Permutation Groups

John Bamberg, Michael Giudici and Cheryl Praeger

Centre for the Mathematics of Symmetry and Computation

The University of Western Australia

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Outline

Notation

- Action of a group G on a set Ω

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- Kernel of the action: $G_{(\Omega)}$

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- Automorphism: Permutation of $V\Gamma$ preserving $E\Gamma$

$$\mathrm{Aut}(\Gamma)$$

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The *permutation group* induced ...

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Equivalent actions

Left and right coset actions

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Equivalent actions

Suppose G acts on Ω and Θ. Then the actions are equivalent if there is a bijection $\beta : \Omega \to \Theta$ such that

$$(\omega^g)\beta = (\omega)\beta^g$$

for all $\omega \in \Omega$ and $g \in G$.

Transitive actions

Recall that G acts transitively on Ω if for any two elements $\omega, \omega' \in \Omega$ there exists $g \in G$ such that

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Theorem (The Fundamental Theorem for Transitive Groups)

Let G act transitively on Ω and let $\alpha \in \Omega$. Then the action of G on Ω is equivalent to the right coset action of G on the right cosets of G_α.

$$\alpha^g \longleftrightarrow G_\alpha g$$

Regular group actions

- G acts transitively on Ω and $G_\alpha = \{1\}$.
- Identify Ω with G:

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Cayley graphs

Suppose G acts regularly on the vertices of Γ. Let $v \in V\Gamma$.

- $v^g \longleftrightarrow g \in G$.
- Let $S \subset G$ be the neighbours of 1.
- Gives rise to $\mathrm{Cay}(G, S)$

$$g_1 \sim g_2 \iff v^{g_1} \sim v^{g_2} \iff v^{g_1 g_2^{-1}} \sim v \iff g_1 g_2^{-1} \in S.$$

Arc-transitive graphs

Arc = directed, Edge = undirected

[1] i.e., an orbit of G on $\Omega \times \Omega$, such that $(\alpha, \beta) \in \mathcal{O} \implies (\beta, \alpha) \in \mathcal{O}$.

Arc-transitive graphs

Arc = directed, Edge = undirected

Lemma

Let Γ *be a connected graph, let* $G \leqslant \mathrm{Aut}(\Gamma)$ *and let* $v \in V\Gamma$*. Then* G *is transitive on arcs of* Γ *if and only if* G *is transitive on vertices and* G_v *is transitive on* $\Gamma(v)$*.*

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Orbital graph

Let G be a transitive permutation group on Ω. Let $\mathcal{O}$ be a nontrivial self-paired orbital[1]. Then the orbital graph $\mathrm{Orb}(\mathcal{O})$ has vertices Ω and edges defined by $\mathcal{O}$.

An orbital graph is always arc-transitive.

[1] i.e., an orbit of G on $\Omega \times \Omega$, such that $(\alpha, \beta) \in \mathcal{O} \implies (\beta, \alpha) \in \mathcal{O}$.

Lemma

Let G be a transitive permutation group on V.

1. *G is an arc-trans. grp. of auts. of a connected graph $\Gamma = (V, E)$ $\iff$ Γ is an orbital graph (for a self-paired orbital of G).*
2. *G is an edge-trans. but not an arc-trans. grp. of auts. of a connected graph $\Gamma = (V, E)$ $\iff$*

$$E = \{\{x, y\} \mid (x, y) \in \mathcal{O} \cup \mathcal{O}^*\},$$

where $\mathcal{O}$ is a nontrivial G-orbital in V with $\mathcal{O}^*$ as its paired orbital and $\mathcal{O} \neq \mathcal{O}^*$.

Blocks

Block

Nonempty subset $\Delta \subseteq \Omega$ such that for all $g \in G$,

$$\Delta^g = \Delta \quad \text{or} \quad \Delta^g \cap \Delta = \varnothing.$$

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Preservation

Intransitive action preserves a proper subset

Imprimitive action preserves a partition

G-invariant partitions, systems of blocks, imprimitivity

G-invariant partition

Suppose G acts transitively on Ω, and let $\mathcal{P}$ be a G-invariant partition of Ω.

- Each part of $\mathcal{P}$ is a block.

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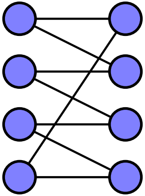
- Each part of $\mathcal{P}$ is a block.
- $\mathcal{P}$ is sometimes called a block system or system of imprimitivity.
- If $\mathcal{P}$ is nontrivial then G is imprimitive.
- Often consider the permutation group $G^{\mathcal{P}}$.

Bipartite graph Γ

Two bi-parts Δ_1 and Δ_2. If $G \leqslant \mathrm{Aut}(\Gamma)$ and G is transitive, then $\{\Delta_1, \Delta_2\}$ is a block system for G.



Primitive groups

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Transitive but not imprimitive.

Lemma

Suppose G is transitive on Ω, let $\omega \in \Omega$. Then there is a lattice isomorphism between

1. *the subgroups of G containing G_ω, and*
2. *the blocks of G containing ω.*

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Corollary

Let G act transitively on Ω and let $\alpha \in \Omega$. If $|\Omega| > 1$, then G is primitive on Ω if and only if G_α is a maximal subgroup of G.

