Normal Cayley Graphs

Ted Dobson

Department of Mathematics & Statistics
Mississippi State University
dobson@math.msstate.edu
http://www2.msstate.edu/~dobson/

Symmetry in Graphs and Networks II, August 2, 2010
Definition

Let $G$ be a group and $S \subseteq G - \{1\}$ such that $S^{-1} = S$. Define a graph
$\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The
graph $\Gamma(G, S)$ is the Cayley graph of $G$ with connection set $S$. 
Definition

Let $G$ be a group and $S \subseteq G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The graph $\Gamma(G, S)$ is the Cayley graph of $G$ with connection set $S$. Note that the group $G_L$ of all bijections $g \rightarrow hg$ (multiplication by $h$ on the left) is a subgroup of $\text{Aut}(\Gamma)$ and is transitive. Thus a Cayley graph is a vertex-transitive graph.
Definition

Let $G$ be a group and $S \subseteq G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The graph $\Gamma(G, S)$ is the Cayley graph of $G$ with connection set $S$. Note that the group $G_L$ of all bijections $g \rightarrow hg$ (multiplication by $h$ on the left) is a subgroup of $\text{Aut}(\Gamma)$ and is transitive. Thus a Cayley graph is a vertex-transitive graph.

A circulant graph of order $n$ is simply a Cayley graph of $\mathbb{Z}_n$. 
Definition

Let $G$ be a group and $S \subseteq G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The graph $\Gamma(G, S)$ is the **Cayley graph of $G$ with connection set $S$**. Note that the group $G_L$ of all bijections $g \rightarrow hg$ (multiplication by $h$ on the left) is a subgroup of $\text{Aut}(\Gamma)$ and is transitive. Thus a Cayley graph is a vertex-transitive graph.

A **circulant graph of order $n$** is simply a Cayley graph of $\mathbb{Z}_n$.

Cayley digraphs and circulant digraphs are defined similarly except that we do not require that $S^{-1} = S$. 
Definition

Let $G$ be a group and $S \subseteq G - \{1\}$ such that $S^{-1} = S$. Define a graph $\Gamma = \Gamma(G, S)$ by $V(\Gamma) = G$ and $E(\Gamma) = \{(g, gs) : g \in G, s \in S\}$. The graph $\Gamma(G, S)$ is the Cayley graph of $G$ with connection set $S$. Note that the group $G_L$ of all bijections $g \rightarrow hg$ (multiplication by $h$ on the left) is a subgroup of $\text{Aut}(\Gamma)$ and is transitive. Thus a Cayley graph is a vertex-transitive graph.

A circulant graph of order $n$ is simply a Cayley graph of $\mathbb{Z}_n$.

Cayley digraphs and circulant digraphs are defined similarly except that we do not require that $S^{-1} = S$.

Symmetry is rare. So, if one wishes to obtain examples of a vertex-transitive graphs, the Cayley graph construction is probably the most common method used.
Definition

A Cayley digraph $\Gamma$ of $G$ is a digraphical regular representation (DRR) of $G$ if $\text{Aut}(\Gamma) = G_L$.
Definition

A Cayley digraph $\Gamma$ of $G$ is a *digraphical regular representation (DRR)* of $G$ if $\text{Aut}(\Gamma) = G_L$.

A similar definition gives a *graphical regular representation (GRR)* of $G$.
Definition

A Cayley digraph \( \Gamma \) of \( G \) is a digraphical regular representation (DRR) of \( G \) if \( \text{Aut}(\Gamma) = G_L \).

A similar definition gives a graphical regular representation (GRR) of \( G \).

One can think of a DRR or GRR as being a Cayley (di)graph \( \Gamma \) in which the Cayley (di)graph construction fails to give any additional automorphisms of \( \Gamma \).
Definition

A Cayley digraph $\Gamma$ of $G$ is a *digraphical regular representation (DRR)* of $G$ if $\text{Aut}(\Gamma) = G_L$.

A similar definition gives a *graphical regular representation (GRR)* of $G$.

One can think of a DRR or GRR as being a Cayley (di)graph $\Gamma$ in which the Cayley (di)graph construction fails to give any additional automorphisms of $\Gamma$.

What if one wanted to construct a Cayley digraph $\Gamma$ of $G$, but wanted to insist that $\text{Aut}(\Gamma)$ was bigger than just $G_L$?
Lemma

Let $\Gamma = \Gamma(G, S)$ be a Cayley digraph of $G$ and $\alpha \in \text{Aut}(G)$. Then $\alpha \in \text{Aut}(\Gamma)$ if and only if $\alpha(S) = S$. 

This is one way to construct a Cayley digraph of $\Gamma(G, S)$ whose automorphism group was bigger than $G_L$, a natural way to achieve this is to insist that $\alpha(S) = S$ for each $\alpha \in H \leq \text{Aut}(G)$. Or, $S$ is a union of orbits of $H$. 

Ted Dobson

Mississippi State University

Normal Cayley Graphs
Lemma

Let $\Gamma = \Gamma(G, S)$ be a Cayley digraph of $G$ and $\alpha \in \text{Aut}(G)$. Then $\alpha \in \text{Aut}(\Gamma)$ if and only if $\alpha(S) = S$.

This if one wanted to construct a Cayley digraph of $\Gamma(G, S)$ whose automorphism group was bigger than $G_L$, a natural way to achieve this is to insist that $\alpha(S) = S$ for each $\alpha \in H \leq \text{Aut}(G)$. 
Lemma

Let $\Gamma = \Gamma(G, S)$ be a Cayley digraph of $G$ and $\alpha \in \text{Aut}(G)$. Then $\alpha \in \text{Aut}(\Gamma)$ if and only if $\alpha(S) = S$.

