The Isomorphism Classes of All Generalized Petersen Graphs

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For integers \( k \) and \( n \) satisfying \( 1 \leq k \leq n - 1, 2k \neq n \), define the **generalized Petersen graph** \( GP(n, k) \) to be the graph with vertex set \( \mathbb{Z}_2 \times \mathbb{Z}_n \) and edge set

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\{(0, i)(0, i + 1), (0, i)(1, i), (1, i)(1, i + k) : i \in \mathbb{Z}_n\}
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![Diagram of the Petersen graph](image_url)

**Figure:** The Petersen graph is $GP(5, 2)$
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*Figure: The generalized Petersen graph GP(10, 4).*
Figure: The Desargues configuration and its Levi graph \( \text{GP}(10, 3) \)
Many famous graphs are generalized Petersen graphs:

- $GP(5, 2)$ is the Petersen graph
- $GP(4, 1)$ is the skeleton of the cube
- $GP(10, 3)$ is the Desargues graph
- $GP(10, 2)$ is the skeleton of the dodecahedron
- $GP(8, 3)$ is the Möbius-Kantor graph

The generalized Petersen graphs have received a great deal of attention — their automorphism groups are known, and exactly which contain Hamilton cycles are known for example.
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Automorphisms of generalized Petersen graphs

Define $\rho, \delta : \mathbb{Z}_2 \times \mathbb{Z}_n \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_n$ by $\rho(i, j) = (i, j + 1)$ and $\delta(i, j) = (i, -j)$.
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Define $\tau : \mathbb{Z}_2 \times \mathbb{Z}_n \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_n$ by $\tau(i, j) = (i + 1, kj)$. In order for $\tau$ to be a bijection, $k \in \mathbb{Z}_n^*$. In order for $\tau$ to be an automorphism of $GP(n, k)$, $\tau^2((i, j)) = (i, kj) \equiv \pm 1 \pmod{n}$ must fix the outside $n$-cycle and the inside vertices, and map spoke edges to spoke edges. As the automorphism group of an $n$-cycle is a dihedral group, we conclude that $k^2 \equiv \pm 1 \pmod{n}$. If $k^2 \equiv \pm 1$, then $\tau((0, j)(0, j + 1)) = (1, kj)(1, kj + k)$, $\tau((1, kj)(1, kj + k)) = (0, \pm j)(0, \pm j \pm 1)$, and $\tau((0, j)(1, j)) = (1, kj)(0, kj)$. Thus $\tau \in \text{Aut}(GP(n, k))$. 

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If $k^2 \equiv 1 \pmod{n}$, then set $B(n, k) = \langle \rho, \delta, \tau \rangle$, if $k^2 \equiv -1 \pmod{n}$, set $B(n, k) = \langle \rho, \tau \rangle$, while if $k^2 \not\equiv \pm 1 \pmod{n}$, set $B(n, k) = \langle \rho, \delta \rangle$. 

In 1971, Frucht, Graver, and Watkins determined the automorphism groups of the generalized Petersen graphs:

Theorem $\text{Aut}(\text{GP}(n, k)) = B(n, k)$ except for the following pairs $(n, k)$, where $2 \leq 2k < n$:

- $(4, 1)$
- $(5, 2)$
- $(8, 3)$
- $(10, 2)$
- $(10, 3)$
- $(12, 5)$
- $(24, 5)$

They also determined the automorphism groups for the seven exceptional pairs, but we will not discuss them.
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Their proof very much makes use of the fact that if $\gcd(n, k) = 1$, then the inside graph is a cycle, and not a disjoint union of cycles.
A general strategy to solve an isomorphism problem

Suppose that $\Gamma$ and $\Gamma'$ are isomorphic graphs with $\phi$ and isomorphism, and $G \leq \text{Aut}(\Gamma) \cap \text{Aut}(\Gamma')$. Then $\phi^{-1}G\phi \leq \text{Aut}(\Gamma)$.

If there exists $\delta \in \text{Aut}(\Gamma)$ such that $\delta^{-1}\phi^{-1}G\phi\delta = G$, then $\phi\delta$ is an isomorphism from $\Gamma$ to $\Gamma'$ that normalizes $G$.

If one then calculates this normalizer, then the isomorphism problem is solved.

Of course this strategy will only work well if $G$ is transitive and small, or intransitive and large.

If one is working with Cayley graphs of a group $G$, then $G_L$ is the obvious small transitive subgroup to work with, and this is exactly how the characterization of which groups are CI-groups with respect to graphs was obtained.
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For us with the generalized Petersen graphs, the obvious choice of $G$ is $\langle \rho \rangle$, which has two orbits of size $n$. 

Lemma

Let $\rho : \mathbb{Z}_2 \times \mathbb{Z}_n \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_n$ by $\rho(i, j) = (i, j + 1)$. Then $N_{S_2}^{\mathbb{Z}_n}(\langle \rho \rangle) = \{(i, j) \rightarrow (i + a, b j) : a \in \mathbb{Z}_2, b \in \mathbb{Z}_n^*, \delta \in \mathbb{Z}_n\}$. 

The only part (sort of) of our strategy remaining is to show that if $\phi^{-1} \langle \rho \rangle \phi \leq \text{Aut}(\text{GP}(n, k))$, then there exists $\delta \in \text{Aut}(\text{GP}(n, k))$ such that $\delta^{-1} \phi^{-1} \langle \rho \rangle \phi \delta = \langle \rho \rangle$. 