Examples of primitive groups

G	G_α
S_n, $n \geqslant 2$	S_{n-1}
A_n, $n \geqslant 3$	A_{n-1}
k-transitive group, $k \geqslant 2$	$(k-1)$-transitive group
$\mathrm{AGL}(V)$	$\mathrm{GL}(V)$
$G \rtimes \mathrm{Aut}(G) \leqslant \mathrm{Sym}(G)$	$\mathrm{Aut}(G)$
prime degree	–
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A diagonal action

Let T be a group, let $G = T \times T$ and let $\Omega = T$. Then G acts on Ω via

$$t^{(x_1,x_2)} := x_1^{-1} t x_2.$$

- Faithful $\iff$ $Z(T) = \{1\}$.
- $G_1 = \{(t,t) | t \in T\}$.
- $N \lhd T \implies N$ is a block for G.

$$N^{(x_1,x_2)} = x_1^{-1} N x_2 = x_1^{-1} N x_1 x_1^{-1} x_2 = N x_1^{-1} x_2$$

A nice exercise

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Proof.

Already have "$\implies$". Suppose B is a block containing 1. Now

$$\begin{aligned} t \in B &\implies B^{(1,t)} = Bt \text{ and } t \in B \cap Bt \\ &\implies Bt = B \end{aligned}$$

Thus B is closed under multiplication.

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Also,

$$\begin{aligned} t \in B &\implies B^{(t,1)} = t^{-1}B \text{ and } 1 \in B \cap t^{-1}B \\ &\implies t^{-1}B = B \\ &\implies t^{-1} \in B \end{aligned}$$

Thus B is closed under inversion, and hence, $B \leqslant T$.

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Thus B is closed under inversion, and hence, $B \leqslant T$.
For $x \in T$, we have

$$B^{(x,x)} = x^{-1}Bx \text{ and } 1 \in B \cap x^{-1}Bx$$

so $B = x^{-1}Bx$. Therefore $B \trianglelefteq T$. □

Normal subgroups of primitive groups

Normal subgroup N of G

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- Suppose G is a primitive permutation group on a set Ω.
- $\implies$ the N-orbits form a block-system for G.
- $\implies$ each N-orbit is either a singleton or the whole of Ω.

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- $\implies$ $N = \{1\}$ or N is transitive.

So G is quasiprimitive (see later).

Lemma

Let G be a group acting transitively on Ω, let $N \trianglelefteq G$ and let $\alpha \in \Omega$. Then N is transitive if and only if $G = G_\alpha N$ $(= NG_\alpha)$.

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Proof.

$$
\begin{aligned}
N \text{ is transitive} &\iff (\forall g \in G)(\exists n \in N) \quad (\alpha^g)^n = \alpha \\
&\iff (\forall g \in G)(\exists n \in N) \quad gn \in G_\alpha \\
&\iff (\forall g \in G) \quad g \in G_\alpha N \\
&\iff G = G_\alpha N
\end{aligned}
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□

From all point-stabilisers

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Connected bipartite graph Γ

- Two bi-parts Δ_1 and Δ_2. If $G \leqslant \mathrm{Aut}(\Gamma)$ and G is transitive, then $\{\Delta_1, \Delta_2\}$ is a block system for G.

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- Two bi-parts Δ_1 and Δ_2. If $G \leqslant \mathrm{Aut}(\Gamma)$ and G is transitive, then $\{\Delta_1, \Delta_2\}$ is a block system for G.
- G^+ stabilises Δ_1 and Δ_2 set-wise, and $|G : G^+| = 2$.

Graphs from groups

- $\mathrm{Cay}(G, S)$
- $\mathrm{Cos}(G, H, HgH)$
- $\mathrm{Cos}(G; \{L, R\})$

Coset graphs

Suppose $H \leqslant G$ and $g \in G$ such that $g \notin H$ and $g^2 \in H$. Then we define the graph $\mathrm{Cos}(G, H, HgH)$ as follows:

Vertices Right cosets of H in G

Adjacency $Hx_1 \sim Hx_2 \iff x_1 x_2^{-1} \in HgH$

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Properties of $\mathrm{Cos}(G, H, HgH)$

- $g^2 \in H \implies g^{-1} \in HgH \implies \mathrm{Cos}(G, H, HgH)$ is undirected
- G acts transitively on the vertices by right multiplication.
- Cayley graph $\mathrm{Cay}(G, \{g\})$ when $H = \{1\}$.
- $\mathrm{Cos}(G, H, HgH)$ is connected $\iff \langle H, g\rangle = G$
- Valency $|H : (H \cap H^g)|$
- Arc-transitive.

Theorem

A vertex-transitive graph Γ is arc-transitive if and only if it is isomorphic to some coset graph $\mathrm{Cos}(G, H, HgH)$.

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Proof.

- Suppose G acts transitively on $V\Gamma$ and let $v \in V\Gamma$.
- Identify $V\Gamma$ with the right cosets of G_v.
- Suppose the arc (v, w) is mapped to (w, v) under $g \in G$.
- Adjacency relation is determined:

$$
\begin{aligned}
G_v x_1 \sim G_v x_2 &\longleftrightarrow v^{x_1} \sim v^{x_2} \\
&\longleftrightarrow x_1 x_2^{-1} \in G_v g G_v
\end{aligned}
$$

□

Coset "incidence geometry"

Let G be a group and let $L, R \leqslant G$. Define a graph $\mathrm{Cos}(G; \{L, R\})$ by

- **Vertices** Right cosets of L in G, and the right cosets of R in G.
- **Adjacency** $Lx \sim Ry \iff Lx \cap Ry \neq \varnothing$

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Properties of $\mathrm{Cos}(G, \{L, R\})$

- Bipartite
- Edge-transitive.

Theorem

If a bipartite graph Γ is edge-transitive but not vertex-transitive, then it is isomorphic to some coset incidence geometry $\mathrm{Cos}(G; \{L, R\})$.

Proof.

Exercise. □