This if one wanted to construct a Cayley digraph of $\Gamma(G, S)$ whose automorphism group was bigger than $G_L$, a natural way to achieve this is to insist that $\alpha(S) = S$ for each $\alpha \in H \leq \text{Aut}(G)$. Or, $S$ is a union of orbits of $H$. 
Ming-Yao Xu introduced the notion of a “normal Cayley graph” in 1998:
Ming-Yao Xu introduced the notion of a “normal Cayley graph” in 1998:

**Definition**

A Cayley (di)graph $\Gamma$ of $G$ is a normal Cayley (di)graph of $G$ if $G_L \vartriangleleft \text{Aut}(\Gamma)$. 
Ming-Yao Xu introduced the notion of a “normal Cayley graph” in 1998:

**Definition**

A Cayley (di)graph $\Gamma$ of $G$ is a *normal Cayley (di)graph of $G$* if $G_L \triangleleft \text{Aut}(\Gamma)$.

The following result is a standard group theoretic result, and can be found in Dixon and Mortimer’s *Permutation Groups*, for example.
Ming-Yao Xu introduced the notion of a “normal Cayley graph” in 1998:

**Definition**

A Cayley (di)graph $\Gamma$ of $G$ is a normal Cayley (di)graph of $G$ if $G_L \triangleleft \text{Aut}(\Gamma)$.

The following result is a standard group theoretic result, and can be found in Dixon and Mortimer’s *Permutation Groups*, for example.

**Lemma**

$N_{S_G}(G_L) = \text{Aut}(G) \cdot G_L$
Ming-Yao Xu introduced the notion of a “normal Cayley graph” in 1998:

**Definition**
A Cayley (di)graph $\Gamma$ of $G$ is a normal Cayley (di)graph of $G$ if $G_L \triangleleft \text{Aut}(\Gamma)$.

The following result is a standard group theoretic result, and can be found in Dixon and Mortimer’s *Permutation Groups*, for example.

**Lemma**
$N_{SG}(G_L) = \text{Aut}(G) \cdot G_L$

So one can think of normal Cayley graphs of $G$ as being those for which we construct the graph so that the automorphism group contains some nontrivial automorphisms of $G$, but for which there are no other graph automorphisms.
Typically, one thinks of being a normal Cayley graph as “normal”, and so one looks for non-normal Cayley graphs. As this is a large problem, there are a few standard approaches that one can take:
Typically, one thinks of being a normal Cayley graph as “normal”, and so one looks for non-normal Cayley graphs. As this is a large problem, there are a few standard approaches that one can take:

- Specify the group $G$ and find all non-normal Cayley graphs of $G$
Typically, one thinks of being a normal Cayley graph as “normal”, and so one looks for non-normal Cayley graphs. As this is a large problem, there are a few standard approaches that one can take:

- Specify the group $G$ and find all non-normal Cayley graphs of $G$
- Specify the degree of a vertex (usually small), and find all corresponding non-normal Cayley graphs (and the their associated automorphism groups)
Typically, one thinks of being a normal Cayley graph as “normal”, and so one looks for non-normal Cayley graphs. As this is a large problem, there are a few standard approaches that one can take:

- Specify the group $G$ and find all non-normal Cayley graphs of $G$
- Specify the degree of a vertex (usually small), and find all corresponding non-normal Cayley graphs (and the their associated automorphism groups)
- Look for interesting families of non-normal Cayley graphs.
Typically, one thinks of being a normal Cayley graph as “normal”, and so one looks for non-normal Cayley graphs. As this is a large problem, there are a few standard approaches that one can take:

- Specify the group $G$ and find all non-normal Cayley graphs of $G$
- Specify the degree of a vertex (usually small), and find all corresponding non-normal Cayley graphs (and their associated automorphism groups)
- Look for interesting families of non-normal Cayley graphs.

We will primarily be interested in the first approach.
Normal Cayley graphs of $\mathbb{Z}_p$

Theorem (Burnside, 1901)

Let $G \leq S_p$, $p$ a prime, contain $(\mathbb{Z}_p)_L$. Then $G \leq AGL(1,p) = \{x \rightarrow ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or $G$ is doubly-transitive.

Note that if $G \leq Aut(\Gamma)$ is doubly-transitive, then $Aut(\Gamma) = S_p$ and $\Gamma$ is complete or has no edges.

As $AGL(1,p) = Aut(\mathbb{Z}_p) \cdot (\mathbb{Z}_p)_L$ (so $(\mathbb{Z}_p)_L \triangleleft AGL(1,p)$), with the exception of the complete graph and its complement, every circulant digraph of prime order is normal.
Normal Cayley graphs of $\mathbb{Z}_p$

**Theorem (Burnside, 1901)**

Let $G \leq S_p$, $p$ a prime, contain $(\mathbb{Z}_p)_L$. Then $G \leq AGL(1,p) = \{x \rightarrow ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or $G$ is doubly-transitive.
Normal Cayley graphs of $\mathbb{Z}_p$

**Theorem (Burnside, 1901)**

Let $G \leq S_p$, $p$ a prime, contain $(\mathbb{Z}_p)_L$. Then $G \leq AGL(1, p) = \{x \rightarrow ax + b : a \in \mathbb{Z}^*_p, b \in \mathbb{Z}_p\}$ or $G$ is doubly-transitive.

Note that if $G \leq Aut(\Gamma)$ is doubly-transitive, then $Aut(\Gamma) = S_p$ and $\Gamma$ is complete or has no edges.
Normal Cayley graphs of $\mathbb{Z}_p$

Theorem (Burnside, 1901)

Let $G \leq S_p$, $p$ a prime, contain $(\mathbb{Z}_p)_L$. Then $G \leq AGL(1, p) = \{x \rightarrow ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or $G$ is doubly-transitive.

