



On an inverse problem to Frobenius' theorem

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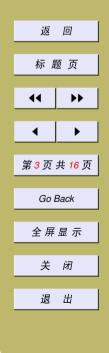
Let G be a finite group and e a positive integer dividing |G|, the order of G. Denoting $L_e(G) = \{x \in G \mid x^e = 1\}.$





Frobenius' Theorem (1895):

For every $e \mid |G|$, there exists a positive integer k such that $|L_e(G)| = ke$.



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Inverse Problem to Frobenius' Theorem:

For a small positive integer k, give a complete classification of all finite groups G with $|L_e(G)| \le ke$ for every $e \mid |G|$.





It is easy to prove that $|L_e(G)| = e$ for every $e \mid |G|$ if and only if G is a cyclic group.



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In this talk, we give a complete classification of finite groups G with $|L_e(G)| \le 2e$ for every $e \mid |G|$.



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Lemma 1.1 Let G be a p-group of order p^n for a prime p. Then $|L_{p^i}(G)| \le 2p^i$ $(0 \le i \le n)$ if and only if one of the following holds:

(1) G is a cyclic group;
(2) G ≅ Z_{2ⁿ⁻¹} × Z₂, where n ≥ 2;
(3) G ≅ Q₈;
(4) G ≅ ⟨a, b | a^{2ⁿ⁻¹} = b² = 1, b⁻¹ab = a^{1+2ⁿ⁻²}⟩, where n ≥ 4.



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Lemma 1.2 Let G be a finite group whose order has at least two distinct prime divisors and e a positive integer dividing |G|. Then $|L_e(G)| \le 2e$ if and only if one of the following holds:

(1) G is a cyclic group;

(2) $G \cong Z_m \times T$, where m > 1 is an odd integer and $T \cong Z_{2^{n-1}} \times Z_2 (n \ge 2)$ or Q_8 or $\langle a, b | a^{2^{t-1}} = b^2 = 1, b^{-1}ab = a^{1+2^{t-2}} \rangle (t \ge 4)$;

(3) $G \cong Z_m \times T$, where $T \cong \langle a, b \mid a^3 = b^{2^n} = 1, b^{-1}ab = a^{-1} \rangle$, $n \ge 1$ and (m, 6) = 1.





By Lemmas 1.1 and 1.2, our main result is as follows:

Theorem 1.3 Let G be a finite group and e a positive integer dividing |G|. Then $|L_e(G)| \leq 2e$ for every $e \mid |G|$ if and only if one of the following statements holds:

(1) G is a cyclic group;

(2) $G \cong Z_m \times T$, where $m \ge 1$ is an odd integer and $T \cong Z_{2^{n-1}} \times Z_2$ $(n \ge 2)$ or Q_8 or $\langle a, b \mid a^{2^{t-1}} = b^2 = 1$, $b^{-1}ab = a^{1+2^{t-2}} \rangle$ $(t \ge 4)$;

(3) $G \cong Z_m \times T$, where $T \cong \langle a, b \mid a^3 = b^{2^n} = 1, b^{-1}ab = a^{-1} \rangle, n \ge 1$ and (m, 6) = 1.





Corollary 1.4 Let G be a finite group and e a positive integer dividing |G|. Then $|L_e(G)| = 2e$ for every proper divisor e of |G| if and only if G is isomorphic to one of the following groups:

(1) $G \cong Z_{2^{n-1}} \times Z_2$, where $n \ge 2$; (2) $G \cong \langle a, b \mid a^{2^{n-1}} = b^2 = 1, b^{-1}ab = a^{1+2^{n-2}} \rangle$, where $n \ge 4$.





Corollary 1.5 Let G be a finite group of odd order and e a positive integer dividing |G|. Then $|L_e(G)| \le 2e$ for every $e \mid |G|$ if and only if G is a cyclic group.

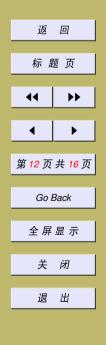
Corollary 1.5' Let G be a finite group of odd order and e a positive integer dividing |G|. If $|L_e(G)| \leq 2e$ for every $e \mid |G|$, then $|L_e(G)| = e$ for every $e \mid |G|$.





The classification of finite groups G with $|L_e(G)| \leq 3e$ for every $e \mid |G|$, is almost finished by W. Meng and J.T. Shi.

Problem 1 When k = 4, 5, 6, ..., give a complete classification of finite groups G with $|L_e(G)| \le ke$ for every $e \mid |G|$.



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Is there an upper bound for k such that a finite group G is solvable if $|L_e(G)| \le ke$ for every $e \mid |G|$?

Theorem 1.6 Let G be a finite group. If $|L_e(G)| \leq 7e$ for every $e \mid |G|$, then G is solvable.





The alternating group A_5 of degree 5 shows that a finite group G with $|L_e(G)| \le 8e$ for every $e \mid |G|$ may be non-solvable.

Problem 2 Classify finite non-solvable groups G with $|L_e(G)| \le 8e$ for every $e \mid |G|$.

Proposition 1.7 Let G be a characteristically simple group. If $|L_e(G)| \le 8e$ for every $e \mid |G|$, then $G \cong A_5$.



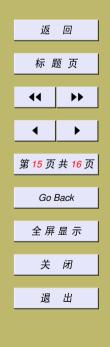


Problem 3 For any positive integer k, does there always exist a finite group G such that $|L_e(G)| = ke$ for some $e \mid |G|$?

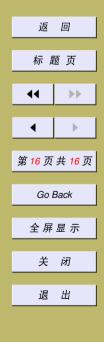
Problem 4 Denoting $\mathcal{K} = \{k \mid |L_e(G)| = ke \text{ for some } e \mid |G|\}.$

(1) If $\mathcal{K} = \{1, m\}$ consists of two distinct positive integers, what can we say about G?

(2) If $\mathcal{K} = \{1, 2, 3, ..., n\}$ consists of continuous positive integers, is there an upper bound for n?







Thank you!

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