The Coxeter graph

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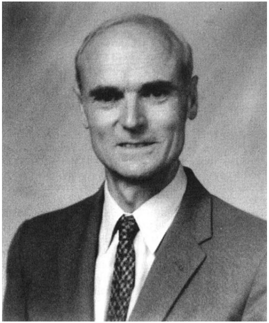
Abstract

Laszlo Lovasz asked whether every finite connected vertex-transitive graph contains a Hamiltonian cycle. This question is still open. However we know five graphs which are finite, connekted and vertex-transitive graphs, but they are not hamiltonian one. They are K_2, Petersen graph, Coxeter graph and truncations of the Petersen graph and Coxeter graph.
We will prove that the Coxeter graph has no Hamiltonian cycle. We will also present some well-know properties of this remarkable graph.

Harold Scott MacDonald Coxeter (9.02.1907 - 31.03. 2003)

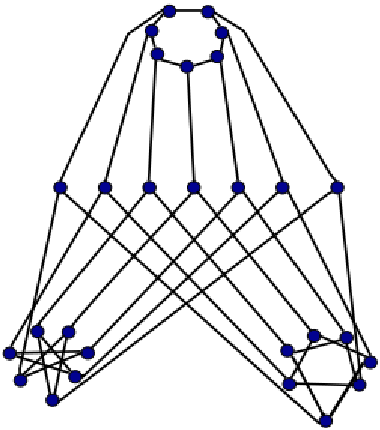


Harold Coxeter is one of the great geometers of the 20th century.

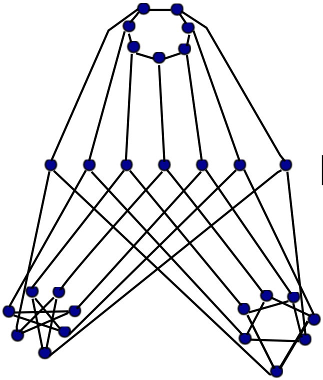
Like any great mathematician, he left a deep mark in different fields of mathematics. Besides of geometry he has publications about group theory and grapn theory.

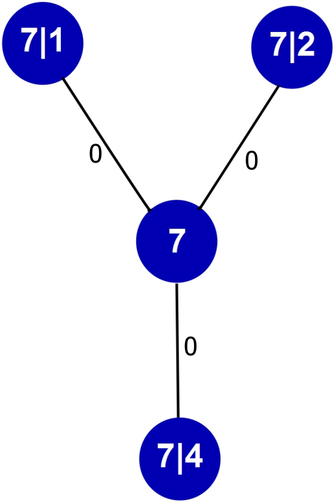
The Coxeter graph

Definition:
The Coxeter graph is a 3-regular graph with 28 vertices and 42 edges. It has chromatic number and chromatic index 3, radius 4, diameter 4 and grith 7. It is also a 3-vertex-connected graph and 3-edge-connected graph.