Note that if $G \leq \text{Aut}(\Gamma)$ is doubly-transitive, then $\text{Aut}(\Gamma) = S_p$ and $\Gamma$ is complete or has no edges.

As $AGL(1, p) = \text{Aut}(\mathbb{Z}_p) \cdot (\mathbb{Z}_p)_L$ (so $(\mathbb{Z}_p)_L \triangleleft AGL(1, p)$)},
Normal Cayley graphs of $\mathbb{Z}_p$

Theorem (Burnside, 1901)

Let $G \leq S_p$, $p$ a prime, contain $(\mathbb{Z}_p)_L$. Then
$G \leq AGL(1, p) = \{x \rightarrow ax + b : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p\}$ or $G$ is doubly-transitive.

Note that if $G \leq Aut(\Gamma)$ is doubly-transitive, then $Aut(\Gamma) = S_p$ and $\Gamma$ is complete or has no edges.

As $AGL(1, p) = Aut(\mathbb{Z}_p) \cdot (\mathbb{Z}_p)_L$ (so $(\mathbb{Z}_p)_L \triangleleft AGL(1, p)$), with the exception of the complete graph and its complement, every circulant digraph of prime order is normal.
More Burnside Type Results

Burnside's Theorem can also be restated as follows:

**Theorem**
A transitive group of prime degree is either doubly-transitive or contains a normal Sylow p-subgroup.

**Theorem (D., D. Witte, 2002)**
There are exactly $2^{p-1}$ transitive p-subgroups $P$ of $S_p^2$ up to conjugation, and all but three have the property that if $G \leq S_p^2$ with Sylow p-subgroup $P$, then either $P \triangleleft G$ or $G$ is doubly-transitive.

**Theorem (D., 2005)**
Let $P$ be a transitive p-subgroup of $S_p^k$, $p$ an odd prime, $k \geq 1$, such that every minimal transitive subgroup of $P$ is cyclic. If $G \leq S_p^k$ with Sylow p-subgroup $P$, then either $P \triangleleft G$ or $G$ is doubly-transitive.
More Burnside Type Results

Burnside’s Theorem can also be restated as follows:

**Theorem**

A transitive group of prime degree is either doubly-transitive or contains a normal Sylow $p$-subgroup.
More Burnside Type Results

Burnside’s Theorem can also be restated as follows:

**Theorem**

*A transitive group of prime degree is either doubly-transitive or contains a normal Sylow p-subgroup.*

**Theorem (D., D. Witte, 2002)**

*There are exactly $2p - 1$ transitive $p$-subgroups $P$ of $S_{p^2}$ up to conjugation, and all but three have the property that if $G \leq S_{p^2}$ with Sylow $p$-subgroup $P$, then either $P \triangleleft G$ or $G$ is doubly-transitive.*
More Burnside Type Results

Burnside’s Theorem can also be restated as follows:

**Theorem**

A transitive group of prime degree is either doubly-transitive or contains a normal Sylow $p$-subgroup.

**Theorem (D., D. Witte, 2002)**

There are exactly $2p - 1$ transitive $p$-subgroups $P$ of $S_{p^2}$ up to conjugation, and all but three have the property that if $G \leq S_{p^2}$ with Sylow $p$-subgroup $P$, then either $P \triangleleft G$ or $G$ is doubly-transitive.

**Theorem (D., 2005)**

Let $P$ be a transitive $p$-subgroup of $S_{p^k}$, $p$ an odd prime, $k \geq 1$, such that every minimal transitive subgroup of $P$ is cyclic. If $G \leq S_{p^k}$ with Sylow $p$-subgroup $P$, then either $P \triangleleft G$ or $G$ is doubly-transitive.
Problems

Problem
What are the overgroups of transitive permutation groups whose only minimal transitive subgroups are a direct product of two cyclic groups of prime-power order?

Perhaps with certain Sylow $p$-subgroups excluded?

What about instead of direct product of two cyclic groups, a semidirect product of two cyclic groups?

Problem
Which transitive groups of (odd) prime-power degree contain a regular cyclic subgroup and some other regular (minimal transitive) subgroup?

J. Morris has solved this if the other regular subgroup is abelian.
Problems

Problem

What are the overgroups of transitive permutation groups whose only minimal transitive subgroups are a direct product of two cyclic groups of prime-power order? Perhaps with certain Sylow $p$-subgroups excluded?

Problem

Which transitive groups of (odd) prime-power degree contain a regular cyclic subgroup and some other regular (minimal transitive) subgroup? J. Morris has solved this if the other regular subgroup is abelian.
Problem

What are the overgroups of transitive permutation groups whose only minimal transitive subgroups are a direct product of two cyclic groups of prime-power order? Perhaps with certain Sylow $p$-subgroups excluded? What about instead of direct product of two cyclic groups, a semidirect product of two cyclic groups?
Problems

Problem
What are the overgroups of transitive permutation groups whose only minimal transitive subgroups are a direct product of two cyclic groups of prime-power order? Perhaps with certain Sylow p-subgroups excluded? What about instead of direct product of two cyclic groups, a semidirect product of two cyclic groups?

Problem
Which transitive groups of (odd) prime-power degree contain a regular cyclic subgroup and some other regular (minimal transitive) subgroup?
Problems

Problem
What are the overgroups of transitive permutation groups whose only minimal transitive subgroups are a direct product of two cyclic groups of prime-power order? Perhaps with certain Sylow $p$-subgroups excluded? What about instead of direct product of two cyclic groups, a semidirect product of two cyclic groups?

