L, \vee_R)$ is a left-handed skew lattice. (See Section 3.4.) Observe that:

i) $x \wedge_L y = y \wedge_L x$ iff $x{\wedge}y = y{\wedge}x$, in which case all four expressions are equal.
ii) $x \vee_R y = y \vee_R x$ iff $x{\vee}y = y{\vee}x$, in which case all four expressions are equal.
iii) Thus $(S; \wedge_L, \vee_R)$ is symmetric iff $(S; \wedge, \vee)$ is symmetric.
iv) $(S; \wedge_L, \vee_R)$ and $(S; \wedge, \vee)$ share the same natural partial order $\geq$ and natural quasi-order $\succeq$.
v) Thus $(S; \wedge_L, \vee_R)$ is normal iff $(S; \wedge, \vee)$ is normal
vi) In particular, $(S; \wedge_L, \vee_R)$ forms skew Boolean algebra iff $(S; \wedge, \vee)$ does.
vii) In which case, $(S; \wedge_L, \vee_R, 0)$ and $(S; \wedge, \vee, 0)$ share a common difference $\setminus$.

Proof of The Bisection Theorem (7.1.1) completed. Given (S; ∨, ∧, \, 0) satisfying (i) – (iii) of Theorem 7.1.1, consider first the left-handed algebra (S; $\wedge_L$, $\vee_R$, \, 0) where $x \wedge_L y = x \wedge y \wedge x$ and $x \vee_R y = y \vee x \vee y$. (S; $\vee_L$, $\wedge_L$, \, 0) satisfies (i) – (iii) in $7.1.1_L$ making it a left-handed skew Boolean algebra. Thus (S; ∨, ∧, \, 0), with the identical partial order, is a skew Boolean algebra. ☐

A stronger version of the above theorem exists; it is Theorem 3.3.21 in Spinks' 2002 dissertation. Its proof, closely modeled after the one above, is left to the reader.

Theorem 7.1.8. (Spinks [2002]) *An algebra* (S; ∨, ∧, \, 0) *of type* <2, 2, 2, 0> *is a skew Boolean algebra if and only if the following conditions hold*:

i) (S; ∨, ∧) *is a symmetric skew lattice*;
ii) (S; \, 0) *is an iBCS-algebra*;
iii) $x \wedge y \wedge x \approx x \setminus (x \setminus y)$ *holds*.

In particular, under these conditions, \ *is the skew Boolean algebra difference*. ☐

When a binary discriminator variety is also additive

A binary operation + on an algebra **A** with a constant element 0 is ***additive*** if for all a in A, $a + 0 = a = 0 + a$. An algebra **A** with a constant element 0 is ***additive*** if an additive operation can be polynomial defined on A. A variety with a constant **0** is said to be ***additive*** if it a binary term $\boldsymbol{x+y}$ can be polynomial defined satisfying the identities $x + 0 \approx x \approx 0 + x$.

Theorem 7.1.9. *If* $\mathcal{V}$ *is a binary discriminator variety with constant term* **0** *and additive term* $\boldsymbol{x+y}$, *then every algebra* **A** *of* $\mathcal{V}$ *has a left-handed skew Boolean algebra term reduct* $\mathbf{A_S}$.

Proof: By Theorem 7.1.6 the binary term $\boldsymbol{x \wedge y = x \setminus (x \setminus y)}$ induces a left normal band operation on every member of $\mathcal{V}$. Let $\boldsymbol{x+y}$ be the additive term of $\mathcal{V}$ and define $\boldsymbol{x \vee y}$ to be the term $\boldsymbol{y + (x \setminus y)}$. We need to show that for any $\mathbf{A} \in \mathcal{V}$, $\mathbf{A_S}$ = (A; ∨, ∧, \, 0) is a left-handed skew Boolean algebra. It is sufficient to show that the left-handed skew Boolean algebra identities hold on any member **A** of $\mathcal{V}$ on which $\boldsymbol{x \setminus y}$ induces the binary discriminator. (That is, the primitive case.) This is done directly by straightforward case-splitting arguments. We consider the associativity of ∨. Let **A** be a member of $\mathcal{V}$ for which \ is the discriminator. Given $a, b, c \in$ A,