The Coxeter Graph in Frucht's notation

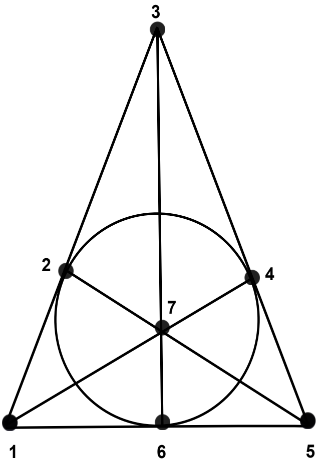




Algebraic properties of the Coxeter graph

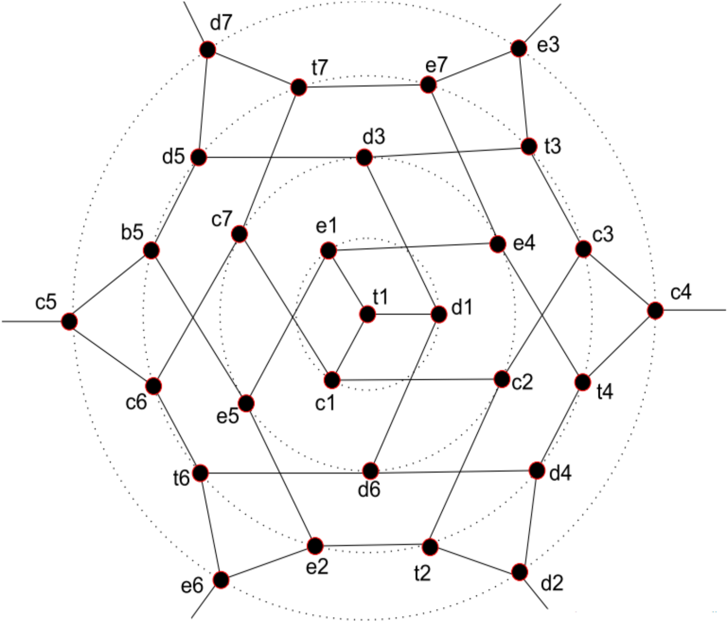
- $Aut(Y) = PGL(2,7), |Aut(Y)| = 336$.
- Y is vertex-transitive, arc-transitive and 3-regular.
- A vertex-stabilizer is of order $2 \cdot 3^{s-1} = 12$ and it is isomorphic to D_{12}.
- A arc-stabilizer is of order $2^{s-1} = 4$ and it isomorphic to $Z_2 \times Z_2$.
- A stabilizer of an edge is of order 8.

Constructing the Coxeter Graph from the Fano Plane

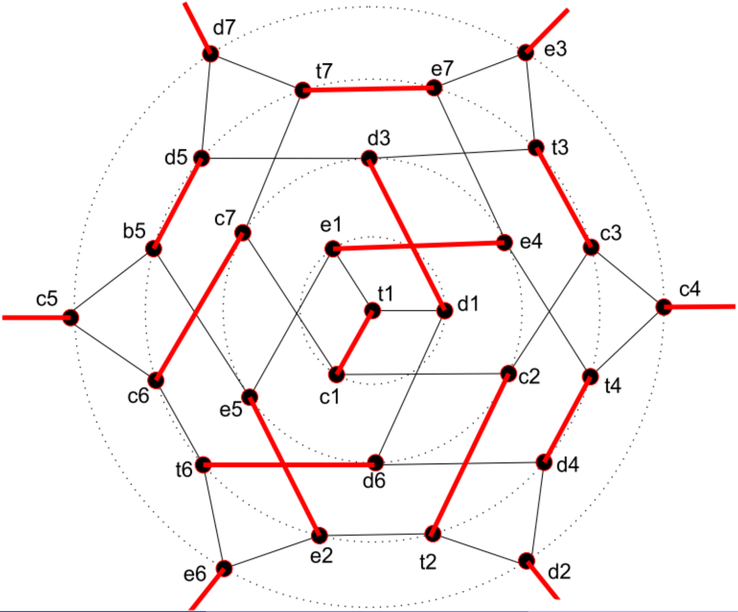


$$X = (V, E)$$
$$V = \{(P, l) \in \mathcal{P} x \mathcal{L} | P \notin l\}$$
$$(P, l) \sim (P', l') \Leftrightarrow \mathcal{P} = l \bigcup l' \bigcup \{P, P'\}$$

