Group actions

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s-arc transitive graphs

An s-arc in a graph is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $v_i \sim v_{i+1}$ and $v_{i-1} \neq v_{i+1}$.

A graph Γ is s-arc transitive if $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs.

K_4 is 2-arc transitive but not 3-arc transitive.

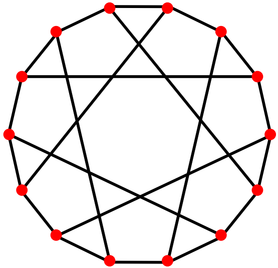
Some basic facts

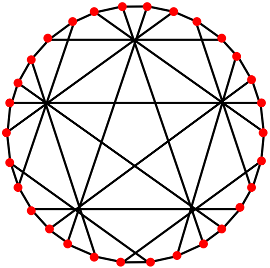
If all vertices have valency at least two then s-arc transitive implies $(s-1)$-arc transitive.

In particular, s-arc transitive $\implies$ arc-transitive $\implies$ vertex-transitive.

Examples

- Cycles are s-arc transitive for arbitrary s.
- Complete graphs are 2-arc transitive.
- Petersen graph is 3-arc transitive.
- Heawood graph (point-line incidence graph of Fano plane) is 4-arc transitive.
- Tutte-Coxeter graph (point-line incidence graph of the generalised quadrangle $\mathrm{W}(3,2)$) is 5-arc transitive.





Bounds on s

Tutte (1947,1959): For cubic graphs, $s \leq 5$.

Weiss (1981): For valency at least 3, $s \leq 7$.

Upper bound is met by the generalised hexagons associated with $G_2(q)$ for $q = 3^n$.

These are bipartite, with $2(q^5 + q^4 + q^3 + q^2 + q + 1)$ vertices and valency $q + 1$.

Locally s-arc transitive graphs

We say that Γ is locally s-arc transitive if for all vertices v, $\mathrm{Aut}(\Gamma)_v$ acts transitively on the set of s-arcs starting at v.

- If Γ is vertex-transitive then Γ is also s-arc transitive.
- s-arc transitive implies locally s-arc transitive.
- locally s-arc transitive implies locally $(s-1)$-arc transitive.
- locally s-arc transitive implies edge-transitve.
- In particular, if Γ is vertex-intransitive then Γ is bipartite.

Examples

- Γ bipartite and (G, s)-arc transitive implies Γ is locally (G^+, s)-arc transitive.
- Γ nonbipartite and (G, s)-arc transitive implies that the standard double cover of Γ is locally (G, s)-arc transitive.
- point-line incidence graph of a projective space is locally 2-arc transitive.

Bounds on s

Stellmacher (1996): $s \leq 9$

Bound attained by classical generalised octagons associated with ${}^2F_4(q)$ for $q = 2^n$, n odd.

These have valencies $\{2^n + 1, 2^{2n} + 1\}$.

Local action

Γ is locally $(G, 2)$-arc transitive if and only if G_v is 2-transitive on $\Gamma(v)$.

In particular, locally 2-arc transitive implies locally primitive.

Quotients of s-arc transitive graphs

The quotient of a 2-arc transitive graph is not necessarily 2-arc transitive.

Babai (1985): Every finite regular graph has a 2-arc transitive cover.

Take normal quotients instead.

Theorem (Praeger 1993)

Let Γ be a (G, s)-arc transitive graph and $N \lhd G$ with at least three orbits on vertices. Then Γ_N is $(G/N, s)$-arc transitive. Moreover, Γ is a cover of Γ_N.

The degenerate quotients are K_1 and K_2.

The basic (G, s)-arc transitive graphs to study are those for which all nontrivial normal subgroups of G have at most two orbits.

Quasiprimitive groups

A permutation group is quasiprimitive if every nontrivial normal subgroup is transitive.

Praeger (1993) proved an O'Nan-Scott Theorem for quasiprimitive groups which classifies them into 8 types.

Only 4 are possible for a 2-arc transitive group of automorphisms.

- Affine (HA): Ivanov-Praeger (1993) $\Longrightarrow$ 2^d vertices and all classified.
- Twisted Wreath (TW): Baddeley (1993)
- Product Action (PA): Li-Seress (2006+)
- Almost Simple (AS):

Li (2001): 3-arc transitive implies AS or PA.

