The Graph Isomorphism Problem and coherent configurations. I, II.

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$$\forall \alpha_1, \beta_1 \in \Omega_1 : \quad (\alpha_1^f, \beta_1^f) \in E_2 \quad \Leftrightarrow \quad (\alpha_1, \beta_1) \in E_1.$$

Such a bijection is called the isomorphism from G_1 to G_2; the set of all of them is denoted by $\mathrm{Iso}(G_1, G_2)$.

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ISO(G_1, G_2): given graphs G_1 and G_2 test whether or not $G_1 \cong G_2$.

Current state of the ISO.

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Theorem (L.Babai, E.Luks and W.Kantor, 1984).

The isomorphism of n-vertex graphs can be tested in time $\exp(O(\sqrt{n \log n}))$.

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a directed edge $\alpha \dashrightarrow \beta$

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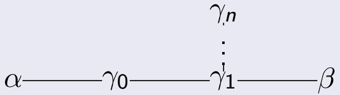

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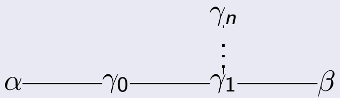

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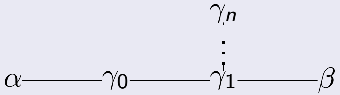

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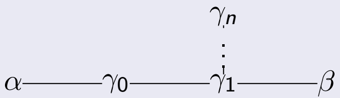

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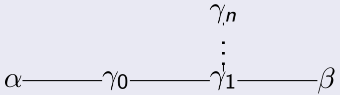

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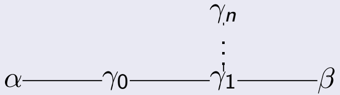

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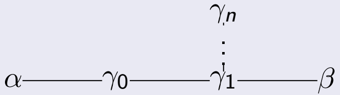

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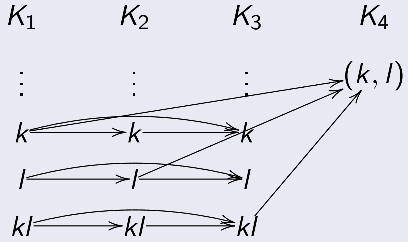

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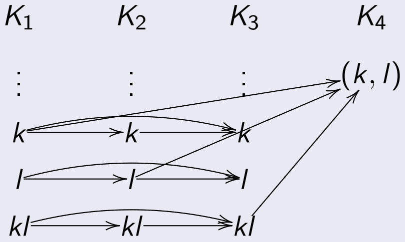

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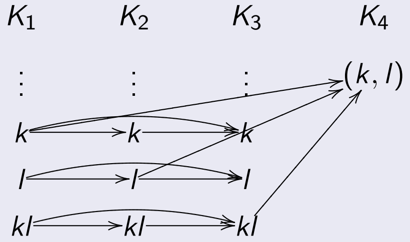

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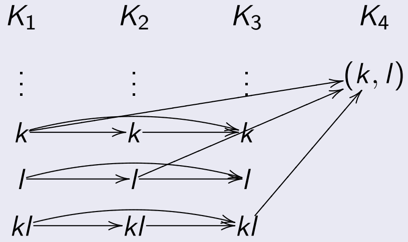

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- The algorithm correctly finds $\Omega/\operatorname{Aut}(G)$ for the class of trees (G.Tinhofer, 1985), for almost all graphs (L.Babai, P.Erdös, S.Selkow, 1980).
- The algorithm fails when G is a regular graphs and the group $\operatorname{Aut}(G)$ is intransitive.

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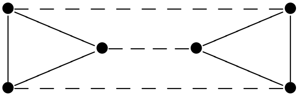

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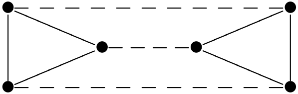

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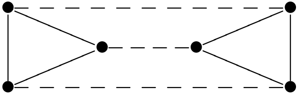

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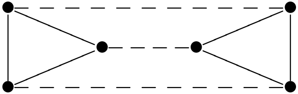

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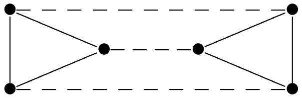

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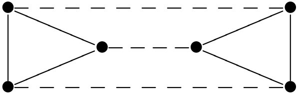

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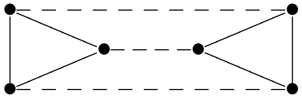

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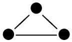
$(r, f) = (2, 1)$: $s_0 =$  $s_1 =$ 

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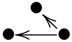
$s_3 =$  $s_4 =$ 

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A coherent configuration $\mathcal{X}$ is called schurian if $\mathcal{X} = \mathrm{Inv}(\Gamma)$ for some group Γ.

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Coherent configurations $\mathcal{X} = (\Omega, S)$ and $\mathcal{X}' = (\Omega', S')$ are isomorphic if there is a bijection $f : \Omega \to \Omega'$, the isomorphism from $\mathcal{X}$ to $\mathcal{X}'$, such that $S^f = S'$.

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Proof. On each step of the Weisfeiler-Leman algorithm $c(\alpha, \beta; r, s) = c(\alpha^f, \beta^f; r^f, s^f)$ for all α, β, r, s.

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One can see that $\mathrm{Aut}(\mathcal{X}) = \mathrm{Iso}(\mathcal{X}, \mathcal{X}, \mathrm{id}_S)$.

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If $\mathcal{X} = \langle\langle G \rangle\rangle$ and $\mathcal{X}' = \langle\langle G \rangle\rangle$ are obtained by the canonical W-L algorithm, then $G \cong G'$ only if $|S| = |S'|$ and the bijection $s_i \mapsto s_i'$ is an algebraic isomorphism.

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Proposition.

The ISO is polynomially equivalent to the following problem: given an algebraic isomorphism $\varphi : \mathcal{X} \to \mathcal{X}'$ test whether $\mathrm{Iso}(\mathcal{X}, \mathcal{X}', \varphi)$ is not empty.

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- If the Babai problem has a positive answer, then probably the Luks algorithm for graphs of bounded valency can be reformulated in terms of bases.