Graph Covers and Applications

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Outline

1. Notations on groups and graphs
2. Covers and automorphisms
3. Voltage graph

Graph automorphisms

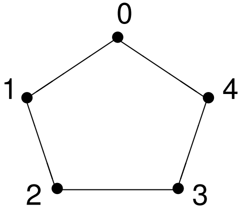
In my talks, all graphs are assumed to be finite and, unless stated otherwise, simple, connected and undirected.

1. Let X be a graph. Denote by $V(X)$ and $E(X)$ the vertex set and edge set of X, respectively.
2. An automorphism of a graph X is a permutation on its vertex set $V(X)$ inducing a bijection on $E(X)$.
3. All automorphisms of X consist of a group, called the automorphism group of X and denoted by $\mathrm{Aut}(X)$.

Automorphism group of a graph

Symmetries of a graph can be measured by its automorphism group.

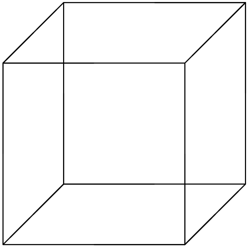
How compute a automorphism group of a graph?



$\mathrm{Aut}(C_5) \cong D_{10}$

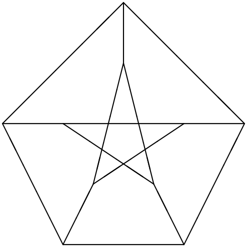
Automorphism group of a graph

Petersen graph O_3 is more symmetry than the three dimensional hypercube Q_3.



$\mathrm{Aut}(Q_3) \cong S_4 \times \mathbb{Z}_2$

Q_3 is 2-arc-transitive



$\mathrm{Aut}(O_3) \cong S_5$

O_3 is 3-arc-transitive

Concepts on groups and graphs

- Let G be a permutation group on Ω, that is, $G \leq S_\Omega$.
- G is transitive on Ω: for any two points in Ω there is a permutation in G mapping one to the other.
- G is semiregular on Ω: for any $\alpha \in \Omega$, $G_\alpha = 1$, that is, only the identity fixes a vertex. G is regular on Ω: G is transitive and semiregular.

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- Let X be a graph.
- X is vertex-transitive or edge-transitive: $\mathrm{Aut}(X)$ is transitive on $V(X)$ or $E(X)$, respectively.

Concepts on graphs

- s-arc: an $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices s.t. $(v_i, v_{i+1}) \in E(X)$, $v_{i-1} \neq v_{i+1}$.
- Let G be a subgroup of $\mathrm{Aut}(X)$.
- X is (G, s)-arc-transitive or (G, s)-regular: G acts transitively or regularly on its s-arc set, respectively.
- X is s-regular, s-arc-transitive, ...: $(\mathrm{Aut}(X), s)$-regular, $(\mathrm{Aut}(X), s)$-arc-transitive, ...
- 0-arc-transitive:vertex-transitive
- 1-arc-transitive: arc-transitive or symmetric

Cayley graph

Let G be a finite group and $S \subset G$ with $1 \notin S$ and $S = S^{-1}$.

- Cayley graph $\mathrm{Cay}(G, S)$: vertex set $V = G$, edge set $E = \{(g, sg) \mid g \in G, s \in S\}$

Cayley graph

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- Cayley graph $\mathrm{Cay}(G, S)$: vertex set $V = G$, edge set $E = \{(g, sg) \mid g \in G, s \in S\}$
- $\mathrm{Cay}(G, S)$ is connected $\Leftrightarrow G = \langle S \rangle$.
- Right regular representation $R(G)$ of G: the permutation group $\{R(g) \mid g \in G\}$ on G, where $R(g) : x \mapsto xg, \ \forall x \in G$ is a permutation on G. Clearly, $R(G)$ is a subgroup of $\mathrm{Aut}(\mathrm{Cay}(G, S))$, which acts regularly on $V(X)$. Thus, any Cayley graph is vertex-transitive. A Cayley graph $X = \mathrm{Cay}(G, S)$ is called normal if $R(G) \trianglelefteq \mathrm{Aut}(X)$.
- Characterization: X is isomorphic to a Cayley graph on G $\Leftrightarrow \mathrm{Aut}(X)$ has a regular subgroup isomorphic to G.