Problem
Which transitive groups of (odd) prime-power degree contain a regular cyclic subgroup and some other regular (minimal transitive) subgroup? J. Morris has solved this if the other regular subgroup is abelian.
Even More Burnside Type Results

Burnside’s Theorem can be restated as follows (using another result of Burnside):

Theorem

Let \( G \leq S_p \) be transitive. Then \( G \) contains a normal subgroup that is either \( Z_p \) or a nonabelian simple group.

Theorem (Jones, 1979)

Let \( G \leq S_{p^2} \) be transitive with Sylow \( p \)-subgroup isomorphic to \( Z_p \times Z_p \).

Then \( G \) contains a normal subgroup \( H \) such that \( H = H_1 \times H_2 \) where \( H_i \) is a transitive nonabelian simple group of degree \( p \) or \( H_i = Z_p \).
Even More Burnside Type Results

Burnside’s Theorem can be restated as follows (using another result of Burnside):

**Theorem**

Let $G \leq S_p$ be transitive. Then $G$ contains a normal subgroup that is either $\mathbb{Z}_p$ or a nonabelian simple group.
Even More Burnside Type Results

Burnside’s Theorem can be restated as follows (using another result of Burnside):

**Theorem**

Let $G \leq S_p$ be transitive. Then $G$ contains a normal subgroup that is either $\mathbb{Z}_p$ or a nonabelian simple group.

**Theorem (Jones, 1979)**

Let $G \leq S_{p^2}$ be transitive with Sylow $p$-subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then $G$ contains a normal subgroup $H$ such that $H = H_1 \times H_2$ where $H_i$ is a transitive nonabelian simple group of degree $p$ or $H_i = \mathbb{Z}_p$. 
Theorem (D. and P. Spiga)

Let $G \leq S_{p^k}$ be transitive with an abelian Sylow $p$-subgroup $P$. Then $G$ contains a normal subgroup permutation isomorphic to a direct product of cyclic groups and doubly-transitive nonabelian simple groups with the canonical (coordinate wise) action, where the number of factors in the direct product is equal to the rank of $P$. Moreover, if $p = 2$, then $P$ is normal in $G$. 
Theorem (D. and P. Spiga)

Let $G \leq S_{p^k}$ be transitive with an abelian Sylow $p$-subgroup $P$. Then $G$ contains a normal subgroup permutation isomorphic to a direct product of cyclic groups and doubly-transitive nonabelian simple groups with the canonical (coordinate wise) action, where the number of factors in the direct product is equal to the rank of $P$. Moreover, if $p = 2$, then $P$ is normal in $G$.

Theorem (D.)

Every transitive group of prime-power degree $p^k$ that contains a transitive metacyclic Sylow $p$-subgroup, $p \geq 5$, contains a normal Sylow $p$-subgroup.
More Problems

Problem

Choose your favorite group $G$ of order a prime-power (not mentioned above). Considering this group as the regular group $G_L$, which subgroups of $S_G$ have $G_L$ as a Sylow $p$-subgroup?

Problem

Suppose you know the groups in $S_{G_1}$ and $S_{G_2}$ that have regular groups $G_1$ and $G_2$ as Sylow $p$-subgroups (both groups of order a power of the prime $p$). Is it true that the only subgroups of $S_{G_1} \times G_2$ with Sylow $p$-subgroup $G_1 \times G_2$ contain a normal subgroup which is a direct product of groups that have Sylow $p$-subgroup $G_1$ in $S_{G_1}$ and $G_2$ in $S_{G_2}$?

Problem

What if $G_2$ is cyclic? Can you find conditions on $G_1$ so that the previous problem is true?
More Problems

Problem
Choose your favorite group $G$ of order a prime-power (not mentioned above). Considering this group as the regular group $G_L$, which subgroups of $S_G$ have $G_L$ as a Sylow $p$-subgroup?

Problem
Suppose you know the groups in $S_{G_1}$ and $S_{G_2}$ that have regular groups $G_1$ and $G_2$ as Sylow $p$-subgroups (both groups of order a power of the prime $p$). Is it true that the only subgroups of $S_{G_1 \times G_2}$ with Sylow $p$-subgroup $G_1 \times G_2$ contain a normal subgroup which is a direct product of groups that have Sylow $p$-subgroup $G_1$ in $S_{G_1}$ and $G_2$ in $S_{G_2}$?
More Problems

Problem
Choose your favorite group $G$ of order a prime-power (not mentioned above). Considering this group as the regular group $G_L$, which subgroups of $S_G$ have $G_L$ as a Sylow $p$-subgroup?

Problem
Suppose you know the groups in $S_{G_1}$ and $S_{G_2}$ that have regular groups $G_1$ and $G_2$ as Sylow $p$-subgroups (both groups of order a power of the prime $p$). Is it true that the only subgroups of $S_{G_1 \times G_2}$ with Sylow $p$-subgroup $G_1 \times G_2$ contain a normal subgroup which is a direct product of groups that have Sylow $p$-subgroup $G_1$ in $S_{G_1}$ and $G_2$ in $S_{G_2}$?

Problem
What if $G_2$ is cyclic? Can you find conditions on $G_1$ so that the previous problem is true?
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_p^k$, $k \geq 1$.
- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.
- $\mathbb{Z}_3^p$, Dobson and I. Kovács, 2009.
- $\mathbb{Z}_p \times \mathbb{Z}_p^2$, Dobson, manuscript.
- Automorphism groups of all vertex-transitive digraphs of order $pq$ are known (many authors).
- Work on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002) give strong constraints on the automorphism group of circulants.

Ponomarenko used these results to give a polynomial time algorithm to compute the full automorphism group of a circulant.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$.  

First proof using method of Schur by Klin and Pöschel, in the early '80's.

Another proof using result above. None of the proofs published...

- $\mathbb{Z}_{p} \times \mathbb{Z}_{p}$, Dobson and D. Witte, 2002.