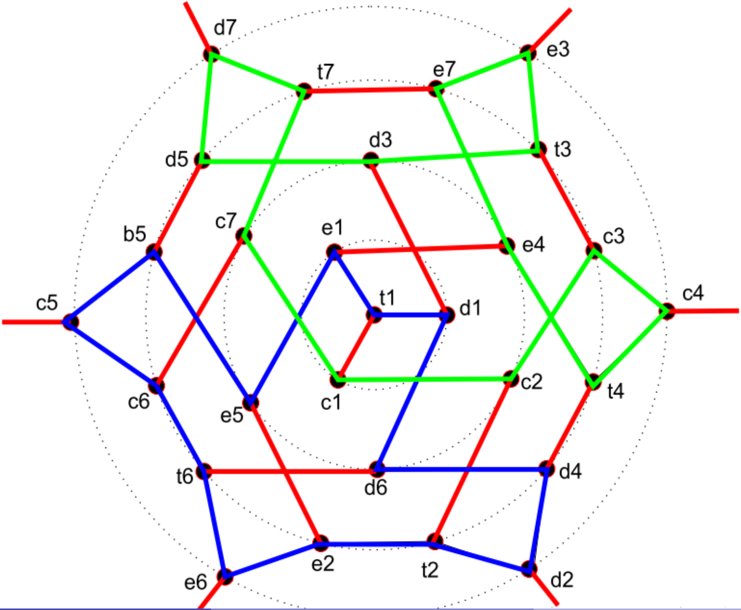
The Coxeter graph in distance-transitive format



A 1-factor $\mathcal{M}$ in the Coxeter graph



A complement of a 1-factor $\mathcal{M}$ in the Coxeter graph

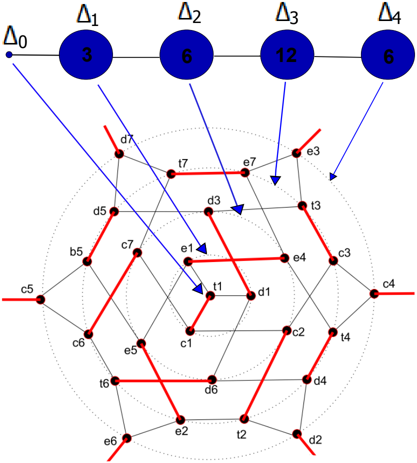


The number of 1-factor in the Coxeter graph

Let $\mathcal{M}_0$ be a 1-factor in Y.
Let μ_{ij} be number of edges in $\mathcal{M}_0$ which join a vertex in Δ_i to one in Δ_j.
Then

$$\mu_{01} = 1, \mu_{12} = 2, \mu_{23} = 4$$

$$4 + 2\mu_{33} + \mu_{34} = 12;$$
$$\mu_{34} + 2\mu_{44} = 6;$$
$$0 \leq \mu_{44} \leq 3.$$



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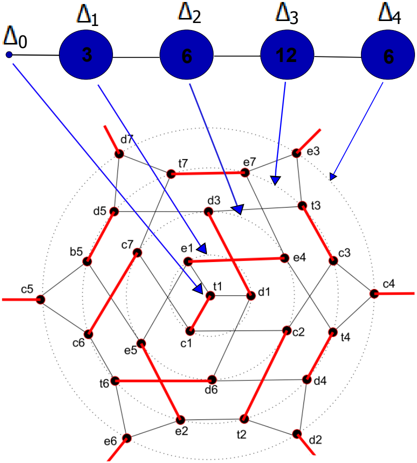
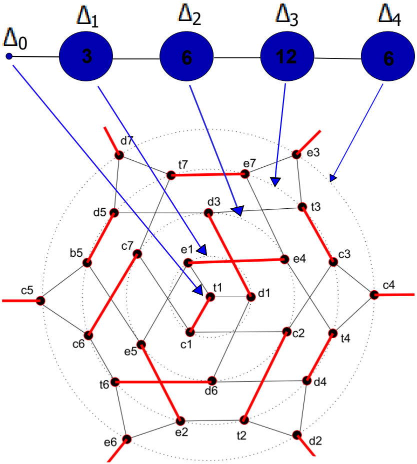
$\Rightarrow$ Y has 84 1-factors.