Bipartite case

Let Γ be a bipartite graph with group G acting transitively on $V\Gamma$.

G has an index 2 subgroup G^+ which fixes the two halves setwise.

In particular, G cannot be quasiprimitive.

The basic graphs to study are those where every normal subgroup of G has at most two orbits, ie G is biquasiprimitive on vertices.

Structure theory of biquasiprimitive groups given by Praeger (2003).

Biquasiprimitive

When G is biquasiprimitive, G^+ may or may not be quasiprimitive on each orbit.

For example

- $G = T \operatorname{wr} S_2$
- acting on set of right cosets of $H = \{(h,h) \mid h \in L\}$ for $L < H$
- $G^+ = T \times T$.

Biquasiprimitive II

Infinite family of $(G, 2)$-arc transitive cubic graphs where G^+ is not quasiprimitive on each orbit recently given by Devillers-Giudici-Li-Praeger.

For these examples, G is not the full automorphism group.

Question

Are there examples where G is the full automorphism group?

Best to consider them as locally (G^+, s)-arc transitive (but remember the graphs are vertex-transitive).

Quotients of locally s-arc transitive graphs

Theorem (Giudici-Li-Praeger (2004))

- Γ a locally (G, s)-arc transitive graph,
- G has two orbits Δ_1, Δ_2 on vertices,
- $N \lhd G$.

1. If N intransitive on both Δ_1 and Δ_2 then Γ_N is locally $(G/N, s)$-arc transitive. Moreover, Γ is a cover of Γ_N.
2. If N transitive on Δ_1 and intransitive on Δ_2 then Γ_N is a star.

Basic graphs

The degenerate quotients are K_2 and $K_{1,r}$.

There are two types of basic locally (G, s)-arc transitive graphs:

(i) G acts faithfully and quasiprimitively on both Δ_1 and Δ_2.

(ii) G acts faithfully on both Δ_1 and Δ_2 and quasiprimitively on only Δ_1. (The star case)

Star case

There are only 5 possibilities for the type of G^{Δ_1}: HA, TW, AS, PA and HS.

In all cases $s \leq 3$.

- HS: all classified.
- PA: Γ is vertex-maximal clique graph of a Hamming graph
- AS: T is one of $\mathrm{PSL}(n, q)$, $\mathrm{PSU}(n, q)$, $\mathrm{P}\Omega^{+}(8, q)$, $E_6(q)$, ${}^2E_6(q)$.
- HA, TW:
  - Identify Δ_1 with regular minimal normal subgroup $N \cong T^k$.
  - Set of neighbours of 1_N is a collection of elementary abelian p-subgroups of N, and elements of Δ_2 are the cosets of these subgroups.

Giudici-Li-Praeger (2006)

Primitive on both sides

Two cases:

- the quasiprimitive types of G^{Δ_1} and G^{Δ_2} are the same and either HA, AS, TW or PA; or
- one is Simple Diagonal (SD) while the other is PA.

Primitive on both sides:

- HA: Iofinova-Ivanov (1993) $\Longrightarrow$ Γ is vertex-transitive. Classified by Ivanov-Praeger.
- PA, TW: only known examples are standard double covers of s-arc transitive graphs
- AS:

The $\{\mathrm{SD}, \mathrm{PA}\}$ case

All characterised by Giudici-Li-Praeger (2006-07).

Either $s \leq 3$ or the following locally 5-arc transitive example:

$\Gamma = \mathrm{Cos}(G; \{L, R\})$ with

- $G = \mathsf{PSL}(2, 2^m)^{2^m} \rtimes \mathsf{AGL}(1, 2^m)$, $m \geq 2$,
- $L = \{(t, \ldots, t) \mid t \in \mathsf{PSL}(2, 2^m)\} \times \mathsf{AGL}(1, 2^m)$,
- $R = (C_2^{2m} \rtimes C_{2^m-1}) \rtimes \mathsf{AGL}(1, 2^m)$

On the set of cosets of R, G preserves a partition into $(2^m + 1)^{2^m}$ parts.

- valencies $\{2^m + 1, 2^m\}$
- $G_v^{\Gamma(v)} = \mathsf{PSL}(2, 2^m)$, $G_w^{\Gamma(w)} = \mathsf{AGL}(1, 2^m)$