- $\mathbb{Z}_{3p}$, Dobson and I. Kovács, 2009.

- $\mathbb{Z}_{p} \times \mathbb{Z}_{p^2}$, Dobson, manuscript.

- Automorphism groups of all vertex-transitive digraphs of order $pq$ are known (many authors).

Work on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002) give strong constraints on the automorphism group of circulants.

Ponomarenko used these results to give a polynomial time algorithm to compute the full automorphism group of a circulant.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...

- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.

- $\mathbb{Z}_{3p}$, Dobson and I. Kovács, 2009.

- $\mathbb{Z}_p \times \mathbb{Z}_p^2$, Dobson, manuscript.

- Automorphism groups of all vertex-transitive digraphs of order $pq$ are known (many authors).

- Work on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002) give strong constraints on the automorphism group of circulants. Ponomarenko used these results to give a polynomial time algorithm to compute the full automorphism group of a circulant.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...
- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known \((p \text{ and } q \text{ are distinct primes})\):

- \(\mathbb{Z}_{p^k}, k \geq 1\). First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...

- \(\mathbb{Z}_p \times \mathbb{Z}_p\), Dobson and D. Witte, 2002.

Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...
- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.
- $\mathbb{Z}_p \times \mathbb{Z}_p^2$, Dobson, manuscript.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...
- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.
- $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, Dobson, manuscript.
- Automorphism groups of all vertex-transitive digraphs of order $pq$ are known (many authors).
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...
- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.
- $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, Dobson, manuscript.
- Automorphism groups of all vertex-transitive digraphs of order $pq$ are known (many authors).
- Work on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002) give strong constraints on the automorphism group of circulants.
Using these results and other techniques, the full automorphism group of a Cayley (di)graph of the following groups are known ($p$ and $q$ are distinct primes):

- $\mathbb{Z}_{p^k}$, $k \geq 1$. First proof using method of Schur by Klin and Pöschel, in the early '80's. Another proof using result above. None of the proofs published...
- $\mathbb{Z}_p \times \mathbb{Z}_p$, Dobson and D. Witte, 2002.
- $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$, Dobson, manuscript.
- Automorphism groups of all vertex-transitive digraphs of order $pq$ are known (many authors).
- Work on Schur rings by Leung and Man (1996 and 1998), and Evdomikov and Ponomarenko (2002) give strong constraints on the automorphism group of circulants. Ponomarenko used these results to give a polynomial time algorithm to compute the full automorphism group of a circulant.
Theoretically, these results should give the corresponding non-normal Cayley digraphs of the appropriate group.
Theoretically, these results should give the corresponding non-normal Cayley digraphs of the appropriate group. This has been done for all groups of order $p^2$ and $pq$. 
Theoretically, these results should give the corresponding non-normal Cayley digraphs of the appropriate group. This has been done for all groups of order $p^2$ and $pq$. However, this seems a very bad way to approach the problem of finding non-normal Cayley digraphs.
Theoretically, these results should give the corresponding non-normal Cayley digraphs of the appropriate group. This has been done for all groups of order $p^2$ and $pq$. However, this seems a very bad way to approach the problem of finding non-normal Cayley digraphs. Given that any “unusual” automorphism group will be non-normal though, this may be the only way.
Theoretically, these results should give the corresponding non-normal Cayley digraphs of the appropriate group. This has been done for all groups of order $p^2$ and $pq$. However, this seems a very bad way to approach the problem of finding non-normal Cayley digraphs. Given that any “unusual” automorphism group will be non-normal though, this may be the only way. So, to obtain interesting results about normal Cayley digraphs without finding all automorphism groups, an asymptotic approach is probably best (as realized by M.Y. Xu).
Asymptotic Problems

Is it true that almost all Cayley digraphs have automorphism group as small as possible? Are normal?

These were conjectured by B. Alspach and M. Y. Xu (1998), respectively.

Imrich, Lovász, Babai, and Godsil in around 1982 conjectured the following:

Conjecture

Let $G$ be a finite group of order $g$ which is neither abelian of exponent at least 3 nor generalized dicyclic. Then

$$\lim_{g \to \infty} \frac{\# \text{of GRR's of } G}{\# \text{of Cayley graphs of } G} = 1.$$

There is also the similar conjecture that "almost all" Cayley digraphs of $G$ are DRR's of $G$.
Asymptotic Problems

Is it true that almost all Cayley digraphs have automorphism group as small as possible? are normal?

These were conjectured by B. Alspach and M. Y. Xu (1998), respectively.

Imrich, Lovász, Babai, and Godsil in around 1982 conjectured the following:

Conjecture

Let $G$ be a finite group of order $g$ which is neither abelian of exponent at least 3 nor generalized dicyclic. Then

$$\lim_{g \to \infty} \frac{\# \text{of GRRs of } G}{\# \text{of Cayley graphs of } G} = 1.$$ 

There is also the similar conjecture that “almost all” Cayley digraphs of $G$ are DRR’s of $G$.

Ted Dobson
Mississippi State University

Normal Cayley Graphs
Asymptotic Problems

Is it true that almost all Cayley digraphs have automorphism group as small as possible? Are normal? These were conjectured by B. Alspach and M. Y. Xu (1998), respectively.

Imrich, Lovász, Babai, and Godsil in around 1982 conjectured the following:

Conjecture

Let $G$ be a finite group of order $g$ which is neither abelian of exponent at least 3 nor generalized dicyclic. Then

$$\lim_{g \to \infty} \frac{\text{# of GRR's of } G}{\text{# of Cayley graphs of } G} = 1.$$
Asymptotic Problems

Is it true that almost all Cayley digraphs have automorphism group as small as possible? Are normal? These were conjectured by B. Alspach and M. Y. Xu (1998), respectively.