Concepts on Covers

1. A homomorphism of graphs $X \mapsto Y$ is map: $V(X) \mapsto V(Y)$ preserving adjacency, that is, mapping each edge of X to an edge of Y.
2. An epimorphism of graphs $X \mapsto Y$ is a homomorphism from X to Y such that it is surjective for $V(X) \mapsto V(Y)$. An isomorphism of graphs $X \mapsto Y$ is an epimorphism from X to Y inducing a bijection from $E(X) \mapsto E(Y)$. (example)
3. An epimorphism $\wp : \widetilde{X} \to X$ of graphs is called a covering projection if $\wp|_{N_{\widetilde{X}}(\tilde{v})} : N_{\widetilde{X}}(\tilde{v}) \to N_X(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in \wp^{-1}(v)$. In this case, $\widetilde{X}$ is called the cover graph, and X is a the base graph of the covering projection $\wp$.

Concepts on Covers

Let $\wp : \widetilde{X} \to X$ be a covering projection.

1. Vertex fibre, edge fibre or arc fibre are the preimages of vertex, edge, or arc of X under $\wp$, respectively. Since the base graph X is connected, all fibres has the same cardinality (Walk action).
2. A covering projection $\wp : \widetilde{X} \to X$ is called a regular covering projection if there is a subgroup $\mathrm{CT}(\wp) \leq \mathrm{Aut}(\widetilde{X})$ such that its orbits on $V(\widetilde{X})$ coincide with the vertex fibres $\wp^{-1}(v)$, $v \in V(X)$. The subgroup $\mathrm{CT}(\wp)$ is called the covering transformation group. Since $\widetilde{X}$ is connected, $\mathrm{CT}(\wp)$ is semiregular.

Fibre preserving group

Let $\wp : \widetilde{X} \to X$ be a regular covering projection with $\mathrm{CT}(\wp)$ as its covering transformation group.

1. An automorphism of $\widetilde{X}$ is said to be fibre-preserving if it maps any vertex fibre to a vertex fibre. All fibre-preserving automorphisms form a group, called the fibre-preserving group, denoted by F. Clearly, $\mathrm{CT}(\wp) \leq F$.
2. $\mathrm{CT}(\wp)$ is the kernel of F acting on the set of vertex fibres. Thus, $\mathrm{CT}(\wp)$ is normal in F, that is, $\mathrm{CT}(\wp) \lhd F$.
3. $F = N_{\mathrm{Aut}(\widetilde{X})}(\mathrm{CT}(\wp))$, the normalizer of $\mathrm{CT}(\wp)$ in $\mathrm{Aut}(\widetilde{X})$. However, it is possible that $F \neq \mathrm{Aut}(\widetilde{X})$. (Example F40A-F20A).

Definition: lift of automorphisms

Let $\wp : \widetilde{X} \to X$ be a regular covering projection with $\mathrm{CT}(\wp)$ as its covering transformation group. $A = \mathrm{Aut}(\widetilde{X})$

1. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\wp\alpha = \widetilde{\alpha}\wp$, we call $\widetilde{\alpha}$ a lift of α, and α a projection of $\widetilde{\alpha}$ ($\Rightarrow \widetilde{\alpha}$ fibre-preserving).
2. If an automorphism $\alpha \in \mathrm{Aut}(X)$ has a lift, say $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$, then $\mathrm{CT}(\wp)\widetilde{\alpha}$ are the all lifts of α. In particular, $\mathrm{CT}(\wp)$ is the lift of the identify group of $\mathrm{Aut}(X)$. ($N_A(\mathrm{CT}(\wp)) \mapsto \mathrm{Aut}(X)$)
3. Maybe two different subgroups of $\mathrm{Aut}(\widetilde{X})$ have the same projection. Let A and B be two subgroups of $\mathrm{Aut}(\widetilde{X})$ in which every element can be projected ($A, B \leq N_A(\mathrm{CT}(\wp))$). Then A and B have the same projection if and only if $\mathrm{CT}(\wp)A = \mathrm{CT}(\wp)B$.