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First, the seven exceptionally pairs \((n, k)\) can be ignored - as isomorphic graphs have isomorphic automorphism groups these generalized Petersen graphs \(GP(n, k)\) are only isomorphic to \(GP(n, -k)\).

Second, show that if \(k \neq \pm 1\), then \(\langle \rho \rangle\) is the unique maximal cyclic subgroup of \(B(n, k)\).
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has order 2, not \(n\).
The case where $k^2 = \pm 1$ but $k \neq \pm 1$, is very similar, with the computations being slightly more complicated.
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So, in the remaining case, we have an isomorphism $\phi$ between $\text{GP}(n, k)$ and $\text{GP}(n, \ell)$ that is contained in the normalizer in $S_{2n}$ of $\langle \rho \rangle$. 
Set $\phi(i, j) = (i + a, \beta j + b_i)$, $a \in \mathbb{Z}_2$, $\beta \in \mathbb{Z}_n^*$, and $b_i \in \mathbb{Z}_n$. 
Set $\phi(i, j) = (i + a, \beta j + b_i)$, $a \in \mathbb{Z}_2$, $\beta \in \mathbb{Z}_n^*$, and $b_i \in \mathbb{Z}_n$. As $\rho \in B(n, k)$, we can and do assume without loss of generality that $b_0 = 0$. 
Set \( \phi(i, j) = (i + a, \beta j + b_i) \), \( a \in \mathbb{Z}_2 \), \( \beta \in \mathbb{Z}^*_n \), and \( b_i \in \mathbb{Z}_n \). As \( \rho \in B(n, k) \), we can and do assume without loss of generality that \( b_0 = 0 \).

Let \( O = \{(0, j) : j \in \mathbb{Z}_n\} \) (the “outer” vertices of \( \text{GP}(n, k) \)).
Set \( \phi(i, j) = (i + a, \beta j + b_i) \), \( a \in \mathbb{Z}_2 \), \( \beta \in \mathbb{Z}_n^* \), and \( b_i \in \mathbb{Z}_n \). As \( \rho \in B(n, k) \), we can and do assume without loss of generality that \( b_0 = 0 \). Let \( O = \{(0, j) : j \in \mathbb{Z}_n\} \) (the “outer” vertices of \( \text{GP}(n, k) \)), and \( I = \{(1, j) : j \in \mathbb{Z}_n\} \) (the “inner” vertices of \( \text{GP}(n, k) \)).
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Let \( O = \{ (0, j) : j \in \mathbb{Z}_n \} \) (the “outer” vertices of \( \text{GP}(n, k) \)), and \( I = \{ (1, j) : j \in \mathbb{Z}_n \} \) (the “inner” vertices of \( \text{GP}(n, k) \)). We refer to edges of the form \((1, i)(0, i)\) as spoke edges, and set \( \phi(\text{GP}(n, k)) = \text{GP}(n, \ell) \).
If \( a = 0 \), then \( \phi \) maps \( \text{GP}(n, k)[\mathcal{O}] \) to \( \text{GP}(n, \ell)[\mathcal{O}] \),
If $a = 0$, then $\phi$ maps $\text{GP}(n, k)[\mathcal{O}]$ to $\text{GP}(n, \ell)[\mathcal{O}]$, and these graphs are equal (as the outside cycles of a generalized Petersen graph are always equal).
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If $a = 0$, then $\phi$ maps $\GP(n, k)[\emptyset]$ to $\GP(n, \ell)[\emptyset]$, and these graphs are equal (as the outside cycles of a generalized Petersen graph are always equal). As the automorphism group of cycle is a dihedral group, we conclude that $\beta = \pm 1$. As $\phi(0, 0) = (0, 0)$ and $\phi$ maps spoke edges to spoke edges, we see that $\phi(1, 0) = (1, 0)$ and so $b_1 = 0$ as well.
If \( a = 0 \), then \( \phi \) maps \( \text{GP}(n, k)[\mathcal{O}] \) to \( \text{GP}(n, \ell)[\mathcal{O}] \), and these graphs are equal (as the outside cycles of a generalized Petersen graph are always equal). As the automorphism group of cycle is a dihedral group, we conclude that \( \beta = \pm 1 \). As \( \phi(0, 0) = (0, 0) \) and \( \phi \) maps spoke edges to spoke edges, we see that \( \phi(1, 0) = (1, 0) \) and so \( b_1 = 0 \) as well. Then \( \phi \in \langle \delta \rangle \), and \( \text{GP}(n, k)[\mathcal{I}] = \text{GP}(n, \ell)[\mathcal{I}] \). Thus \( \ell = \pm k \).
If $a = 1$, then $\phi$ maps $\text{GP}(n, k)[O]$ to $\text{GP}(n, \ell)[I]$. 

As $\phi((0, 0)(0, 1)) = (1, 0)(1, \ell)$ or $(1, 0)(1, -\ell)$, we have that $\beta = \pm \ell$.

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The Isomorphism Classes of All Generalized Petersen Graphs

Ted Dobson
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If $a = 1$, then $\phi$ maps $\text{GP}(n, k)[O]$ to $\text{GP}(n, \ell)[I]$. As $\phi((0, 0)(0, 1)) = (1, 0)(1, \ell)$ or $(1, 0)(1, -\ell)$, we have that $\beta = \pm \ell$. 
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Theorem

The generalized Petersen graphs $\text{GP}(n, k)$ and $\text{GP}(n, \ell)$ are isomorphic if and only if either $k \equiv \pm \ell \pmod{n}$ or $k\ell \equiv \pm 1 \pmod{n}$. 