Imrich, Lovász, Babai, and Godsil in around 1982 conjectured the following:

Conjecture

Let $G$ be a finite group of order $g$ which is neither abelian of exponent at least 3 nor generalized dicyclic. Then

$$\lim_{g \to \infty} \frac{\# \text{ of GRR's of } G}{\# \text{ of Cayley graphs of } G} = 1.$$
Asymptotic Problems

Is it true that almost all Cayley digraphs have automorphism group as small as possible? are normal? These were conjectured by B. Alspach and M. Y. Xu (1998), respectively.

Imrich, Lovász, Babai, and Godsil in around 1982 conjectured the following:

Conjecture

Let $G$ be a finite group of order $g$ which is neither abelian of exponent at least 3 nor generalized dicyclic. Then

$$\lim_{g \to \infty} \frac{\# \text{ of GRR's of } G}{\# \text{ of Cayley graphs of } G} = 1.$$ 

There is also the similar conjecture that “almost all” Cayley digraphs of $G$ are DRR’s of $G$. 

Ted Dobson  Mississippi State University

Normal Cayley Graphs
Asymptotic Results

The first results were obtained by Godsil in 1981:
Asymptotic Results

The first results were obtained by Godsil in 1981:

**Theorem**

Let $G$ be a group of prime-power order with no homomorphism onto $\mathbb{Z}_p \wr \mathbb{Z}_p$ (a full Sylow $p$-subgroup of $S_{p^2}$). Then almost all Cayley digraphs of $G$ are DRR's of $G$.
Asymptotic Results

The first results were obtained by Godsil in 1981:

**Theorem**

Let $G$ be a group of prime-power order with no homomorphism onto $\mathbb{Z}_p \wr \mathbb{Z}_p$ (a full Sylow $p$-subgroup of $S_{p^2}$). Then almost all Cayley digraphs of $G$ are DRR’s of $G$.

**Theorem**

Let $G$ be a non-abelian group of prime-power order with no homomorphism onto $\mathbb{Z}_p \wr \mathbb{Z}_p$ (a full Sylow $p$-subgroup of $S_{p^2}$). Then almost all Cayley graphs of $G$ are GRR’s of $G$. 
These results were improved by Babai and Godsil in 1982:
These results were improved by Babai and Godsil in 1982:

**Theorem**

Let $G$ be a nilpotent group of odd order $g$. Let $\Gamma$ be a random Cayley digraph or Cayley graph of $G$. In the undirected case, assume additionally that $G$ is not abelian. Then the probability that $\text{Aut}(\Gamma) \neq G$ is less than $(0.91 + o(1))^{\sqrt{g}}$.

Let $G$ be an abelian group of order $g \equiv -1 \pmod{4}$. Then, for almost all Cayley graphs $\Gamma$ of $G$, $|\text{Aut}(\Gamma)| = 2|G|$. 

Ted Dobson
Mississippi State University

Normal Cayley Graphs
These results were improved by Babai and Godsil in 1982:

**Theorem**

Let $G$ be a nilpotent group of odd order $g$. Let $\Gamma$ be a random Cayley digraph or Cayley graph of $G$. In the undirected case, assume additionally that $G$ is not abelian. Then the probability that $\text{Aut}(\Gamma) \neq G$ is less than $(0.91 + o(1))\sqrt{g}$.

**Theorem**

Let $G$ be an abelian group of order $g \equiv -1 \pmod{4}$. Then, for almost all Cayley graphs $\Gamma$ of $G$, $|\text{Aut}(\Gamma)| = 2|G|$. 
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

Theorem (D.)

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2^{|G|}$.

How are these results proven?

The basic idea is to show that if $Aut(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of $G$ contained in $Aut(\Gamma)$. The connection set of $\Gamma$ must then be a union of orbits of this automorphism. The number of orbits are then counted, as well as the possible “extra” automorphisms. This number is typically very small in comparison with the number of possible Cayley (di)graphs of $G$.
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

**Theorem (D.)**

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2 \cdot |G|$. 

How are these results proven?

The basic idea is to show that if $\text{Aut}(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of $G$ contained in $\text{Aut}(\Gamma)$. The connection set of $\Gamma$ must then be a union of orbits of this automorphism. The number of orbits are then counted, as well as the possible “extra” automorphisms. This number is typically very small in comparison with the number of possible Cayley (di)graphs of $G$. 

Ted Dobson

Mississippi State University

**Normal Cayley Graphs**
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

**Theorem (D.)**

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2 \cdot |G|$.

How are these results proven?
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

**Theorem (D.)**

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2 \cdot |G|$.

How are these results proven? The basic idea is to show that if $\text{Aut}(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of $G$ contained in $\text{Aut}(\Gamma)$. 

Ted Dobson
Normal Cayley Graphs
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

**Theorem (D.)**

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2 \cdot |G|$.

How are these results proven? The basic idea is to show that if $\text{Aut}(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of $G$ contained in $\text{Aut}(\Gamma)$. The connection set of $\Gamma$ must then be a union of orbits of this automorphism.
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

**Theorem (D.)**

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2 \cdot |G|$.

How are these results proven? The basic idea is to show that if $\text{Aut}(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of $G$ contained in $\text{Aut}(\Gamma)$. The connection set of $\Gamma$ must then be a union of orbits of this automorphism. The number of orbits are then counted, as well as the possible “extra” automorphisms.
The following result extends Babai and Godsil’s result for graphs, provided that the order of the group is an odd prime power.

**Theorem (D.)**

Let $G$ be an abelian group of odd prime-power order. Then almost every Cayley graph of $G$ has automorphism group of order $2 \cdot |G|$.