$$a \vee (b \vee c) \;=\; a \vee (c + (b \setminus c)) \;=\; (c + (b \setminus c)) + (a \setminus (c + (b \setminus c)))$$

and similarly

$$(a \vee b) \vee c \;=\; c + ((b + (a \setminus b)) \setminus c).$$

Denoting the two expressions on the right above in succession by (L) and (R), we consider four cases that together cover all possibilities.

$a = 0$: L $= (c + (b \setminus c)) + (0 \setminus (c + (b \setminus c)) = (c + (b \setminus c)) + 0 = c + (b \setminus c)$.
R $= c + ((b + (0 \setminus b)) \setminus c) = c + ((b + 0) \setminus c) = c + (b \setminus c) =$ L.

$b = 0$. L $= (c + 0) + ((a \setminus (c + 0)) = c + (a \setminus c)$.
R $= c + (0 + (a \setminus 0) \setminus c) = c + ((0 + a) \setminus c) = c + (a \setminus c) =$ L

$c = 0$: In a similar fashion, L $= b + (a \setminus b) =$ R.

$a, b, c \neq 0$. Thus $a \setminus b = b \setminus c = a \setminus c = 0$ so that
L $= (c + 0) + (a \setminus (c + 0)) = c + (a \setminus c) = c + 0 = c$, while
R $= c + ((b + 0) \setminus c) = c + (b \setminus c) = c + 0 = c$.

Thus $a \vee (b \vee c) = (a \vee b) \vee c$ in all possible cases. The absorption identities are similarly checked for \ being the binary discriminator. We consider only $a \wedge (a \vee b) = a$. Here there are three cases.

$a = 0$: $0 \wedge (0 \vee b) = 0 \wedge (b + 0\setminus b) = 0 \wedge (b + 0) = 0 \wedge b = 0$.

$b = 0$: $a \wedge (a \vee 0) = a \wedge (0 + a\setminus 0) = a \wedge (0 + a) = a \wedge a = a$.

$a, b \neq 0$: $a \wedge (a \vee b) = a \wedge (b + a\setminus b) = a \wedge (\mathrm{b} + 0) = a \wedge b = a$.

Thus (S; ∨, ∧, 0) is at least a normal skew lattice with a zero. To show symmetry we verify that $a\wedge b = b\wedge a$ implies $a\vee b = b\vee a$. (The converse holds for all normal skew lattices.) Since S is primitive, two main (nonexclusive) cases of $a\wedge b = b\wedge a$ occur. *Either a or b is 0*, say $a = 0$. Here

$$0 \vee b = b + (0 \setminus b) = b$$

while

$$b \vee 0 = 0 + (b \setminus 0) = 0 + b = b.$$

The other case is *a equals b*. Here commutativity of ∨ is trivial. Thus (S; ∨, ∧) is symmetric, normal skew lattice. The rest follows from Theorem 7.1.6. ☐

Example 7.1.1. Let **SA** denote the variety of Stone algebras. A member of **SA** has the form (A; ∩, ∪, *, 0,1), where (A; ∩, ∪, 0,1) is a bounded distributive lattice and * is a relative pseudo-complementation operation; thus a^* is the largest element of A such that $a \cap a^* = 0$. Stone algebras form a subvariety of the variety of pseudo-complemented distributive lattices. They are distinguished from other pseudo-complemented distributive lattices by the identity $\boldsymbol{x^* \approx x^{***}}$. Since they are generated by the two and three-element chains, it is not difficult to see that Stone algebras form a binary discriminator variety with binary discriminator term $x \setminus y = x \cap y^*$. For $a, b \in$ A let $a \vee b$ be $b \cup (a \cap b^*)$ and $a \wedge b$ be $a \cap (a \cap b^*)^*$. Then by theorem above, the derived algebra (A; ∨, ∧, \, 0) is a left-handed skew Boolean algebra. ☐