Isomorphic covers

Basic Problem: Determine non-isomorphic covers for a given base graph (sometimes, a covering transformation group is given, or a group of automorphisms lift).

1. Two regular covering projections $\wp : \widetilde{X} \to X$ and $\wp' : \widetilde{X}' \to X$ of a graph X are isomorphic if there exist an automorphism $\alpha \in \mathrm{Aut}(X)$ and an isomorphism $\widetilde{\alpha} : \widetilde{X} \to \widetilde{X}'$ such that $\alpha\wp = \wp'\widetilde{\alpha}$. In particular, if $\alpha = id$ then $\wp$ and $\wp'$ are equivalent.
2. The covering graphs corresponding to isomorphic (equivalent) covering projections are isomorphic. However, maybe two covering graphs corresponding to non-isomorphic covering projection are isomorphic.

Voltage graph

Regular covers can be realized (reconstructed) by voltage graph, which was given by Gross and Tucker.
J.L. Gross and T.W. Tucker, *Topological Graph Theory*, Wiley-Interscience, New York, 1987.

1. Let X be a graph and K a finite group. Let $A(X)$ be the arc set of X. By a^{-1} we mean the reverse arc to an arc a. A voltage assignment (or, K-voltage assignment) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called voltages, and K is the voltage group.
2. The derived graph $X \times_\phi K$ from a voltage assignment $\phi : A(X) \to K$ is the graph with vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e, g) of $X \times_\phi K$ joins a vertex (u, g) to $(v, g\phi(a))$ for $a = (u, v) \in A(X)$ and $g \in K$, where $e = \{u, v\}$.

Equivalent regular covering

1. The first coordinate projection $\wp : X \times_\phi K \to X$ is called the natural projection.
2. By defining $(u, g')^g := (u, g^{-1}g')$ for any $g \in K$ and $(u, g') \in V(X \times_\phi K)$, we have $K \leq \mathrm{Aut}(X \times_\phi K)$, which is semiregular. Thus, the natural projection $\wp : X \times_\phi K \to X$ is a regular covering projection with $K = \mathrm{CT}(\wp)$.
3. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be T-reduced if the voltages on the tree arcs are the identity.
4. Gross and Tucker showed that every regular covering projection $\wp : \widetilde{X} \to X$ with covering transformation group $\mathrm{CT}(\wp)$ is equivalent to a natural projection $X \times_\phi \mathrm{CT}(\wp) \to X$ such that ϕ is a T-reduced voltage assignment with respect to an arbitrary fixed spanning tree T.

automorphism lifting conditions

Let $X \times_\phi K \to X$ be a K-covering projection derived from a T-reduced voltage assignment.

1. Given $\alpha \in \mathrm{Aut}(X)$, the function $\overline{\alpha}$ is defined by

$$\overline{\alpha} : \phi(C) \mapsto \phi(C^\alpha), (\text{ voltages on fcw } (\subseteq K) \mapsto K)$$

   where C ranges over all fundamental closed walks at v.

2. By Malnič [1, Theorem 4.2], An automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.

3. Let $\wp_1 : X \times_{\phi_1} K \to X$ and $\wp_2 : X \times_{\phi_2} K \to X$ be two regular covering projection. By Malnič, Marušič and Potočnik [2], $\wp_1$ and $\wp_2$ are isomorphic if and only if there is an automorphism $\alpha \in \mathrm{Aut}(X)$ and an automorphism $\beta \in \mathrm{Aut}(N)$ such that $W^{\phi_1\beta} = W^{\alpha\phi_2}$ for all fundamental closed walks W at some base vertex of X.

- A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182(1998) 203-218.
- A. Malnič, D. Marušič, P. Potočnik, Elementary Abelian Covers of Graphs, Journal of Algebraic Combinatorics 20(2004) 71-97.

Thank you!