How are these results proven? The basic idea is to show that if $\text{Aut}(\Gamma)$ does not have automorphism group as small as possible, then there is some extra automorphism of $G$ contained in $\text{Aut}(\Gamma)$. The connection set of $\Gamma$ must then be a union of orbits of this automorphism. The number of orbits are then counted, as well as the possible “extra” automorphisms. This number is typically very small in comparison with the number of possible Cayley (di)graphs of $G$. 
Note that if a $G$ is of prime-power order $p^k$ and a Sylow $p$-subgroup of $\text{Aut}(\Gamma)$ has order greater than $p^k$, then $N_{\text{Aut}(\Gamma)}(G_L) \neq G_L$. 

Using a previous permutation group theoretic result, this produces an “extra” graph and group automorphism of order $p^k$. So the earlier results about overgroups of regular $p$-groups are the only automorphism groups that need to be considered.

One can prove more results though...

Theorem (D.)

Let $G$ be an abelian group of prime-power order $p^k$. Then almost every Cayley digraph of $G$ that is not a DRR is a normal Cayley digraph of $G$. In particular, 

$$\lim_{p \to \infty} |\text{NorCayDi}(G) - \text{DRR}(G)| = 1.$$
Note that if a $G$ is of prime-power order $p^k$ and a Sylow $p$-subgroup of $\text{Aut}(\Gamma)$ has order greater than $p^k$, then $N_{\text{Aut}(\Gamma)}(G_L) \neq G_L$. Using a previous permutation group theoretic result, this produces an "extra" graph and group automorphism of order $p$. 

**Theorem (D.)**

Let $G$ be an abelian group of prime-power order $p^k$. Then almost every Cayley digraph of $G$ that is not a DRR is a normal Cayley digraph of $G$. In particular, 

$$\lim_{p \to \infty} |\text{NorCayDi}(G) - \text{DRR}(G)| = 1.$$
Note that if a $G$ is of prime-power order $p^k$ and a Sylow $p$-subgroup of $\text{Aut}(\Gamma)$ has order greater than $p^k$, then $N_{\text{Aut}(\Gamma)}(G_L) \neq G_L$. Using a previous permutation group theoretic result, this produces an “extra” graph and group automorphism of order $p$. So the earlier results about overgroups of regular $p$-groups are the only automorphism groups that need to be considered.
Note that if a $G$ is of prime-power order $p^k$ and a Sylow $p$-subgroup of $\text{Aut}(\Gamma)$ has order greater than $p^k$, then $N_{\text{Aut}(\Gamma)}(G_L) \neq G_L$. Using a previous permutation group theoretic result, this produces an “extra” graph and group automorphism of order $p$. So the earlier results about overgroups of regular $p$-groups are the only automorphism groups that need to be considered. One can prove more results though...
Note that if a $G$ is of prime-power order $p^k$ and a Sylow $p$-subgroup of $\text{Aut}(\Gamma)$ has order greater than $p^k$, then $N_{\text{Aut}(\Gamma)}(G_L) \neq G_L$. Using a previous permutation group theoretic result, this produces an “extra” graph and group automorphism of order $p$. So the earlier results about overgroups of regular $p$-groups are the only automorphism groups that need to be considered.

One can prove more results though...

**Theorem (D.)**

Let $G$ be an abelian group of prime-power order $p^k$. Then almost every Cayley digraph of $G$ that is not a DRR is a normal Cayley digraph of $G$. In particular,

$$\lim_{p \to \infty} \frac{|\text{NorCayDi}(G) - \text{DRR}(G)|}{|\text{CayDi}(G) - \text{DRR}(G)|} = 1.$$
Theorem (D.)

Let $G$ be an abelian group of prime-power order $p^k$. Then almost every Cayley graph of $G$ that does not have automorphism group of order $2 \cdot |G|$ is a normal Cayley graph of $G$. 
Even More Problems

Conjecture

Almost every Cayley (di)graph whose automorphism group is not as small as possible is a normal Cayley (di)graph.
Even More Problems

Conjecture

Almost every Cayley (di)graph whose automorphism group is not as small as possible is a normal Cayley (di)graph.

There are two additional families of (di)graphs that can be considered, namely “semiwreath products” and “deleted wreath products”
Even More Problems

Conjecture

*Almost every* Cayley (di)graph whose automorphism group is not as small as possible *is a normal* Cayley (di)graph.

There are two additional families of (di)graphs that can be considered, namely “semiwreath products” and “deleted wreath products” - such graphs are not normal Cayley graphs provided $p \neq 2$. 
Conjecture

Almost every Cayley (di)graph whose automorphism group is not as small as possible is a normal Cayley (di)graph.

There are two additional families of (di)graphs that can be considered, namely “semiwreath products” and “deleted wreath products” - such graphs are not normal Cayley graphs provided $p \neq 2$.

Definition

A Cayley graph $\Gamma$ of an abelian group $G$ is a semiwreath product if there exist subgroups $H \leq K < G$ such that $S - K$ is a union of cosets of $H$. 
Even More Problems

Conjecture

Almost every Cayley (di)graph whose automorphism group is not as small as possible is a normal Cayley (di)graph.

There are two additional families of (di)graphs that can be considered, namely “semiwreath products” and “deleted wreath products” - such graphs are not normal Cayley graphs provided $p \neq 2$.

Definition

A Cayley graph $\Gamma$ of an abelian group $G$ is a semiwreath product if there exist subgroups $H \leq K < G$ such that $S - K$ is a union of cosets of $H$.