Ternary discriminators and ternary discriminator varieties

A ***ternary discriminator*** on a set S is a function d: $S^3 \to S$ defined on a given set S by

$$d(x, y, z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise} \end{cases}.$$

An algebra **A** is a ***ternary discriminator algebra*** if d can be polynomial-defined on its underlying set A. *Ternary discriminator algebras are simple*. Indeed given elements $a \neq b$ in such an algebra **A** and let θ be a congruence on **A** for which $a\ \theta\ b$. Then for all $c \in$ A,

$$a = d(a, b, c)\ \theta\ d(a, a, c) = c.$$

Thus the only non-identity congruence on A is the universal congruence. Murskii [1975] proved that in a certain sense *almost all finite algebras are ternary discriminator algebras*.

If $\mathcal{K}$ is a class of algebras of common type having a common ternary discriminator term, the variety $\mathcal{V}$ generated from $\mathcal{K}$ is a ***ternary discriminator variety***. Burris and Sankappanavar [1981] described such a variety as "the most successful generalization of Boolean algebras to date, successful because we obtain Boolean product representations." Examples of these varieties include Boolean algebras, n-dimensional cylindric algebras, p-rings and skew Boolean ∩-algebras. The remaining results in this section are from Bignall and Leech [1995].

Theorem 7.1.10. *In the variety of skew Boolean ∩-algebras the polynomial term*

$$d(x, y, z) = (x / y) \vee [z \setminus ((x / y) \vee (y / x))]$$

is a ternary discriminator on any primitive algebra. Thus skew Boolean ∩-algebras form a ternary discriminator variety. As such they are both congruence distributive and congruence permutable. (The congruence distributive property was already observed in Theorem 4.4.3.)

Proof. If $x = y$, then clearly $d(x, y, z) = z$ on any algebra, primitive or otherwise. If $x \neq y$ on some primitive algebra P, then $x \cap y = 0$, so that $x / y = x$ and $y / x = y$ on P. Thus the displayed polynomial reduces to $x \vee [z \setminus (x \vee y)]$. If $x \neq 0$, the latter reduces to $x \vee 0 = x$. If $x = 0$, then $y \neq 0$ so that $x \vee [z \setminus (x \vee y)]$ reduces to $z \setminus y = 0 = x$ on P. The congruence distributive property has already been seen in the case of ∩-algebras. That it and the congruence permutable property hold on all discriminator varieties follows from results of Bulman-Flemming, Keimel and Werner. (See Theorem IV.9.4 in Burris and Sankppanavar [1981].) ☐

A ***pointed ternary discriminator variety*** is a ternary discriminator variety with a constant term **0**. $\mathcal{PD}_0$ denotes the pointed ternary discriminator variety generated by the class of all ternary discriminator algebras (A; d, 0) with 0 as a nullary operation, i.e., a constant.

Theorem 7.1.11. *$\mathcal{PD}_0$ is term equivalent to the variety of all right-handed skew Boolean $\cap$-algebras. Given* (A; d, 0) *in $\mathcal{PD}_0$, right-handed skew Boolean $\cap$-operations are given by*

$$x \vee y \;=\; d(x, 0, y), \quad x \wedge y \;=\; d(x, d(x, 0, y), y) \quad \text{and} \quad x \,/\, y \;=\; d(x, y, 0).$$

Conversely, given a right-handed skew Boolean $\cap$-algebra (S; $\vee$, $\wedge$, /, 0), *an algebra* (S; d, 0) *in $\mathcal{PD}_0$ is given by*

$$d(x, y, z) \;=\; x\,/\,y \;\vee\; (x \wedge z) \vee (y \vee z)/y.$$

Finally both processes, $\{d, 0\} \to \{\vee, \wedge, /, 0\}$ *and* $\{\vee, \wedge, /, 0\} \to \{d, 0\}$, *are reciprocal*.

