The Isomorphism Classes of All Generalized Petersen Graphs

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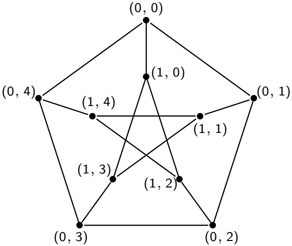

Figure: The Petersen graph is $\mathrm{GP}(5,2)$

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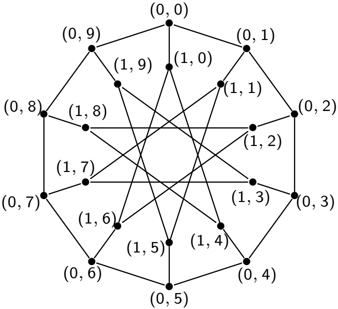

Figure: The generalized Petersen graph $\mathrm{GP}(10, 4)$.

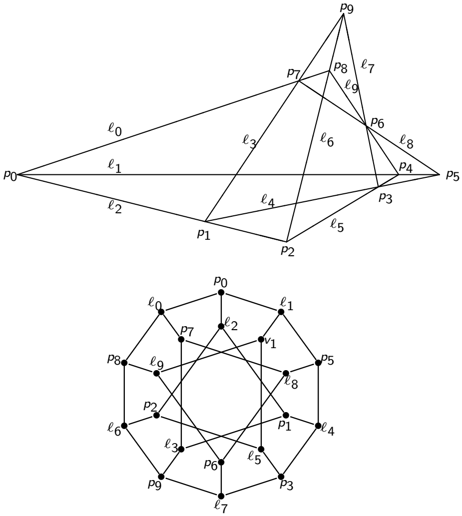

Figure: The Desargues configuration and its Levi graph $\mathrm{GP}(10,3)$

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Automorphisms of generalized Petersen graphs

Define $\rho, \delta : \mathbb{Z}_2 \times \mathbb{Z}_n \to \mathbb{Z}_2 \times \mathbb{Z}_n$ by $\rho(i,j) = (i, j+1)$ and $\delta(i,j) = (i, -j)$.

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If $k^2 \equiv 1 \pmod{n}$, then set $B(n,k) = \langle \rho, \delta, \tau \rangle$, if $k^2 \equiv -1 \pmod{n}$, set $B(n,k) = \langle \rho, \tau \rangle$, while if $k^2 \not\equiv \pm 1 \pmod{n}$, set $B(n,k) = \langle \rho, \delta \rangle$.

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$\mathrm{Aut}(\mathrm{GP}(n,k)) = B(n,k)$ except for the following pairs (n,k), where $2 \leq 2k < n$:

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They also determined the automorphism groups for the seven exceptional pairs, but we will not discuss them.

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Their proof very much makes use of the fact that if $\gcd(n,k) = 1$, then the inside graph is a cycle, and not a disjoint union of cycles.

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If one is working with Cayley graphs of a group G, then G_L is the obvious small transitive subgroup to work with, and this is exactly how the characterization of which groups are CI-groups with respect to graphs was obtained.

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Lemma

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The only part (sort of) of our strategy remaining is to show that if $\phi^{-1}\langle\rho\rangle\phi \leq \mathrm{Aut}(\mathrm{GP}(n,k))$, then there exists $\delta \in \mathrm{Aut}(\mathrm{GP}(n,k))$ such that $\delta^{-1}\phi^{-1}\langle\rho\rangle\phi\delta = \langle\rho\rangle$.

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$$(\delta\rho^b)^2(i, j) = \delta\rho^b\delta(i, j + b) = \delta\rho^b(i, -j - b) = \delta(i, -j) = (i, j)$$

First, the seven exceptionally pairs (n, k) can be ignored - as isomorphic graphs have isomorphic automorphism groups these generalized Petersen graphs $\mathrm{GP}(n, k)$ are only isomorphic to $\mathrm{GP}(n, -k)$.

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So, in the remaining case, we have an isomorphism ϕ between $\mathrm{GP}(n, k)$ and $\mathrm{GP}(n, \ell)$ that is contained in the normalizer in S_{2n} of $\langle \rho \rangle$.

Set $\phi(i,j) = (i + a, \beta j + b_i)$, $a \in \mathbb{Z}_2$, $\beta \in \mathbb{Z}_n^*$, and $b_i \in \mathbb{Z}_n$.

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If $a = 0$, then ϕ maps $\mathrm{GP}(n,k)[\mathcal{O}]$ to $\mathrm{GP}(n,\ell)[\mathcal{O}]$, If $a = 0$, then ϕ maps $\mathrm{GP}(n, k)[\mathcal{O}]$ to $\mathrm{GP}(n, \ell)[\mathcal{O}]$, and these graphs are equal (as the outside cycles of a generalized Petersen graph are always equal).

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Theorem

The generalized Petersen graphs $\mathrm{GP}(n,k)$ *and* $\mathrm{GP}(n,\ell)$ *are isomorphic if and only if either* $k \equiv \pm\ell \pmod{n}$ *or* $k\ell \equiv \pm 1 \pmod{n}$.