Note that if $H = K$, a semiwreath product is in fact a wreath product.
Definition

A Cayley graph $\Gamma$ of an abelian group $G$ is a deleted wreath product if $\Gamma = (\Gamma_1 \wr \bar{K}_m) - m\Gamma_1$, where $\Gamma_1$ is a Cayley graph of an abelian group of order $|G|/m$, and $m\Gamma_1$ is $m$ vertex-disjoint copies of $\Gamma_1$. 

Conjecture

Let $\Gamma$ be a Cayley (di)graph of an abelian group $G$. Then one of the following is true:

$\Gamma$ is a normal Cayley graph of $G$,

$\Gamma$ is a semiwreath product, or

the automorphism group of $\Gamma$ has a normal subgroup which is the same as the automorphism group of a deleted wreath product.

The preceding conjecture is known to be true if $G$ is cyclic, and is false for some nonabelian groups.
Definition

A Cayley graph $\Gamma$ of an abelian group $G$ is a deleted wreath product if
$\Gamma = (\Gamma_1 \wr \tilde{K}_m) - m\Gamma_1$, where $\Gamma_1$ is a Cayley graph of an abelian group of order $|G|/m$, and $m\Gamma_1$ is $m$ vertex-disjoint copies of $\Gamma_1$.

Conjecture

Let $\Gamma$ be a Cayley (di)graph of an abelian group $G$. Then one of the following is true:

- $\Gamma$ is a normal Cayley graph of $G$,
Definition
A Cayley graph $\Gamma$ of an abelian group $G$ is a deleted wreath product if $\Gamma = (\Gamma_1 \wr \bar{K}_m) - m\Gamma_1$, where $\Gamma_1$ is a Cayley graph of an abelian group of order $|G|/m$, and $m\Gamma_1$ is $m$ vertex-disjoint copies of $\Gamma_1$.

Conjecture
Let $\Gamma$ be a Cayley (di)graph of an abelian group $G$. Then one of the following is true:

- $\Gamma$ is a normal Cayley graph of $G$,
- $\Gamma$ is a semiwreath product, or
- the automorphism group of $\Gamma$ has a normal subgroup which is the same as the automorphism group of a deleted wreath product.

The preceding conjecture is known to be true if $G$ is cyclic, and is false for some nonabelian groups.
Definition

A Cayley graph $\Gamma$ of an abelian group $G$ is a **deleted wreath product** if

$$\Gamma = (\Gamma_1 \wr \bar{K}_m) - m\Gamma_1,$$

where $\Gamma_1$ is a Cayley graph of an abelian group of order $|G|/m$, and $m\Gamma_1$ is $m$ vertex-disjoint copies of $\Gamma_1$.

Conjecture

Let $\Gamma$ be a Cayley (di)graph of an abelian group $G$. Then one of the following is true:

- $\Gamma$ is a normal Cayley graph of $G$,
- $\Gamma$ is a semiwreath product, or
- the automorphism group of $\Gamma$ has a normal subgroup which is the same as the automorphism group of a deleted wreath product.

The preceding conjecture is known to be true if $G$ is cyclic, and is false for some nonabelian groups.
Definition

A Cayley graph $\Gamma$ of an abelian group $G$ is a **deleted wreath product** if

$$\Gamma = (\Gamma_1 \wr \overline{K}_m) - m\Gamma_1,$$

where $\Gamma_1$ is a Cayley graph of an abelian group of order $|G|/m$, and $m\Gamma_1$ is $m$ vertex-disjoint copies of $\Gamma_1$.

Conjecture

Let $\Gamma$ be a Cayley (di)graph of an abelian group $G$. Then one of the following is true:

- $\Gamma$ is a normal Cayley graph of $G$,
- $\Gamma$ is a semiwreath product, or
- the automorphism group of $\Gamma$ has a normal subgroup which is the same as the automorphism group of a deleted wreath product.

The preceding conjecture is known to be true if $G$ is cyclic,
Definition
A Cayley graph \( \Gamma \) of an abelian group \( G \) is a deleted wreath product if
\( \Gamma = (\Gamma_1 \wr \bar{K}_m) - m\Gamma_1 \), where \( \Gamma_1 \) is a Cayley graph of an abelian group of order \( |G|/m \), and \( m\Gamma_1 \) is \( m \) vertex-disjoint copies of \( \Gamma_1 \).

Conjecture
Let \( \Gamma \) be a Cayley (di)graph of an abelian group \( G \). Then one of the following is true:
- \( \Gamma \) is a normal Cayley graph of \( G \),
- \( \Gamma \) is a semiwreath product, or
- the automorphism group of \( \Gamma \) has a normal subgroup which is the same as the automorphism group of a deleted wreath product.

The preceding conjecture is known to be true if \( G \) is cyclic, and is false for some nonabelian groups.
Problem

For an abelian group $G$, does there exist a natural collection $\mathcal{F}$ of families of Cayley (di)graphs of $G$ and a partial order $\leq$ on $\mathcal{F}$ such that every Cayley (di)graph of $G$ is contained in some element of $\mathcal{F}$ and if $F_1 \leq F_2$ and there is no $F_3$ such that $F_1 \leq F_3 \leq F_2$, then almost every Cayley (di)graph of $G$ that is not in $F_1$ is in $F_2$?

Conjecture

Almost every vertex-transitive (di)graph is a Cayley (di)graph.
Problem
For an abelian group $G$, does there exist a natural collection $\mathcal{F}$ of families of Cayley (di)graphs of $G$ and a partial order $\preceq$ on $\mathcal{F}$ such that every Cayley (di)graph of $G$ is contained in some element of $\mathcal{F}$ and if $F_1 \preceq F_2$ and there is no $F_3$ such that $F_1 \preceq F_3 \preceq F_2$, then almost every Cayley (di)graph of $G$ that is not in $F_1$ is in $F_2$?

Conjecture
Almost every vertex-transitive (di)graph is a Cayley (di)graph.