Proof. If (A; d, 0) is a pointed ternary discriminator algebra, then

$$d(x, 0, y) = \begin{cases} x & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}; \qquad d(x, d(x, 0, y), y) = \begin{cases} y & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}; \qquad d(x, y, 0) = \begin{cases} 0 & \text{if } x = y \\ x & \text{if } x \neq y \end{cases}.$$

But these identities describe respectively $\vee$, $\wedge$ and / on the primitive right-handed Boolean $\cap$-algebra with upper $\mathcal{D}$-class A \ {0} and zero-class {0}. Hence all algebras in $\mathbf{PD_0}$ induce right-handed skew Boolean $\cap$-algebras. Conversely, given (S; $\vee$, $\wedge$, /, 0) and $d(x, y, z)$ as stated, then

$$d(x, x, z) \;= (x \wedge z) \vee (x \vee z)/x \;= z \qquad \text{and} \qquad d(0, y, z) \;= (y \vee z)/y = \begin{cases} 0 & \text{if } y \neq 0 \\ z & \text{if } y = 0 \end{cases}.$$

When x is neither y nor 0, then $d(x, y, z) \;=\; x \;\vee (x \wedge z) \vee (y \vee z)/y = x$. Thus in all cases $d(x, y, z)$ is indeed the ternary discriminator on S and (S; d, 0) is a pointed discriminator algebra. Thus all algebras (S; d, 0) induced from right-handed skew Boolean $\cap$-algebras lie in $\mathbf{PD_0}$. That the operations are reciprocal is easily checked at the "entry level" of pointed discriminator algebras and primitive right-handed Boolean $\cap$-algebras. ☐

Corollary 7.1.12. *Any skew Boolean $\cap$-algebra* **A** *has a right-handed skew Boolean $\cap$-algebra reduct* $\mathbf{A_R}$ *with the property that its congruences and its $\cap$, / and \ operations coincide with those of* **A**. *The new skew lattice operations $\vee_R$ and $\wedge_R$ are defined in terms of the old by*

$$x \vee_R y = x \vee y \vee x \quad \text{and} \quad x \wedge_R y = y \wedge x \wedge y.$$

Proof. Indeed one has

$$x \vee_R y \;=\; d(x, 0, y) \;=\; (x\,/\,0) \vee \;[y \setminus (\,(x\,/\,0) \vee (0\,/\,x)\,)] = \; x \vee \;(y \setminus x) = x \vee y \vee x$$

with the last identity holding first on all primitive $\cap$-algebras and hence on all $\cap$-algebras. Also,

$$x \wedge_R y \;=\; d(x, d(x, 0, y), y) \;=\; d(x, x \vee y \vee x, y) \;= y \setminus [(x \vee y \vee x)\,/\,x] = y \wedge x \wedge y$$

with the last identity holding again first on all primitive $\cap$-algebras and thus on all $\cap$-algebras. That the standard differences agree is clear. Intersections also agree since the natural partial ordering is unchanged: $x \wedge y = y \wedge x = x$ iff $x \wedge y \wedge x = y \wedge x \wedge y = x$. Thus BCK differences are also unchanged. By Theorem 4.4.13 the congruence structure is unchanged since both algebras share the same maximal lattice image. ☐

In view of Corollary 7.1.12 and McKenzie [1975] Theorem 1.3, we also have:

Corollary 7.1.13. *Any algebra* **A** *in a pointed ternary discriminator variety has a skew Boolean* $\cap$*-algebra polynomial reduct whose congruences coincide with those of* **A**.