The action of $AutY$ on the set of 1-factors

Let $\mathcal{M}$ be the set of all 1-factors in Y.
Then Aut$Y$ acts on $\mathcal{M}$.
Let $\mathcal{M}_0$ be particular 1-factor.

c1/t1, e1/e4, d1/d3, e2/e5, d6/t6, c6/c7, c2/t2, t7,e/7, t3/c3, t4/d4, t5/d5, d2/d7, e3/e6, c4/c5.



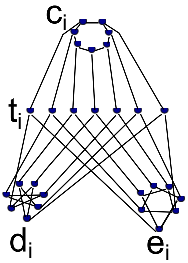
Suppose that $\phi \in AutY$ fixes $\mathcal{M}_0$ setwise.
Since $\mathcal{M}_0$ contains the three "extreme" edges with respect to $t1$, $\mathcal{M}_0$ also contain the "extreme" edges with respect to $\phi(t1)$.

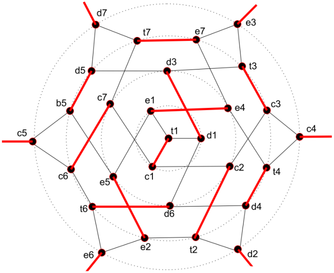
The "extreme" edges

t, i	c, i	d, i	e, i
$c, i-3/c, i+3$	$t, i-3/d, i-3$	$t, i-1/e, i-1$	$t, i-2/c, i-2$
$d, i-1/d, i+1$	$t, i+3/d, i+3$	$t, i+1/e, i+1$	$t, i+2/c, i+2$
$e, i-2/e, i+2$	$e, i-2/e, i+2$	$c, i-3/c, i+3$	$d, i-1/d, i+2$

$M =$

$c1/t1, e1/e4, d1/d3, e5/e5, d6/t6, c6/c7, c2/t2, t7/e7, t3/c3, t4/d4, t5/d5, d2/d7$





The stabilizer of $\mathcal{M}_0$.

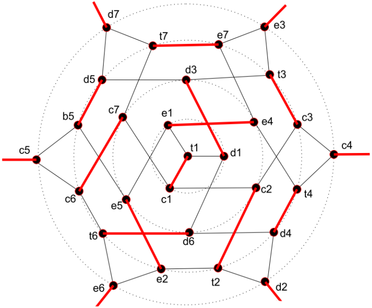
Automorphism fixing an edge from a group of order 8. One of such automorphism is that which is induced by the permutation

$$(1)(27)(36)(45)$$

of the numerical parts of the vertex-labels. This automorphism does not fix $\mathcal{M}_0$ and so stabilizer of $\mathcal{M}_0$ has order at most 4.
But the following automorphism of Y fixes $\mathcal{M}_0$ and has order 4:

$$\theta = (t1c1)(t2d3c6e4)(d1c7e1c2)(d4d7t5c4)$$
(e3 e6)(d6 t7 e5 c3)(t3 t6 e7 e2)(t4 d2 d5 c5)

The stabilizer of $\mathcal{M}_0$.



Automorphism induced by permutation (1)(27)(36)(45) is not in $Stab_{\mathcal{M}_0}$

$$(t1c1)(t2d3c6e4)(d1c7e1c2)(d4d7t5c4)$$
$$(\text{e3 e6})(\text{d6 t7 e5 c3})(\text{t3 t6 e7 e2})(\text{t4 d2 d5 c5}) \in Stab_{\mathcal{M}_0}$$

The stabilizer of $\mathcal{M}_0$.

By Orbit-Stabilizer property the 1-factor $\mathcal{M}_0$ has exactly

$$\frac{|Aut(Y)|}{|Stab_{\mathcal{M}_0}|} = |Orb_{Aut(Y)}(\mathcal{M}_0)| = \frac{336}{4} = 84$$

distinct images under the action of $AutY$. Since $|M| = 84$ it follows that

$\Rightarrow AutY$ is transitive on the set of 1-factors.

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$\Rightarrow AutY$ is transitive on the set of 1-factors.

$\Rightarrow Y$ does not have a Hamiltonian cycle.

Thank you!