For more on skew Boolean algebras and discriminator varieties, see the paper by Karin Cvetko-Vah and Antonino Salibra in the following references. Also of interest is the paper by Murskii which, among other things, shows that almost all finite algebras are discriminator algebras (with "almost all" under stood in a certain sense).

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Addendum 2020

This survey has focused on developments in skew lattices research up through much of 2017. The research does not end there. To pick up the trail, we begin with a conference held in Slovenia in May of 2018. Its official title was *Noncommutative Structures 2018: a Workshop in Honor of Jonathan Leech*. I gave the opening address, which was later published as the following article:

J. Leech,
My journey into noncommutative lattices and their theory, *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.01.

This paper provides the interested seeker with a fairly thorough overview of much that transpired in the first thirty years of the renewed study of noncommutative lattices. In the same issue of this online journal were contributions by other workshop participants:

K. Cvetko-Vah, M. Kinyon, J. Leech & T. Pisanski,
Regular antilattices . *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.06.

D. G. FitzGeraltd:
Groupoids on a skew lattice of objects. *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.03.

D. Ellerman:
A graph-theoretic method to define any Boolean operation on partitions. *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.02.

J. Jovanović & A. Tepavčević:
Ω-lattices from skew lattices. *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.04.

A. Bucciarelli & A. Salibra:
On noncommutative generalisations of Boolean algebras. *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.07.

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On the coset structure of distributive skew lattices, *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.05
Open problems from NCS 2018. *The Art of Discrete and Applied Mathematics*, 2 (2019) #P2.09

More recent skew lattice research has come from Karin CvetkoVah and friends:

K. CvetkoVah:
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K. CvetkoVah, J. Hemelaer & J. Leech:
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K. CvetkoVah, Jens Hemelaer & Lieven Le Bruyn:
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Until 2017 most of the research on skew lattices has, to my knowledge, occurred in Australia, North America and Europe. But that is changing and with the change comes research that pushes the theory in new directions.

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Onward!

Acknowledgement:

I would like to express my appreciation to all who have done research on skew lattices. In particular I am grateful to those I have worked directly with: Robert J. Bignall, Karin Cvetko-Vah, Michael Kinyon, Ganna Kudryavtseva, Gratiela Laslo, Tomaž Pisanski, Joao Pita Costa, and Matthew Spinks. I would also like to express my gratitude to the late Charles Wells, who in 1983 provided the venue at Case Western Reserve University where I began my research on skew lattices and skew Boolean algebras. Charles was very supportive of my initial research. In addition, many thanks to those involved with bringing this survey of skew lattice theory to publication: the editorial team (Tomaž Pisanski, Karin Cvetko Vah and Michael Kinyon), the technical support, Nino Bašić, the University of Primorska and its Rector Klavdija Kutnar and the University of Primorska Press Editor-In-Chief Jonatan Vinkler To ALL, thank you!

Jonathan E. Leech
NONCOMMUTATIVE LATTICES:
Skew Lattices, Skew Boolean Algebras and Beyond

The extended study of non-commutative lattices was begun in 1949 by Ernst Pascual Jordan, a theoretical and mathematical physicist and co-worker of Max Born and Werner Karl Heisenberg. Jordan introduced noncommutative lattices as algebraic structures potentially suitable to encompass the logic of the quantum world. The modern theory of noncommutative lattices began forty years later with Jonathan Leech's 1989 paper "Skew lattices in rings." Recently, noncommutative generalizations of lattices and related structures have seen an upsurge in interest, with new ideas and applications emerging, from quasilattices to skew Heyting algebras. Much of this activity is derived in some way from the initiation of Jonathan Leech's program of research in this area. The present book consists of seven chapters, mainly covering skew lattices, quasilattices and paralattices, skew lattices of idempotents in rings and skew Boolean algebras. As such, it is the first research monograph covering major results due to this renewed study of noncommutative lattices. It will serve as a valuable graduate textbook on the subject, as well as a handy reference to researchers of noncommutative algebras.

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ISBN 978-961-293-027-1